Chap 4. Local Bifurcation Theory

Let $F : U \subset \mathbb{F} \times X \to Y$ be Fréchet-differentiable and

$F(\lambda, 0) = 0 \quad \forall_{\lambda \in \mathbb{F}}$.

Question: for which $\lambda_0 \in \mathbb{F}$ is there a sequence $\{(\lambda_n, x_n)\} \subset \mathbb{F} \times X \setminus \{0\}$ of solutions converging to $(\lambda_0, 0)$ in $\mathbb{F} \times X$?

Such $(\lambda_0, 0)$ are called bifurcation points on the line of trivial solutions $\{(\lambda, 0) : \lambda \in \mathbb{F}\}$; λ_0 is also referred as the bifurcation point.

4.1. A necessary condition

Assume that F is cont. diff. in a neighborhood of $(\lambda_0, 0)$

Implicit Function Theorem

I.F.T: $\partial_x F[\lambda_0, 0]$ is a homeomorphism $\Rightarrow$ in a nbhd of $(\lambda_0, 0)$, all solutions of $F(\lambda, x) = 0$ lie on a unique curve $\{(\lambda, x) : x = \phi(\lambda),\ \lambda \in (\lambda_0 - \varepsilon, \lambda_0 + \varepsilon)\}$.

$$\xrightarrow{F(\lambda,0)=0\ \forall_{\lambda \in \mathbb{F}}} F^{-1}(0) \cap \underset{\text{of } (\lambda_0,0)}{\text{a nbhd}} = \{(\lambda, 0) : \lambda_0 - \varepsilon < \lambda < \lambda_0 + \varepsilon\} \Rightarrow (\lambda, 0) \text{ is } \underline{\text{no}} \text{ bf. p}$$

In other words,

$(\lambda_0, 0)$ is a bifurcation point $\Rightarrow \partial_x F[\lambda_0, 0]$ is not a homeom.

$$\underset{\partial_x F[\lambda_0,0] \in \mathcal{L}(X,Y)}{\overset{2.3.2}{\Longleftrightarrow}} \partial_x F[\lambda_0, 0] \text{ is not a bijection.}$$

if $\partial_x F[\lambda_0, 0]$ is Fredholm of index 0 $\Rightarrow \{0\} \neq \ker \partial_x F[\lambda_0, 0] : X \to Y$

Example 4.1.1 Let $X=Y$ and $F: \mathbb{F}\times X\to Y$ is a C^1-function (w.r.t. x) of form $F(\lambda,x)=x-G(\lambda,x)$, where $G(\lambda,\cdot): X\to Y$ is a compact nonlinear operator with $G(\lambda,0)=0$ $\forall_{\lambda\in\mathbb{F}}$. Then,

$$\partial_x F[\lambda_0,0] = I - \partial_x G[\lambda_0,0],$$

where $\partial_x G[\lambda_0,0]$ is a compact linear operator on X by Lemma 3.1.8. By Theorem 2.4.7, $\partial_x F[\lambda_0,0]$ is a Fredholm operator of index 0. Thus, $\partial_x F[\lambda_0,0]$ is a homeo. $\Leftrightarrow$ $\ker \partial_x F[\lambda_0,0]=\{0\}$. So

$$(\lambda_0,0) \text{ is a bf. point } \Rightarrow \exists\, \xi\neq 0 \text{ s.t. } \partial_x F[\lambda_0,0]\xi=0.$$

#4.1.1.

The condition "$\partial_x F[\lambda_0,0]$ is not a bijection" is however, not sufficient for $(\lambda_0,0)$ to be a bf. point:

Example 4.1.2 $X=Y=\mathbb{C}$, regarded as a Banach space over $\mathbb{R}$.

$$F: \mathbb{R}\times X\to Y$$
$$(\lambda, z)\mapsto z-\lambda z-i|z|^2 z.$$

Then, for $\lambda_0=1$, $\partial_z F[\lambda_0,0]$ is not a bijection, however, $(\lambda_0,0)=(1,0)$ is not a bifurcation point. H.

