1.7. Equivalence relations and structural stability

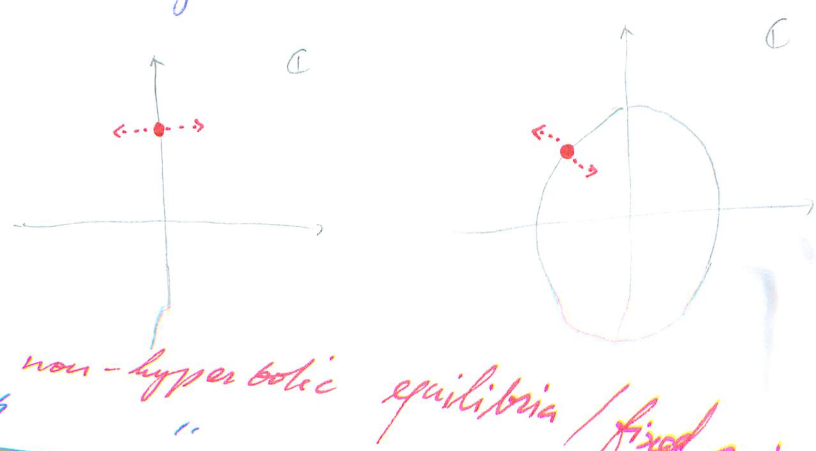
Def. 1.7.1 If $f \in C^r(\mathbb{R}^n)$, $r, k \in \mathbb{Z}_+$, $k \le r$, and $\varepsilon > 0$, then g is a C^k perturbation of size ε if there is a compact set $K \subset \mathbb{R}^n$ s.t. $f = g$ on the set $\mathbb{R}^n \setminus K$ and for all $(i_1, \dots, i_n)$ with $i_1 + \dots + i_n =: i \le k$, we have $\left| \frac{\partial^i}{\partial x_1^{i_1} \dots \partial x_n^{i_n}} (f - g) \right| < \varepsilon$.

Remark: in this definition f, g can be either vec. fields or maps.

Recall def. of topo. equivalence, topo conjugacy, C^k-conjugacy.

Def. 1.7.4. A C^r-vec. field f (resp. a map $F \in C^r(\mathbb{R}^n)$) is structurally stable, if there is an $\varepsilon > 0$ s.t. all C^1, ε perturbations of f (resp. of F) are topo. equivalent to f (resp. to F).

We have shown in HW 2.1 that
"every vec. field is locally around hyperbolic equilibrium structurally stable." $\sigma(Df(x_0)) \cap i\mathbb{R} = \phi$ Similary one shows that
"every map is locally around hyperbolic fixed point structurally stable." $\sigma(DF(\xi_0)) \cap S^1 = \phi$



Systems with non-hyperbolic equilibria / fixed points
"
"
"

(Hw) a C^k-equivalence for $k \geq 1$ (insted of C^0-equivalence) would be too strict for Def. 1.7.4.

$\dot{\xi} = Df(p)\xi$ and $\dot{\eta} = Dg(q)\eta$, eigenvalues must be in the same ratios.

1.8. 2-D flows (planary flows in $\mathbb{R}^2$, and some flows on M^2).

Consider a flow in $\mathbb{R}^2$ generated by

$$(*) \quad \begin{cases} \dot{x} = f(x,y) \\ \dot{y} = g(x,y) \end{cases} \qquad (x,y) \in U \subseteq \mathbb{R}^2, \text{ wh. } f, g \text{ suff. smooth.}$$

We usually do the following:

(I) find equilibria of $(*)$ by $f(\bar{x},\bar{y}) = g(\bar{x},\bar{y}) = 0$

(II) analyse stability of $(\bar{x},\bar{y})$ by $L_{(\bar{x},\bar{y})} := \left.\begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix}\right|_{(\bar{x},\bar{y})}$:

if $\sigma(L) \cap i\mathbb{R} = \phi$, then hyperbolic (sink, source or saddle)

otherwise perhaps using Liapunov functions.

(III) find per. orbits of $(*)$:

Thm 1.8.1 (Poincaré-Bendixson) A nonempty compact ω- or α-limit set of a planar flow, which contains no fixed points, is a closed orbit.

