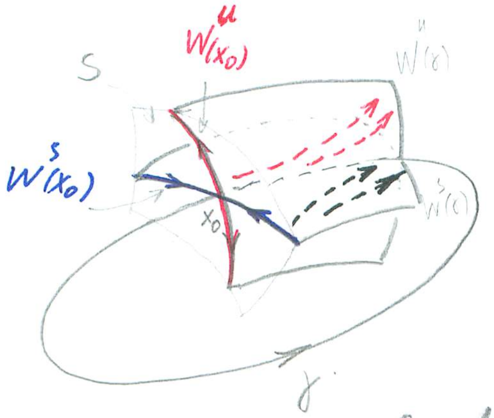


$$P: S \to S, \quad P(x_0) = x_0$$
$$x \mapsto P(x)$$

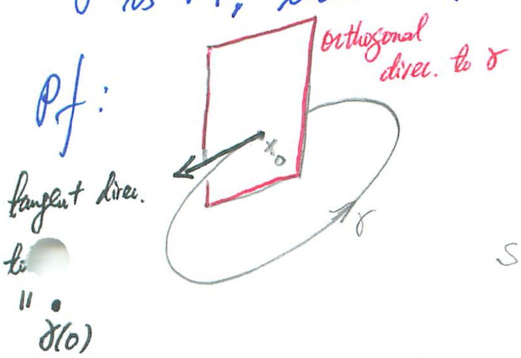
$$\dim W^s(x_0) = \dim W^u(x_0) = 1 \quad \left(\text{for discret system } x_{n+1} = P(x_n)\right)$$

for γ of the conts flow from $\dot{x} = f(x)$ $\rightsquigarrow$ $\begin{cases} \dim W^s(\gamma) = \dim W^s(x_0) + 1 \\ \dim W^u(\gamma) = \dim W^u(x_0) + 1 \end{cases}$

Theorem 1.5.5 (connection between linear stability of γ using Floque multipliers and using Poincaré maps)

Let γ be a per. orbit of a C^1 flow φ, S be a local section at $x_0 \in \gamma$, and $P: S \to S$ be the Poincaré map. If the monodromy matrix of γ is M, then $\sigma(M) = \sigma(DP(x_0)) \cup \{1\}$.

Pf:



$$M \underset{\substack{\text{w.r.t. eigenbasis} \\ \{v_1, v_2, \dots, v_n\} \\ \mu_1, \mu_2, \dots, \mu_n}}{=\!=\!=} \begin{pmatrix} M|_S & 0 \\ * & 1 \end{pmatrix}$$

since $M \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$

$S \leftarrow v_1, \dots, v_{n-1}$; $v_n \to \dot{\gamma}(0)$; $\mu_n = 1$

It suffices to show $DP(x_0) = M|_S$.

Define for x in a nbhd of γ, $Q(x) = \varphi_{\tau(x)}(x)$, where $\tau(x) > 0$ is the first return time to S.

Then $P = Q|_S \Rightarrow DP(x) = DQ(x)|_S$ for $x \in S$.

$$Q(x) = \varphi_{\tau(x)}(x) \xrightarrow[\text{w.r.t. } x]{\text{diff}} DQ(x) = D_x \varphi_{\tau(x)}(x) + \left(\frac{d}{dt} \varphi_{\tau(x)}(x)\right) \left(D_x \tau(x)\right)^T$$

$$\xrightarrow[x=x_0]{\text{eval. at}} \quad DQ(x_0) = \underbrace{D_x\varphi_T(x_0)}_{M} + \underbrace{\left(\frac{d}{dt}\varphi_T(x_0)\right)}_{f(x_0)} \left(D_x\tau(x_0)\right)^T$$

$$\Rightarrow \quad DQ(x_0) = M + \underbrace{f(x_0)}\left(D_x\tau(x_0)\right)^T$$

perpendicular to S $\Rightarrow$ same direction as $\dot{\gamma}(0)$ $\Rightarrow f(x_0) = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ f_n(x_0) \end{pmatrix}$

$$= \begin{pmatrix} M|_S & 0 \\ * & 1 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ f_n(x_0) \end{pmatrix} \left(\frac{\partial\tau}{\partial x_1}(x_0), \dots, \frac{\partial\tau}{\partial x_n}(x_0) \right)$$

$$= \begin{pmatrix} M|_S & 0 \\ * & 1 \end{pmatrix} + \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 0 \\ f_n(x_0)\frac{\partial\tau}{\partial x_1}(x_0) & \cdots & f_n(x_0)\frac{\partial\tau}{\partial x_n}(x_0) \end{pmatrix}$$

$$\Longrightarrow \quad DQ(x_0)\big|_S = M\big|_S \quad \Rightarrow \quad DP(x_0) = M\big|_S$$

