Chap 5. Global bifurcation theory

Let X, Y be Banach spaces over $\mathbb{R}$, $U \subset \mathbb{R} \times X$ open and $F: U \to Y$ be a $\mathbb{R}$-analytic such that

(G1) $(\lambda, 0) \in U$ and $F(\lambda, 0) = 0$ for all $\lambda \in \mathbb{R}$

(G2) $\partial_x F[\lambda, x]$ is a Fredholm operator of index zero when $F(\lambda, x) = 0$, $(\lambda, x) \in U$.

(G3) for some $\lambda_0 \in \mathbb{R}$, $\left\{ \begin{array}{l} \ker \partial_x F[\lambda_0, 0] = \{ s\xi_0 : s \in \mathbb{R} \},\ \xi_0 \neq 0, \\ \partial^2_{\lambda, x} F[\lambda_0, 0](1, \xi_0) \notin \text{range}(\partial_x F[\lambda_0, 0]) \end{array} \right.$

By Theorem 4.3.1, there exists a ($\mathbb{R}$-analytic) function $(\Lambda, \kappa): (-\varepsilon, \varepsilon) \to \mathbb{R} \times X$ such that $(\lambda, x) = (\Lambda(s), \underbrace{\kappa(s)}_{= s\chi(s)})$ satisfies $F(\lambda, x) = 0$ $\forall_{s \in (-\varepsilon, \varepsilon)}$, $\Lambda(0) = \lambda_0$ and $\kappa'(0) = \xi_0$.

Let
$$\mathcal{R}^+ = \{ (\Lambda(s), \kappa(s)) : s \in (0, \varepsilon) \}$$
$$\mathcal{S} = \{ (\lambda, x) \in U : F(\lambda, x) = 0 \}$$
$$\mathcal{J} = \{ (\lambda, x) \in \mathcal{S} : x \neq 0 \},$$
where $\varepsilon > 0$ is so small that $\kappa'(s) \neq 0$ for all $s \in (-\varepsilon, \varepsilon)$ and $\mathcal{R}^+ \subset \mathcal{J}$.

We want to "extend" $\mathcal{R}^+$ inside $\mathcal{S}$.

5.1. Analytic varieties

Recall that

$U \subset X$ ← $\mathbb{F}$-Banach spaces, $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$

Def. 5.1.1. A map $F : U \to Y$ is $\mathbb{F}$-analytic at $x_0 \in U$ if, for all $x \in U$ with $\|x - x_0\|$ sufficiently small,

$$F(x) = \sum_{k=0}^{\infty} m_k (x - x_0)^k ,$$

where $F(x_0) = m_0 \in Y$, $m_k \in \mathcal{M}^k(X, Y)$ is symmetric and there exists $r > 0$ such that

$$\sup_{k \geq 0} r^k \|m_k\| = M < \infty .$$

The map F is called $\mathbb{F}$-analytic on U if it is so at every point of U.

⌈ A mapping $m : X_1 \times \dots \times X_p \to Y$ is called multilinear, if

$$x \longmapsto m(x_1, x_2, \dots, x_{l-1}, x, x_{l+1}, \dots, x_p)$$

is linear for all $l = 1, \dots p$. Moreover, it is called bounded multilinear, if

$$\sup \{ \|m(x_1, x_2, \dots, x_p)\| : \|x_1\|, \dots, \|x_p\| \leq 1 \} = M < \infty .$$

The notation $\mathcal{M}^p(X, Y) = \{ m : \underbrace{X \times \dots \times X}_{p \text{ times}} \to Y : m \text{ is bounded multilinear} \}$.

An $m \in \mathcal{M}^p(X, Y)$ is called symmetric, if

$$m(x_1, \dots, x_p) = m(x_{\sigma(1)}, \dots, x_{\sigma(p)}) \quad \forall \sigma \in S_p$$

← the permutation group of p symbols.

⌋

Theorem 5.1.2 (analytic implicit function theorem)

Let X, Y, Z be Banach spaces, $U \subset X \times Y$ open. Let $(x_0, y_0) \in U$ be such that $\partial_x F[x_0, y_0] \in \mathcal{L}(X, Z)$ is a homeomorphism for an analytic function $F: U \to Z$. Then,

$$\exists \underset{\nearrow \atop \text{open nb. of } y_0}{V \subset Y}, \quad \underset{\uparrow \atop \text{open}}{W \subset U}, \quad \underset{\uparrow \atop \mathbb{F}\text{-analytic}}{\phi: V \to X} \quad \text{s.t. } (x_0, y_0) \in W, \quad F^{-1}(z_0) \cap W = \{(\phi(y), y) : y \in V\}$$

Def. 5.1.3 Let $O \subset \mathbb{F}^n$ be an open (non-empty) set and G be a finite collection of $\mathbb{F}$-analytic functions $g: O \to \mathbb{F}$.

The set

$$\mathrm{var}(O, G) := \{ x \in O : g(x) = 0 \ \forall g \in G \}$$

is called the $\mathbb{F}$-analytic variety generated by G on O.

If $O \subset \mathbb{C}^n$ and every $g \in G$ is real-on-real, then we call $\mathrm{var}(O, G)$ real-on-real.

