3. Main equation

$\ulcorner (1-2\lambda w(t))\left(w'(t)^2+(1+\mathcal{C}w'(t))^2\right)=1 \quad \forall_{a.e.}\ t\in[-\pi,\pi].$

We replace the equation (CP) with another one of more convenient form.

Let Y denote the space of 2π-periodic absolutely continuous even functions $u:\mathbb{R}\to\mathbb{R}$ with $u'\in L^2[-\pi,\pi]$, endowed with

$$\|u\|_Y^2 = \|u\|^2_{L^2_{[-\pi,\pi]}} + \|u'\|^2_{L^2[-\pi,\pi]}.$$

Let X (resp. Z) be the space of even (resp. odd) functions in $L^2_{[-\pi,\pi]}$.

Theorem 10.3.2 Suppose that $w\in Y$ is a solution of

(C) $$\mathcal{C}w' = \lambda\{w + w\mathcal{C}w' + \mathcal{C}(ww')\}.$$

Then, we have

(a) w satisfies (CP).

(b) $1-2\lambda w>0$ on $[-\pi,\pi] \iff w'\in L^3[-\pi,\pi]$.

(c) If $w\in C^{2,\alpha}$, $1-2\lambda w>0$ and $1+\mathcal{C}w'\geq 0$ on $[-\pi,\pi]$, then Z defined by (Z) is injective (and by our previous analysis is the solution w of (C) gives rise to a symmetric Stokes wave with profile

$$S=\{(t+\mathcal{C}w(t),\, w(t)) : t\in\mathbb{R}\},\ \text{wh. } 1+\mathcal{C}w'>0 \text{ on } [-\pi,\pi].)$$

pf: (a) We rewrite (C) as

$$(1-2\lambda w)(1+\mathcal{C}w') + \mathcal{C}((1-2\lambda w)w') = 1 \qquad (*)$$

applying $\mathcal{C}$ to both sides gives

$$(1-2\lambda w)w' = \mathcal{C}((1-2\lambda w)(1+\mathcal{C}w')) \qquad \ulcorner \mathcal{C}^2u=-u \ \forall_{2\pi\text{-per. } u}.\lrcorner$$

So that

$$\mathcal{C}u = (1-2\lambda w)w' \quad , \text{ where } \quad u = (1-2\lambda w)(1+\mathcal{C}w').$$

By Theorem 10.2.3, there exists a holomorphic function U on D with

$(w \in Y \Rightarrow w \in L^2, w' \in L^2 \Rightarrow u \in L^2)$

$$U\big|_{S^1} = u + i\mathcal{C}u = (1-2\lambda w)\left(1+\mathcal{C}w'+iw'\right)$$

$$= i(1-2\lambda w)\left(w' - i(1+\mathcal{C}w')\right).$$

Again by Thm 10.2.3, there exists a holomorphic function W on D such that

$$W\big|_{S^1} = w' + i(1+\mathcal{C}w').$$

Thus, we have

$$UW\big|_{S^1} = i(1-2\lambda w)(w'^2 + (1+\mathcal{C}w')^2) \in i\mathbb{R}.$$

By the last part of Thm 10.2.3, UW is constant on D.

Also,

$$\underset{\uparrow \text{ Cauchy's integral formula}}{U(0)} = \frac{1}{2\pi}\int_{-\pi}^{\pi}(u+i\mathcal{C}u)\,dt = \frac{1}{2\pi}\int_{-\pi}^{\pi}u\,dt + \frac{i}{2\pi}\underbrace{\int_{-\pi}^{\pi}\mathcal{C}u\,dt}_{=\int_{-\pi}^{\pi}(1-2\lambda w)w'\,dt = \int_{w(-\pi)}^{w(\pi)}(1-2\lambda w)\,dw \overset{!}{=} 0 \ (w\ 2\pi\text{-per.})}$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi}u\,dt = \frac{1}{2\pi}\int_{-\pi}^{\pi}(1-2\lambda w)(1+\mathcal{C}w')\,dt$$

$$\overset{(*)}{=} \frac{1}{2\pi}\int_{-\pi}^{\pi}1\,dt - \frac{1}{2\pi}\underbrace{\int_{-\pi}^{\pi}\mathcal{C}((1-2\lambda w)w')\,dt}_{\overset{!}{=}\mathcal{C}\int_{-\pi}^{\pi}(1-2\lambda w)w'\,dt = \mathcal{C}(0) = 0} = 1$$

