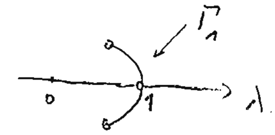
11.2. Global bifurcation theory

- **global bifurcation from $\lambda=1$**

Recall that Theorem 11.1.1 gives a parametrization of local bifurcating solution $(\lambda,w)\in(0,\infty)\times Y$ of (C) from $(1,0)$ given by

$$\Gamma_1=\left\{\left(\Lambda_1(s),\ s(\varphi_1+\Phi_1(s))\right):\ s\in(-\varepsilon,\varepsilon)\setminus\{0\}\right\}\subset\mathcal{E},$$



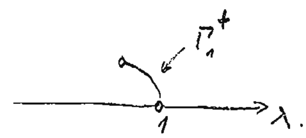
where $(\Lambda_1,\Phi_1):(-\varepsilon,\varepsilon)\to O_1$ is a real-analytic function and $\varphi_1(t)=\text{cost}$;

O_1: nbhd of $(1,0)$ in $(0,\infty)\times Y$

$$\mathcal{E}=\{(\lambda,w)\in(0,\infty)\times Y:\ (\lambda,w)\text{ is a solution of (C)},\ w\neq 0\}$$ ← all non-trivial solutions of (C).

$$\mathcal{U}=\{(\lambda,w)\in(0,\infty)\times Y:\ 1-2\lambda w>0\}$$ ← open in $(0,\infty)\times Y$

$$\Gamma_1^+=\{(\Lambda_1(s),\ s(\varphi_1+\Phi_1(s)):\ s\in(0,\varepsilon)\}$$



the upper symmetric half of Γ_1

$$\mathcal{E}_1^+=\overline{\text{max. connected set containing }\Gamma_1^+\text{ in }\mathcal{E}\cap\mathcal{U}}$$ ← the "branch" of solutions extended from Γ_1^+

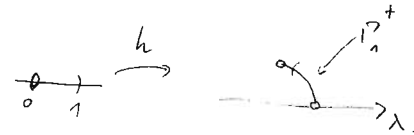
$$S=\{(\lambda,w)\in\mathcal{E}_1^+:\ \partial_w G(\lambda,w):Y\to X\text{ is a homeomorphism}\}$$ ← contains only non-bifurcating points

Theorem 11.2.1 (Existence of global continuum)

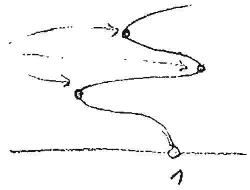
For equation (C), there exists a function $h=(\Lambda,W):(0,\infty)\to\mathbb{R}\times Y$ s.t.

(i) $h:(0,\infty)\to\overline{S}\cap\mathcal{U}$ is continuous;

(ii) $h((0,1))\subset\Gamma_1^+$ and $\lim\limits_{s\to 0}h(s)=(1,0)$;



turning points; $\Lambda'(s)=0$



(iii) h is injective on $h^{-1}(S)$; no overlapping.

(iv) at all points $s \in h^{-1}(S)$, h is real-analytic with $\Lambda'(s) \neq 0$; no turning.

(v) the set $h^{-1}(\bar{S} \setminus S) \subset (0,\infty)$ consists of isolated values;

(vi) locally near every point $s_0 \in h^{-1}(\bar{S} \setminus S)$, there exists an injective and continuous re-parametrization $\gamma : [-1,1] \to (s_0-\varepsilon, s_0+\varepsilon)$ s.t. $s=\gamma(t)$ for $t \in [-1,1]$, $s_0 = \gamma(0)$ and $h \circ \gamma$ is a real-analytic function.

11.2.1.

Let $\mathcal{R} = \{ h(s) : s \in [0,\infty) \} \overset{(i)}{\subset} \bar{S} \cap \mathcal{U} \subset \mathcal{U} \subset (0,\infty) \times Y$

- $\mathcal{R}$ is path-connected in $(0,\infty) \times C^k$, for all $k \in \mathbb{N}$

$\left.\begin{array}{l}\mathcal{R} \text{ is path-conn. in } \mathcal{U} \\ \mathcal{U} \text{ is open in } (0,\infty)\times Y\end{array}\right\} \Rightarrow \mathcal{R}$ is path-conn. in $(0,\infty)\times Y \xRightarrow[10.5.4]{\text{Theorem}} \mathcal{R}$ is path-conn. in $(0,\infty)\times C^k$, $\forall k \in \mathbb{N}$

Let $\mathcal{W} = \{ (\lambda, w) \in \mathbb{R} \times C^{2,\alpha} : (\lambda,w)$ is a solution of (C) and $1+[w'>0$ on $[-\pi,\pi]$, $\underbrace{w'<0 \text{ on } (0,\pi)}_{w(0)=\max\{w(t)\,:\,t\in[-\pi,\pi]\}}$, $w''(0)<0<w''(\pi) \}$.

