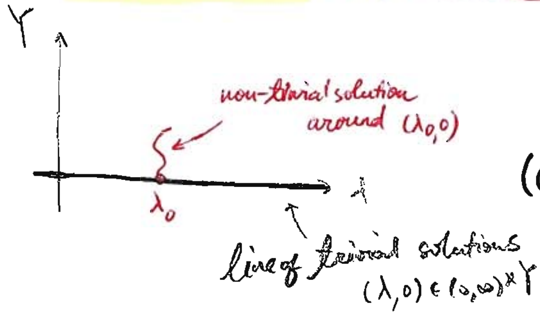
# Global existence of Stokes waves

## 11.1 Local bifurcation theory

Recall that (C) $\mathcal{C}w' = \lambda(w + w\mathcal{C}w' + \mathcal{C}(ww'))$, $\lambda > 0$, $w \in Y$ has a trivial solution $w = 0$ for all $\lambda$. By a "local" bifurcation, we mean the existence of non-trivial solutions of (C) in a neighborhood of the trivial solution $w=0$ for certain $\lambda = \lambda_0 > 0$.




(C) can be rewritten as

$$\text{(G)} \qquad G(\lambda, w) = 0, \quad (\lambda, w) \in (0,\infty) \times Y,$$

where

$$G : (0,\infty) \times Y \to X, \quad G(\lambda, w) = \mathcal{C}w' - \lambda(w + w\mathcal{C}w' + \mathcal{C}(ww')),$$

which has the following properties:

(A) $G(\lambda, 0) = 0 \quad \forall \lambda > 0$

(B) $\partial_w G(\lambda, w) \cdot \varphi = \mathcal{C}\varphi' - \lambda\varphi - \lambda(w\mathcal{C}\varphi' + \varphi\mathcal{C}w' + \mathcal{C}(\varphi w)')$.

(C) $\partial_\lambda G(\lambda, w) \cdot \mu = -\mu(w + w\mathcal{C}w' + \mathcal{C}(ww'))$

Also, $\partial_w G : (0,\infty) \times Y \to \mathcal{L}(Y, X)$ and $\partial_\lambda G : (0,\infty) \times Y \to \mathcal{L}(\mathbb{R}, X)$ are both continuous. So, by Lemma 3.17, $G : (0,\infty) \times Y \to X$ is continuously Fréchet differentiable. In fact, all its partial Fréchet derivatives of orders $> 3$ are zero everywhere. Thus, $G$ is a real-analytic operator.

$G$ is continuously Fréchet diff.

$\partial_w G(\lambda,0)$ is Fredholm with index zero

$\Big\}$ Ex. 4.1.1 ~~Ex. 8.1.1~~ $\Longrightarrow$

a $\boxed{\text{necessary condition}}$ for $(\lambda_0, 0)$ to be a bifurcation point is

$$\partial_w G(\lambda,0) : Y \to X \text{ is NOT injective,}$$

i.e.

$$\partial_w G(\lambda,0)\varphi = \ell\varphi' - \lambda\varphi = 0 \text{ has a nonzero solution } \varphi \in Y$$

## Bifurcation from a simple eigenvalue

$$\ell\varphi' - \lambda\varphi = 0, \quad \varphi \in Y\setminus\{0\} \iff \begin{cases} \varphi(t) = a\cos nt & \text{for some } a \in \mathbb{R},\ n \in \mathbb{N}_0 = \{0,1,2\ldots\} \\ \lambda = n & \text{non-negative integer.} \end{cases}$$

We show that every non-negative integer $\lambda_0 = n$ is a bifurcation point.

$\lambda_0 = 0$ is clearly a bifurcation point, since $(\lambda, w) = (0, c)$ is a solution of (G) for $\forall c \in \mathbb{R}$.

