Let H_k be the linear space of vec. fields whose coefficients are homogeneous polyn. of degree k, i.e.

$$H_k = \left\{ \begin{pmatrix} P_1 \\ P_2 \\ \vdots \\ P_N \end{pmatrix} : \begin{array}{l} P_i \text{ is a homog. polyn. of degree } k \text{ in } (x_1, \dots, x_N) \\ \text{i.e. } P_i \text{ is a lz. comb. of monomials of deg. } k. \end{array} \right\}$$

Let $L = Df(0) \cdot x$, the linear part of „ $\dot{x} = f(x)$, $f(0)=0$."

Define a linear operator on H_k using L:

$$\begin{aligned} \operatorname{ad} L : \quad H_k &\longrightarrow H_k \\ Y &\longmapsto \underset{\uparrow \text{ Lie Bracket}}{[Y, L]} = DL\,Y - DY\,L. \end{aligned}$$

<u>Example 2.3.2</u> (part I) Let $Df(0) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, then $L = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -y \\ x \end{pmatrix}$.

Consider $H_2 = \left\{ \begin{pmatrix} P_1 \\ P_2 \end{pmatrix} : P_i = P_i(x,y) = a_i x^2 + b_i xy + c_i y^2, \; i = 1,2, \; a_i, b_i, c_i \in \mathbb{R} \right\} \cong \mathbb{R}^6$, a 6-dim$^{\underline{l}}$

real vector space spanned by $\begin{pmatrix} x^2 \\ 0 \end{pmatrix}, \begin{pmatrix} xy \\ 0 \end{pmatrix}, \begin{pmatrix} y^2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ x^2 \end{pmatrix}, \begin{pmatrix} 0 \\ xy \end{pmatrix}, \begin{pmatrix} 0 \\ y^2 \end{pmatrix}$.

write as $\quad x^2\frac{\partial}{\partial x} \quad xy\frac{\partial}{\partial x} \quad y^2\frac{\partial}{\partial x} \quad x^2\frac{\partial}{\partial y} \quad xy\frac{\partial}{\partial y} \quad y^2\frac{\partial}{\partial y}$

$$\operatorname{ad}L\left(x^2\frac{\partial}{\partial x}\right) = \operatorname{ad}L\begin{pmatrix} x^2 \\ 0 \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}}_{DL}\begin{pmatrix} x^2 \\ 0 \end{pmatrix} - \underbrace{\begin{pmatrix} 2x & 0 \\ 0 & 0 \end{pmatrix}}_{DY}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2xy \\ x^2 \end{pmatrix} = 2xy\frac{\partial}{\partial x} + x^2\frac{\partial}{\partial y}$$

$$\operatorname{ad}L = 6\times 6\text{-matrix} = \begin{array}{c} \\ x^2\frac{\partial}{\partial x} \\ xy\frac{\partial}{\partial x} \\ y^2\frac{\partial}{\partial x} \\ x^2\frac{\partial}{\partial y} \\ xy\frac{\partial}{\partial y} \\ y^2\frac{\partial}{\partial y} \end{array} \begin{array}{c} \begin{array}{cccccc} x^2\frac{\partial}{\partial x} & xy\frac{\partial}{\partial x} & y^2\frac{\partial}{\partial x} & x^2\frac{\partial}{\partial y} & xy\frac{\partial}{\partial y} & y^2\frac{\partial}{\partial y} \end{array} \\ \left(\begin{array}{ccc|ccc} 0 & -1 & 0 & -1 & 0 & 0 \\ 2 & 0 & -2 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 \\ \hline 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 2 & 0 & -2 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{array}\right) \end{array}$$

Theorem 2.3.3 Let $\dot{x} = f(x)$ be a C^r-system of diff. equations with $f(0)=0$ and $Df(0)x = L$. Let G_k be a complement for the range $\operatorname{ad}L(H_k)$ of $\operatorname{ad}L$ in H_k, i.e. $H_k = \operatorname{ad}L(H_k) + G_k$. Then,