Thm 1.8.2 (Bendixson's criterion) If on a simply connected region $D \subseteq \mathbb{R}^2$ the expression $\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y}$ is not identically zero and does not change sign, then $(*)$ has no closed orbits lying entirely in D.

pf: (Green's theorem) for any solution curve of $(*)$, we have

$$\frac{dy}{dx} = \frac{g}{f} \quad \text{or} \quad \int_\gamma (f(x,y)\,dy - g(x,y)\,dx) = 0 \quad \forall_{\text{closed orbit } \gamma}.$$

$\xRightarrow[\text{thm}]{\text{Green's}}$ (**) $\displaystyle\iint_S \left(\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} \right) dx\,dy = 0$ wh. S is interior of γ.

But if $\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} > 0$ or < 0 on entire D, then $\nexists\, S \subseteq D$ s.t. (**) holds. $\boxed{↯}$

(IV) stability of per. orbits using Poincaré maps or Floquet theory.

(V) for planar flows, all the possible nonwandering sets fall into:

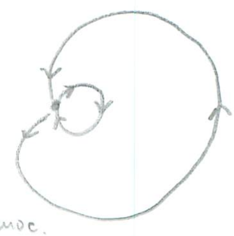
(i) equilibria / fixed points

(ii) per. orbits / closed orbits

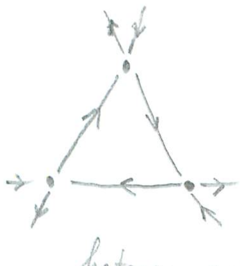
(iii) the unions of fixed points and trajectories connecting them.
(e.g. heteroclinic orbits, homoclinic orbits, heteroclinic cycles)

heteroc.

homoc.



double homoc.

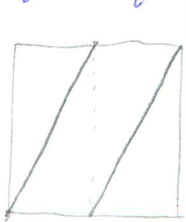


heteroc. cycles

bands of per. orbits

(VI) in non planary systems, other types of nonwandering sets can arise.

$$\begin{cases} \dot\theta = 1 \\ \dot\phi = \pi \end{cases} \qquad (\theta, \phi) \in T^2 \in \mathbb{R}^2/\mathbb{Z}^2$$

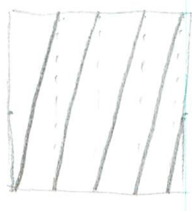


2-per

3-per

... $\mathbb{Q}$-angle

for irrational flows on T^2, every point is nonwandering.



non-per.

$\mathbb{Q}^c$-angle.

In general, it is difficult to show the existence of such hetero-, homo-clinic orbits or cycles, but systems with special structure or symmetries it may be easy to find.

Consider

$$(W) \quad \begin{cases} \dot{x} = -\zeta x - \lambda y + xy \\ \dot{y} = \lambda x - \zeta y + \frac{1}{2}(x^2 - y^2) \end{cases},$$

which arose in wind induced oscillation studies (Holmes, 1979) where $0 \le \zeta \ll 1$ is a damping factor, and $|\lambda| \ll 1$ is a detuning parameter.

- when $\zeta = 0$, (W) becomes a Hamiltonian system:

<u>Def.</u> $H: \mathbb{R}^{2n} \to \mathbb{R}$ is a smooth function, let $J: \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ be $\begin{pmatrix} 0 & \mathbb{1}_n \\ -\mathbb{1}_n & 0 \end{pmatrix}$, then $\dot{x} = J \cdot \nabla H(x)$ is called an <u>Hamiltonian</u> system.

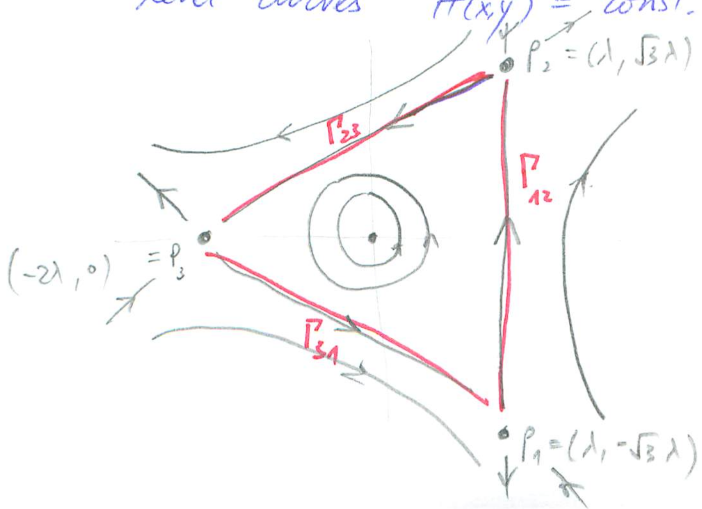
$$(W) \text{ with } \zeta = 0 \rightsquigarrow \begin{cases} \dot{x} = \dfrac{\partial H}{\partial y} \\ \dot{y} = -\dfrac{\partial H}{\partial x} \end{cases} \quad \text{where } H = H(x,y) = -\frac{\lambda}{2}(x^2+y^2) + \frac{1}{2}\left(xy^2 - \frac{x^3}{3}\right).$$

