Chap 1. Diff. equations and dynamical systems

1.0. Existence and uniqueness of solutions (a review)

Consider

$$(1.0.1) \qquad \frac{dx}{dt} \overset{\text{def}}{=} \dot{x} = f(x),$$

where $x = x(t) \in \mathbb{R}^n$, $t \in \mathbb{R}$, $f : U \to \mathbb{R}^n$ is smooth on U. ($U \subseteq \mathbb{R}^n$)

We say that the vector field f generates a flow

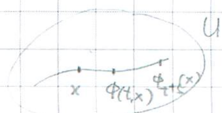
$$\phi_t : U \to \mathbb{R}^n,$$

where $\phi_t(x) = \phi(x,t)$ is a smooth function defined on U and for $t \in I = (a,b) \subseteq \mathbb{R}$ for some interval I and ϕ satisfies (1.0.1) in the sense:

$$\frac{d}{dt}\left(\phi(x,t)\right)\Big|_{t=\tau} = f(\phi(x,\tau)),$$

for all $x \in U$ and $\tau \in I$.

Note: (i) in its domain of definition, ϕ_t satisfies the group properties: $\phi_0 = id$; $\phi_{t+s} = \phi_t \circ \phi_s$.



(ii) Systems of form (1.0.1), in which the vec. field does not contain t explicitly, are called autonomous.

Often given is an <u>initial condition</u>

$$x(0) = x_0 \in U,$$

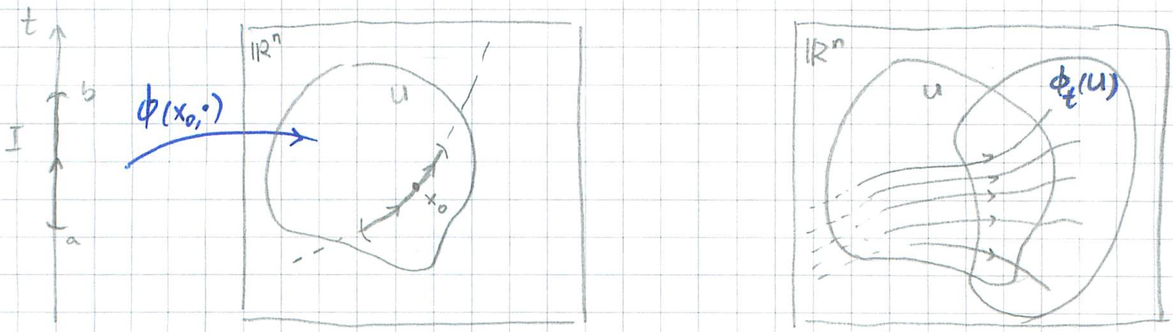
for which we seek a solution $\phi(x_0,t)$ s.t. $\cancel{\phi}$

$$\phi(x_0,0) = x_0$$

(we also write such a solution as $x(x_0,t)$, or simply $x(t)$.).

In this case, $\phi(x_0,\cdot): I \to \mathbb{R}^n$ defines a <u>solution curve</u>, <u>trajectory</u>, or <u>orbit</u> of the diff. eqn. (1.0.1) <u>based at</u> x_0.

<u>Note</u>: autonomous system (1.0.1) is invariant under translations in t ($y(t) := x(t+\tau) \rightsquigarrow \dot{y}\big|_t = \dot{x}\big|_{t+\tau} = f(x(t+\tau)) = f(y(t))$), so solutions based at $t_0 \neq 0$ can be always translated to $t_0 = 0$.



(a) the solution curve $\phi_t(x_0)$; (b) the flow ϕ_t .

the stress of ODE

the stress of dynam. sys.

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<u>Remark</u>: we also deal with dynamical systems defined by vector fields on manifolds M, regarded as

$$f: M \to TM,$$

where TM is the tangent bundle of M. But most times we work with "simple" manifolds like (a single chart is possible)

$$T^2 = \mathbb{R}^2/\mathbb{Z}^2, \quad \text{or} \quad S^1 = \mathbb{R}/\mathbb{Z} \quad \text{or} \quad S^1 \times \mathbb{R} = \mathbb{R}^2/\mathbb{Z}.$$

torus circle cylinder

Such systems arise typically when the vec. field f is periodic in some of its components.