$\exists$ an analytic change of coord. in a nbhd of 0 s.t. $\dot{x} = f(x)$ is transformed to

$$\dot{y} = g(y) = g^{(1)}(y) + g^{(2)}(y) + \dots + g^{(r)}(y) + R_r$$

where $L = g^{(1)}(y)$, $g^{(k)} \in G_k$ for $2 \le k \le r$ and $R_r = o(|y|^r)$

Pf: (Sketch), By induction, assume $\dot{x} = f(x)$ is transformed s.t.

$$\dot{x} = f^{(1)}(x) + f^{(2)}(x) + \dots + f^{(s)}(x) + R \quad , \quad L = f^{(1)}(x), \text{ and } f^{(k)} \in G_k \ \forall_{2 \le k < s}$$

Let $x = y + P(y)$ with P being a homg. polyn. of degree s. Then

$$(I + DP(y))\dot{y} = f^{(1)}(y) + f^{(2)}(y) + \dots + f^{(s)}(y) + Df(0)P(y) + o(|y|^s)$$

$\ulcorner$

$$\dot{x} = (I + DP(y))\dot{y} = f^{(1)}(x) + \dots + f^{(s)}(x) + R = f^{(1)}(y+P(y)) + \dots + f^{(s)}(y+P(y)) + R$$

$$= f^{(1)}(y) + \dots + f^{(s)}(y) + Df(0)P(y) + R' \lrcorner$$

$$\xrightarrow{(I+DP)^{-1} \approx I - DP} \quad \dot{y} = f^{(1)}(y) + \dots + f^{(s)}(y) + \underbrace{Df(0)}_{=DL} P(y) - DP(y) \underbrace{f^{(1)}(y)}_{=L} + o(|y|^s)$$

i.e. the term of degree s is

$$f^{(s)}(y) + DLP(y) - DP(y)L = f^{(s)}(y) + \operatorname{ad}L(P)(y)$$

choose P s.t. the above lies in G_s. #.

Example 2.3.2. (part II) Consider the system

$$\begin{cases} \dot{x} = -y + o(|x|,|y|) \\ \dot{y} = x + o(|x|,|y|) \end{cases} \quad \text{or} \quad \dot{x} = f(x) \text{ with } f(0)=0,\ Df(0) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

and we derive the norm form for it:

Let $L = Df(0)\begin{pmatrix} x \\ y \end{pmatrix}$ and $ad L : H_k \to H_k$ as defined above.

$k=2$: as calculated in Example 2.3.2 (part I), $ad L : H_2 \to H_2$ is given by a 6×6-matrix, which is nonsingular, thus $G_2 = \{0\}$. By Thm 2.3.3, all degree-2 terms can be removed from f by change of coord.

$k=3$: Similar calculation shows:

$$ad L = \begin{array}{c} \\ x^3\frac{\partial}{\partial x} \\ x^2y\frac{\partial}{\partial x} \\ xy^2\frac{\partial}{\partial x} \\ y^3\frac{\partial}{\partial x} \\ x^3\frac{\partial}{\partial y} \\ x^2y\frac{\partial}{\partial y} \\ xy^2\frac{\partial}{\partial y} \\ y^3\frac{\partial}{\partial y} \end{array} \begin{array}{c} \begin{array}{cccccccc} x^3\frac{\partial}{\partial x} & x^2y\frac{\partial}{\partial x} & xy^2\frac{\partial}{\partial x} & y^3\frac{\partial}{\partial x} & x^3\frac{\partial}{\partial y} & x^2y\frac{\partial}{\partial y} & xy^2\frac{\partial}{\partial y} & y^3\frac{\partial}{\partial y} \end{array} \\ \left(\begin{array}{cccc|cccc} 0 & -1 & 0 & 0 & -1 & 0 & 0 & 0 \\ 3 & 0 & -2 & 0 & 0 & -1 & 0 & 0 \\ 0 & 2 & 0 & -3 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 \\ \hline 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 3 & 0 & -2 & 0 \\ 0 & 0 & 1 & 0 & 0 & 2 & 0 & -3 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \end{array}\right) \end{array}$$

is singular with 2-dimensional kernal. $\Rightarrow G_3 \cong \mathbb{R}^2$, choices of G_3:

Choice 1: $G_3 = \ker(ad L) = \langle v_1, v_2 \rangle = \left\langle \begin{pmatrix} 3 \\ 0 \\ 1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 3 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 3 \\ -3 \\ 0 \\ -1 \\ 0 \end{pmatrix} \right\rangle$, v_1, v_2 eigenvectors of 0.

