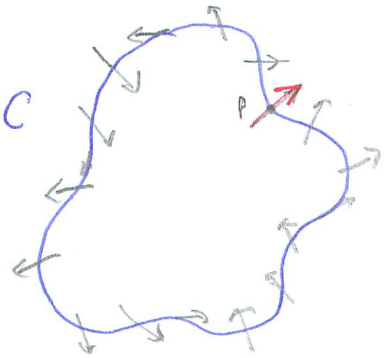
Back to 2D flows, we give a useful result which relates the existence of closed orbits and fixed points : Poincaré index of fixed points.



a simple closed curve in $\mathbb{R}^2$ that contains no fixed points of a vec. field.

following the directions (arrows) of the vec. field along C, thinking of their images on S^1 in $\mathbb{R}^2$, the image must have winded around the origin for k times counting orientation

$\begin{cases} k > 0 & \text{if counterclockwise} \\ k \leq 0 & \text{if anti ..} \end{cases}$

k is called the index of C.

$$\boxed{K = \frac{1}{2\pi}\int_C d\left(\arctan\frac{dy}{dx}\right) = \frac{1}{2\pi}\int_C \frac{f\,dg - g\,df}{f^2+g^2}}$$

If C is chosen to enclose a single isolated fixed point $\bar{x}$, then k is called the index of $\bar{x}$.

Prop. 1.8.4

(i) ind of a sink, a source or a center is +1.

(ii) ind of a hyperb. saddle point is −1.

(iii) ind of a closed orbit is +1.

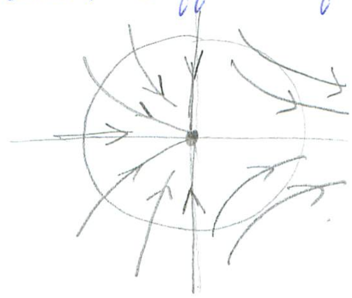
(iv) ind of a closed ~~orbit~~ curve not containg any fixed points is 0.

(v) ind of a closed curve $= \sum_{\bar{x} \in \mathrm{int}(C)}$ ind of fixed point $\bar{x}$.
C

Corollary 1.8.5 Inside any closed orbit γ there ~~is~~ must be at least one fixed point. If there is only one, then it must be a sink or a source. If all the fixed points within γ are hyperb, then there must be an odd number, $2n+1$, of which n are saddles and $n+1$ either sinks or sources.

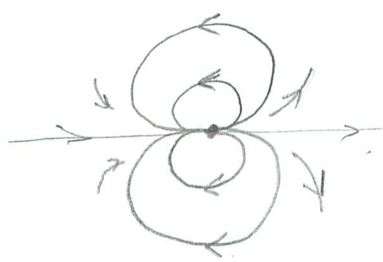
Example 1.8.6 Systems having degerate fixed points, i.e. fixed points with index different from ± 1.

$$\begin{cases} \dot{x} = x^2 \\ \dot{y} = -y \end{cases}$$



0

$$\begin{cases} \dot{x} = x^2 - y^2 \\ \dot{y} = 2xy \end{cases}$$



2.

1.9. Peixoto's theorem for 2D flows

Theorem 1.9.1 (Andronov-Pontryagin) A vector field in $\mathbb{R}^2$ is structurally stable, if and only if:

(i) all its fixed points are hyperbolic; and

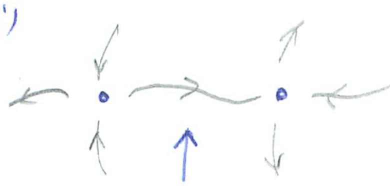
(ii) all its periodic orbits are hyperbolic; and

(iii) there are no trajectories connecting saddles to saddles.

Pf: " $\Rightarrow$ " easier direction.

(i), (ii) ✓ vector field with nonhyperb. fixed points or per. orbits can not be s. stable.

(iii)



perturbation - sensitive

" $\Leftarrow$ " Recall the Poincaré-Bendixson theorem:

A nonempty compact ω- or α-limit set of a planar flow, which contains no fixed points, is a closed orbit.

Thus, an orbit of $x \in B_R$ (some disc of radius R):

- either leaves B_R in finite time
- or converges to a fixed point or a per. orbit inside B_R

By continuous dependence of initial value, this behavior remains the same, if x is perturbed slightly.

Theorem 1.9.2 (Peixoto) A C^r-vector field on a compact 2dim.l manifold M^2 is s. stable, if and only if

