1.2. Flows and invariant subspaces

Given a flow $\phi: U \times I \subseteq \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$, we say a subset $X \subseteq U$ is *flow-invariant*, or simply *invariant*, if

$$\forall_{x \in U}: \phi(x,t) \in U \text{ for all } t \in I,$$

i.e. $\quad \phi(U,t) \subseteq U, \ \forall_{t \in I}.$

This subset can be linear subspaces (then called *invariant subspaces*), or manifolds (then called *invariant manifolds*).

Resume from the discussion of linear systems

$$\dot{x} = Ax, \qquad x \in \mathbb{R}^n,$$

this generates a globally defined flow

$$\phi: \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n, \quad \phi(x,t) = e^{tA} \cdot x$$

It can be directly checked that eigenspaces of A are invariant subspaces for ϕ. (HW)

We divide the spectrum of A into

$$\sigma(A) = \underbrace{\sigma_s(A)}_{\text{stable}} \cup \underbrace{\sigma_u(A)}_{\text{unstable}} \cup \underbrace{\sigma_c(A)}_{\text{center}},$$

where
$$\sigma_s(A) = \{\lambda \in \sigma(A) : Re(\lambda) < 0\}$$
$$\sigma_u(A) = \{\lambda \in \sigma(A) : Re(\lambda) > 0\}$$
$$\sigma_c(A) = \{\lambda \in \sigma(A) : Re(\lambda) = 0\}$$

Corresponding eigenspaces are defined as:

(generalized) eigenvectors for those eigenvalues $\lambda \in \sigma_s(A)$

$$E^s = \text{span}\{\underbrace{v^1, \dots, v^{n_s}}\} = \bigoplus_{\lambda \in \sigma_s} E(\lambda)$$

eigenspace of λ.

$$E^u = \text{span}\{u^1, \dots, u^{n_u}\} = \bigoplus_{\lambda \in \sigma_u} E(\lambda)$$

$$E^c = \text{span}\{w^1, \dots, w^{n_c}\} = \bigoplus_{\lambda \in \sigma_c} E(\lambda).$$

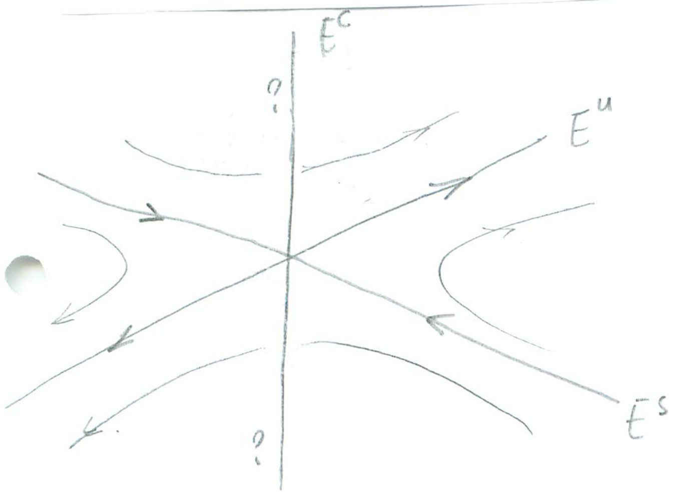
$E(\lambda)$ is invariant $\forall \lambda \in \sigma(A)$, $\rightsquigarrow$ E^s, E^u, E^c are invariant, respectively.

Moreover:

in E^s, solutions : exponential decay

in E^u, " : exponential growth

in E^c, depends on the multiplicity of $\lambda \in \sigma_c(A)$ (HW)

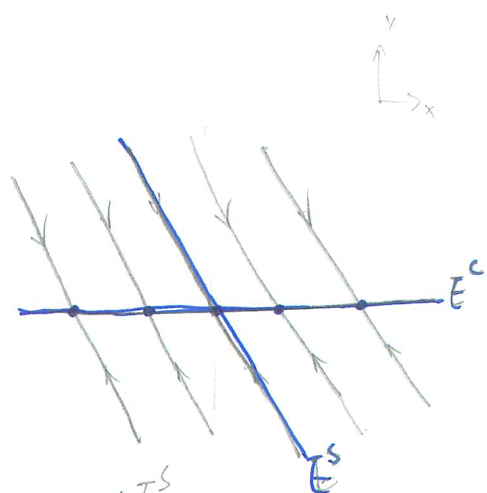


Examples on E^c:

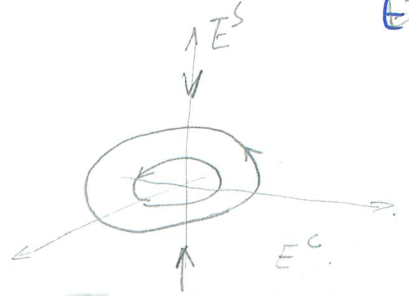
(a) $A = \begin{pmatrix} 0 & 1 \\ 0 & -4 \end{pmatrix} \rightsquigarrow E^s = \langle \begin{pmatrix} 1 \\ -4 \end{pmatrix} \rangle, \; E^c = \langle \begin{pmatrix} 1 \\ 0 \end{pmatrix} \rangle$

$\lambda = -4$ $\qquad$ $\lambda = 0$

remain constant



(b) $A = \begin{pmatrix} 1 & -2 & 0 \\ 1 & -1 & 0 \\ -\frac{1}{2} & 0 & -1 \end{pmatrix}$ $\quad \sigma(A) = \{-1, \pm i\}$



1.3. The nonlinear system $\dot{x} = f(x)$

Idea of linearization: for a smooth function f,

$$f(x) = f(\bar{x}) + Df(\bar{x})(x-\bar{x}) + o(|x-\bar{x}|),$$

where the linear part "$f(\bar{x}) + Df(\bar{x})(x-\bar{x})$" gives an approximation of f, which is a linear function. Now applying this idea to dynamical systems leads to using

$$\dot{x} = f(\bar{x}) + Df(\bar{x})(x-\bar{x}) \xrightarrow[??]{\text{apprx.}} \dot{x} = f(x).$$

in a nbhd of $\bar{x}$

In a nbhd of a fixed point: consider a fixed point $\bar{x}$ of f, i.e. $f(\bar{x}) = 0$. We discuss the relation between dynamics of the following equations:

$$\dot{x} = Df(\bar{x})(x-\bar{x})$$

or equivalently, $\xi = x-\bar{x}$:

$$\dot{\xi} = Df(\bar{x})\xi, \quad \xi \in \mathbb{R}^n \qquad (1.3.2)$$

linear eq.

and

$$\dot{x} = f(x), \quad x \in \mathbb{R}^n \qquad (1.3.1)$$

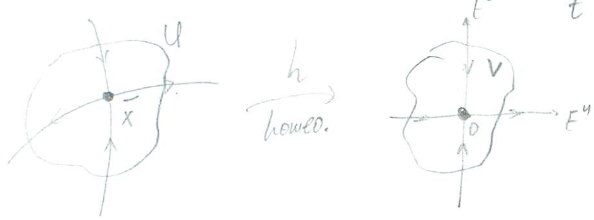
nonlinear eq.

locally around $\bar{x}$.

↑ means $\dot{x} = f(x)$, $x \in B_\varepsilon(\bar{x})$
vs. $\dot{\xi} = Df(\bar{x})\xi$, $\xi \in B_\varepsilon(0)$ for some $\varepsilon > 0$.

Theorem 1.3.1. (Hartman-Grobman) If $Df(\bar{x})$ has no zero or purely imaginary eigenvalues i.e. $\sigma_c(Df(\bar{x}))=\phi$, then there is a homeomorphism h defined on some nbhd U of $\bar{x}$ in $\mathbb{R}^n$ locally taking orbits (i.e. trajectories) of the nonlinear flow ϕ_t of (1.3.1) to those of the linear flow $e^{tDf(\bar{x})}$ of (1.3.2). i.e. let $\psi_t = e^{tDf(\bar{x})}$, then

$\exists$ homeom. h s.t. $\psi_t = h \circ \varphi_t \circ h^{-1}$ locally at $\bar{x}$.



topo. ~~equivalent~~ conjugate

Remark: (i) On the differentiability of h: to obtain a diffeomorphism between the nonlinear and linear flows, one requires further nonresonance conditions on eigenvalues of $Df(\bar{x})$:

if $\lambda_i = \sum_{j=1}^{n} a_j \lambda_j$ for some nonnegative integers $a_j \geq 0$, then we say

$\lambda_1, \lambda_2, \dots, \lambda_n$ are resonant of order $\sum_{j=1}^{n} a_j =: k$

C^r- ~~equivalent~~ conjugate.

e.g.

λ_1	λ_2	λ_3	res.	relation	order
0	0	1	✓	$\lambda_1 = k\lambda_2$	any $k \geq 2$
i	$-i$	2	✓	$\lambda_1 = 2\lambda_1 + \lambda_2 = 3\lambda_1 + 2\lambda_2 = \dots$	$k = 3, 5, 7, \dots$
2	2	3	×		
2	2	0	✓	$\lambda_2 = \lambda_1 + k\lambda_2$	any $k \geq 2$
0	1	2	✓	$\lambda_3 = 2\lambda_2 + k\lambda_3$	any $k \geq 3$

Sternberg's Theorem '58: under assumptions of Thm 1.3.1, and additionally, if eigenvalues $\lambda_1, \dots, \lambda_n$ of $Df(\bar{x})$ are not resonant of any order >1, then $\exists$ a diffeomorphism h s.t. ....

(ii) $\bar{x}$ is called a <u>hyperbolic</u> or <u>nondegenerate</u> fixed point, if $Df(\bar{x})$ has no eigenvalues with zero real part.

By Thm 1.3.1, the asymptotic behavior of solutions near a hyperbolic fixed point (hence its stability type) is determined by the linearization.

<u>Example</u>: for a degenerate fixed point:

$$(*)\begin{cases} \dot{x} = -(x^2+y^2)x - y \\ \dot{y} = -(x^2+y^2)y + x \end{cases} \xrightarrow[\text{at } (0,0)]{\text{linearization}} \begin{cases} \dot{x} = -y \\ \dot{y} = x \end{cases}$$

(HW)

$\bar{x} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, $Df(\bar{x}) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ has eigenvalues $\pm i$.

<u>However, (0,0) is globally asymptotically stable, "attacting sink".</u>

#

We can describe the dynamics around hypebolic fixed points more presisely using:

$$W^s_{loc}(\bar{x}) = \{ x \in U : \phi_t(x) \to \bar{x} \text{ as } t \to \infty, \text{ and } \phi_t(x) \in U \text{ for all } t \geq 0 \},$$

$$W^u_{loc}(\bar{x}) = \{ x \in U : \phi_t(x) \to \bar{x} \text{ as } t \to -\infty, \text{ and } \phi_t(x) \in U, \ \forall t \leq 0 \},$$

called the <u>local stable</u> and <u>unstable manifolds</u> of $\bar{x}$, respectively, where U is a nbhd of $\bar{x}$ in $\mathbb{R}^n$.

<u>Note</u>: By definition, both $W^s_{loc}(\bar{x})$ and $W^u_{loc}(\bar{x})$ are flow invariant.

$W^s_{loc}(\bar{x})$ and $W^u_{loc}(\bar{x})$ provide nonlinear analogues of the flat stable and unstable eigenspaces E^s and E^u of the linear problem:

<u>Theorem 1.3.2</u> (Stable manifold for a fixed point).

Let $\bar{x}$ be a hyperbolic fixed point for $\dot{x} = f(x)$. Then, there exist local stable and unstable manifolds $W^s_{loc}(\bar{x})$ and $W^u_{loc}(\bar{x})$, of $\dim E^s$ and $\dim E^u$, respectively, being tangent to E^s and E^u at $\bar{x}$. $W^s_{loc}(\bar{x})$ and $W^u_{loc}(\bar{x})$ are as smooth as the function f.

Globally, one can define

$$W^s(\bar{x}) = \bigcup_{t \le 0} \phi_t(W^s_{loc}(\bar{x})),$$

called <u>global stable mnfd of $\bar{x}$</u>

$$W^u(\bar{x}) = \bigcup_{t \ge 0} \phi_t(W^u_{loc}(\bar{x})),$$

<u>" unstable " of $\bar{x}$.</u>

<u>Rmk</u>: $W^s(\bar{x})$ and $W^u(\bar{x})$ are as well flow invariant.

(HW) $W^s_{loc}(\bar{x}) \neq W^s(\bar{x}) \cap U$, $W^u_{loc}(\bar{x}) \neq W^u(\bar{x}) \cap U$ in general;

(HW) $W^s(\bar{x})$ and $W^u(\bar{x})$ are in general not submnfds of $\mathbb{R}^n$.

Example Consider

$$(*) \quad \begin{cases} \dot{x} = x \\ \dot{y} = -y + x^2 \end{cases} \xrightarrow[\text{at } (0,0)]{\text{linearization}} \begin{cases} \dot{x} = x \\ \dot{y} = -y \end{cases} \qquad \begin{aligned} E^s &= \{(x,y) \in \mathbb{R}^2 : x = 0\} \\ E^u &= \{(x,y) \in \mathbb{R}^2 : y = 0\} \end{aligned}$$

$(*)$ rewritten : $\dfrac{dy}{dx} = \dfrac{-y}{x} + x$

$$\frac{dy}{dx} = \frac{-y}{x} \Rightarrow y = \frac{c}{x} \quad \forall c$$

$$\left.\begin{aligned} y = \frac{c(x)}{x} &\Rightarrow y' = \frac{x c'(x) - c(x)}{x^2} \\ -\frac{y}{x} + x &= -\frac{c(x)}{x^2} + x \end{aligned}\right\}$$

$$\Rightarrow x c'(x) - \cancel{c(x)} = -\cancel{c(x)} + x^3$$

$$\Rightarrow c'(x) = x^2 \Rightarrow c(x) = \frac{x^3}{3}$$

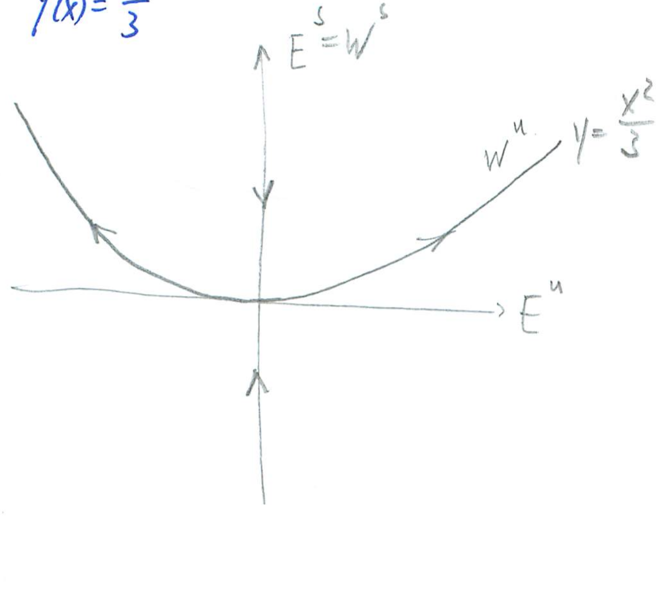
$$\Rightarrow y = \frac{c(x)}{x} + \frac{c}{x} = \frac{x^2}{3} + \frac{c}{x}$$

solvable $\Longrightarrow y(x) = \dfrac{x^2}{3} + \dfrac{c}{x}$

choose our initial condition

$x(0) = y(0) = 0 \Rightarrow c = 0$

$\Rightarrow$ solution lies on the graph of $y(x) = \dfrac{x^2}{3}$



1.4. Linear and nonlinear maps

$\dot{x} = f(x)$, $x \in \mathbb{R}^n$ | $x_{n+1} = G(x_n)$, $n \in \mathbb{Z}$

$\rightsquigarrow$ (smooth) flow ϕ_t, $t \in \mathbb{R}$ $\xrightarrow{\text{fix } t=\tau}$ diffeomorphism $G = \phi_\tau$

"conts. dyn. system" | "discrete dyn. system"

For diffeomorphisms or discrete dyn. systems, one can define similarly, fixed points, stable/unstable/center subspaces, mnfds

Any initial point $x = x_0$ generates a unique orbit, since G is a diffeomorphism.

Linear maps $G(x) = Bx$, where B is an isom. i.e. $0 \notin \sigma(B)$, e.g. $B = e^{\tau A}$

$$\sigma(B) = \sigma_s \cup \sigma_u \cup \sigma_c, \quad \text{with} \begin{cases} \sigma_s = \{\lambda \in \sigma(B) : |\lambda| < 1\} \\ \sigma_u = \{\lambda \in \sigma(B) : |\lambda| > 1\} \\ \sigma_c = \{\lambda \in \sigma(B) : |\lambda| = 1\} \end{cases}$$

$$E^s = \bigoplus_{\lambda \in \sigma_s} E(\lambda), \quad E^u = \bigoplus_{\lambda \in \sigma_u} E(\lambda), \quad E^c = \bigoplus_{\lambda \in \sigma_c} E(\lambda)$$

contraction | expansion | ?

Nonlinear maps $G : x \mapsto G(x)$

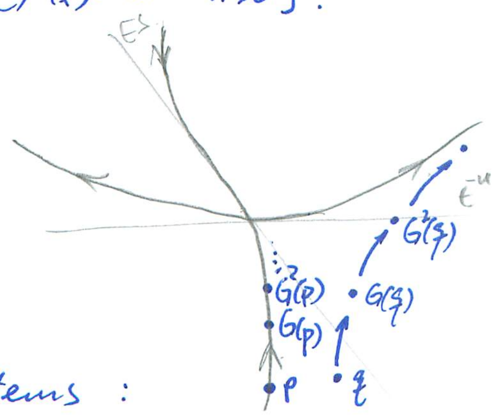
A fixed point $\bar{x}$ with $\bar{x} = G(\bar{x})$ is hyperbolic, if $\sigma(G) \cap S^1 = \emptyset$.

Theorem 1.4.1 (Hartman-Grobman) Let $G: \mathbb{R}^n \to \mathbb{R}^n$ be a C^1-diffeomorphism with a hyperbolic fixed point $\bar{x}$. Then there exists a homeomorphism h defined on some nbhd U of $\bar{x}$ such that
$$h(G(\xi)) = DG(\bar{x})\, h(\xi), \quad \forall \xi \in U.$$

Theorem 1.4.2. (Stable mnfd for a fixed point) Let $G: \mathbb{R}^n \to \mathbb{R}^n$ be a C^1-diffeom. with a hyperbolic fixed point $\bar{x}$. Then there exist local stable and unstable mnfds $W^s_{loc}(\bar{x})$, $W^u_{loc}(\bar{x})$, tangent to the eigenspaces E^s, E^u of $DG(\bar{x})$ at $\bar{x}$, of corresponding dimensions. $W^s_{loc}(\bar{x})$ and $W^u_{loc}(\bar{x})$ are as smooth as the map G.

Note: More precisely,
$$W^s_{loc}(\bar{x}) = \{ x \in U : G^n(x) \to \bar{x} \text{ as } n \to \infty, \text{ and } G^n(x) \in U \ \forall n \geq 0 \}$$
$$W^u_{loc}(\bar{x}) = \{ x \in U : G^{-n}(x) \to \bar{x} \text{ as } n \to \infty, \text{ and } G^{-n}(x) \in U \ \forall n \geq 0 \}.$$

Globally,
$$W^s(\bar{x}) = \bigcup_{n \geq 0} G^{-n}(W^s_{loc}(\bar{x}))$$
$$W^u(\bar{x}) = \bigcup_{n \geq 0} G^{n}(W^u_{loc}(\bar{x})).$$



Distinction between conts and discrete systems:

- an orbit is a curve for conts. systems, and is a sequence of points for discrete systems.
- invariant mnfds are composed of unions of sol. curves for conts. systems; and are " " " discrete orbit points.