Then, $G_3 = \left\{ a\left(3x^2\frac{\partial}{\partial x} + xy^2\frac{\partial}{\partial x} + x^2y\frac{\partial}{\partial y} + 3y^3\frac{\partial}{\partial y}\right) + b\left(x^2y\frac{\partial}{\partial x} + 3y^3\frac{\partial}{\partial x} - 3x^3\frac{\partial}{\partial y} - xy^2\frac{\partial}{\partial y}\right) : a, b \in \mathbb{R} \right\}$

$$= \left\{ \begin{pmatrix} a(3x^3 + xy^2) + b(x^2y + 3y^3) \\ \\ a(x^2y + 3y^3) + b(-3x^3 - xy^2) \end{pmatrix} : a, b \in \mathbb{R} \right\}$$

<u>Choice 2</u>: $G_3 = \left\langle \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 0 \\ -1 \\ 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \right\rangle$ is also a complement of $ad L(H_3)$, then

$$G_3 = \left\{ a\left(x^3\frac{\partial}{\partial x} + xy^2\frac{\partial}{\partial x} + x^2y\frac{\partial}{\partial y} + y^3\frac{\partial}{\partial y}\right) + b\left(-x^2y\frac{\partial}{\partial x} - y^3\frac{\partial}{\partial x} + x^3\frac{\partial}{\partial y} + xy^2\frac{\partial}{\partial y}\right) : a, b \in \mathbb{R} \right\}$$

$$= \left\{ \begin{pmatrix} (ax - by)(x^2 + y^2) \\ \\ (ay + bx)(x^2 + y^2) \end{pmatrix} : a, b \in \mathbb{R} \right\}$$

$\xRightarrow{\text{Thm 2.3.3}}$ the system can be transformed to

$$\begin{cases} \dot{x} = -y + (ax - by)(x^2 + y^2) + o(\|(x,y)\|^3) \\ \\ \dot{y} = x + (ay + bx)(x^2 + y^2) + o(\|(x,y)\|^3) \end{cases},$$

where the values of a, b are determined by higher derivatives of f.

#.

Normal forms for parametrised system:

$$\tilde{f}(x,\mu) := \begin{pmatrix} f(x,\mu) \\ 0 \end{pmatrix}$$

$\dot{x} = f(x,\mu)$, $x \in \mathbb{R}^N$, $\mu \in \mathbb{R}$ $\xrightarrow{\text{extend}}$ $\left\{ \begin{array}{l} \dot{x} = f(x,\mu) \\ \dot{\mu} = 0 \end{array} \right\}$ $x \in \mathbb{R}^N$, $\mu \in \mathbb{R}$.

$f(0,\mu) = 0$ $\forall_\mu$ in a nbhd of $\mu_0 \in \mathbb{R}$

Consider $\begin{pmatrix} x \\ \mu \end{pmatrix} = H(y,\mu) = \begin{pmatrix} h(y,\mu) \\ \mu \end{pmatrix} = \begin{pmatrix} y + P(y,\mu) \\ \mu + 0 \end{pmatrix}$ for some $\overbrace{\text{homog. polyn.}}^{P = P(y,\mu)}$ of deg. k,

in order to reduce the parametrised system to its "normal form".

