2.3. Normal forms (simplified forms of flows)

Consider $\dot{x} = f(x)$, $x \in \mathbb{R}^N$, with $f(0)=0$.

Depending on eigenvalues of $Df(0)$, we consider 2 cases:

<u>Case A</u> : non-resonance

<u>Case B</u> : resonance

<u>Def 2.3.1</u> Let $\lambda_1, \dots, \lambda_N \in \mathbb{C}$. If there exist $a_1, \dots, a_N \in \mathbb{R}^+ \cup \{0\}$ with $\sum_{j=1}^{N} a_j \geq 2$ such that for some λ_i :

$$\lambda_i = \sum_{j=1}^{N} a_j \lambda_j ,$$

then $\lambda_1, \dots, \lambda_N$ are said to be <u>resonant</u> of <u>order</u> $|a| := \sum_{j=1}^{N} a_j$

Classical examples of resonance :

(I) zero eigenvalue : $\lambda_i = \lambda_i + 0 = \lambda_i + 2 \cdot 0 = \dots$ any order ≥ 2.

(II) pure imaginary eigenvalues $\pm i\beta$: $\lambda_i = \lambda_i + (i\beta + (-i\beta)) = \lambda_i + 2 \cdot (i\beta + (-i\beta)) = \dots$

any odd order ≥ 3.

Case A. $Df(0)$ has nonresonant eigenvalues $\lambda_1, \dots, \lambda_N$.

Using eigenbasis of $Df(0)$, we write $\dot{x} = f(x)$ as

$$\begin{cases} \dot{x}_1 = \lambda_1 x_1 + g_1(x_1, \dots, x_N) \\ \dot{x}_2 = \lambda_2 x_2 + g_2(x_1, \dots, x_N) \\ \vdots \\ \dot{x}_N = \lambda_N x_N + g_N(x_1, \dots, x_N) \end{cases} \quad \text{or} \quad \dot{x} = \Lambda x + g(x),$$

where $g(0)=0$, $Dg(0)=0$, i.e. the smallest degree of a nonzero derivative of any g_i is at least 2.

Let k = the smallest degree of nonzero derivative of all $g_1, \dots, g_N$ $(k \geq 2)$

We want to find a coordinate transformation (nonlinear)

$x = h(y) = y + P(y)$, where P is a polynomial of degree k,

such that the resulting system. $\dot{x} = (I + DP(y))\dot{y}$

$$(*) \quad \dot{y} = (I + DP(y))^{-1}(\Lambda y + \Lambda \cdot P(y) + g(y + P(y)))$$

$$:= \Lambda \cdot y + \eta(y)$$

satisfies:

$k' :=$ the smallest degree of nonzero derivative of all $\eta_1, \dots, \eta_N \geq k+1$.

Considering $(*)$ but only keeping terms of degree $\leq k$:

$$(I+DP)(I-DP) = I \cancel{- DP + DP} - DPDP = I - \underbrace{DPDP}_{\text{lowest degree} = k-1+k-1 = 2k-2 \geq k \ (k \geq 2)}$$

$$\Rightarrow (I+DP)^{-1} = I - DP + O(y^k)$$

$$\dot y = \underbrace{\left(I+DP(y)\right)^{-1}}_{\substack{\| \\ I-DP(y)+O(y^k)}} \underbrace{\left(\Lambda y + \Lambda P(y) + g(y+P(y))\right)}_{O(y)}$$

$$= \left(I-DP(y)\right)\left(\Lambda y + \Lambda P(y) + g(y+P(y))\right) + \cancel{O(y^{k+1})}$$

$$\doteq \underset{1}{\Lambda y} + \underset{k}{\Lambda P(y)} + \underbrace{\underset{k}{g}(\underset{1}{y}+\underset{k}{P(y)})}_{\substack{\| \\ g^k(y)+O(y^{k+1})}} - \underset{k-1}{DP(y)}\,\underset{1}{\Lambda y}\ \boxed{-\underset{k-1}{DP(y)}\,\underset{k}{\Lambda P(y)} - \underset{k-1}{DP(y)}\,\underset{k}{g(y+P(y))}}$$

smallest degree:

degree higher than k

g^k: leading term in g for the degree k

$$\dot y = \Lambda y + \underbrace{\Lambda P(y) + g^k(y) - DP(y)\cdot\Lambda\cdot y}_{\text{let } = 0} + O(y^{k+1})$$

let $\overset{\|}{0}$, s.t. $k' \geq k+1$

That is, we seek for $P = \begin{pmatrix} P_1 \\ \vdots \\ P_N \end{pmatrix}$ satisfying

$$(*_p) \qquad F(P_i) := \lambda_i P_i(y) - \sum_{j=1}^{N} \frac{\partial P_i}{\partial y_j} \lambda_j y_j = -g_i^k(y) \quad , \quad i=1,\dots,N$$

where P_i is a polynomial of degree k.

$(*_p)$ is linear in coefficients of P_i $\Rightarrow$ only need to consider monomials P_i

Let $P_i = y_1^{a_1} y_2^{a_2} \cdots y_N^{a_N}$ with $\sum_{j=1}^{N} a_j = k$, a monomial of degree k.

