Supplementary: on <u>topo. conjugacy</u> and <u>topo. equivalence</u>

- two flows $\varphi_t : A \to A$ and $\psi_t : B \to B$ are <u>topo. conjugate</u>, if
  (resp. diffeomorphic)

  $\exists$ homeom. $h : A \to B$ s.t. $h \circ \varphi_t \circ h^{-1} = \psi_t$,
  (resp. diffeom.)

  i.e. $h(\varphi_t(x)) = \psi_t(h(x)) \quad \forall x \in A.$

<u>Ex 1.</u> two linear flows of $\dot{x} = -x$ and $\dot{y} = -2y$ are topo. conjugate.

$\varphi_t(x) = x e^{-t}$, $\psi_t(y) = y e^{-2t}$, needs to find $h : \mathbb{R} \to \mathbb{R}$ s.t.

$$(*) \qquad h(x e^{-t}) = h(x) e^{-2t} .$$

let $h(x) = \begin{cases} x^2 & x \geq 0 \\ -x^2 & x \leq 0 \end{cases}$ then check $(*)$ is satisfied.

#

([cf. Aufgabe 2.1 (iii) Hinweis] in general, we have :

two linear flows $\varphi_t(x) = e^{tA} \cdot x$ and $\psi_t(y) = e^{tB} \cdot y$ with A, B hyperbolic,

are topo. conjugate iff $\dim E^s(A) = \dim E^s(B)$ and $\dim E^u(A) = \dim E^u(B)$.)

diffeom: $\varphi_t(x) = e^{tA} \cdot x$ and $\psi_t(y) = e^{tB} \cdot y$ are diffeom. $\Leftrightarrow$ A is similar to B. (HW)

A more relaxed concept to describe "equivalency" between flows is:

- two flows $\varphi_t : A \to A$ and $\psi_t : B \to B$ are <u>topo. equivalent</u>, if

  $\exists$ homeom. $h : A \to B$ s.t. h maps orbits of φ onto orbits of ψ

  and preserves direction of time.

$\exists$ an increasing map $\tau : A \times \mathbb{R} \to \mathbb{R}$ s.t. $h(\varphi_{\tau(x,t)}(x)) = \psi_t(h(x)) \quad \forall x \in A$
on $\mathbb{R}$

Ex 2. on $\mathbb{R}^+$ the flow of $\dot{x} = -x$ and the flow of $\dot{y} = -y^2$ are topo. equiv.

$\varphi_t(x) = x e^{-t}$, $\psi_t(y) = \frac{y}{1+ty}$. to find $h: \mathbb{R}^+ \to \mathbb{R}^+$ s.t.

(*) $h(x e^{-\tau(x,t)}) = \frac{h(x)}{1 + t h(x)}$ for some increasing $\tau : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$

try $h(x) = x$ and solve $x e^{-\tau} = \frac{x}{1+tx}$, then

$\tau(x,t) = \ln(1+tx)$, which is increasing in t, for fixed $x \in \mathbb{R}^+$.

Theorem (1D-equivalence) two flows φ and ψ in $\mathbb{R}$ are topo. equiv. iff their equilibria, ordered on the line, can be put into one-to-one correspondence and have the same topo. type (sink, source or semistable).

Ex 3. (corollary) the flows of Ex. 2 are NOT topo. equivalent on $\mathbb{R}$.

1.5. Closed orbits, Floquet multipliers and Poincaré maps

Recall: for nonlr. system $\dot{x} = f(x)$, qualitative dynamics around a (hyperbolic) fixed point can be described by Thm 1.3.1 (Hartman-Grobman) – Thm 1.3.2 (stable mnfd). We want to achieve the same for periodic orbits (also called closed orbits)

Consider $\dot{x} = f(x)$, $f \in C^1$. Assume that $\gamma = \gamma(t)$ is a T-per. orbit, i.e. $\gamma(t) = \gamma(t+T)$ $\forall_{t \in \mathbb{R}}$, for $\dot{x} = f(x)$. Then,

$$\dot{\gamma}(t) = f(\gamma(t)).$$

[Floquet theory]

Stability of per. orbits let $x(t) = \gamma(t) + y(t)$ and expand $\dot{x}$ about the orbit γ:

deviation

$$\dot{x} = \dot{\gamma} + \dot{y} = f(x) = f(\gamma + y) = f(\gamma) + Df(\gamma)y + o(y)$$

$$\rightsquigarrow \quad \dot{y} = Df(\gamma)y + o(y(t))$$

linearize

$$\rightsquigarrow \quad \dot{y} = A(t)y, \text{ where } A(t) := Df(\gamma(t)), \quad \underline{\text{Note}}: A(t+T) = A(t)$$

Theorem 1.5.1 (Floque) Let $A: \mathbb{R} \to M_{n \times n}$ be cont. with $A(t+T) = A(t)$ $\forall_{t \in \mathbb{R}}$. Then every fundamental matrix of $\dot{y} = A(t)y$ has the form

$$\Phi(t) = P(t)e^{tB},$$

where B is a constant matrix, and $P(t) = P(t+T)$, $P(0) = I$.

pf: $A(t+T)=A(t) \Rightarrow$ if $\Phi(t)$ is a fundamental matrix, then so is $\Phi(t+T)$.