For $\lambda_0 = n > 0$ (let $\varphi_n(t) = \cos nt$, $n = 1, 2 \ldots$), note that

- $\ker \partial_w G(n,0) = \operatorname{Span}\{\varphi_n\}$ is one-dimensional
- $\varphi_n \notin \operatorname{range} \partial_w G(n,0)$

$\Big\} \Rightarrow \lambda_0 = n \in \mathbb{N}$ is a simple eigenvalue of $\partial_w G(n,0) : Y \to X$

By Theorem 4.4.1 ~~8.4.1~~, $\lambda_0 = n \in \mathbb{N}$ is a bifurcation point for (G).

Thus, the conclusion of Theorem 4.3.1 ~~8.3.1~~ holds, which becomes the following theorem :

$\{0,1,2,3,\dots\}$

<u>Theorem 11.1.1</u> Let $n \in \mathbb{N}_0$. Then there exists a neighborhood $O_n$ of $(n,0)$ in $\mathbb{R}\times Y$, $\varepsilon>0$ and a unique real-analytic function $(\Lambda_n, \Phi_n) : (-\varepsilon,\varepsilon) \to O_n$ with $\int_{-\pi}^{\pi} \varphi_n(t)\, \Phi_n(s)(t)\, dt = 0 \ \forall s\in(-\varepsilon,\varepsilon)$ such that

$(\lambda,w)\in O_n$ is a solution of (C) with $w\neq 0$

$\Longleftrightarrow \exists\, s\in(-\varepsilon,\varepsilon)\setminus\{0\}$ s.t. $\underbrace{(\lambda,w) = (\Lambda_n(s), s(\varphi_n+\Phi_n(s)))}_{(*)}$ with $(\Lambda_n(0),\Phi_n(0))=(n,0)$.

- <u>Bifurcation from $\lambda=1$ (a closer look)</u>

Let $\mathcal{E}$ denote the set of all non-trivial solutions of (C), i.e.

$$\mathcal{E} = \{(\lambda,w)\in(0,\infty)\times Y : G(\lambda,w)=0,\ (\lambda,w)\neq(0,0)\}$$

Let $\Gamma_1 = \{(\Lambda_1(s), s(\varphi_1+\Phi_1(s))) : s\in(-\varepsilon,\varepsilon)\setminus\{0\}\}\subset\mathcal{E}$

denote the curve of non-triv. solutions bifurcating from $(1,0)$ (cf. Thm. 11.1.1).

Let $\mathcal{E}_1$ be the maximal connected subset of $\overline{\mathcal{E}}$ that contains $\Gamma_1$, i.e.

$$\Gamma_1 \subset \underbrace{\mathcal{E}_1}_{\text{max. connected}} \subset \overline{\mathcal{E}}$$

Note that $(\lambda,w)\in\mathcal{E} \Leftrightarrow (\lambda,\tilde{w})\in\mathcal{E}$, where $\tilde{w}(t) := w(t+\pi)$, $t\in(-\pi,\pi)$.

Let $(\lambda,w)\in\Gamma_1$, then $(\lambda,w(t)) = (\Lambda_1(s), s(\varphi_1(t)+\Phi_1(s)(t)))$ for some $s\in(-\varepsilon,\varepsilon)\setminus\{0\}$, thus

① $(\lambda,\tilde{w}(t)) = (\lambda, w(t+\pi)) = (\Lambda_1(s), s(\varphi_1(t+\pi)+\Phi_1(s)(t+\pi)))$. On the other hand, by the uniqueness of the solution form (*),

② $(\lambda,\tilde{w}(t)) = (\Lambda_1(s'), s'(\varphi_1(t)+\Phi_1(s')(t)))$ for some $s'\in(-\varepsilon,\varepsilon)\setminus\{0\}$.

Comparing ①-②, we have (since $\varphi_1(t+\pi) = -\varphi_1(t)$) : $s'=-s$ and

$\Lambda_1(-s) = \Lambda_1(s)$, $\widetilde{\Phi_1(s)} = -\Phi_1(-s)$.

In particular, $\underbrace{\dot{\Lambda}_1(0)}_{\frac{d}{ds}\Lambda(s)\big|_{s=0}} = 0$, since $\Lambda_1(-s) = \Lambda_1(s)$.