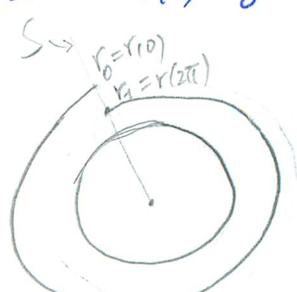
#

<u>Thm 1.5.6</u> If γ is a per. orbit of a C^2 flow which is linearly asymp. stable ($\sigma(DP(x_0)) \subset \{z \in \mathbb{C} : |z| < 1\}$), then it is asymptotically stable.

<u>Ex. 1.5.7</u> (cf. Ex. 1.5.3) $\dot{r} = ar(1-r)$, $\dot{\theta} = 1$ in polar coord.

$$\xrightarrow[\text{of variables}]{\text{separation}} \quad r(t) = \frac{r_0}{r_0 + (1-r_0)e^{-at}}$$

for ini. cond. $r(0) = r_0$



$$\xrightarrow{T=2\pi} \quad r_1 = \frac{r_0}{r_0 + (1-r_0)e^{-2\pi a}}$$

Thm 1.5.9 (Stability of orbits using Poincaré map)

Suppose γ is a per. orbit of $\dot{x} = f(x)$, for a Lipschitz conts map $f: U \to \mathbb{R}^n$. Let $x_0 \in \gamma$ and $P: S \to S$ be a Poincaré map on the section S around x_0. Then,

(i) γ is stable $\Longleftrightarrow$ $\forall_{\varepsilon > 0}$ $\exists_{\delta > 0}$ s.d. whenever $x \in S$ with $|x - x_0| < \delta$, we have $P^n(x)$ is defined for $n \in \mathbb{N}$ and $|P^n(x) - x_0| < \varepsilon$ $\forall_{n \in \mathbb{N}}$.

i.e. x_0 is a stable fixed point of P.

(ii) γ is asymptotically stable $\Longleftrightarrow$ γ is stable and $\exists$ an open nbhd U_1 of x_0 in S s.t. whenever $x \in U_1$, we have $P^n(x)$ is defined for $n \in \mathbb{N}$ and $P^n(x) \to x_0$ als $n \to \infty$.

i.e. x_0 is an asym. stable fixed point of P.

(iii) γ is unstable, $\Longleftrightarrow$ $\exists$ $x \in S$ s.d. $P^{-n}(x)$ is defined for all $n \in \mathbb{N}$, and $P^{-n}(x) \to x_0$ als $n \to \infty$.

pf: "$\Longrightarrow$" easier direction (Hw)

"$\Longleftarrow$" conts. dependence of initial conditions (cf. Thm 1.0.4) and compactness of γ.

Thm. 1.5.10 γ is asymp. stable $\iff$

γ is stable and asymp. linearly stable.

Pf. (Hw)

? γ is stable $\iff$ γ is linearly stable ?

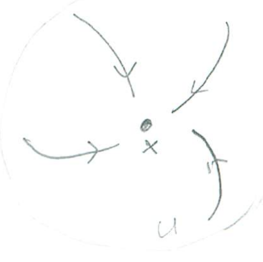
1.6. Asymptotic behavior

Def 1.6.1 Recall that an *ω-limit set* of x is defined by

$$\omega(x) = \bigcap_{t\geq 0} \overline{\mathcal{O}^+(\varphi(x,t))} = \{ \underset{\substack{\uparrow \\ \text{a limit point of } \mathcal{O}^+(x)}}{y} \mid \exists \{t_n\}_{n=1}^{\infty} \subset \mathbb{R} \text{ s.t. } t_n \to \infty \text{ as } n\to\infty \text{ and } y = \lim_{n\to\infty} \varphi(x,t_n) \}$$