4.2. Lyapunov-Schmidt Reduction

existence of sol. of $F(\lambda,x)=0$ in a nbhd of $(\lambda_0,0)$ $\underbrace{}_{(\lambda,x)\in\mathbb{F}\times X}$ ↖ infinit dim. $\xrightarrow{L\text{-}S}$ existence of sol. of $h(\lambda,\xi)=0$ in a nbhd of $(\lambda_0,0)$. $\underbrace{}_{(\lambda,\xi)\in\mathbb{F}\times\mathbb{F}^p}$ ↑ finite dim.

Theorem 4.2.1 (Lyapunov-Schmidt Reduction) Suppose $F\in C^k(U,Y)$ for $U\subset\mathbb{F}\times X$ open and $F(\lambda_0,x_0)=0$ for some $(\lambda_0,x_0)\in U$. such that

$$L:=\partial_x F[\lambda_0,x_0]:\ X\to Y$$

is a Fredholm operator with $\ker(L)\neq\{0\}$, $q:=\operatorname{codim}\operatorname{range}(L)$.

Then, there exist $U_0\subset U$, $V\subset\mathbb{F}\times\ker(L)$, $\psi\in C^k(V,X)$, $h\in C^k(V,\mathbb{F}^q)$ s.t.

- $(\lambda_0,x_0)\in U_0$, $(\lambda_0,0)\in V$, $\psi(\lambda_0,0)=x_0$ and
- $F(\lambda,x)=0$ for some $(\lambda,x)\in U_0$ $\iff$ $\begin{cases}\psi(\lambda,\xi)=x\\ h(\lambda,\xi)=0\end{cases}$ for some $(\lambda,\xi)\in V$

Remark 4.2.2 The infinite-dimensional problem $F(\lambda,x)=0$ is "reduced" to the equivalent finite-dimensional problem "find $(\lambda,\xi)\in V\subset\mathbb{F}\times\ker(L)$ such that $h(\lambda,\xi)=0$".

pf. 4.2.1 L is Fredholm $\Rightarrow$ $X=\ker(L)\oplus \overset{\text{closed}}{W}$, $Y=\overset{\cong\mathbb{F}^q}{Z}\oplus\operatorname{range}(L)$
$L|_W$ is a bijection

Let $P:Y\to Y$ be a (bounded) projection s.t. $\begin{cases}\ker P=\operatorname{range}(L)\\ \operatorname{range}P=Z\end{cases}$.

$\ulcorner$ $P:\ Z\oplus\operatorname{range}(L)\to Z\oplus\operatorname{range}(L)$
$(x,\ y)\mapsto(x,\ 0)$ $\lrcorner$

Then, $(I-P)\underbrace{Lx}_{\in\operatorname{range}(L)=\ker(P)}=Lx$ for all $x\in X$.

Thus, $(I-P)L$ is a bijection, thus a homeom. (since P, L are bounded) from W to $\text{range}(L)$.

Consider $(\lambda, x) \in U$ and write $(\lambda, x) = (\lambda, x_0 + \xi + \eta)$ for $\xi \in \ker(L), \eta \in W$, since $U \subset \mathbb{F} \times X = \mathbb{F} \times \ker(L) \oplus W$. Define $G : \{(\lambda, \xi, \eta) \in \mathbb{F} \times \ker(L) \times W : (\lambda, x_0 + \xi + \eta) \in U\} \to Y$ by

$$G(\lambda, \xi, \eta) = (I-P) F(\lambda, x_0 + \xi + \eta) \underset{\substack{\uparrow \\ \text{rang}(I-P) = \ker(P) = \text{range}(L)}}{\in} \text{rang}(L)$$

Then, we have

- $G(\lambda_0, 0, 0) = (I-P) \underbrace{F(\lambda_0, x_0)}_{\overset{\shortparallel}{0}} = 0$
- $\partial_\eta G[\lambda_0, 0, 0] \eta = (I-P) \partial_x F[\lambda_0, x_0] \eta = (I-P) L \eta \quad \forall \eta \in W$

  thus, $\partial_\eta G[\lambda_0, 0, 0]$ is a homeom. from W to $\text{range}(L)$.

I.F.T. : $\exists\, U_0 \subset U$, $V \subset \mathbb{F} \times \ker(L)$, $\phi \in C^k(V, W)$ s.t.