Similarly, $W(0) = i$ (check it!). Therefore, $UW \equiv i$ on D. Restricted on S^1, this gives (CP).

"$\Rightarrow$"

(b) Let $w \in Y$ satisfy (C). Since $1-2\lambda w > 0$ and w is continuous, there exists $\varepsilon > 0$ such that $1-2\lambda w \geq \varepsilon$. On the other hand, w satisfies (CP) by (a), i.e. $(1-2\lambda w(t))(w'(t)^2 + (1+\mathcal{C}w'(t))^2) = 1$. Thus,

$$w'(t)^2 + (1+\mathcal{C}w'(t))^2 \leq \frac{1}{\varepsilon} \quad \Rightarrow \quad w' \in L^\infty(-\pi,\pi) \subset L^3[-\pi,\pi].$$

"$\Leftarrow$" Let $w \in Y$ be such that $w' \in L^3[-\pi,\pi]$. By Thm of M. Riesz, we have $\mathcal{C}w' \in L^3[-\pi,\pi]$. Since by (a), w satisfies (CP), it is clear that $1-2\lambda w \geq 0$. Assume now to the contrary that $1-2\lambda w(a) = 0$ for some $a \in [-\pi,\pi]$. Then,

$$|1-2\lambda w(t)| = 2\lambda \left| \int_a^t w'(s)\,ds \right| \underset{\uparrow}{\leq} 2\lambda \|w'\|_{L^3} \cdot |t-a|^{2/3}$$

Hölder's ineq. $p=3$, $q=\frac{3}{2}$.

By (CP), then

$$w'(t)^2 + (1+\mathcal{C}w'(t))^2 \geq \frac{1}{2\lambda \|w'\|_{L^3} \cdot |t-a|^{2/3}}$$

Since $|t-a|^{-1} \notin L^1[-\pi,\pi]$, $w'^2 + (1+\mathcal{C}w')^2 \notin L^{\frac{3}{2}}[-\pi,\pi]$ $\lightning$ $w' \in L^3[-\pi,\pi]$

$L^q_{[-\pi,\pi]} \subset L^p_{[-\pi,\pi]}$ $\forall\, 1 \leq p < q \leq \infty$.

(c) We use Lemma 10.2.4.

$Z|_{\partial D} : t \mapsto (t + \mathcal{C}w(t), w(t))$ is injective, since $\left(\frac{d}{dt}(t+\mathcal{C}w(t)), \frac{d}{dt}w(t)\right) = (1+\mathcal{C}w'(t), w'(t)) \neq (0,0)$,

the image of Z when restricted to ∂D is S.

$w'(t)^2 + (1+\mathcal{C}w'(t))^2 \neq 0$ by (CP).

also $Z \in C^{2,\alpha}(\bar{D})$, since $w \in C^{2,\alpha}$ and $\mathcal{C}w \in C^{2,\alpha}$ by Theorem Privalov.

Thus, by lemma 10.2.4, Z is a gobal bijection.

Finally, by lemma 10.1.2, $1 + \mathcal{C}w' > 0$ on $[-\pi,\pi]$.

$\#$ 10.3.2.