- $\mathcal{R} \subset \mathcal{W}$

By the local expression of $w=w_s$ in the end of Section 11.1, one shows that $h(s) \in \mathcal{W}$ for $s \in (0,\varepsilon)$, if $\varepsilon>0$ is small.

Thus, $\underline{\mathcal{R} \cap \mathcal{W} \neq \phi}$. On the other hand, $\underline{\mathcal{R}\cap\mathcal{W} \text{ is both open and closed in } \mathcal{R}}$, Since
  - $\mathcal{R} \cap \mathcal{W}$ is open in $\mathcal{R}$ by definition of $\mathcal{W}$
  - $\overline{\mathcal{R} \cap \mathcal{W}} = \mathcal{R} \cap \mathcal{W}$ by Lemma 10.4.1.

Since $\mathcal{R}$ is path-connected, this is only possible, if $\mathcal{R} \cap \mathcal{W} = \mathcal{R}$, i.e. $\mathcal{R} \subset \mathcal{W}$.

Thus, Theorem 10.4.4 can be applied to $\mathcal{R}$, so

$$\forall_{(\lambda,w)\in\mathcal{R}} \quad 0<k\le\lambda\le K<\infty, \quad -K\le w\le K, \quad \text{for some } k,K>0.$$

- $\mathcal{R}$ is unbounded in $(0,\infty)\times Y$.

Assume that $\mathcal{R}$ is bounded in $(0,\infty)\times Y$. Then, by Theorem 9.1.1 (e), $\mathcal{R}$ forms a closed loop in $(0,\infty)\times Y$. Let

$$\mathcal{K}=\{w\in Y : w'\le 0 \text{ on } (0,\pi)\},$$

which is a cone in Y. Also, $\mathcal{R}\subset\mathbb{R}\times\mathcal{K}$. Thus, Theorem 9.2.2's conditions (a), (b) hold. Moreover, (c) holds for $\xi_0=\text{cst}\in\mathcal{K}$. For (d), let $(\lambda,w)\in\mathcal{R}$, then $w\in C^{2,\alpha}$ and $\vartheta\le\pi/3$ on $(0,\pi)$, also all solutions of (C) close to (λ,w) in $\mathbb{R}\times Y$, are close in $\mathbb{R}\times C^{2,\alpha}$, thus $\vartheta\le\pi/3$ as well. So (d) holds, Consequently, by Thm 9.2.2, $\mathcal{R}$ is not a closed loop $\lightning$

- $1-2\lambda w(0)\to 0$, as $s\to\infty$, when $h(s)=(\lambda,w)$.

Otherwise, $\exists\,\delta>0$ s.t. $1-2\lambda w(0)\ge\delta>0$ $\left(\mathcal{R}\subset\mathcal{U}_{\alpha\, 1-2\lambda w>0}\right)$ $\forall_{(\lambda,w)\in\mathcal{R}}$

$$\Rightarrow \nu=(1-2\lambda w(0))^{3/2}/3\lambda \ge \delta^{3/2}/3\lambda \underset{\substack{\uparrow\\ 0<k\le\lambda\le K<\infty}}{\ge} \delta^{3/2}/K$$

for $t\in(0,\pi)$

$$(1-2\lambda w(t))^{3/2}=3\lambda\Big(\nu+\underbrace{\int_0^t \sin\vartheta(s)\,ds}_{>0 \text{ since } 0<\vartheta(s)<\pi/3,\ s\in(0,\pi)}\Big) > 3\lambda\nu \ge 3k\,\delta^{3/2}/K$$

(CP): $(1-2\lambda w(t))\big(w'(t)^2+\underbrace{(1+\ell w(t))^2}_{>0 \text{ by def of } \mathcal{U}}\big)=1$

$$\Rightarrow w'(t)^2 < w'(t)^2+(1+\ell w(t))^2=\frac{1}{1-2\lambda w(t)} \boxed{< \frac{1}{(3k\delta^{3/2}/K)^{2/3}}=\frac{K^{2/3}}{(3k)^{2/3}\cdot\delta}} \le \frac{1}{1-2\lambda w(0)}\le\frac{1}{\delta}$$