$(\lambda_0, 0) \in V$, $(\lambda_0, x_0) \in U_0$, $\phi(\lambda_0, 0) = 0$ and $G(\lambda, \xi, \phi(\lambda, \xi)) = 0 \ \forall (\lambda, \xi) \in V$ and

$$\{(\lambda, x_0 + \xi + \eta) \in U_0 : \underline{(I-P) F(\lambda, x_0 + \xi + \eta) = 0}\} = \{(\lambda, x_0 + \xi + \eta) : (\lambda, \xi) \in V, \underline{\eta = \phi(\lambda, \xi)}\}.$$

Thus, let $\psi(\lambda, \xi) = x_0 + \xi + \phi(\lambda, \xi)$ and $h(\lambda, \xi) = PF(\lambda, \psi(\lambda, \xi)) \underset{\substack{\uparrow \\ \text{range } P = Z}}{\in} Z$.

Then, for all $(\lambda, \xi) \in V$,

$$h(\lambda, \xi) = 0 \iff PF(\lambda, x_0 + \xi + \phi(\lambda, \xi)) = 0$$

$$\iff F(\lambda, x_0 + \xi + \phi(\lambda, \xi)) = 0$$

$\ulcorner$ $G(\lambda, \xi, \phi(\lambda, \xi)) = 0 \ \forall (\lambda, \xi) \in V \overset{\partial_\xi}{\Rightarrow} \underbrace{\partial_\xi G[\lambda_0, 0, 0] \cdot h}_{\substack{\shortparallel \\ (I-P) \underbrace{L h}_{\substack{\shortparallel \\ 0 \text{ since } h \in \ker(L)}} = 0}} + \underbrace{\partial_\eta G[\lambda_0, 0, 0] \cdot \partial_\xi \phi[\lambda_0, 0] \cdot h}_{\underbrace{(I-P) L}_{\text{homeo. on } W} (\underbrace{\partial_\xi \phi[\lambda_0, 0] \cdot h}_{\in W})} = 0 \quad \forall h \in \ker(L)$ **4.2.1.**

$\Rightarrow \partial_\xi \phi[\lambda_0, 0] \cdot h = 0 \quad \forall h \in \ker(L)$

$\Rightarrow \partial_\xi \phi[\lambda_0, 0] = 0. \quad (*)$ $\lrcorner$

4.3 Crandall-Rabinowitz transversality

We give a sufficient condition for $(\lambda_0,0)$ to be a b.f. point.

Theorem 4.3.1 Suppose that $F: \mathbb{F}\times X \to Y$ is of C^k, $k\geq 2$, and $F(\lambda,0)=0 \ \forall_{\lambda\in\mathbb{F}}$. Assume that

- $L=\partial_x F[\lambda_0,0]$ is a Fredholm operator of index 0;
- $\ker L = \{\xi\in X: \xi = s\xi_0 \text{ for some } s\in\mathbb{F}\}$ for some $\xi_0\in X\setminus\{0\}$ is one-dimensional;
- the transversality condition holds: $\partial^2_{\lambda,x}F[\lambda_0,0](1,\xi_0)\notin \operatorname{range}(L)$.

Then, $(\lambda_0,0)$ is a bifurcation point. More precisely, $\exists\ \varepsilon>0$ and a branch of solutions

$$\{(\lambda,x)=(\Lambda(s), s\chi(s)) : s\in\mathbb{F},\ |s|<\varepsilon\}\subset \mathbb{F}\times X$$

such that $\Lambda(0)=\lambda_0$, $\chi(0)=\xi_0$, $F(\Lambda(s), s\chi(s))=0\ \forall_s$ with $|s|<\varepsilon$, and

Λ, $s\mapsto s\chi(s)$ are of class C^{k-1}, χ is of class C^{k-2} on $(-\varepsilon,\varepsilon)$;

$\exists\ U_0\subset \mathbb{F}\times X$ s.t. $(\lambda_0,0)\in U_0$ and

$$\{(\lambda,x)\in U_0 : F(\lambda,x)=0,\ x\neq 0\} = \{(\Lambda(s), s\chi(s)) : 0<|s|<\varepsilon\}.$$