Now substituting $(\lambda, w) = (\Lambda_1(s), s(\varphi_1 + \Phi_1(s)))$ into $(\mathcal{C})$ leads to

i.e. $\lambda = \Lambda_1(s)$

$w(t) = s(\varphi_1(t) + \Phi_1(s)(t))$

$\mathcal{C}w' = \lambda(w + w\mathcal{C}w' + \mathcal{C}(ww'))$

$$\mathcal{C}(\cancel{s}(\varphi_1' + \Phi_1(s)')) = \Lambda_1(s)\Big(\cancel{s}(\varphi_1 + \Phi_1(s)) + \cancel{s}(\varphi_1 + \Phi_1(s))\cdot\mathcal{C}(s(\varphi_1' + \Phi_1(s)')) + \mathcal{C}(\cancel{s}(\varphi_1 + \Phi_1(s))\cdot s(\varphi_1' + \Phi_1(s)'))\Big)$$

$$\overset{s\neq 0}{\Rightarrow} \mathcal{C}(\varphi_1' + \Phi_1(s)') = \Lambda_1(s)\Big(\varphi_1 + \Phi_1(s) + (\varphi_1 + \Phi_1(s))\cdot\mathcal{C}(s(\varphi_1' + \Phi_1(s)') + \mathcal{C}((\varphi_1 + \Phi_1(s))\cdot s\cdot(\varphi_1' + \Phi_1(s)'))\Big) \quad (**)$$

$\frac{d}{ds}\Big|_{s=0} (**)$

using $\Lambda_1(0) = 1$, $\dot{\Lambda}_1(0) = 0$, $\Phi_1(0) = 0$, $\Phi_1(0)' = 0$

$$\Rightarrow \left.\begin{aligned} \mathcal{C}\dot{\Phi}_1(0)' &= \dot{\Phi}_1(0) + \varphi_1\mathcal{C}\varphi_1' + \mathcal{C}(\varphi_1\varphi_1') \\ &= \dot{\Phi}_1(0) + \varphi_2 + \tfrac{1}{2} \end{aligned}\right\} \Rightarrow \mathcal{C}\dot{\Phi}_1(0)' - \dot{\Phi}_1(0) = \varphi_2 + \tfrac{1}{2}$$

Therefore, $\boxed{\dot{\Phi}_1(0) = \underset{\underset{\cos(2t)}{\parallel}}{\varphi_2} - \frac{1}{2}}$ (*1), since $\underline{\dot{\Phi}_1(0) = \dot{\Phi}_1(0)(t) \text{ is even and } \int_{-\pi}^{\pi} \varphi_1(t)\dot{\Phi}_1(0)(t)\,dt = 0}$

$\Uparrow$ $w = w(t)$ is even, $\int_{-\pi}^{\pi} \varphi_1(t)\Phi_1(s)(t)\,dt = 0 \;\; \forall s$.

Now $\frac{d}{ds}\Big|_{s=0}$ once more,

$\frac{d^2}{ds^2}\Big|_{s=0} (**)$

$$\Rightarrow \mathcal{C}\ddot{\Phi}_1(0)' = \ddot{\Lambda}_1(0)\varphi_1 + \ddot{\Phi}_1(0) + 2\dot{\Phi}_1(0)\mathcal{C}\varphi_1' + 2\varphi_1\mathcal{C}(\dot{\Phi}_1(0)') + 2\mathcal{C}(\dot{\Phi}_1(0)\varphi_1') + 2\mathcal{C}(\varphi_1\dot{\Phi}_1(0)')$$

Substituting $\dot{\Phi}_1(0) = \varphi_2 - \frac{1}{2} = \cos(2t) - \frac{1}{2}$, $\varphi_1 = \cos t$ gives

$$\mathcal{C}\ddot{\Phi}_1(0)' - \ddot{\Lambda}_1(0)\varphi_1 - \ddot{\Phi}_1(0) - 2\varphi_1 - 6\varphi_3 = 0$$