$\Rightarrow \mathcal{R}$ is bounded in $(0,\infty)\times Y$ $\lightning$

We therefore conclude that

$$\begin{cases} h(s) \to \partial\mathcal{U} \text{ as } s\to\infty \quad \text{i.e. } \operatorname{dist}(h(s),\partial\mathcal{U}) \to 0, \text{ as } s\to\infty \\ \mathcal{R} \text{ lies in a bounded subset of } [k,\infty)\times W^{1,p}_{[-\pi,\pi]}, \text{ for some } k>0, \ \forall p\in[1,3) \\ 1-2\lambda w(0) \to 0 \text{ as } s\to\infty, \text{ for } h(s)=(\lambda,w), \end{cases}$$

(cf. Thm. 10.4.4)

where $W^{1,p}_{[-\pi,\pi]} = \{ u:\mathbb{R}\to\mathbb{R} : u \text{ is } 2\pi\text{-periodic}, \ u' \in L^p_{[-\pi,\pi]} \}$. Consequently,

weak derivative

every sequence in $\mathcal{R}$ has a subsequence that is

- convergent weakly in $[k,\infty)\times W^{1,p}_{[-\pi,\pi]}$ $\forall_{1<p<3}$, and
  convergent strongly in $[k,\infty)\times C^\alpha$ $\forall_{\alpha\in(0,\frac{2}{3})}$.

by Banach-Alaoglu Thm (cf. Coro. 10.5.7)

let $(\lambda_n, w_n) \xrightarrow{w} (\lambda_*, w_*) \in [k,\infty)\times W^{1,p}_{[-\pi,\pi]}$, we show $w_* \in C^\alpha$ $\forall_{\alpha\in(0,\frac{2}{3})}$.

$w_* \in W^{1,p} \Rightarrow w_*' \in L^p \Rightarrow \int_{-\pi}^{\pi} |w_*'|^p \, dt < \infty \Rightarrow |w_*'(t)|^p < A$ a.e.

$$\left(\frac{|w_*(x)-w_*(y)|}{|x-y|^\alpha}\right)^p = \frac{|w_*'(\xi)(x-y)|^p}{|x-y|^{\alpha p}} = |w_*'(\xi)|^p |x-y|^{(1-\alpha)p}$$

$(1-\alpha)p > 0$, $< (1-\frac{2}{3})\cdot 3 = 1$

$< A\cdot(\pi-(-\pi))^1 = A\cdot 2\pi$ bounded ✓.

Let $\mathcal{H} = \overline{\mathcal{R}} \setminus \mathcal{R}$ in $[k,\infty)\times C^{\frac{1}{2}}$.

Lemma 11.2.2 $\mathcal{H} \subset \mathbb{R}\times W^{1,p}_{[-\pi,\pi]}$, $1<p<3$ and every $(\lambda,w)\in\mathcal{H}$ satisfies (C) with $1-2\lambda w(0)=0$. Moreover, $\mathcal{H}$ is a compact, connected subset of $\mathbb{R}\times C^{\frac{1}{2}}$.

Pf: • $\mathcal{H} \subset \mathbb{R}\times W^{1,p}_{[-\pi,\pi]}$

$(\lambda,w)\in\mathcal{H} \Leftrightarrow \exists (\lambda_k,w_k)\in\mathcal{R}$ s.t. $(\lambda_k,w_k)\to(\lambda,w)$ in $[k,\infty)\times C^{\frac{1}{2}}$,

$\Rightarrow$ Since every sequence in $\mathcal{R}$ has a weakly conv. subsequence in $[k,\infty)\times W^{1,p}_{[-\pi,\pi]}$, we have $(\lambda,w)\in[k,\infty)\times W^{1,p}_{[-\pi,\pi]} \subset \mathbb{R}\times W^{1,p}_{[-\pi,\pi]}$.

- every $(\lambda,w)\in\mathcal{H}$ satisfies (C) with $1-2\lambda w(0)=0$.