A point $x \in \mathrm{var}(O, G)$ is called m-regular if there exists an open neighborhood $\tilde{O}$ of x such that $\tilde{O} \cap \mathrm{var}(O, G)$ is an $\mathbb{F}$-analytic manifold of dimension m. Note that

$$\mathrm{var}(O, G_1) \cap \mathrm{var}(O, G_2) = \mathrm{var}(O, G_1 \cup G_2)$$

$$\mathrm{var}(O, G_1) \cup \mathrm{var}(O, G_2) = \mathrm{var}(O, \underbrace{G_1 \cdot G_2}_{= \{g = g_1 g_2 \,:\, g_1 \in G_1, g_2 \in G_2\}})$$

Def. 5.1.4 Let $a \in \mathbb{F}^n$. Two subsets S and T of $\mathbb{F}^n$ are said to be equivalent at a, if there exists an open nbhd of O of a such that $O \cap S = O \cap T$. We denote the equivalence relation as $S \sim_a T$. The equivalence class $[S]_{\sim_a}$ will be denoted by $\gamma_a(S)$ and called the germ of S at a.

Convention: $\gamma_a(\{a\}) = \{a\}$; $\gamma_a(\phi) = \phi$; $\gamma_a(S) = \phi$ if $a \notin \bar{S}$.

H. $$\gamma_a(S_1 \cap T_1) = \gamma_a(S_2 \cap T_2)$$
$$\gamma_a(S_1 \cup T_1) = \gamma_a(S_2 \cup T_2)$$
$$\gamma_a(S_1^c) = \gamma_a(S_2^c) \qquad \forall\, S_1 \sim_a S_2,\ T_1 \sim_a T_2$$

The germ at a of an $\mathbb{F}$-analytic variety is called an $\mathbb{F}$-analytic germ; the germ of a real-on-real $\mathbb{C}$-analytic variety is called a real-on-real germ.

$V_a(\mathbb{F}^n)$ = the set of all $\mathbb{F}$-analytic germs at a.

For $\alpha \in V_a(\mathbb{F}^n)$, the dimension of α, $\dim_{\mathbb{F}} \alpha$, is defined as the largest integer m such that every representative of α contains an m-regular point (the point a itself needs not be m-regular). If no such integer exists, than $\dim_{\mathbb{F}} \alpha = -1$.

Theorem 5.1.5 (a structure theorem of $\mathbb{C}$-analytic varieties)

Let $n \geq 2$ and $\alpha \in V_0(\mathbb{C}^n) \setminus \{0\}$ be such that $\{0\} \subset \alpha \neq \gamma_0(\mathbb{C}^n)$.

Then there exists sets $B_1, \ldots, B_N$ such that

(a) $\alpha = \gamma_0(B_1 \cup \ldots \cup B_N \cup \{0\})$

(b) each B_j, $1 \leq j \leq N$, after a linear change of coordinates (depending on j), is a branch of a Weierstrass analytic variety (depending, including its dimension, on j)

(c) $\dim_{\mathbb{C}} \alpha = \max_{1 \leq j \leq N} \{\dim_{\mathbb{C}} B_j\}$.

(d) if $L \subset \mathbb{C}^n$, $\gamma_0(L) \neq \phi$, is a connected $\mathbb{C}$-analytic manifold of dimension $l \in \{1, \ldots, n\}$, whose points are l-regular points of a representative of α, then there exists $j \in \{1, \ldots, N\}$ such that $\gamma_0(L) \subset \gamma_0(\overline{B_j})$ and $\dim_{\mathbb{C}} B_j = l$

(e) if α is real-on-real, then it can be arranged that each branch B_j with $B_j \cap \mathbb{R}^n \neq \phi$ is real-on-real.

(f) $\alpha \cap \gamma_0(\mathbb{R}^n) = \gamma_0(\tilde{B}_1 \cup \ldots \cup \tilde{B}_K \cup \{0\})$, where the $\tilde{B}_j$ denotes those branches which intersect $\mathbb{R}^n$ non-trivially.

(g) $\dim_{\mathbb{R}} (\alpha \cap \mathbb{R}^n) = \max_{1 \leq j \leq K} \dim_{\mathbb{R}} (\tilde{B}_j \cap \mathbb{R}^n)$.

⌈ A Weierstrass analytic variety is an analytic variety generated by Weierstrass polynomials. A Weierstrass polynomial defined on $\mathbb{C}^n$ is of form

$$W(z) = W(z, z_2, z_3, \ldots, z_n) = z^k + g_{k-1} z^{k-1} + \ldots + g_1 z + g_0,$$

where $g_i = g_i(z_2, z_3, \ldots, z_n)$ is analytic and $g_i(0, \ldots 0) = 0$. ⌋

By analytic implicit function theorem, a Weierstrass analytic variety in $\mathbb{C}^n$ determined by $(n-m)$ Weierstrass polynomials is a connected $\mathbb{C}$-analytic manifold of dimension m, where the polynomials are of form $W_k(z_1, \dots, z_n) = A_k(z_k; z_1, \dots, z_m)$ for $k = m+1, \dots, n$,

W-polynomial on $\mathbb{C}^{m+1}$.