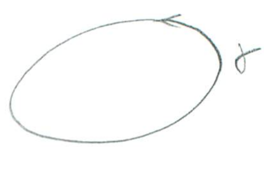
Define an *α-limit set* of x to be

$$\alpha(x) = \bigcap_{t\leq 0} \overline{\mathcal{O}^-(\varphi(x,t))} = \{ \underset{\substack{\uparrow \\ \text{a limit point of } \mathcal{O}^-(x)}}{y} \mid \exists \{t_n\}_{n=1}^{\infty} \subset \mathbb{R} \text{ s.t. } t_n \to -\infty \text{ and } y = \lim_{n\to\infty} \varphi(x,t_n) \}$$



$\omega(y) = \{x\}$ $\forall_{y\in U}$

$\alpha(y) = \{x\}$ $\forall_{y\in U}$



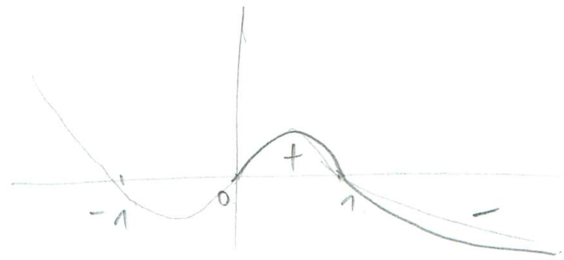
$\omega(x) = \alpha(x) = \gamma$ $\forall_{x\in\gamma}$

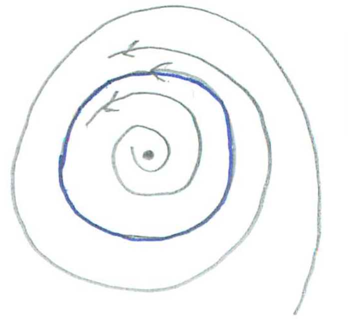
A *limit cycle* γ is a per. orbit that is the α- or ω-limit set of some point $x \notin \gamma$.

Example 1.6.2. $\begin{cases} \dot{x} = x(1-x^2-y^2)-y \\ \dot{y} = y(1-x^2-y^2)+x \end{cases}$

Polar coordinates: $r = x^2+y^2$, $\tan\theta = \frac{y}{x}$ $\longrightarrow$

$\begin{cases} \dot{r} = r(1-r^2) \\ \dot{\theta} = 1 \end{cases}$





$\gamma = \{ r = 1 \}$ is the ω-limit set of every point in $\mathbb{R}^2 - \{0\}$.

$\diamondsuit$

A point x is nonwandering if

$\forall_{\text{o. nbhd } W \text{ of } x} \ \forall_{T>0} \ \exists_{t>T}$ s.t. $\varphi_t(W) \cap W \neq \emptyset$

Examples Equilibria, per. Orbits, heteroclinic orbits, homoclinic orbits, irrational flow on Torus T^2.

A set S is minimal, if it is closed, nonempty, invariant and does not contain any such set as a proper subset.

Thm 1.6.3 Suppose S to be a compact set. Then,

S is minimal $\Leftrightarrow$ $S = \omega(x) \ \forall_{x \in S}$.

(Hw).

Stability of periodic orbits

Def 1.5.6 A compact invariant set K is stabil, if

$\forall$ open nbhd U of K $\exists$ an open nbhd V of K s.t. $O^+(x) \subset U$ for all $x \in V$

$$O^+(x) = \{\varphi(x,t) \mid t \geq 0\} \quad \text{positive orbit}$$

Def 1.5.7 A compact invariant set K is attractive, if

$\exists$ an open nbhd U of K s.t. $\omega(x) \subset K$ for all $x \in U$

$$\omega(x) = \bigcap_{t \geq 0} \overline{O^+(\varphi(x,t))} \quad \omega\text{-limit set}$$

$$\overset{(Hw)}{=} \{y \mid \exists \{t_n\}_{n=1}^{\infty} \subset \mathbb{R} \text{ s.t. } \lim_{n\to\infty} t_n = \infty \wedge y = \lim_{n\to\infty} \varphi(x,t_n)\}$$

? is a stable set always attractive or/and

is an attractive set always stable ?

Observation: a periodic orbit is a compact invariant set.

Def 1.5.8 A closed orbit (a per. orbit) is stabil, if it is stable as a compact inv. set (cf. Def 1.5.6); a closed orbit is asymptotically stable, if it is stable and attractive as a compact inv. set (cf. Def 1.5.7); it is unstabil, if it is not a stabil orbit.