Remark 4.3.2 The notation $\partial^2_{\lambda,x}F[\lambda_0,0](1,\xi_0)$ means the following:

$$\underset{(\lambda_0,\,x_0)}{F: U\subset \mathbb{F}\times X\to Y} \rightsquigarrow \partial_x F[\lambda_0,x_0]\in \mathcal{L}(X,Y)$$

$$\rightsquigarrow G=\partial_x F: \mathbb{F}\times X \to \mathcal{L}(X,Y) \underset{\partial_\lambda G}{\rightsquigarrow} \partial^2_{\lambda,x}F[\lambda_0,x_0]\in \mathcal{L}(\mathbb{F},\mathcal{L}(X,Y))$$

let $1\in\mathbb{F}$, $\xi_0\in X$, then $\partial^2_{\lambda,x}F[\lambda_0,x_0]\cdot 1\in\mathcal{L}(X,Y)$, $\partial^2_{\lambda,x}F[\lambda_0,x_0](1,\xi_0)\in Y$. So:

H. $$\partial^2_{\lambda,x}F[\lambda_0,0](1,\xi_0)=\lim_{t\to 0}\frac{\partial_x F[\lambda_0+t,0]\xi_0-\partial_x F[\lambda_0,0]\xi_0}{t}\in Y.$$

Pf 4.3.1: Let U_0, V, ϕ, ψ and h be given by the Lyapunov-Schmidt reduction of $F(\lambda,x)=0$ in a neighborhood of $(\lambda_0,0)\in\mathbb{F}\times X$. Then,

$$\left\{\begin{array}{l} U_0\subset\mathbb{F}\times X,\ V\subset\mathbb{F}\times\ker(L),\ \phi: V\to W \text{ s.t.} \\ \{(\lambda, x_0+\xi+\eta)\in U_0 : (I-P)F(\lambda, x_0+\xi+\eta)=0\} = \{(\lambda, x_0+\xi+\eta) : (\lambda,\xi)\in V,\ \eta=\phi(\lambda,\xi)\} \\ h(\lambda,\xi) := PF(\lambda, x_0+\xi+\phi(\lambda,\xi)) \end{array}\right.$$

where $P: Y\to Y$ proj. $\ker P = \text{range } L$, where $x_0=0$.

Therefore, we have $\phi(\lambda,0)=0$ $\forall_{\lambda\in\mathbb{F}}$ and H

- $h(\lambda,0)=0$ $\forall_{(\lambda,0)\in V}$
- $\partial_\xi h[\lambda_0,0]=0$
- $\partial^2_{\lambda,\xi} h[\lambda_0,0](1,\xi_0)\neq 0$

Define $g: \underset{\subset\mathbb{F}\times\ker(L)=\mathbb{F}\times\langle\xi_0\rangle}{V}\to\mathbb{F}$ by $g(\lambda,\xi)=\int_0^1 \partial_\xi h[\lambda,t\xi]\xi_0\,dt$.

Then, g is of class C^{k-1} (since h is C^{k-1}) and

- $g(\lambda_0,0)=0$
- $\partial_\lambda g(\lambda_0,0)=\partial^2_{\lambda,\xi} h[\lambda_0,0](1,\xi_0)\neq 0$

$$\partial_\lambda g[\lambda_0,0]=\int_0^1 \partial^2_{\lambda,\xi} h[\lambda_0,0](1,\xi_0)\,dt = \partial^2_{\lambda,\xi} h[\lambda_0,0](1,\xi_0)\cdot 1$$

I.F.T: $\exists\ \Lambda\in C^{k-1}(\{s\in\mathbb{F}: |s|<\varepsilon\},\mathbb{F})$ for some $\varepsilon>0$ s.t.

$\Lambda(0)=\lambda_0$, $g(\Lambda(s), s\xi_0)=0$ if $|s|<\varepsilon$.

$$g(\lambda,s\xi_0)=\int_0^1 \partial_\xi h[\lambda,ts\xi_0]\xi_0\,dt$$
$$(\text{if } s\neq 0) = \frac{1}{s}\int_0^{s\xi_0}\partial_\xi h[\lambda,\underbrace{ts\xi_0}]\,d(ts\xi_0)$$
$$=\frac{1}{s}\Big(h(\lambda,s\xi_0)-\underbrace{h(\lambda,0)}_{=0}\Big)=\frac{1}{s}h(\lambda,s$$

Moreover, since $h(\lambda,0)=0$ $\forall_\lambda$,

$$(*)\quad g(\lambda,\xi)=\begin{cases}\dfrac{h(\lambda,\xi)}{s} & \text{if } s\neq 0\\ \partial_\xi h[\lambda,0]\xi_0 & \text{if } s=0\end{cases}\quad \text{for } (\lambda,\xi)=(\lambda,s\xi_0)\in V$$