Some convenient properties of $\underbrace{(C)}_{}$ are:

$$\mathcal{C}w' = \lambda\left(w + w\mathcal{C}w' + \mathcal{C}(ww')\right)$$

(i) It is the Euler-Lagrange equation of the functional

$$\mathcal{J}(w) = \frac{1}{2}\int_{-\pi}^{\pi}\left(w\mathcal{C}w' - \lambda w^2(1+\mathcal{C}w')\right)dt, \quad w \in Y$$

(using the fact that $u \mapsto \mathcal{C}u'$ is self-adjoint, i.e. $\langle \mathcal{C}u', x\rangle = \langle u, \mathcal{C}x'\rangle$.)

(ii) It is a quadratic equation (with no higher order terms), whose non-zero solutions give rise to exact solutions of the steady periodic water-wave problem (without approximation).

(iii) It has the trivial solution $w=0$ for all λ. Linearizing w.r.t. w at the trivial solution yields the self-adjoint eigenvalue problem

$$\mathcal{C}w' = \lambda w, \quad w \in Y,$$

which has a complete set of eigenfunctions $\{\cos(nt)\}_{n\in\mathbb{N}\cup\{0\}}$ corresponding to eigenvalues $\mathbb{N}\cup\{0\}$ (since Y consists of even functions).

(iv) It can be rewritten as

$$(1-2\lambda w)\,\mathcal{C}w' = \lambda\left(w - \mathcal{Q}(w)\right) \quad (10.18) \quad \text{or} \quad \mathcal{C}\left((1-2\lambda w)w'\right) = \lambda\left(w + \mathcal{Q}(w)\right), \quad (10.19)$$

where $\mathcal{Q}(u) = u\mathcal{C}u' - \mathcal{C}(uu')$ (properties of $\mathcal{Q}$ are to be discussed)

(v) It can be rewritten as $G(\lambda, w) = 0$ where $G: (0,\infty)\times Y \to X$ is defined by

$$G(\lambda, w) = \mathcal{C}w' - \lambda\left(w + w\mathcal{C}w' + \mathcal{C}(ww')\right).$$

For $(\lambda, w) \in \mathcal{U}$, where $\mathcal{U} = \{(\lambda,w)\in(0,\infty)\times Y : 1-2\lambda w > 0\}$,

$\partial_w G(\lambda, w): Y \to X$ is a Fredholm operator with index 0 (to be elaborated).

10.5 Weak solutions are classical

At the beginning of the discussion on Stokes waves, we described the profil of a „classical" solution by

$S := \{(x, w(x)) : x \in \mathbb{R}\}$, where $w \in C^{2,\alpha}$, $\alpha \in (0,1)$, 2π-per, even.

In 10.2-10.3, we reduced the original water wave equation to finding „weak" solutions of (CP) $(1-2\lambda w(t))(w'(t)^2 + (1+\mathcal{C}(w'(t)))^2) = 1$, $t \in [-\pi,\pi]$, a.e.

or equivalently (C) $\mathcal{C}w' = \lambda(w + w\mathcal{C}w' + \mathcal{C}(ww'))$, where $w \in Y$ $\leftarrow$ 2π-per, even, $w' \in L_2[-\pi,\pi]$.

We show that if $1-2\lambda w > 0$, then every weak solution is classical.

For a per. function u with $u' \in L_2[-\pi,\pi]$, define

$$\mathcal{Q}(u)(t) := u(t)\mathcal{C}u'(t) - \mathcal{C}(uu')(t).$$

Lemma 10.5.1 If u is 2π-per. and $u' \in L_2[-\pi,\pi]$, then for almost all $t \in \mathbb{R}$,

$$0 \le \mathcal{Q}(u)(t) = \frac{1}{8\pi}\int_{-\pi}^{\pi}\left(\frac{u(t)-u(s)}{\sin\frac{t-s}{2}}\right)^2 ds \le \frac{2}{\pi}\|u'\|^2_{L_2[-\pi,\pi]}.$$

Pf: based on the formula $\mathcal{C}u(t) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{u(s)}{\tan\frac{t-s}{2}}ds$. #

Lemma 10.5.2 $u' \in L_p[-\pi,\pi]$, for some $2<p<\infty$ $\Rightarrow$ $\mathcal{Q}(u) \in C^{1-\frac{2}{p}}$.