$(\lambda,w)\in\mathcal{H} \Rightarrow \exists (\lambda_k,w_k)\in\mathfrak{R} \overset{11.2.1}{\subset}$ solution set of (C), so

$$\mathcal{C}w_k' = \lambda_k\left(w_k + w_k\,\mathcal{C}w_k' + \mathcal{C}(w_k w_k')\right)$$

$\Rightarrow \forall_{2\pi\text{-per. smooth}}\ \phi$

$$0=\int_{-\pi}^{\pi}\phi\left(\mathcal{C}w_k' - \lambda_k(w_k + w_k\,\mathcal{C}w_k' + \mathcal{C}(w_k w_k'))\right)dt$$

$$\to \int_{-\pi}^{\pi}\phi\left(\mathcal{C}w - \lambda(w + w\,\mathcal{C}w' + \mathcal{C}(ww'))\right)dt,$$

since $\mathcal{C}w_k'\to\mathcal{C}w'$, $w_k'\to w'$ weakly in Y and $w_k\to w$ uniformly on $[-\pi,\pi]$, $k\to\infty$.

$\Rightarrow (\lambda,w)$ satisfies (C).

Also, since $\left.\begin{array}{l} 1-2\lambda_k w_k(0)\to 0 \text{ as } k\to\infty \\ 1-2\lambda_k w_k(0) \underset{\substack{\uparrow \\ w_k\to w \text{ uniformly}}}{\to} 1-2\lambda w(0)\end{array}\right\}$ we have $1-2\lambda w(0)=0$.

- $\mathcal{H}$ is compact

Consider $\{\overset{x^m}{\overset{\shortparallel}{(\lambda^m,w^m)}}\}\subset\mathcal{H}$ a sequence in $\mathcal{H}$, and convergent sequences $\{\overset{x_k^m}{\overset{\shortparallel}{(\lambda_k^m,w_k^m)}}\}\subset\mathfrak{R}$

such that $x_1^m, x_2^m, \dots, x_k^m \dots \to x^m \quad \forall m=1,2\dots$

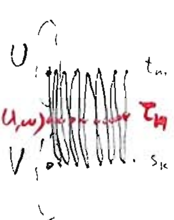
then the sequence $\{x_1^1,\dots,x_k^k\dots\}\subset\mathfrak{R}$ as a sequence in $\mathfrak{R}$ has a convergent subsequence, say itself, and $x_k^k \underset{\text{in } C^{\frac12}}{\longrightarrow} x_*$ as $k\to\infty$. Then $x^m \underset{\text{in } C^{\frac12}}{\to} x_*$ as $m\to\infty$.

Also, $\mathcal{H}=\partial\mathfrak{R}$ is closed, thus $x_*\in\mathcal{H}$. This shows $\mathcal{H}$ is sequentially cpt. ✓

- $\mathcal{H}$ is connected.

Assume not, then $\exists\ \overset{\phi}{\overset{\#}{U}}, \overset{\phi}{\overset{\#}{V}}\subset\mathcal{H}$ distinct compact subsets of $\mathcal{H}$, s.t. $\mathcal{H}=U\cup V$.

Let $\operatorname{dist}(U,V)=d>0$. Take $u\in\mathcal{H}\cap U$, $v\in\mathcal{H}\cap V$. Note that



$\exists\, t_m\to\infty$ s.t. $h(t_m)\to u$ (since $u\in\mathcal{H}\subseteq\overline{\mathfrak{R}}$, and every sequence in $\mathfrak{R}$ has conv. subsq in $C^{\frac12}$)

$\exists\, s_k\to\infty$ s.t. $h(s_k)\to v$.

Since $\mathfrak{R}$ is path connected, every $\overset{\in\mathfrak{R}}{h(t_m)}$ and $\overset{\in\mathfrak{R}}{h(s_k)}$ can be connected by path.

Thus $\exists\,\tau_n\to\infty$ s.t. $\operatorname{dist}(h(\tau_n),U)>d/4$, $\operatorname{dist}(h(\tau_n),V)>d/4$. $h(\tau_n)\underset{\text{for some }(\lambda,w)}{\to}(\lambda,w) \Rightarrow (\lambda,w)\in\mathcal{H}$ ↯ #.

Remark 11.2.3 The set $\mathcal{H}$ consists of Stokes waves of greatest height which arise as the limit as $s \to \infty$ of points on $\mathcal{R}$. It is a connected set, which can not be derived by a topological bifurcation theory.

Remark 11.2.4 Let $\mathcal{R}_k = \{ (k\lambda, w(kt)) : (\lambda, w) \in \mathcal{R} \}$, for $k \in \mathbb{N}$. Then, $\mathcal{R}_k$ contain solutions to (C) that correspond to Stokes waves of period $2\pi/k$. It is the global curve of solutions bifurcating from $(k, 0)$ that would be obtained (using the same method for $\mathcal{R}$) by a study of global bifurcation from the simple eigenvalue $\lambda = k$.