Now let $\chi(s) = s^{-1}\psi(\Lambda(s), s\xi_0)$ for $0<|s|<\varepsilon$ and $\chi(0)=\xi_0$

then
$$\lim_{0\neq s\to 0}\chi(s) = \lim_{s\to 0}\frac{\frac{d}{ds}\psi(\Lambda(s), s\xi_0)}{\frac{d}{ds}(s)} = \lim_{s\to 0}\Big(\partial_\lambda\psi[\Lambda(s), s\xi_0]\Lambda'(s) + \partial_\xi\psi[\Lambda(s), s\xi_0]\xi_0\Big)$$
$$= \underbrace{\partial_\lambda\psi[\lambda_0,0]}_{=0}\Lambda'(0) + \underbrace{\partial_\xi\psi[\lambda_0,0]\xi_0}_{} = \xi_0$$

$\psi(\lambda,0) = \phi(\lambda,0) = 0 \ \forall \lambda$

$\psi(\lambda,\xi) = \xi + \phi(\lambda,\xi)$

$\Rightarrow \partial_\xi\psi[\lambda_0,0] = I_\xi + \underbrace{\partial_\xi\phi[\lambda_0,0]}_{\overset{(*)}{=}0} = I_\xi$

$\Rightarrow \partial_\xi\psi[\lambda_0,0]\xi_0 = \xi_0.$

Also, for all $|s|<\varepsilon$,

$$g(\Lambda(s), s\xi_0) = 0 \overset{(*)}{\Rightarrow} h(\Lambda(s), s\xi_0) = 0 \overset{L\text{-}S}{\Rightarrow} F(\Lambda(s), \psi(\Lambda(s), s\xi_0)) = 0$$
$$\Rightarrow F(\Lambda(s), s\chi(s)) = 0 \quad \forall\, |s|<\varepsilon.$$

4.3.1

Example 4.3.2 (Concerning transversality) H.

(a) $\mathbb{F}=\mathbb{R}$, $X=Y=\mathbb{R}$, $F(\lambda,x) := x(\lambda^2+x^2)$, then at $(\lambda_0,0)=(0,0)$, we have

$\partial^2_{\lambda,x}F[0,0](1,\xi_0) \in \text{range}(L)$, and $(0,0)$ is not a bifurcation point.

(b) $\mathbb{F}=\mathbb{R}$, $X=Y=\mathbb{R}$, $F(\lambda,x) = x(\lambda+x^2)$, then at $(0,0)$,

$\partial^2_{\lambda,x}F[0,0](1,\xi_0) \notin \text{range}(L)$ and $(0,0)$ is a bifurcation point.

(c) $\mathbb{F}=\mathbb{R}$, $X=Y=\mathbb{R}$, $F(\lambda,x) = x(\lambda^3+x^3)$, then at $(0,0)$,

$\partial^2_{\lambda,x}F[0,0](1,\xi_0) \in \text{range}(L)$ but $(0,0)$ is a bifurcation point.

Proposition 4.3.4

(a) Let U_0, V, h and Ψ be given in Theorem 4.2.1. Then, U_0 and V can be chosen small enough so that for all $(\lambda, \xi) \in V$,

$$\dim \ker\left(\partial_x F[\lambda, \Psi(\lambda,\xi)]\right) = \dim \ker\left(\partial_\xi h[\lambda,\xi]\right)$$

(b) Let $\Lambda, \chi, \varepsilon$ be given in Theorem 4.3.1 and $U = \mathbb{F} \times X$, $(\lambda_0, x_0) = (\lambda_0, 0)$.