Let $L = D_{\boxed{(x,\mu)}} \tilde{f}\boxed{(0,\mu_0)} \begin{pmatrix} x \\ \mu \end{pmatrix}$ "the linear part of $\left\{ \begin{array}{l} \dot{x} = f(x,\mu) \\ \dot{\mu} = 0 \end{array} \right.$ "

Let H_k be the lin. space of vec. fields whose coefficients are homogeneous polyn. in $\boxed{(x,\mu)}$ of degree k; i.e.

$$H_k(x,\mu) = \left\{ \begin{pmatrix} P_1 \\ \vdots \\ P_N \\ 0 \end{pmatrix} : P_i \text{ is a homog. polyn. of deg. } k \text{ in } (x_1, \dots, x_N, \mu) \right\}$$

Define $\mathrm{ad}L : \quad H_k(x,\mu) \longrightarrow H_k(x,\mu)$

$$Y = Y(x,\mu) \longmapsto DLY - DYL$$

Theorem 2.3.3' Let $\dot{x} = f(x,\mu)$ be a C^r-system with $f(0,\mu) = 0$ $\forall_{\mu \text{ near } \mu_0}$

and $D_{(x,\mu)} \tilde{f}(0,\mu_0) \begin{pmatrix} x \\ \mu \end{pmatrix} = L$. If G_k is a complement for $\mathrm{ad}L(H_k)$ in H_k, then $\exists$ analytic change of coord. (near 0) s.t. $\dot{x} = f(x,\mu)$ becomes

$\dot{y} = g(y,\mu) = g^{(1)}_{(y,\mu)} + \dots + g^{(r)}_{(y,\mu)} + o(|y|^r)$ wh. $L = g^{(1)}_{(y,\mu)}$, $g^k \in G_k$ $\forall_{2 \le k \le r}$.

Example 2.3.2 (part II) Consider the param. system

$$\begin{cases} \dot{x} = \mu x - \omega y + o(|x|,|y|) \\ \dot{y} = \omega x + \mu y + o(|x|,|y|) \end{cases}$$

$(x,y) \in \mathbb{R}^2$, $\omega \in \mathbb{R}$ fixed constant, $\mu \in \mathbb{R}$ parameter

The linearization $\begin{pmatrix} \mu & -\omega \\ \omega & \mu \end{pmatrix}$ has eigenvalues $\mu \pm i\omega$, so has purely imaginary eigenvalues $\pm i\omega$, for $\mu = \mu_0 = 0$.

Let $L = \left(\begin{array}{cc|c} \cancel{\mu}0 & -\omega & 0 \\ \omega & 0\cancel{\mu} & 0 \\ \hline 0 & 0 & 0 \end{array}\right)\begin{pmatrix} x \\ y \\ \mu \end{pmatrix}$ and $ad L : H_k \to H_k$. One can show that the system can be brought to the form:

$$(*_p) \begin{cases} \dot{x} = \mu x - \omega y + (ax - by)(x^2+y^2) + O(|(x,y)|^5) \\ \dot{y} = \omega x + \mu y + (ay + bx)(x^2+y^2) + O(|(x,y)|^5) \end{cases}$$

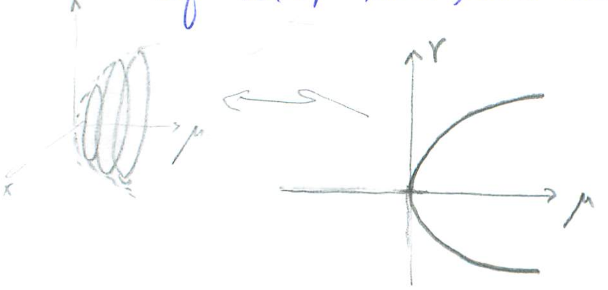
where $a, b \in \mathbb{R}$ are determined by the nonlinearity $o(|(x,y)|^3)$.