11.3 Gradients, Morse index and bifurcation

We give a brief sketch of how real-analytic bifurcation theory interacts with the gradient structure of (C) to conclude the existence of multiple secondary-bifurcation points on the global branch. The existence of a path, not just a connected set, plays an essential role here.

Def 11.3.1 Let Y be a dense subset of a Hilbert space $(X, \langle\cdot,\cdot\rangle)$ and $G(\lambda,\cdot)$ be the gradient of a C^2-functional $g(\lambda,\cdot)$ such that for some $y \in Y$:

- $G(\lambda, y) = 0 \quad \forall_{\lambda \in \mathbb{R}}$
- $\mu \cdot \iota - \partial_y G(\lambda,y) : Y \to X$ is a homeomorphism for all $\mu \notin \mathcal{S}(\lambda,y)$.

  embedding of Y in X ... a discrete set in $\mathbb{R}$
- for $\mu \in \mathcal{S}(\lambda,y)$, $\left(\mu\cdot\iota - \partial_y G(\lambda,y)\right)$ is a Fredholm operator of index 0.

The Morse index of the solution (λ,y) is

$$M(\lambda,y) := \# \left\{ \mu < 0 : \ker\left(\mu\cdot\iota - \partial_y G(\lambda,y)\right) \neq \emptyset \right\}$$

counted with multiplicity, where $m(\mu) := \dim \ker(\mu\cdot\iota - \partial_y G(\lambda,y))$.

Proposition 11.3.2 Let $\mathcal{U} \subset (0,\infty) \times Y$ be an open set, $G : \mathcal{U} \to X$ be C^2 and such that $M(\lambda,y)$ is well-defined for all $(\lambda,y) \in \mathcal{U}$ with $G(\lambda,y) = 0$. Suppose that for compact sets of solutions in $\mathcal{U}$, the sets $\mathcal{S}(\lambda,y)$ are uniformly bounded below. Let

$$S := \{(\lambda(s), y(s)) : s \in (-\varepsilon,\varepsilon)\} \subset \mathcal{U}$$

be a continuous curve of solutions to $G(\lambda,y)=0$ s.t. $0 \notin \mathcal{S}(\lambda(s),y(s))$ for all $s \in (-\varepsilon,\varepsilon)\setminus\{0\}$ and

$$\lim_{s\to 0+} M(\lambda(s),y(s)) \neq \lim_{s \to 0-} M(\lambda(s),y(s)).$$

Then, $(\lambda(0), y(0))$ is a bifurcation point.

for the same λ_k.

meaning $\exists\, (\lambda_k, \hat{y}_k), (\lambda_k, \tilde{y}_k)$ two sequences of solutions s.t. $\hat{y}_k \neq \tilde{y}_k \ \forall_{k \in \mathbb{N}}$ and $(\lambda_k, \tilde{y}_k) \to (\lambda_0, y_0)$, $(\lambda_k, \hat{y}_k) \to (\lambda_0, y_0)$ as $k \to \infty$.

Recall that (C) has a gradient structure:

$$J(\lambda,w)=\frac{1}{2}\int_{-\pi}^{\pi}\left(w\mathcal{C}w'-\lambda w^2(1+\mathcal{C}w')\right)dt$$

with $\nabla J=G$, $G:\mathbb{R}\times Y\to X$ and (C) corresponds to $G(\lambda,w)=0$, where both J and G are real-analytic. Moreover, for $(\lambda,w)\in\mathcal{U}$, and $G(\lambda,w)=0$, the Morse index $M(\lambda,w)$ is well-defined.

Since a change of Morse index indicates bifurcation points on the curve $\mathcal{R}$, the following is significant:

Proposition 11.3.3 If $(\lambda_k,w_k)\in\mathcal{U}$ is a sequence of solutions of (C) in Theorem 11.2.1 with $1-2\lambda_k w_k(0)\to 0$ as $k\to\infty$, then $M(\lambda_k,w_k)\to\infty$.

Corollary 11.3.4 There is an infinite set Σ of values $s>0$ at which $h(s)\in\overline{S}\setminus S$ is a bifurcation point for equation $G(\lambda,w)=0$. The set $\{h(s)\in\overline{S}\setminus S,\ s\in\Sigma\}\subset\mathcal{U}$ is infinite.