Then,

$$\dim \ker\left(\partial_x F[\Lambda(s), s\chi(s)]\right) \in \{0,1\}$$

and for s with $0 < |s| < \varepsilon$,

$$\dim \ker\left(\partial_x F[\Lambda(s), s\chi(s)]\right) = 1 \iff \Lambda'(s) = 0.$$

pf: (a) $\quad (I-P)\, F(\lambda, x_0 + \xi + \phi(\lambda,\xi)) \equiv 0 \quad \forall (\lambda,\xi) \in V \subset \mathbb{F} \times \ker(L)$

$$\xRightarrow[\text{w.r.t. } \xi]{\text{diff.}} \quad (I-P)\, \partial_x F[\lambda, x_0 + \xi + \phi(\lambda,\xi)] \left(I_\xi + \underbrace{\partial_\xi \phi[\lambda,\xi]}_{\in \mathcal{L}(\ker(L), Y)}\right) \equiv 0 \quad \forall (\lambda,\xi) \in V$$

$$\Rightarrow \quad (I-P)\, \partial_x F[\lambda, x_0 + \xi + \phi(\lambda,\xi)] \left(v + \partial_\xi \phi[\lambda,\xi] v\right) \equiv 0 \quad \forall v \in \ker(L).$$

Consider $(v, w) \in V \times W \subseteq \mathbb{F} \times \underbrace{\ker(L) \times W}_{= X}$ such that

$$\partial_x F[\lambda, x_0 + \xi + \phi(\lambda,\xi)]\, (v + w) = 0$$

$$\Rightarrow \quad (I-P)\, \partial_x F[\lambda, x_0 + \xi + \phi(\lambda,\xi)]\, (v + w) = 0$$

$$\Rightarrow \quad (I-P)\, \partial_x F[\lambda, x_0 + \xi + \phi(\lambda,\xi)]\, \underbrace{(w - \partial_\xi \phi[\lambda,\xi] v)}_{\in W} = 0.$$

$$\xRightarrow[\text{is a bijection,}]{(I-P)\partial_x F[\lambda, x_0 + \xi + \phi(\lambda,\xi)]|_W} \quad w = \partial_\xi \phi[\lambda,\xi] v$$

if U_0, V small enough, since $(I-P)\partial_x F[\lambda_0, x_0]$ is a bij.

Thus,

$$\partial_x F[\lambda, x_0 + \xi + \phi(\lambda,\xi)](v+w) = 0 \quad \text{for some } w \in W$$

$$\iff P\partial_x F[\lambda, x_0+\xi+\phi(\lambda,\xi)]\,(v + \partial_\xi \phi[\lambda,\xi]v) = 0$$

$$\xLeftrightarrow{h(\lambda,\xi) = P\cdot F(\lambda, x_0+\xi+\phi(\lambda,\xi))} \partial_\xi h[\lambda,\xi]\, v = 0$$

$\#$(a)

(b)

$$\dim\ker \partial_x F[\Delta(s), s\chi(s)] \overset{(a)}{=} \dim\ker \partial_\xi h[\lambda,\xi] \in \{0,1\}$$

$$h: V \subset \mathbb{F}_\lambda \times \ker(L)_\xi \to \mathbb{F}$$

$$\partial_\xi h[\lambda,\xi]: \quad \ker(L) = \langle \xi_0 \rangle \to \mathbb{F}$$

1-dim 1-dim

$$\dim\ker \partial_x F[\Delta(s), s\chi(s)] = 1$$

$$\iff \dim\ker \partial_\xi h[\lambda,\xi] = 1 \iff \partial_\xi h[\lambda,\xi]\xi_0 = 0$$

Since $h(\Delta(s), s\xi_0) = 0 \quad \forall_s$ with $|s| < \varepsilon$,

$$\xRightarrow[\text{w.r.t. } s]{\text{diff.}} \underbrace{\partial_\lambda h[\Delta(s), s\xi_0]}_{= s\partial_\lambda g[\Delta(s), s\xi_0]} \Delta'(s) + \underbrace{\partial_\xi h[\Delta(s), s\xi_0]\cdot\xi_0}_{=0, \text{ if } \dim\ker \partial_x F[\cdot,\cdot] = 1} = 0 \quad \forall_{s,\ |s|<\varepsilon}$$

$\partial_\lambda g[\Delta(s), s\xi_0] \neq 0$
s small, $\Longrightarrow$

$$\Delta'(s) = 0$$

Since $\partial_\lambda g[\lambda_0, 0] = \partial^2_{\lambda,\xi} h[\lambda_0, 0](1,\xi_0) \neq 0$.

$\#$(b).