Using polar coordinates, we have

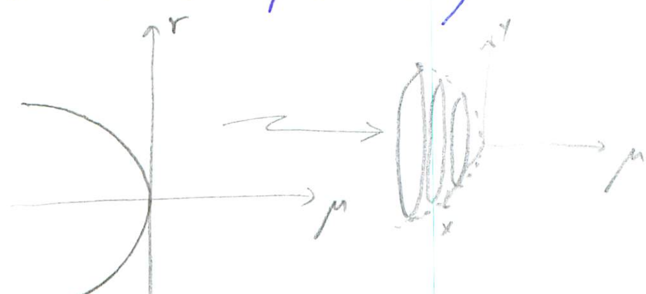
$$\begin{cases} \dot{r} = (\mu + ar^2) r \\ \dot{\theta} = \omega + br^2 \end{cases},$$

So periodic solutions can be obtained from $\dot{r} = 0$.

If $a \neq 0$, then there solutions lie on the parabola $\mu = -ar^2$



$a<0$.



$a>0$.

It is worthwhile noting that (*) is neatly expressed in complex varibles as

$$\dot{w} = \lambda w + c w^2 \bar{w} + O(|w|^5),$$

where $\lambda = \mu + i\omega$, $c = a + ib$, $w = x + iy$.

#.

2.4. Codimension-One bifurcations of equilibria

Recall "codimension one bifurcations" are related to either (i) a simple zero eigenvalue $D_x f_\mu = \begin{pmatrix} 0 & 0 \\ 0 & A \end{pmatrix}$

or (ii) a simple pure ima. pair $D_x f_\mu = \left(\begin{array}{c|c} \begin{pmatrix} 0 & -\omega \\ \omega & 0 \end{pmatrix} & 0 \\ \hline 0 & A \end{array} \right)$,

where A is a square matrix s.t. $\sigma(A) \cap i\mathbb{R} = \phi$.

We give some simplest examples of codim-1 bifurcations of equilibria:

(i1) saddle-node bifurcation

(i2) transcritical bifurcation

(i3) pitchfork bifurcation

(ii) Hopf bifurcation

2.4.1 Codim-One bifurcations related to a simple zero eigenvalue

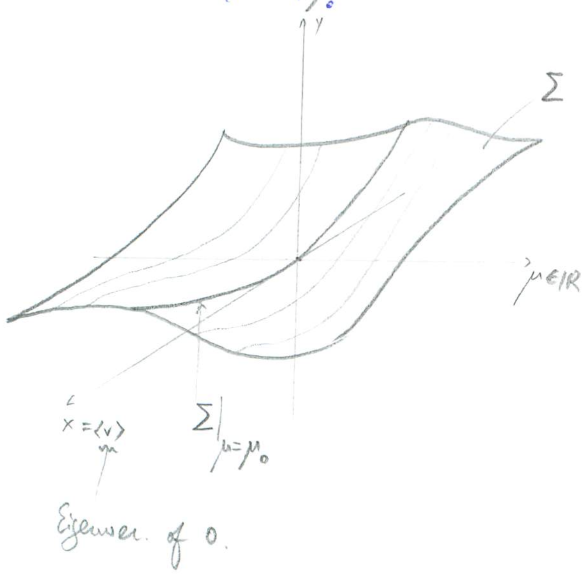
Consider $\dot{x} = f(x,\mu)$, $x \in \mathbb{R}^n$, $\mu \in \mathbb{R}$, f_μ smooth.

Assume $f(x_0,\mu_0)=0$ and $\partial_x f(x_0,\mu_0)$ has a simple eigenvalue 0 (other than that no other eigenvalues on $i\mathbb{R}$).

Center Mfld Theorem $\longrightarrow$ $\exists$ 2-dimensional center mfld $\Sigma \subset \mathbb{R}^n \times \mathbb{R}$ passing through (x_0,μ_0)

(which is spanned by an eigenvector of 0 and the μ-axis)

s.t. Σ and the flow on Σ describe the qualitative structure of trajectories which remain close to (x_0,μ_0) (in forwards time or backwards time).



$\xrightarrow[\text{to } \Sigma]{\text{restriction}}$ a one-parameter family of equations on the one-dimensional curves Σ_μ in Σ obtained by fixing μ.

"restriction of our bif. problem".

Therefore, it suffices to consider: