2.4. Codimension-1 Bifurcations from an equilibrium

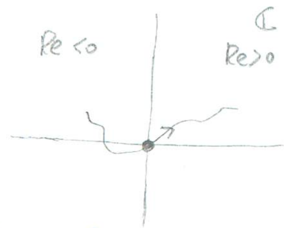
Consider $\qquad$ smooth f throughout.

$$\dot{x} = f(x,\mu) \quad , \quad x \in \mathbb{R}^n, \ \mu \in \mathbb{R} \qquad (*)$$

around an equilibrium $(x_0, \mu_0) = (0,0)$ (i.e. $f(0,0)=0$). We say that

(i) the system $(*)$ undergoes a bifurcation of steady states, or a fold bifurcation around (x_0,μ_0) as μ crosses μ_0, if there is new equilibrium appearing or existing equilibrium disappearing, as μ crosses μ_0.

$\boxed{\text{HW}}$ fold BF $\Rightarrow$ $D_x f(x_0,\mu_0)$ has a zero eigenvalue.



The simplest / codimension-1 fold bifurcation is a bifurcation related to

$$D_x f(x_0,\mu_0) \simeq \left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & A_{n-1} \end{array}\right) \quad \text{simple} \quad \text{with} \quad \sigma(A) \cap i\mathbb{R} = \phi \qquad (**)$$

We describe in detail several different codim-1 fold BFs:

(i1) Saddle-Node Bifurcation (generic scenario under $(*)$-$(**)$)

Start by considering $(*)$ in 1D:

$$\dot{x} = f(x,\mu) \quad , \quad x \in \mathbb{R}, \ \mu \in \mathbb{R}. \qquad (*_{1D})$$

Assume that $f(0,0)=0$ and $f_x(0,0)=0$ (codim-1 condition)

If the following non-degeneracy conditions hold:

(SN1) $\quad \underbrace{f_\mu(0,0)}_{=:a} \neq 0$

(SN2) $\quad \underbrace{\tfrac{1}{2} f_{xx}(0,0)}_{=:b} \neq 0$

then $(*_{1D})$ is topo. equivalent (locally near $(0,0)$) to the normal form

$$\dot{x} = \operatorname{sign}(a)\cdot\mu + \operatorname{sign}(b)\cdot x^2 .$$

Pf: Taylor expansion (f smooth) around $(0,0)$:

$$f(x,\mu) = \underbrace{f(0,0)}_{=0} + \underbrace{f_x(0,0)}_{=0}\, x + \underbrace{f_\mu(0,0)}_{\neq 0}\,\mu + \underbrace{\tfrac{1}{2} f_{xx}(0,0)}_{\neq 0}\, x^2 + \tfrac{1}{2} f_{x\mu}(0,0)\, x\mu + \tfrac{1}{2} f_{\mu\mu}(0,0)\,\mu^2 + O(|(x,\mu)|^3)$$

Moreover, $f_\mu(0,0) \neq 0 \xrightarrow{\text{I.F.T.}} f(x,\mu) = 0 \underset{\text{near } (0,0)}{\iff} \mu = \mu(x)$ for $\mu(0)=0$

$\Rightarrow f(x,\mu(x)) = 0 \Rightarrow f_x + f_\mu \mu' = 0 \Rightarrow \underbrace{f_x(0,0)}_{=0} + \underbrace{f_\mu(0,0)}_{\neq 0}\cdot \mu'(0) = 0 \Rightarrow \mu'(0) = 0$ $\Big\} \Rightarrow \mu = O(x^2)$

