9.3 BENDING AN ELASTIC ROD III

We now show how the theory of global bifurcation in cones can be applied to the boundary-value problem in §8.5:

$$\phi''(x) + \lambda \sin\phi(x) = 0 \text{ for } x \in [0, L], \quad \phi'(0) = \phi'(L) = 0, \tag{9.5}$$

where L is fixed and $\lambda > 0$ is the parameter in the problem. As before let

$$\mathbb{F} = \mathbb{R}, \quad X = \{\phi \in C^2[0, L] : \phi'(0) = \phi'(L) = 0\}, \quad Y = C[0, L],$$

and define $F(\lambda, \phi) = \phi'' + \lambda \sin\phi$. Then $F : \mathbb{R} \times X \to Y$ is $\mathbb{R}$-analytic,

$$\partial_\phi F[(\lambda, 0)]\phi = 0$$

if and only if

$$\phi'' + \lambda\phi = 0 \in Y \text{ and } \phi'(0) = \phi'(L) = 0$$

and the bifurcation points form the set $\{\lambda_K = (K\pi/L)^2 : K \in \mathbb{N}\}$.

Here we focus on finding a global extension of the local bifurcation at the point $(\pi/L)^2$ corresponding to $K = 1$. In keeping with the notation of the last section let λ_0 denote $(\pi/L)^2$ and let $\xi_0(x) = \cos(\pi x/L)$, $x \in [0, L]$. Next we verify the hypotheses of Theorem 9.2.2. We have already seen that (G1) and (G3) hold. To check (G2) let $(\lambda, \psi) \in \mathbb{R} \times X$ be a solution of (9.5). Then

$$d_\phi F[(\lambda, \phi)](\psi) = \psi'' + \lambda\psi \cos\phi, \quad \psi \in X.$$

By the theory of ordinary differential equations, $\psi'' + \lambda\psi \cos\phi = 0$ has two linearly independent solutions at most one of which is in X. If there are no solutions in X the problem

$$\psi'' + \lambda\psi \cos\phi = f, \quad \psi \in X \tag{9.6}$$

has a solution ψ for every $f \in Y$. If, on the other hand, it has a solution $\hat{\psi} \in X$, then (9.6) has a solution if and only if

$$\int_0^L \hat{\psi}(x) f(x)\, dx = 0.$$

In both cases the range is closed, the codimension of the range and the dimension of kernel of $d_\phi F[(\lambda, \phi)]$ coincide.

This shows that in all cases $d_\phi F[(\lambda, \phi)]$ is a Fredholm operator of index zero and so (G2) holds.

Now let $\mathcal{K} \subset X$ be the cone defined by

$$\mathcal{K} = \{u \in X : u \text{ is odd about } L/2 \text{ and } u \geq 0 \text{ on } [0, L/2]\}.$$

We have seen that in this example hypothesis (a) of Theorem 9.2.2 holds, and (c) is obvious since the (unique up to normalization) eigenfunction corresponding the eigenvalue $(\pi K/L)^2$ is $\cos(K\pi x/L)$ and only when $K = 1$ is it in $\mathcal{K}$.

To see that (d) holds suppose that $(\lambda, \phi) \in \mathbb{R} \times (\mathcal{K} \setminus \{0\})$ satisfies (9.5). Then clearly $\lambda \neq 0$, $\sin\phi(0) \neq 0$ and $\sin\phi(L) \neq 0$. (If any one of them is zero then ϕ is a constant, by the uniqueness theorem for the initial-value problems for second order ordinary differential equations, and so ϕ is not odd about $L/2$.) Also any solution $(\hat{\lambda}, \hat{\phi})$ of (9.5) satisfies

$$\tfrac{1}{2}\hat{\phi}'(x)^2 + \hat{\lambda}\cos\hat{\phi}(0) - \hat{\lambda}\cos\hat{\phi}(x) \equiv 0 \text{ on } [0, L] \tag{9.7}$$

and, if $\hat{\lambda} \neq 0$, $\cos\hat{\phi}(0) = \cos\hat{\phi}(L)$.

Since $\lambda \neq 0$ and the derivative of cosine at $\phi(0)$ and at $\phi(L)$ is not zero, it follows that if $(\hat{\lambda}, \hat{\phi})$ is a solution of (9.5) which is sufficiently close to (λ, ϕ) then $\hat{\phi}(0) = -\hat{\phi}(L)$. Hence the functions $\hat{\phi}(x)$ and $-\hat{\phi}(L - x)$ solve the same initial value problem, and so are equal. This shows that $\hat{\phi}$ is odd about $L/2$.