<u>Theorem</u> (local existence and uniqueness) <u>1.0.1</u>:

Let $U \subset \mathbb{R}^n$ be an open subset of real Euclidean space (or of a differentiable manifold M), let $f: U \to \mathbb{R}^n$ be a conts. diff. (C^1) map and $x_0 \in U$. Then $\exists$ some constant $c > 0$ and $\exists!$ solution $\phi(x_0, \cdot): (-c, c) \to U$ satisfying the diff. eqn. $\dot{x} = f(x)$ with initial condition $x(0) = x_0$.

<u>Remark</u>: in fact, f need only be locally Lipschitz, i.e.

$$|f(y) - f(x)| \le K|x - y| \text{ for some } K < \infty, \ \forall x, y \in U.$$

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The Thm 1.0.1 is a "local" theorem, since any solution may leave U after sufficient time.

<u>Example</u>:

$$\dot{x} = 1 + x^2$$

has the general solution $x(t) = \tan(t+c)$, so $x(t)$ leaves any subset $U \subset \mathbb{R}$ in a finite time.

global existence for compact mnf.

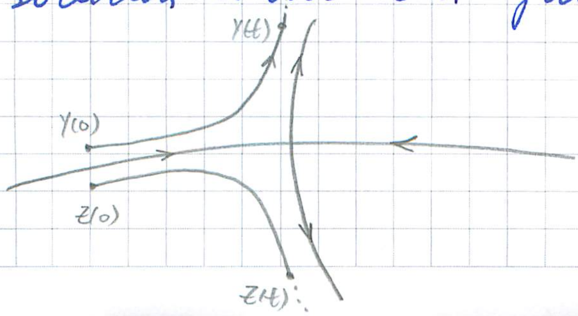
<u>Theorem 1.0.3</u> The diff. equ. $\dot{x} = f(x)$, $x \in M$, with M being a compact manifold, and $f \in C^1$, has a solution curves defined ~~on~~ for all $t \in \mathbb{R}$.

<u>Theorem 1.0.4</u> (continuous dependence on initial conditions)

Let $U \subset \mathbb{R}^n$ be open and suppose $f: U \to \mathbb{R}^n$ has a Lipschitz constant K. Let $y(t)$, $z(t)$ be solutions to $\dot{x} = f(x)$ on the closed interval $[t_0, t_1]$. Then, for all $t \in [t_1, t_2]$,

$$|y(t) - z(t)| \leq |y(t_0) - z(t_0)| \, e^{K(t-t_0)}.$$

<u>Note</u>: solution curves can grow exponentially apart (typical for chaotic flows)



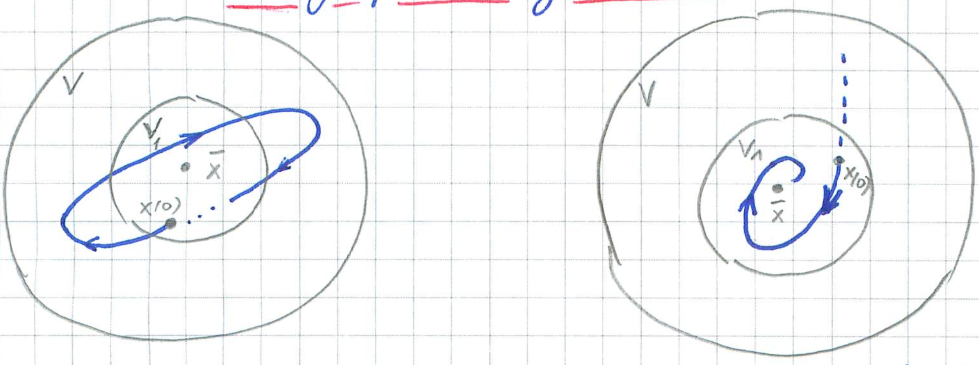
Fixed points, also called equilibria or zeroes, are an important class of solutions of a diff. eqn., they are defined by $\bar{x}$ such that $f(\bar{x})=0$.

A fixed point $\bar{x}$ is called stable, if any solution $x(t)$ based nearby remains close to $\bar{x}$ for all time, i.e.