Let C be s.t. $\Phi(t+T)=\Phi(t)\cdot C$, note: C is nonsingular

(Hw) $\exists B$ s.t. $C=e^{TB}$

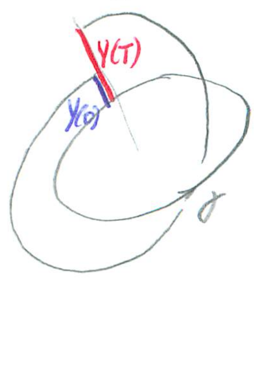
but $\Phi(t+T)=\Phi(t)\Phi(T)$, thus $C=\Phi(T)$

Write $$P(t) := \Phi(t)e^{-tB}.$$

Then,
$$P(t+T)=\Phi(t+T)e^{-(t+T)B} = \Phi(t)\underbrace{e^{TB}}_{C} \cdot e^{-(t+T)B} = \Phi(t)e^{-tB} = P(t) \qquad \#$$

Thus, every solution $y=y(t)$ to $\dot{y}=A(t)y$ has the form
$$y(t)=\Phi(t)y(0) \quad \text{for } \Phi(t)=P(t)e^{tB}, \quad P(t)=P(t+T).$$

Define $M:=\underbrace{\Phi(T)}_{P(T)e^{TB}=P(0)e^{TB}=e^{TB}}$ to be the monodromy matrix for γ.



Geom. meaning of M:

$$y(T)=M\cdot y(0),$$

If $y(0)$ is an eigenvec. of M for the eigenvalue μ, then
$$y(T)=M\cdot y(0)=\mu\cdot y(0). \qquad y(k\cdot T)=\Phi(k\cdot T)y(0)=M^k y(0)=\mu^k y(0)$$

- if $|\mu|>1$, then $|y(T)|>|y(0)|$, deviation grows.
- if $|\mu|<1$, then $|y(T)|<|y(0)|$, deviation decays.

Eigenvalues μ of M are called Floque multipliers. $\quad M=e^{TB}$

" λ of B " " Floque exponents $\quad \Rightarrow \mu=e^{\lambda T}$.

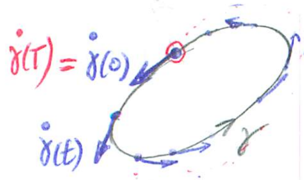
<u>Thm 1.5.2</u> The monodromy matrix M for the linearization of $\dot{x}=f(x)$ about a per. orbit $\gamma=\gamma(t)$ has at least one unit eigenvalue.

pf: $\quad \dot{\gamma} = f(\gamma(t)) \xrightarrow[\text{w.r.t. } t]{\text{diff.}} \frac{d}{dt}\dot{\gamma}(t) = Df(\gamma(t))\,\dot{\gamma}(t) = A(t)\,\dot{\gamma}(t)$

$\Rightarrow \quad \dot{\gamma}(t)$ is a sol. to $\dot{y} = A(t)\,y$

$\Rightarrow \quad \dot{\gamma}(t) = \Phi(t)\,\dot{\gamma}(0) \overset{t=T}{\Rightarrow} \dot{\gamma}(T) = M\dot{\gamma}(0) \xrightarrow{\gamma \text{ is } T\text{-per.}} \dot{\gamma}(0) = M\,\dot{\gamma}(0)$

i.e. $\dot{\gamma}(0)$ is an eigenvec. of M for the eigenvalue 1. #



Geometrically, $\dot{\gamma}(t)$ gives the direction of phase shift at time t on the orbit γ.

M has $\mu = 1$ for eigenvec. $\dot{\gamma}(0) = \dot{\gamma}(T)$ $\Leftarrow$ The phase shift remains constant after one period time.

We say a per. orbit is <u>linearly stable</u>, if $|\mu_i| \le 1, \ \forall_{\mu_i \in \sigma(M)}, i = 1, \dots, n$

" " <u>asymptotically linearly stable</u>, if $|\mu_i| < 1 \ \forall_{\mu_i \in \sigma(M)}$ $i=2 \dots n$ $\mu_1 = 1$.

<u>Ex</u> 1.5.3: Find the fundamental matrix and discuss the stability of the per. orbit (in polar coordinates) of $\dot{r} = r(1-r)$ and $\dot{\theta} = 1$.