$\Rightarrow$ critical points of H = fixed points of flow of $(W)_{\zeta=0}$. $P_1, P_2, P_3, 0$

also, we have

$$\frac{\partial H}{\partial t} = \frac{\partial H}{\partial x}\dot{x} + \frac{\partial H}{\partial y}\dot{y} = \frac{\partial H}{\partial x}\frac{\partial H}{\partial y} - \frac{\partial H}{\partial y}\frac{\partial H}{\partial x} \equiv 0$$

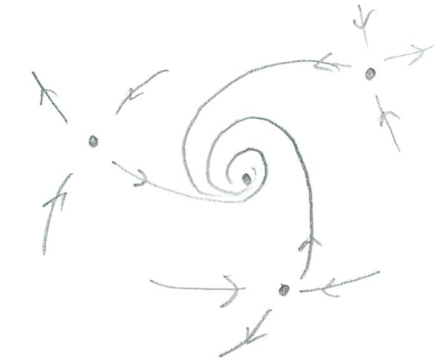
$\Rightarrow$ level curves $H(x,y) = \text{const.}$ are solution curves for $(W)_{\zeta=0}$.



$$\Gamma_{ij} = W^u(p_i) \cap W^s(p_j)$$

$$\text{heteroc. cycle} = \Gamma_{12} \cup \Gamma_{23} \cup \Gamma_{31}$$

• when $\xi > 0$



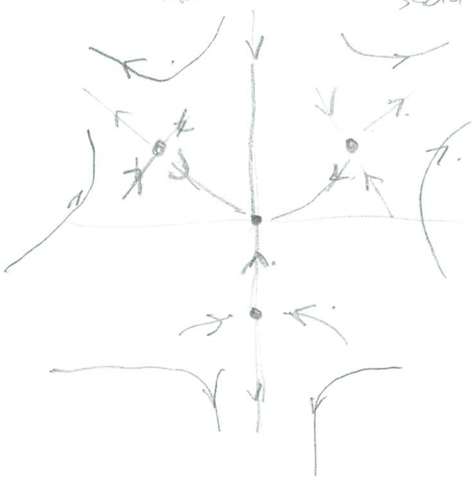
• when $\lambda = 0$, (W) becomes a gradient system:

$$(W)_{\lambda=0} \rightsquigarrow \begin{cases} \dot{x} = -\dfrac{\partial V}{\partial x} \\ \dot{y} = -\dfrac{\partial V}{\partial y} \end{cases} \quad \text{wh. } V = V(x,y) = \frac{\xi}{2}(x^2+y^2) + \frac{1}{2}\left(\frac{y^3}{3} - x^2 y\right)$$

$\Rightarrow$ critical points of V = fixed points of $(W)_{\lambda=0}$.

$(0,0)$, $(0,-2\xi)$, $(\pm\sqrt{3}\,\xi, \xi)$

sink saddle saddle



In general,

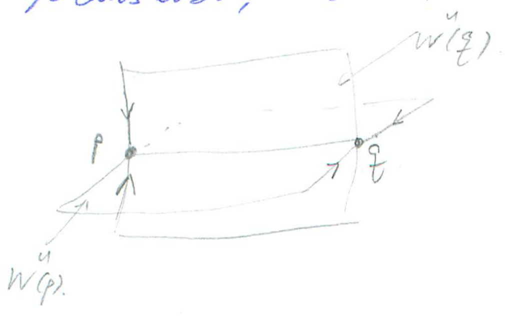
$$\dot{x} = -\operatorname{grad} V(x)$$

for general potential function $V : \mathbb{R}^n \to \mathbb{R}$.

$\forall_{x \in V^{-1}(h)}$ at which $\operatorname{grad} V(x) \neq 0$,

the vector $-\operatorname{grad} V(x)$ points at the steepest decrease in value of V.

<u>Thm 1.8.3</u> Gradient systems for which all fixed points are hyperbolic and all intersections of stable and unstable manifolds transversal, are structually stable.



$W^u(p) \pitchfork W^s(q)$. if $\langle T_x W^u(p), T_x W^s(q) \rangle = \mathbb{R}^n$.