Now to show that $\hat{\phi} \geq 0$ on $[0, L/2]$ suppose that there is a sequence (λ_k, ϕ_k) of solutions of (9.5) which converges to (λ, ϕ) in $\mathbb{R} \times X$ such that $\phi(x_k) < 0$, $x_k \in [0, L/2)$. Since $\phi_k(L/2) = 0$ and $\phi_k(0) > 0$ for k sufficiently large, we may assume that x_k is a minimizer of ϕ_k on $[0, L/2]$ and hence $\phi_k'(x_k) = 0$. In the limit as $k \to \infty$ we find that there exists $x \in [0, L/2]$ with $\phi(x) = \phi'(x) = 0$. By the uniqueness theorem for initial value problems this means that $\phi \equiv 0$, which is false. This contradiction establishes (d).

It remains to show (b), that $\mathcal{R}^+ \subset \mathbb{R} \times \mathcal{K}$. First we show that if $(\lambda, \phi) \in \mathcal{R}^+$, for $\epsilon > 0$ sufficiently small, then ϕ is odd about $L/2$. Recall from Theorem 8.4.1 that

$$\mathcal{R}^+ = \{(\Lambda(s), s(\xi_0 + \tau(s)) : s \in (0, \epsilon)\},$$

where $\Lambda(s) \rightarrow 1$ and $\tau(s) \rightarrow 0$ in X as $s \rightarrow 0$. To complete the proof that hypothesis (b) is satisfied recall that $\xi_0(x) = \cos(\pi x/L)$ and hence $\xi_0(0) = 1 = -\xi_0(L)$. So (9.7) gives

$$\cos\left(s(1 + \tau(s)(0))\right) = \cos\left(s(-1 + \tau(s)(L))\right) = \cos\left(s(1 - \tau(s)(L))\right)$$

whence $s(1 + \tau(s)(0)) = \pm s(1 - \tau(s)(L))$ for $s > 0$ sufficiently small. It follows that the sign must be plus, and $\tau(s)(0) = -\tau(s)(L)$. Thus $s(\xi_0 + \tau(s))$ is odd about $L/2$ for $s > 0$ sufficiently small. Now $\kappa(s) = s(\xi_0 + \tau(s)) \geq 0$ on $[0, L/2]$ follows since $\kappa(L/2) = 0$, $\kappa(s)'(L/2) = s(-\pi/L + \tau(s)'(L/2))$ and $\tau(s) \rightarrow 0$ in X as $s \rightarrow 0$. Hence hypothesis (b) is satisfied.

Thus Theorem 9.2.2 gives the existence of a curve

$$\mathfrak{R} = \{(\Lambda(s), \kappa(s) : s \in [0, \infty)\}$$

with $(\Lambda(0), \kappa(0)) = ((\pi/L)^2, 0)$, $\kappa(s) \in \mathcal{K}$ for $s > 0$ and

$$\|(\Lambda(s), \kappa(s))\| \rightarrow \infty \text{ as } s \rightarrow \infty.$$

If now $(\lambda, \phi) = (\Lambda(s), \kappa(s)) \in \mathfrak{R}$ satisfies (9.5) it is obvious that $\lambda \neq 0$ and, by connectedness, $\lambda > 0$ for all $(\lambda, \phi) \in \mathfrak{R}$. Multiplying (9.5) by ξ_0 and integrating by parts gives

$$0 = \int_0^L \xi_0\,(\phi'' + \sin\phi)\,dx = \int_0^L \phi\,\xi_0\left(-\left(\frac{\pi}{L}\right)^2 + \frac{\lambda\sin\phi}{\phi}\right)dx$$

Since ϕ, $\xi_0 \in \mathcal{K}$, the product $\phi\xi_0$ is non-negative and not identically zero. Since $\lambda > 0$ and $(\lambda \sin\phi)/\phi < \lambda$, it follows that $\lambda > (\pi/L)^2$ for all $(\lambda, \phi) \in \mathfrak{R}$, $\phi \neq 0$. Hence the global curve lies to the right of the bifurcation point. Since $\phi'(0) = 0 = \phi(L/2)$ for all solutions of (9.5), it is immediate that the set

$$\{(\lambda, \phi) \in \mathfrak{R} : \lambda \leq M\}$$

is bounded in $\mathbb{R} \times X$ for all finite M. Since $\mathfrak{R}$ is unbounded,

$$\{\lambda : (\lambda, \phi) = (\Lambda(s), \kappa(s)) : s > 0\} = ((\pi/L)^2, \infty).$$

Finally, if (λ, ϕ) is a solution of (9.5) ϕ can be extended as a smooth $2L$ periodic function on the real line. When this has been done, let

$$\mathfrak{R}_K = \{(K^2\lambda, \phi(Kx)) : (\lambda, \phi) \in \mathfrak{R}\}.$$

It is an easy matter to check that $\mathfrak{R}_K$ is a global branch of solutions bifurcating from $(KL/\pi)^2$, $K \in \mathbb{N}$.

Thus many qualitative features of the *global* bifurcation of solutions of (9.5), observed originally in the introduction, are a consequence of abstract considerations based on the theory of real analytic varieties and it is clear that the abstract method has much greater applicability. In the remaining chapters we give a substantial example to which the global theory makes a vital contribution.