Thus, $f(x,\mu) = \underbrace{f_\mu(0,0)}_{=a \sim \pm 1}\,\mu + \underbrace{\tfrac{1}{2} f_{xx}(0,0)}_{=b \sim \pm 1}\, x^2 + O(x^3)$

and $(*_{1D})$ is topo. equiv. to

$$\dot{x} = \operatorname{sign}(a)\cdot\mu + \operatorname{sign}(b)\cdot x^2$$

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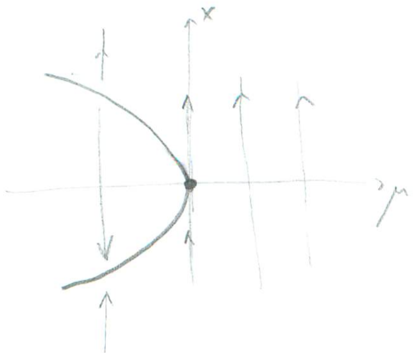
e.g. $\quad \dot{x} = \mu + x^2$



as μ crosses 0, the two equilibria $\pm\sqrt{-\mu}$ (one stable, the other unstable) collide and disappear.

$\boxed{\text{HW}}$ analyze other 3 cases: $\quad \mu - x^2$, $\quad -\mu + x^2$, $\quad -\mu - x^2$.

Back to n-D.

$$\dot{x} = f(x,\mu), \quad x \in \mathbb{R}^n, \ \mu \in \mathbb{R}, \qquad (*)$$

where $f(0,0)=0$ and $D_x f(0,0)$ has a simple eigenvalue 0.

inner product in $\mathbb{R}^n$

Let v be an eigenvector of 0 s.t. $D_x f(0,0) \cdot v = 0$, and $\langle v, v\rangle = 1$,

let w be a <u>left eigenvector</u> of 0 s.t. $w^T D_x f(0,0) = 0$ and $\langle w, v\rangle = 1$.

If $w^T A = \lambda w^T$, $w \neq 0$, then w is a left eigenvector of λ for A.

Recall if $v_1, \dots, v_n$ are eigenvectors of $\lambda_1, \dots, \lambda_n$, then

$$T^{-1} A T = J \leftarrow \text{Jordanform}, \quad T = (v_1, \dots, v_n)$$

columns of T

Indeed, T^{-1} is composed of left eigenvectors as row vectors, i.e.

$$T^{-1} = \begin{pmatrix} w_1^T \\ \vdots \\ w_n^T \end{pmatrix} \leftarrow \text{rows of } T^{-1}$$

In particular, $\mathbb{1} = T^{-1} T \Rightarrow w_i^T v_j = \delta_{ij} = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$

The left and right eigenvectors chosen this way are called <u>dual bases</u>.

Based on Taylor expansion for multivariable functions:

$$f(x) = f(a) + Df(a)(x-a) + \underbrace{\frac{1}{2}(x-a)^T \overset{\text{bilinear form}}{D^2 f(a)} (x-a)}_{j\text{th component} = \frac{1}{2}\sum_{k,l=1}^{n} \frac{\partial^2 f_j}{\partial x_k \partial x_l}(x_k - a_k)(x_l - a_l)} + O(|x-a|^3), \quad x, a \in \mathbb{R}^n$$

We can replace (SN1)-(SN2) for 1D case with:

(SN1) $\langle w, D_\mu f(0,0) \rangle \neq 0$

(SN2) $\langle w, \frac{1}{2} v^T D_{xx} f(0,0) v \rangle \neq 0$

and state a similar result for $(*)$:

Consider

$$\dot{x} = f(x, \mu), \quad x \in \mathbb{R}^n, \mu \in \mathbb{R} \qquad (*)$$

Such that

$$f(0,0)=0 \text{ and } D_x f(0,0) \text{ has a simple e.v. } 0, \qquad (**)$$

and

(SN1) $\langle w, D_\mu f(0,0) \rangle \neq 0$

(SN2) $\langle w, \frac{1}{2} v^T D_{xx} f(0,0) v \rangle \neq 0.$

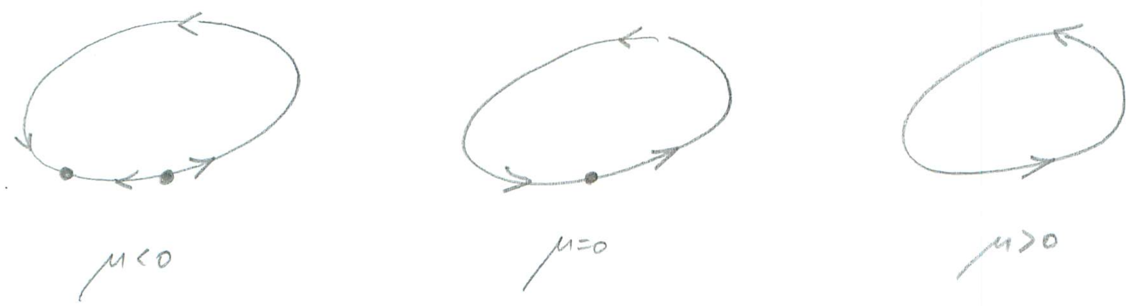
Then, $(*)$ undergoes a saddle-node bifurcation around $x_0=0$ as μ crosses $\mu_0=0$, which is characterized by collision and disappearance of two equilibria with opposite stability of $(*)$"

Remark: we also call the saddle-node bifurcation to be a generic pheonomenon for systems of form $(*)$ satisfying $(**)$.

That is, if a system, say $(*)$, under some condition given by finitely many equalities of derivatives, say $(**)$, will definitely undergo certain pheonomenon, saddle-node BF, unless further equalities of derivatives have to be satisfied, then we will call this pheonomenon generic.

In other words, we can formulate the above result as:

"A system of form $(*)$ satisfying condition $(**)$ generically undergoes a saddle-node bifuration."



Saddle-node (local) bifurcation on $W^c \simeq S^1$ leads to a new periodic orbit $(\mu>0)$ through homoclinic orbit $(\mu=0)$. This is a global change of portraits. — global bifurcation.

Sometimes different setting of $(*)$ may prevent (SN1) or (SN2) to be satisfied.

(ii) Transcritical Bifurcation (generic scenario for $(*)$-$(**)$ with $f(0,\mu)\equiv 0 \ \forall_\mu$)

In classical bifurcation theory, $(*)$ is assumed to have a trivial solution from which bifurcation occurs, i.e.

$$f(0,\mu) \equiv 0 \qquad \forall_\mu \text{ near } \mu_0 = 0 \qquad \text{(T)}$$

$\Rightarrow D_\mu f(0,0) = 0 \Rightarrow$ ~~(SN1)~~ . Instead, we assume

(T1) $\langle w, \frac{1}{2} v^T D_{\mu x} f(0,0) v \rangle \neq 0$ | $\frac{1}{2} f_{\mu x}(0,0) \neq 0$

in 1D

(SN2) $=$ (T2) $\langle w, \frac{1}{2} v^T D_{xx} f(0,0) v \rangle \neq 0$ | $\frac{1}{2} f_{xx}(0,0) \neq 0$

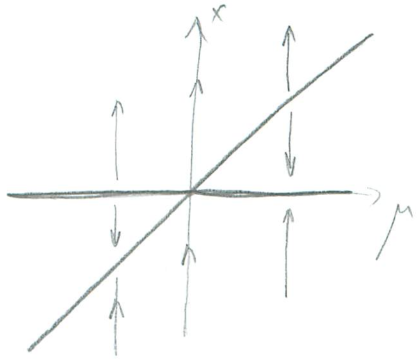
Consider 1D again, let $g(x,\mu) = f_x(x,\mu)$, then $g(0,0) = 0$, $g_x = f_{xx}$, $g_\mu = f_{\mu x}$

$$g(x,\mu) = 0 + \underset{\neq 0}{g_x(0,0)}\, x + \underset{\neq 0}{g_\mu(0,0)}\, \mu + \ldots$$

$g_x(0,0) \neq 0$, $g_\mu(0,0) \neq 0$ $\Rightarrow \mu = O(x)$

$$= \pm 2x \pm \mu \quad \Rightarrow f(x,\mu) = \pm x^2 \pm \mu x .$$

e.g. $f(x,\mu) = x^2 + \mu x$ $\boxed{HW}$ analyse the other 3 cases.



One shows in a similar way as for SN-BF, that:

"A system of form (*) satisfying conditions (**) and (T), generically undergoes a transcritical bifurcation, which is characterized by the exchange of stability of two equilibria as μ crosses $\mu_0 = 0$"

(i3) Pitchfork Bifurcation (generic scenario for (*)-(**) with $\underbrace{f(-x,\mu) = -f(x,\mu)}_{\mathbb{Z}_2\text{-symmetry}}$)

Many systems possess symmetry. (such as Duffing, Lorenz-equations)

Assume

$$f(-x,\mu) = -f(x,\mu) \qquad \forall_{x\in\mathbb{R}^n,\, \mu\in\mathbb{R}} \qquad (S)$$

$\Rightarrow f$ is odd in $x \Rightarrow f_x$ is even in $x \Rightarrow f_{xx}$ is odd in $x \Rightarrow f_{xx}(0,0) = 0 \Rightarrow \cancel{(SN2)}$

Also, (P) $\Rightarrow f(0,\mu) = 0 \ \forall_\mu \Rightarrow \cancel{(SN1)}$. Instead, we assume

(T1)=(P1) $\langle w, \frac{1}{2} v^T D_{\mu x} f(0,0)\, v\rangle \neq 0$ | $\frac{1}{2} f_{\mu x} \neq 0$

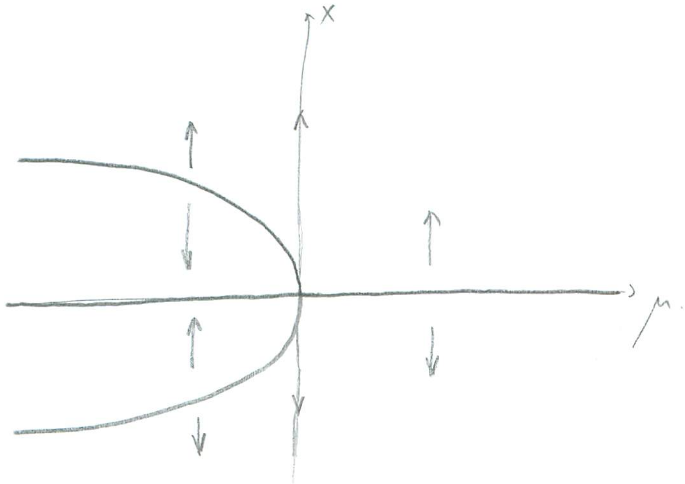
(P2) $\langle w, \frac{1}{6} D_{xxx} f(0,0)(v,v,v)\rangle \neq 0$ | $\frac{1}{6} f_{xxx} \neq 0$

in 1D

Similarly, one shows in 1D, we have

$$f(x,\mu) = \pm \mu x \pm x^3$$

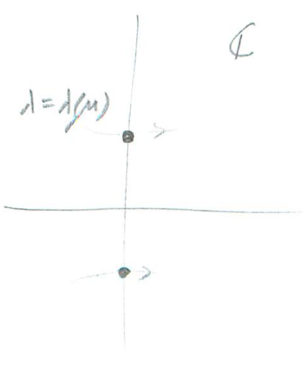
e.g. $f(x,\mu) = \mu x + x^3 = x(\mu + x^2)$.



"A system of form $(*)$ satisfying $(**)$ and (S), generically undergoes a <u>pitchfork bifurcation</u>, which is characterized by appearance or disappearance of two equilibria with the same stability of $(*)$ as μ crosses $\mu_0 = 0$",

(ii) the system $(*)$ undergoes a <u>bifurcation of oscillating states</u>, or a <u>Hopf bifurcation</u> around (x_0, μ_0) as μ crosses μ_0, if there are non-constant periodic solutions appearing or disappearing, as μ crosses μ_0. We assume (codim-1 BF)

$$D_x f(x_0,\mu_0) \simeq \left(\begin{array}{cc|c} 0 & -\omega & 0 \\ \omega & 0 & \\ \hline \multicolumn{2}{c|}{0} & A_{n-2} \end{array}\right), \quad \sigma(A) \cap i\mathbb{R} = \emptyset. \qquad (**_H)$$



- $D_x f(x_0,\mu_0)$ remains invertible $\xrightarrow{\text{I.F.T.}}$ $x = x(\mu)$ $(x(0) = 0)$ unique equilibrium for every μ near μ_0.
- stability of the unique equilibrium <u>does</u> change, as $\dim W^u$ and $\dim W^s$ change.