Lemma 10.5.3 $u' \in C^{\alpha}$, for some $\alpha \in (0,1)$ $\Rightarrow$ $\mathcal{Q}(u) \in C^{1,\delta}$, for some $0<\delta<\alpha$.

Pf of 10.5.2, 10.5.3 are based on the formula of $\mathcal{Q}(u)$ in 10.5.1. #

Corollary of regularity: Suppose $u' \in L_2[-\pi,\pi]$. Then,

(a) $\mathcal{Q}(u)(x) > 0$ for almost all $x \in [-\pi,\pi]$, unless $u \equiv$ constant. (10.5.1)

(b) $\mathcal{Q}(u) \in L_\infty[-\pi,\pi]$, i.e. bounded (10.5.1)

(c) $\mathcal{Q}(u) \in C^\alpha$, $\forall_{\alpha\in(0,1)}$, if $u' \in L_p[-\pi,\pi]$, $\forall_{p<\infty}$. (10.5.2)

(d) $\mathcal{Q}(u) \in C^{1,\alpha}$ $\forall_{\alpha\in(0,1)}$, if $u' \in C^\beta$, $\forall_{\beta\in(0,1)}$ (10.5.3)

Pf: 10.5.1 – 10.5.3. #.

Let $\mathcal{U} = \{(\lambda,w) \in (0,\infty)\times Y : 1-2\lambda w > 0\}$.

$\mathcal{C}w' = \lambda(w + w\mathcal{C}w' + \mathcal{C}(ww'))$

Theorem 10.5.4 Suppose $(\lambda,w) \in \mathcal{U}$ is a solution of (C). Then,

$$w \in C^{2,\alpha}, \quad \forall_{\alpha\in(0,1)}.$$

Pf: Recall (C) can be rewritten as (cf. (10.18))

$$(*) \qquad (1-2\lambda w)\,\mathcal{C}w' = \lambda(w - \mathcal{Q}(w))$$

$$\xRightarrow{1-2\lambda w>0} \mathcal{C}w' = \frac{\lambda(w-\mathcal{Q}(w))}{1-2\lambda w} \overset{(b)}{\in} L_\infty[-\pi,\pi] \subset L_p[-\pi,\pi], \ \forall_{p\geq 1}.$$

$$\xRightarrow{\text{Riesz}} w' \in L_p[-\pi,\pi] \ \forall_{1<p<\infty} \xRightarrow[\text{inequality}]{\text{Hölder}} w \in C^\beta, \ \forall_{\beta\in(0,1)}$$

$$\xRightarrow{(c)} \mathcal{Q}(w) \in C^\beta \ \forall_{\beta\in(0,1)}$$

$$\xRightarrow{(*)} \mathcal{C}w' \in C^\beta \ \forall_{\beta\in(0,1)}$$

$$\xRightarrow{\text{Privalov}} w' \in C^\beta \ \forall_{\beta\in(0,1)} \xRightarrow{(d)} \mathcal{Q}(w) \in C^{1,\alpha} \ \forall_{\alpha\in(0,1)}$$

$$\Longrightarrow w \in C^{1,\alpha} \ \forall_{\alpha\in(0,1)}$$

$$\xRightarrow{(*)} \mathcal{C}w' \in C^{1,\alpha} \ \forall_{\alpha\in(0,1)}$$

$$\xRightarrow{\text{Privalov}} w' \in C^{1,\alpha} \ \forall_{\alpha\in(0,1)} \Rightarrow w \in C^{2,\alpha}, \ \forall_{\alpha\in(0,1)}$$

10.5.4.

Clearly, the proof of Thm. 10.5.4 can be continued infinitely to prove by induction that w can be extended as infinitely differentiable 2π-periodic function on $\mathbb{R}$. And we see from Thm. 10.3.2 & Thm. 10.5.4 that the assumption $1-2\lambda w>0$ (strict inequality) everywhere is essential to this conclusion.