Then $\frac{\partial P_i}{\partial y_j} \cdot y_j = a_j P_i \quad \Rightarrow \quad F(P_i) = \lambda_i P_i - \sum_{j=1}^{N} a_j \lambda_j P_i = \left(\lambda_i - \sum_{j=1}^{N} a_j \lambda_j\right) P_i$

i.e. monomial $(a_1,\dots,a_N)$ is an eigenvector of operator F for eigenvalue $\left(\lambda_i - \sum_{j=1}^{N} a_j \lambda_j\right)$.

$\Rightarrow (*_p)$ can be solved, if $\underbrace{\lambda_i - \sum_{j=1}^{N} a_j \lambda_j \neq 0 \quad \forall i \quad \forall (a_1,\dots,a_N) \text{ with } \sum_{j=1}^{N} a_j = k, \ a_j \geq 0,}_{\text{satisfied due to nonresonance assumption}}$

Conclusion: if the eigenvalues $\lambda_1, \dots, \lambda_N$ of $Df(0)$ are non resonant of all order $k \geq 2$, then the system $\dot{x} = f(x)$, $f(0) = 0$ can be linearized (using polynomial change of coordinates) to any desired algebraic order.

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case B: $Df(0)$ has resonant eigenvalues $\lambda_1, \dots, \lambda_N$

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more interesting case for bifurcation problems.

Assume $\lambda_1, \dots, \lambda_N$ are resonant of order k with

$$\lambda_i - \sum_{j=1}^{N} a_j \lambda_j = 0 \quad \text{for some } \lambda_i \text{ and } \sum_{j=1}^{N} a_j = k,\ a_j \geq 0,$$

then in $(*_p)$, LHS=0 for monomial $(a_1, \dots, a_N)$, thus if $g_i^k = g_i^k(y)$ has nonzero monomial $(a_1, \dots, a_N)$-term, then this term can NOT be removed by any change of coordinates; "essential term" or "resonance term"

Goal: to find change of coordinates, so only essential terms remain in the form of f, and this form is called the normal form of f.

To state the theorem of normal forms, a couple of notations:

Let H_k be the linear space of vec. fields whose coefficients are homogeneous polyn. of degree k, i.e.

$$H_k = \left\{ \begin{pmatrix} P_1 \\ P_2 \\ \vdots \\ P_N \end{pmatrix} : \begin{array}{l} P_i \text{ is a homog. polyn. of degree } k \text{ in } (x_1, \dots, x_N) \\ \text{i.e. } P_i \text{ is a lin. comb. of monomials of deg. } k. \end{array} \right\}$$

Let $L = Df(0) \cdot x$, the linear part of "$\dot{x} = f(x)$, $f(0) = 0$."

Define a linear operator on H_k using L:

$$\begin{aligned} \mathrm{ad}L : \quad & H_k \longrightarrow H_k \\ & Y \longmapsto \underset{\substack{\uparrow \\ \text{Lie bracket}}}{[Y, L]} = DLY - DYL. \end{aligned}$$

<u>Example 2.3.2</u> (part I) Let $Df(0) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, then $L = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -y \\ x \end{pmatrix}$.

Consider $H_2 = \left\{ \begin{pmatrix} P_1 \\ P_2 \end{pmatrix} : P_i = P_i(x,y) = a_i x^2 + b_i xy + c_i y^2,\ i = 1,2,\ a_i, b_i, c_i \in \mathbb{R} \right\} \cong \mathbb{R}^6$, a 6-dim'l real vector space spanned by

$$\begin{array}{lcccccc} & \begin{pmatrix} x^2 \\ 0 \end{pmatrix}, & \begin{pmatrix} xy \\ 0 \end{pmatrix}, & \begin{pmatrix} y^2 \\ 0 \end{pmatrix}, & \begin{pmatrix} 0 \\ x^2 \end{pmatrix}, & \begin{pmatrix} 0 \\ xy \end{pmatrix}, & \begin{pmatrix} 0 \\ y^2 \end{pmatrix}. \\ & \| & \| & \| & \| & \| & \| \\ \text{write as} & x^2 \frac{\partial}{\partial x} & xy \frac{\partial}{\partial x} & y^2 \frac{\partial}{\partial x} & x^2 \frac{\partial}{\partial y} & xy \frac{\partial}{\partial y} & y^2 \frac{\partial}{\partial y} \end{array}$$

$$\mathrm{ad}L\left(x^2 \frac{\partial}{\partial x}\right) = \mathrm{ad}L \begin{pmatrix} x^2 \\ 0 \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}}_{DL} \begin{pmatrix} x^2 \\ 0 \end{pmatrix} - \underbrace{\begin{pmatrix} 2x & 0 \\ 0 & 0 \end{pmatrix}}_{DY} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2xy \\ x^2 \end{pmatrix} = 2xy \frac{\partial}{\partial x} + x^2 \frac{\partial}{\partial y}$$

$$\mathrm{ad}L = 6\times 6\text{-matrix} = \begin{array}{c|l} & x^2\frac{\partial}{\partial x} \quad xy\frac{\partial}{\partial x} \quad y^2\frac{\partial}{\partial x} \ \vdots \ x^2\frac{\partial}{\partial y} \quad xy\frac{\partial}{\partial y} \quad y^2\frac{\partial}{\partial y}, \\ \hline x^2\frac{\partial}{\partial x} & 0 \\ xy\frac{\partial}{\partial x} & 2 \\ y^2\frac{\partial}{\partial x} & 0 \\ x^2\frac{\partial}{\partial y} & 1 \\ xy\frac{\partial}{\partial y} & 0 \\ y^2\frac{\partial}{\partial y} & 0 \end{array}$$