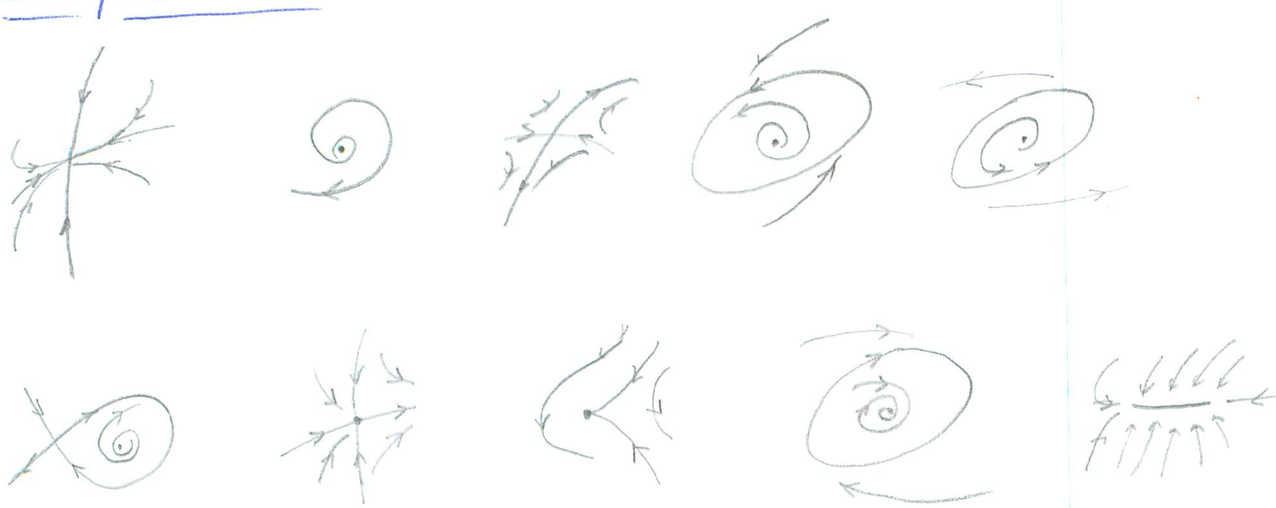
(i)–(iii) of Thm 1.9.1. hold and

(iv) the nonwandering set consists of fixed points and per. orbits alone.

Moreover, if M^2 is orientable, the set of s. stable vec. fields is open and dense in $\mathfrak{X}^r(M^2) =:$ the set of all C^r vec. fields.

Pf: (sketch) (iv) is to exclude things like the irrational flow on T^2. #

Examples 1.9.3



Def. 1.9.4 A Morse-Smale system is a system for which:

(i) the number of fixed points and per. orbits is finite and each is hyperbolic;

(ii) all stable and unstable manifolds intersect transversally

(iii) the non-wandering set consists of fixed points and per. orbits alo

Remark 1.9.5 (i) Morse-Smale $\Rightarrow$ s. stable, $\not\Leftarrow$

(ii) Morse-Smale systems are not dense in $\mathfrak{X}^r(M^m)$ in general, if $m > 2$.
S. stable " are not dense in $\mathfrak{X}^r(M^m)$

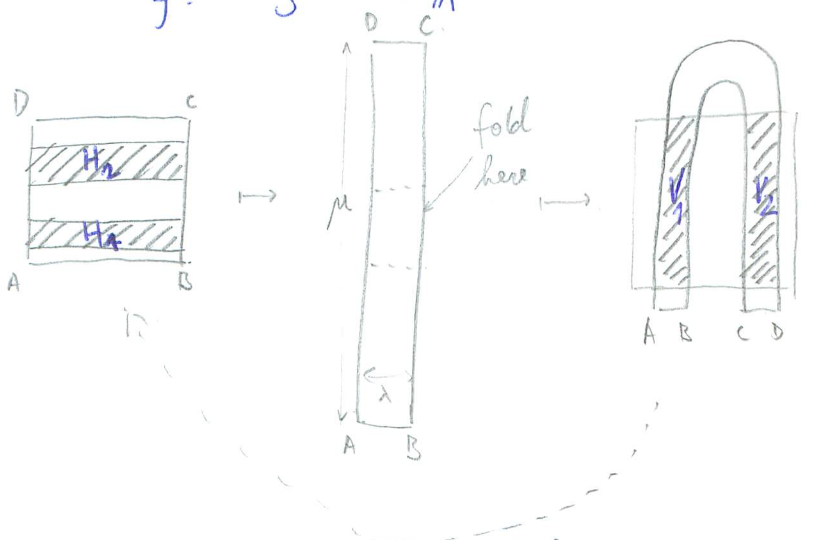
Example 1.9.6. (The Smale Horseshoe)

Let $S = [0,1]\times[0,1]$ be the unit square in $\mathbb{R}^2$. Define

$$f : S \to \mathbb{R}^2$$

$0 < \lambda < \frac{1}{2}$

$\mu > 2$.



$f^{-1}(S \cap f(S))$

$$f : f^{-1}(S\cap f(S)) \to S \cap f(S)$$

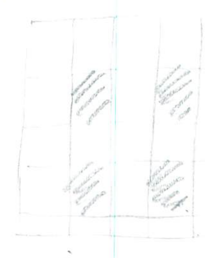
Jacobian of $f|_{H_1} : H_1 \to V_1$ is $\begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}$

" " $f|_{H_2} : H_2 \to V_2$ is $\begin{pmatrix} -\lambda & 0 \\ 0 & -\mu \end{pmatrix}$

Consider the iteration of f: $\{f^i : -\infty < i < \infty\}$ and those points that do not leave S for all time:

$$\Lambda = \{x : f^i(x) \in S \ \forall_{-\infty < i < \infty}\}$$

e.g.



$\Lambda^1 = \{x : f^i(x) \in S : i = \pm 1, 0\}$.