Remark 10.5.5 If $\{(\lambda_k, w_k)\}$ is a sequence of solutions of (C) with $1-2\lambda_k w_k(t) \geq d > 0$ $\forall t \in [-\pi,\pi]$, and $\{\lambda_k\}$ is bounded, then (CP) gives that $\{w_k'\}$ is bounded in $L_2[-\pi,\pi]$, thus by the argument in the proof of Thm 10.5.4, $\{w_k\}$ is bounded in $C^{2,\alpha}$, $\forall \alpha \in (0,1)$. Since $C^{2,\alpha} \hookrightarrow C^2$ is compactly embedded (by Ascoli-Arzelà), $\{w_k\}$ is compact in C^2.

To proceed, we need

Theorem (dominated convergence theorem)

Suppose that $\{f_n\}$ is a sequence of measurable functions, that $f_n \to f$ pointwise almost everywhere, as $n\to\infty$, and that $|f_n| \leq g$ for all n, where g is integrable. Then, f is integrable, and

$$\int f\,d\mu = \lim_{n\to\infty} \int f_n\,d\mu .$$

Fredholm property

Recall that (C) can be rewritten as $G(\lambda,w)=0$, where

$$G: (0,\infty)\times Y \to X$$
$$(\lambda, w) \mapsto \ell w' - \lambda\big(w + w\ell w' + \ell(ww')\big).$$

We show that for $(\lambda,w)\in \underset{\{(\lambda,w)\in(0,\infty)\times Y:\ 1-2\lambda w>0\}}{\mathcal{U}}$, $\partial_w G(\lambda,w): Y\to X$ is Fredholm with index 0.

We rewrite G in terms of Q as

$$G(\lambda,w) = (1-2\lambda w)(\ell w'+w) + \lambda Q(w) - (1-2\lambda w+\lambda)w.$$

Thus,

$$\partial_w G(\lambda,w)\cdot h = \underbrace{(1-2\lambda w)(\ell h'+h)}_{\text{homeom.}}$$
$$\underbrace{+\lambda\, dQ(w)\cdot h}_{\text{compact}} \underbrace{-2\lambda h(\ell w'+w) - \lambda h + 4\lambda w h - h}_{\text{compact}},$$

where $h\mapsto \underbrace{(1-2\lambda w)}_{>0}\underbrace{(\ell h'+h)}_{}$ is a homeomorphism,

homeom, since $h\mapsto \ell h'+h$ is linear, and $\ell h'+h=0 \Rightarrow h=0$

and $h\mapsto 2\lambda h(\ell w'+w)$, $h\mapsto -\lambda h$, $h\mapsto 4\lambda w h$, $h\mapsto -h$ are compact linear operators from Y to X, since $Y\hookrightarrow X$ is a compact embedding. We show that $Q: Y\to X$ is compact, which then implies $h\mapsto \lambda dQ(w)\cdot h$ is compact,

So $\partial_w G(\lambda,w) = T+K$ for a homeom. T and a compact operator K from Y to X.

then by Thm 2.7.6, $\partial_w G(\lambda,w)$ is Fredholm with index 0.

Recall:

We show a general result for $\mathcal{Q}$. $\quad \mathcal{Q}: Y \to L_\infty[-\pi,\pi] \subset L_p[-\pi,\pi]$ $\forall p \geq 1$

<u>Theorem 10.5.6</u> $\mathcal{Q}: Y \to L_p[-\pi,\pi]$ is sequentially continuous, with respect to the weak topo. on Y, and strong topo. on L_p, for all $1<p<\infty$.