4.4. Bifurcation from a simple eigenvalue

Theorem 4.4.1 *Let X be a real Banach space, which is continuously embedded in a real Banach space Y and $\{(\lambda,0) : \lambda \in \mathbb{R}\} \subset U \subset \mathbb{R} \times X$, where U is open. Suppose that*

$F \in C^k(U, Y)$, $k \geq 2$ and for all $\lambda \in \mathbb{R}$,

$$F(\lambda, 0) = 0 \quad \text{and} \quad \partial_x F[\lambda, 0] = \lambda \iota - A$$

where $\iota : X \hookrightarrow Y$ is the embedding.

Then, every simple eigenvalue λ_0 of A is a bifurcation point and the conclusion of Theorem 4.3.1 holds.

pf: 4.

Suppose the assumptions of Theorem 4.4.1 hold.

Let $y^* \in Y^*$ s.t. $y^*(\xi_0) = \|\xi_0\|$, $\|y^*\| = 1$, and $\ker y^* = \operatorname{range}(\lambda_0 \iota - A)$,

where $\ker(\lambda_0 \iota - A) = \langle \xi_0 \rangle$.

Let $(\Lambda(s), s\chi(s))$ be given by Theorem 4.4.1 and let

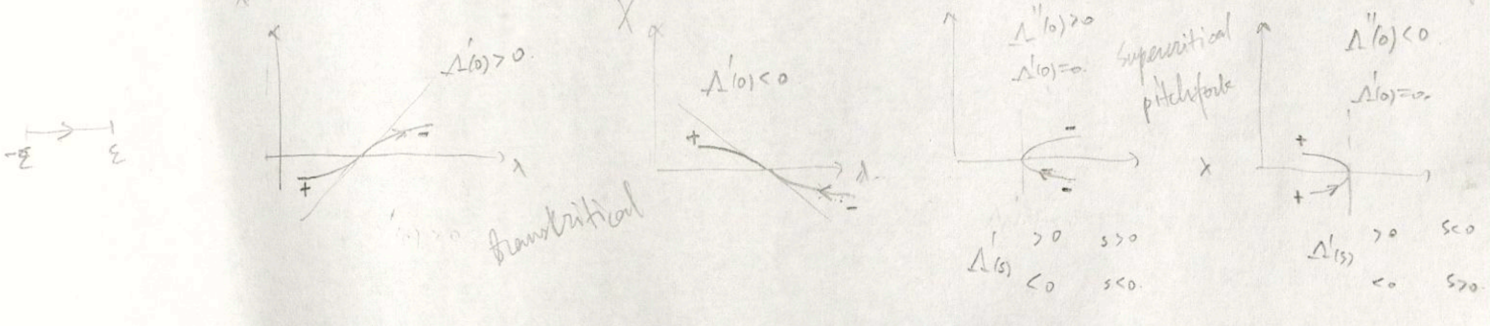
$$L(s) = \partial_x F[\Lambda(s), s\chi(s)] \in \mathcal{L}(X, Y), \quad \text{for } s \in (-\varepsilon, \varepsilon).$$

By Theorem 3.3.1, there exists a C^{k-1}-curve $\{(\mu(s), \xi(s)) : s \in (-\varepsilon, \varepsilon)\}$ $\subset \mathbb{R} \times X$ such that $(\mu(0), \xi(0)) = (0, \xi_0)$ and

$$L(s)\xi(s) = \mu(s)\iota\xi(s), \quad y^*(\iota\xi(s)) = 1, \qquad s \in (-\varepsilon, \varepsilon)$$

where $\xi(s) = \xi_0 + \eta(s)$, $\eta(0) = 0$, $\iota\eta(s) \in \operatorname{range} L(0)$.

<u>Proposition 4.4.2</u> $\displaystyle \lim_{\substack{s \to 0 \\ s\Lambda'(s) \neq 0}} \frac{\mu(s)}{s\Lambda'(s)} = -1.$



4.5. Bending an elastic rod II

We apply the local bifurcation theory to the boundary-value problem (cf. Sec. 1.1):

$$(*)\begin{cases}\phi''(x)+\lambda\sin\phi(x)=0 & \text{for } x\in[0,L]\\ \phi'(0)=\phi'(L)=0\end{cases},$$

where $L>0$ is fixed length of the rod, $\lambda>0$ is the bifurcation parameter (related to the force P).