$\int_{-\pi}^{\pi} \varphi_1(t)\Phi_1(s)(t)\,dt = 0, \;\forall s$ $\Rightarrow$ $\ddot{\Lambda}_1(0) = -2$ and $\mathcal{C}\ddot{\Phi}_1(0)' - \ddot{\Phi}_1(0) = 6\varphi_3$

So $\varphi_1$ is orthogonal to $\ddot{\Phi}_1(0)$, and $\mathcal{C}\ddot{\Phi}_1(0)'$ $\Rightarrow$ $\boxed{\ddot{\Lambda}_1(0) = -2}$ (*2) and $\boxed{\ddot{\Phi}_1(0) = 3\varphi_3 = 3\cos(3t)}$ (*3)

Moreover, consider the eigenvalue problem for the linearized $G$ (with respect to $w$) at the local bifurcating solution $(\lambda, w) = (\Lambda_1(s), w_s)$, where $w_s = s(\varphi_1 + \Phi_1(s))$ :

$$\partial_w G(\Lambda_1(s), w_s)\, v_s = \eta_s \cdot v_s \,, \qquad v_s \in Y, \qquad (EG)$$

In the notation of Thm. ~~8.4.1~~ 4.4.1, $G = -F$, since $\partial_w G(\lambda, 0) = A - \lambda\iota$, for $\iota : Y \to X$ embedding, $A : Y \to X$, $\varphi \mapsto \ell\varphi'$.

Let $\tilde{L}(s) = \partial_w G(\Lambda_1(s), w_s)$, then $\tilde{L}(s) = -L(s) = -\partial_x F(\Lambda(s), s\chi(s))$.

Thus, if $L(s)\xi(s) = \mu(s)\iota\xi(s)$, then $\tilde{L}(s)\xi(s) = \underbrace{-\mu(s)}_{\eta_s}\iota\underbrace{\xi(s)}_{v_s}$.

By Proposition ~~8.4.2~~ 4.4.2, we have $\boxed{\lim_{s\to 0} \frac{s\,\dot{\Lambda}_1(s)}{\eta_s} = 1.}$ (*4)

By the symmetry of $\Gamma_1$, we have $\eta_s = \eta_{-s}$, $\tilde{v}_s = v_{-s}$.

In summary, by (*1)–(*4),

$$\left.\begin{aligned} w_s &= s(\varphi_1 + \Phi_1(s)) = s\varphi_1 + s^2\left(\varphi_2 - \tfrac{1}{2}\right) + \tfrac{3}{2}s^3\varphi_3 \\ \Lambda_1(s) &= 1 - s^2 \\ \eta_s &= -2s^2 \end{aligned}\right\} + O(|s|^4),$$

as $|s| \to 0$.

<u>Observation</u>: When $|s| > 0$ small enough, (EG) has exactly two negative eigenvalues and all the others exceed $1/2$.

at $s=0$, $\partial_w G(\Lambda_1(0), w_0)\varphi = \partial_w G(1,0)\varphi = \ell\varphi' - \varphi$, so $\partial_w G(1,0)\varphi = \underset{\sim}{\alpha}\cdot\varphi \Leftrightarrow \ell\varphi' = (\alpha+1)\varphi \Leftrightarrow \alpha + 1 \in \mathbb{N}_0$ (eigenvalue of $\partial_w G(1,0)$; $\{0,1,2,\dots\}$)

thus $\alpha = -1, 0, 1, 2, 3, \dots$

now for $0 < |s|$ small, the eigenvalues $\eta_s$ of $\partial_w G(\Lambda_1(s), w_s)$ are close to the eigenvalues of $\partial_w G(1,0)$, i.e. $\eta_s \approx -1, 0, 1, 2, 3, \dots$ so all except two $\eta_s$'s exceed $1/2$, also the $\eta_s$ close to $0$ is negative, since $\#$.