Pf: $\mathcal{Q}: Y \to L_p$ is <u>sequentially conts</u>, if $v_n \xrightarrow{\omega} v$ in Y implies $\mathcal{Q}(v_n) \to \mathcal{Q}(v)$ in L_p.
weak strong

Let $\{v_n\}_{n=1}^\infty$ be a sequence in Y such that

$v_n \xrightarrow{\omega} v$ in Y, i.e. $\langle v_n, y\rangle_Y \to \langle v, y\rangle_Y$ $\forall y \in Y$, since Y is Hilbert, thus reflexive.

i.e. $\langle v_n - v, y\rangle_Y \to 0$ $\forall y \in Y$

i.e. $\int^\pi (v_n - v)y + (v_n' - v')y'\,dt \to 0$

Then, $v_n' \to v'$ in L_2.

Let $u_n(t) = \int_0^t v_n'(s) - v'(s)\,ds = v_n(t) - v(t) - v_n(0) + v(0)$.

$\Rightarrow \mathcal{Q}(u_n)(t) = \mathcal{Q}(v_n - v)(t)$

$$\Rightarrow \int_{-\pi}^{\pi} \mathcal{Q}(v_n - v)(t)\,dt = \int_{-\pi}^{\pi} \mathcal{Q}(u_n)(t)\,dt = \int_{-\pi}^{\pi} u_n(t)\mathcal{C}u_n'(t) - \mathcal{C}(u_n u_n')(t)\,dt$$

$$= \int_{-\pi}^{\pi} u_n(t)\mathcal{C}u_n'(t)\,dt - \underbrace{\int_{-\pi}^{\pi} \mathcal{C}(u_n u_n')(t)\,dt}_{=0}, \text{ since } \int_{-\pi}^{\pi} \mathcal{C}v(t)\,dt = 0 \ \forall v \in L_2[-\pi,\pi]$$

$$= \int_{-\pi}^{\pi} u_n(t)\,\mathcal{C}(v_n' - v')(t)\,dt \longrightarrow 0,$$ since $\mathcal{C}$ is bounded, $v_n' \to v'$ in L_2, and $u_n \to 0$ in L_2.

By Lemma 10.5.1, $\mathcal{Q}(v_n - v)(t) \geq 0$ for almost all t, thus

$$\int_{-\pi}^{\pi} |\mathcal{Q}(v_n - v)(t)|\,dt = \int_{-\pi}^{\pi} \mathcal{Q}(v_n - v)(t)\,dt \to 0,$$

i.e. $\mathcal{Q}(v_n - v) \to 0$ in L_1 $\Rightarrow$ $\mathcal{Q}(v_n)(t) \to \mathcal{Q}(v)(t)$ for almost all t.

dominated convergence thm $\xRightarrow[\text{by Lemma 10.5.1.}]{\{\mathcal{Q}(v_n)\} \text{ is bdd}}$ $\mathcal{Q}(v_n) \to \mathcal{Q}(v)$ in L_p, $\forall 1<p<\infty$.

#10.5.6

Corollary 10.5.7 $\mathcal{Q} : Y \to X$ is a compact map.

pf: $\mathcal{Q}$ is compact, if $\overline{\mathcal{Q}(B)}$ is compact, for every bounded set $B \subset Y$.

or equivalently (since X is metric space), $\{\mathcal{Q}(y_n)\}$ has a convergent subsequence, for every bounded sequence $\{y_n\} \subset Y$.

Let $\{y_n\} \subset Y$ be bounded, $\xRightarrow{Y \text{ is Hilbert}}$ $\{y_n\}$ contains a weakly convergent subsequence $\{y_{n_k}\}$ $\xrightarrow{\text{Thm } 10.5.6}$ $\mathcal{Q}(y_{n_k}) \to \mathcal{Q}(y)$ ✓

$\{y_{n_k}\} \rightharpoonup y$ #.

Banach-Alaoglu theorem:

the closed unit ball of the dual of a normed vector space is compact in weak topology

$\Rightarrow$ in a Hilbert space, every bounded and closed set is weakly relatively compact.

$\Rightarrow$ every bounded sequence has a weakly convergent subsequence.

#