Aufgabe 11.2 $\Rightarrow$ (*) satisfying $(**_H)$ in 2D can be brought to the normal form: (skipping the $\mu^2 x$ and $\mu^2 y$ terms in Aufgabe 11.2, why can it be skipped(?)

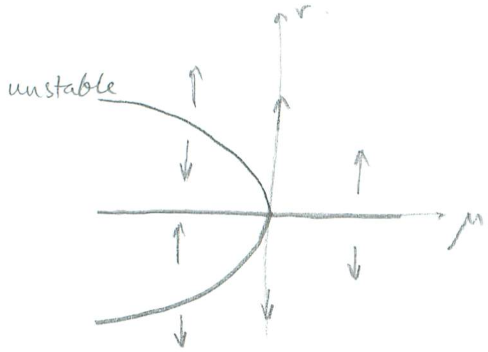
$$\begin{cases} \dot{x} = (-\omega - b\mu)y + a\mu \cdot x + (cx - dy)(x^2+y^2) \\ \dot{y} = a\mu \cdot y + (\omega + b\mu)x + (dx + cy)(x^2+y^2) \end{cases} \qquad a,b,c,d \in \mathbb{R}.$$

Polar-

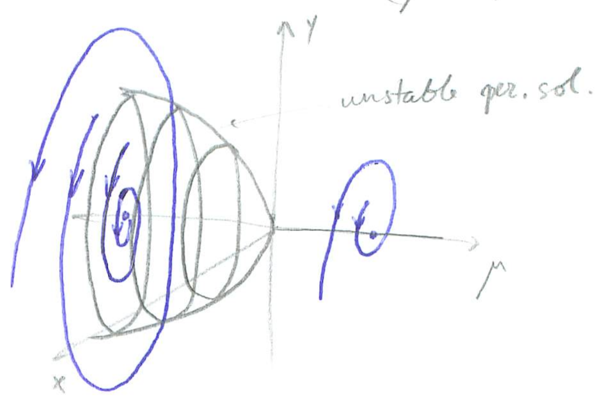
$$\begin{cases} \dot{r} = (a\mu + cr^2)r \\ \dot{\theta} = \omega + b\mu + d \cdot r^2 \end{cases}$$

generic conditions

$a \neq 0$, $c \neq 0$ $\Rightarrow$ $\dot{r} = (\pm\mu \pm r^2) r$



e.g. $\dot{r} = (\mu + r^2) r$



For n-D, using center manifold + normal form, one shows the generic conditions are:

(H1) $\quad a = \frac{d}{d\mu} \operatorname{Re}(\lambda(\mu)) \Big|_{\mu = \mu_0 = 0} \neq 0$

(H2) $\quad c = \frac{1}{16}\left(f_{xxx} + f_{xyy} + g_{xxy} + g_{yyy}\right) + \frac{1}{16\omega}\left[f_{xy}(f_{xx} + f_{yy}) - g_{xy}(g_{xx} + g_{yy}) - f_{xx} g_{xx} + f_{yy} g_{yy}\right] \neq 0$;

for the normal form on W^c by $\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 0 & -\omega \\ \omega & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} f(x,y) \\ g(x,y) \end{pmatrix}$, $(x,y) \in E^c$.

"A system of form (*) with $(**_H)$ generically undergoes a Hopf bifurcation, which is characterized by a birth or disappearance of a surface of per. solutions which lie on the paraboloid $\mu = -\frac{c}{a}(x^2+y^2)$

up to the 3rd order of $|(x,y)|$. If $c>0$, they are repelling; if $c<0$, they are attacting.