Attractors and Basins

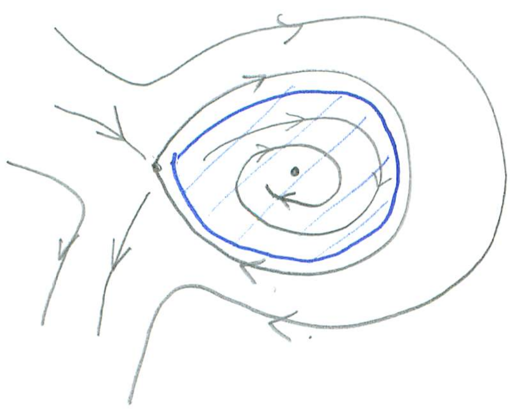
Def. 1.6.4 A compact invariant set K is asymp. stable, if it is stable and attractive (cf. Def. 1.5.6-1.5.7)

The limit cycle in Example 1.6.2 is asymp. stable.

A set $N \neq \emptyset$ is a trapping region of the flow φ_t, if

N is compact and $\varphi_t(N) \subset \mathrm{int}(N)$ for $t > 0$.

Example. N is trapping region if the vector field of φ_t points inwards everywhere on the boundary of N.



A set Δ is an attracting set, if there is a trapping region N containing Δ s.t. $\Delta = \bigcap_{t \geq 0} \varphi_t(N)$

(HW) attracting set is compact and nonempty.

Observation: $\Lambda = \bigcap_{t \geq 0} \varphi_t(N) \Rightarrow$

$\forall_{x \in N}$ we have $\varphi_t(x) \to \Lambda$ as $t \to \infty$, $t > 0$.

⌜ Otherwise, $\exists x_0 \in N$ s.t. $\text{dist}(\varphi_t(x_0), \Lambda) > \varepsilon$ $\forall_{t>0}$, ↯

$\varphi_t(x_0) \xrightarrow{\text{subseq}} x^*$: cpt of N.

$\Rightarrow x^* = \varphi_t(x^*)$ $\forall_{t>0}$.

$\Rightarrow x^* \in \bigcap_{t\geq 0} \varphi_t(N) = \Lambda$ ↯ $\text{dist}(x^*, \Lambda) \geq \varepsilon$ ⌟

Lemma 1.6.7. (i) An attracting set is asymp. stable;

(ii) every compact, asymp. stable set contains an attracting set.

Pf: (i) let Λ be an attracting set with trapp. region N.

- Λ is compact and invariant ✓ $\Lambda = \bigcap_{t\geq 0} \varphi_t(N)$.
- Λ is stable.

  if not, $\exists\, U$ of Λ $\forall\, V$ of Λ $\exists x \in V$, $t > 0$ s.t. $\varphi_t(x) \notin U$.

  let $V_0 = \text{int}(N)$, $V_1 = \varphi_1(\text{int}(N))$, ... $V_k = \varphi_k(\text{int}(N))$

  $\exists$ $x_0, x_1, \ldots, x_k$; $t_0, t_1, \ldots, t_k$ s.t. $\varphi_{t_k}(x_k) \notin U$ $\forall_k$.

  but $x_k \in \underset{\subset N}{\text{int}(N)}$ $\forall_k$, $\Rightarrow \varphi_t(x_k) \xrightarrow[t\to\infty]{} \Lambda$ $\forall_k$ ↯

- Λ is attractive.

  take $U = \text{int}(N)$, then $\omega(x) \subset \Lambda$ $\forall_{x \in U}$.

#(i)

(ii) let A be compact, asymp stable.

$\Rightarrow \quad \forall_{x \in \bar{B}_\delta(A)} \quad \exists_{T(x)} \quad$ s.t. $\varphi_t(x) \in \bar{B}_\delta(A) \quad \forall_{t \geq T(x)}$.

Let $T_{max} = \max_{x \in \bar{B}_\delta(A)} T(x)$. Then $N = \varphi_{T_{max}}(\bar{B}_\delta(A))$ is a trapping region. #(ii)

The <u>basin of attraction</u> or <u>stable set</u> $W^s(\Lambda)$ of an inv. set Λ is the set of all points x for which $\varphi_t(x) \to \Lambda$ as $t \to \infty$.

If Λ is an attracting set with trapping region N, then $W^s(\Lambda) = \bigcup_{t \leq 0} \varphi_t(N)$.

A set Λ is an <u>attractor</u> if it is an attracting set and $\exists\, x$ s.t. $\Lambda = \omega(x)$.