$\forall$ nbhd V of $\bar{x}$ in U, $\exists$ a nbhd $V_1 \subset V$ s.t.

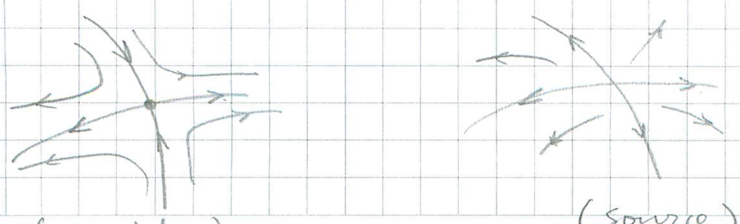
every solution $x(x_0, t)$ with $x_0 \in V_1$ is defined and lies in V for all $t > 0$.

If in addition, V_1 can be chosen s.t. $x(t) \to \bar{x}$ as $t \to \infty$, then $\bar{x}$ is called asymptotically stable.



(a) Stability (b) asymp. stability. (sink)

A fixed point is called unstable, if it is not stable.



(saddle) (source)

Note: stability is a local concept

A useful way to show stability : Liapunov functions

Theorem 1.0.2 (stability of fixed points using Liapunov functions) Let $\bar{x}$ be a fixed point for (1.0.1) and $V: W \to \mathbb{R}$ be a diff. function defined on some nbhd $W \subseteq U$ of $\bar{x}$ s.t.

(i) $V(\bar{x}) = 0$ and $V(x) > 0$ if $x \neq \bar{x}$; and

(ii) $\dot{V}(x) \leq 0$ in $W - \{\bar{x}\}$.

Then, $\bar{x}$ is stable. Moreover, if

(iii) $\dot{V}(x) < 0$ in $W - \{\bar{x}\}$;

then $\bar{x}$ is asymptotically stable.

Here,
$$\dot{V} = \sum_{j=1}^{n} \frac{\partial V}{\partial x_j} \dot{x}_j = \sum_{j=1}^{n} \frac{\partial V}{\partial x_j} f_j(x)$$
is the derivative of V along solution curves of (1.0.1)

Remark : (i) if we can choose $W = \mathbb{R}^n$ in case (iii), then x is called globally asymp. stable, and we can conclude that all solutions remain bounded and approach ~~x~~ $\bar{x}$ as $t \to \infty$. (without solving the equation)

(ii) however, there is no general methods for finding suitable Liapunov functions (in mechanical problems, the energy is often a good candidate)

1.1. The linear system $\dot{x} = Ax$

Consider

$$\frac{dx}{dt} \stackrel{\text{def}}{=} \dot{x} = Ax, \qquad x \in \mathbb{R}^n \tag{1.1.1}$$

where A is an $n \times n$ matrix with constant components. The eqn. (1.1.1) is called a linear eqn, since $\forall a \in \mathbb{R}$, $y(t) := ax(t)$,

$\dot{y} = a\dot{x} = aAx = A(ax) = Ay \rightsquigarrow y$ is a sol, if x is.

Given an initial condition $x(0) = x_0$, (1.1.1) has a global solution given by

$$x(x_0, t) = e^{tA} x_0,$$

where e^{tA} is the $n \times n$ matrix defined by

$$e^{tA} = I + tA + \frac{t^2}{2!} A^2 + \ldots + \frac{t^n}{n!} A^n + \ldots$$

A general solution of (1.1.1) can be obtained from a set of n linearly independent solutions $\{x^1(t), \ldots, x^n(t)\}$:

$$x(t) = \sum_{j=1}^{n} c_j x^j(t),$$

where $c_1, \ldots, c_n$ are unknown constants to be determined by the initial condition. Denote by

$$X(t) = [x^1(t), x^2(t), \ldots, x^n(t)]$$

the fundamental solution matrix, then $e^{tA} = X(t)X(0)^{-1}$.

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(1.1.1) can also be solved by first diagonalizing A:
Let T be an invertible matrix s.t.

$$J = T^{-1} A T$$

is a Jordan form of A. Then (1.1.1) becomes:

$$\dot{y} = Jy \quad \text{for} \quad x = Ty.$$

Then, $$e^{tA} = T e^{tJ} T^{-1}.$$