$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases} \Rightarrow \begin{cases} \dot{x} = x - rx - y =: f_1(x,y) \\ \dot{y} = x - ry + y =: f_2(x,y) \end{cases}$

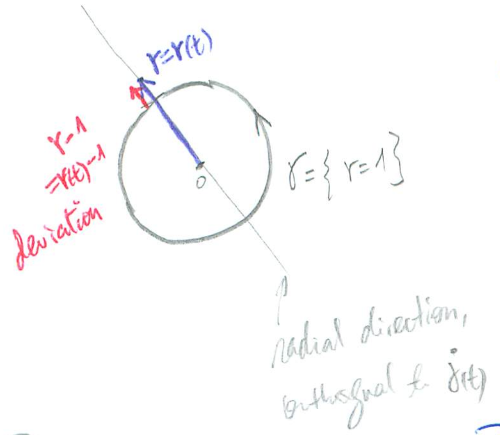
per. Orbit
$\gamma = \{(x,y) \in \mathbb{R}^2 : r = 1\}$
$= \{(x,y) \in \mathbb{R}^2 : x = \cos t, y = \sin t\}_{t \in \mathbb{R}}$.

$$Df(\gamma(t)) = \begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{pmatrix}\Bigg|_{\substack{x=\cos t \\ y = \sin t}} = \begin{pmatrix} -\cos^2 t & -1-\sin t\cos t \\ 1 - \sin t \cos t & -\sin^2 t \end{pmatrix} = A(t)$$

$\dot{y} = A(t)y$ has always a solution $y(t) = \dot{\gamma}(t)$ (cf. Pf of Thm. 1.5.2),

$$\dot{\gamma}(t) = (-\sin t, \cos t)$$

to find another one ; which is orthogonal to $\dot{\gamma}(t)$



let $g(r) = r(1-r)$, and $\boxed{\delta r} = r-1$, be [illegible]

the deviation from the orbit in the radial direction

$$\Rightarrow \boxed{\delta \dot{r}} = \dot{r} = g(r) = \underbrace{g(1)}_{=0} + \underbrace{g'(1)}_{=-1} \cdot \underbrace{\delta r}_{=r-1} + \cancel{O(\delta r^2)}$$

$$\rightsquigarrow \boxed{\delta \dot{r}} = -\boxed{\delta r} \Rightarrow \boxed{\delta r} = \boxed{\delta r_0} e^{-t}$$

deviation in x-direction:

$$\boxed{\delta x} = x(t) - \cos t = r(t)\cos t - \cos t = \boxed{\delta r(t)} \cos t$$

in y-direction:

$$\boxed{\delta y} = \boxed{\delta r(t)} \sin t$$

Thus, $\begin{pmatrix} \delta x \\ \delta y \end{pmatrix} = \delta r(t) \begin{pmatrix} \cos t \\ \sin t \end{pmatrix} = \delta r_0 \begin{pmatrix} e^{-t}\cos t \\ e^{-t}\sin t \end{pmatrix}$ represents the deviation vector, and is a sol. to $\dot{y} = A(t)y$.

$\Rightarrow$ ~~the~~ a fundamental matrix $\Phi(t) = \begin{pmatrix} e^{-t}\cos t & -\sin t \\ e^{-t}\sin t & \cos t \end{pmatrix}$.

$$M = \Phi(T) = \Phi(2\pi) = \begin{pmatrix} e^{-2\pi} & 0 \\ 0 & 1 \end{pmatrix} \rightsquigarrow \text{asympto. linearly stable.} \quad \#$$

<u>Theorem 1.5.4</u> (Abel) $\det M = \exp \int_0^T tr(Df(\gamma(s)))\,ds$.

is useful for 2D flows.

<u>Ex. 1.5.3</u> $\left.\begin{array}{l} \det M = \exp \int_0^T -1\,ds = e^{-T} < 1 \\ \sigma(M) = \{\mu_1 = 1, \mu_2\} \quad \det M = \mu_1\mu_2 = \mu_2 \end{array}\right\} \Rightarrow \mu_2 < 1 \Rightarrow$ asym. lr. stable $\quad \#$

Poincaré map

let γ be a per. orbit in $\mathbb{R}^n$ of some flow ϕ_t from a nonlinear vec. field $f(x)$.

let $S \subset \mathbb{R}^n$ be a hypersurface of dimension $n-1$, s.t.

- $\gamma \cap S = \{x_0\}$ unique intersection point x_0 and $f(x_0) \underset{\substack{\uparrow \\ \text{perpendicular}}}{\perp} S$
- the flow is transverse to S everywhere, i.e.

  $f(x) \cdot \nu(x) \neq 0 \quad \forall_{x \in S}$, where $\nu(x)$ is the unit normal vector to S at x.

S is called a <u>local cross section</u> at x_0.

The <u>Poincaré map</u> $P: S \to S$ is defined by

$P(x) = \varphi_{\tau(x)}(x)$, where $\tau(x) > 0$ is the first time x (following flow φ) returns to S.

Existence of locally def. Poincaré map:

$$\left.\begin{array}{l} x_0 \in \gamma \cap S \\ f(x_0) \perp S \end{array}\right\} \xrightarrow[\text{of } f]{\text{cont.}} \exists \text{ nbhd of } x_0 \text{ for which } f \text{ is transverse to } S$$

Let T be the period of x_0, then $\varphi_T(x_0) = x_0$.

Let $x \in S$ near x_0 $\xrightarrow[\text{of } \varphi \text{ in } x]{\text{cont.}}$ $\varphi_T(x)$ is near $x_0 \Rightarrow \varphi_{\tau(x)}(x) \in S$ for $\tau(x)$ close to T.

Significance of Poincaré map:

stability of γ in $\dot{x} = f(x)$ $\xleftrightarrow[1\text{-}1]{}$ stability of x_0 in $P: x \mapsto P(x)$.