Let $\mathbb{F}=\mathbb{R}$, $X=\{\phi\in C^2[0,L]:\phi'(0)=\phi'(L)=0\}$, $Y=C[0,L]$, and

$$F:\mathbb{R}\times X\to Y,\quad F(\lambda,\phi):=\phi''+\lambda\sin\phi.$$

<u>Note</u>: F is <u>$\mathbb{R}$-analytic</u>, i.e. $F(x)=\sum_{n=0}^{\infty}\frac{F^{(n)}(x_0)}{n!}(x-x_0)^n \quad \forall_{x\in\mathbb{R}\times X}$.

Indeed, $F=F_1+F_2$, where $F_1(\lambda,\phi)=\phi''$, $F_2(\lambda,\phi)=\lambda\sin\phi$.

Clearly, F_1 is $\mathbb{R}$-analytic, since $\partial_\phi F_1[\lambda_0,\phi_0]:h\mapsto h''$ and $\partial_\lambda F_1[\lambda_0,\phi_0]:\lambda\mapsto 0$,

so $\underset{\phi''}{F_1(\lambda,\phi)}=\underset{\phi_0''}{F_1(\lambda_0,\phi_0)}+\underbrace{\frac{\partial_\phi F_1[\lambda_0,\phi_0]}{1!}(\phi-\phi_0)}_{(\phi-\phi_0)''=\phi''-\phi_0''}$

Also, F_2 is $\mathbb{R}$-analytic, since $f_2:\mathbb{R}^2\to\mathbb{R}$, $f_2(\lambda,x)=\lambda\sin x$ is $\mathbb{R}$-analytic.

⌈In general, if $f:\mathbb{F}^N\to\mathbb{F}^M$ is a C^k- (resp. analytic) function, then the <u>Nemytskii operator</u> $F:C([0,L],\mathbb{F}^N)\to C([0,L],\mathbb{F}^M)$

$$u\longmapsto F(u):t\mapsto f(u(t))$$

is C^k- (resp. analytic). ⌋

Thus, F is C^∞-function from $\mathbb{R}\times X$ to Y.

Moreover, $F(\lambda,0)=0\ \forall_{\lambda\in\mathbb{R}}$ and $\partial_\phi F[\lambda,0]\phi=\phi''+\lambda\phi\ \ \forall_{\phi\in X}$.

Consider $\phi''+\lambda\phi=0$ with $\phi'(0)=\phi'(L)=0$.

This linear problem has non-zero solutions for $\lambda > 0$

$\Longleftrightarrow$ $\lambda \in \{ \lambda_k := \left(\frac{k\pi}{L}\right)^2 : k \in \mathbb{N} \}$ and for $\lambda = \lambda_k$, $\phi := c \cdot \cos \frac{k\pi s}{L}$, for some $c \in \mathbb{R}$.

i.e. $\ker \partial_\phi F[\lambda_k, 0] = \langle \phi_k \rangle$, where $\phi_k(s) = \cos \frac{k\pi s}{L}$

So $\dim \ker \partial_\phi F[\lambda_k, 0] = 1$.

Furthermore,

$$\text{range}\left(\partial_\phi F[\lambda_k, 0]\right) = \left\{ v \in C[0,L] : \int_0^L v(s)\, \phi_k(s)\, ds = 0 \right\}$$ ← codim $= 1$.

"$\subseteq$" if $\exists\, u \in X$ s.t. $u'' + \lambda_k u = v \in Y$, then

$$\int_0^L v(s)\, \phi_k(s)\, ds = \int_0^L \left(u''(s) + \lambda_k u(s)\right) \phi_k(s)\, ds \overset{\text{integration by parts}}{=} \int_0^L \left(\phi_k''(s) + \lambda_k \phi_k(s)\right) u(s)\, ds = 0$$

"$\supseteq$" variation of constants

Therefore, for $A : X \to Y$ be given by $A\phi = -\phi''$, every λ_k, for $k \in \mathbb{N}$, is a simple eigenvalue of A. Thus, by Theorem 4.4.1, there is a bifurcation from a simple eigenvalue for $(*)$ at every point $(\lambda_k, 0)$ on the line of trivial solutions.

4.6. Bifurcation of periodic solutions

We give a simple example of the existence, via bifurcation from a line of trivial solutions, of non-constant, but periodic, solutions of differential equations.

Let $\delta > 0$ and $A, B \in C^2((-\delta,\delta)\times\mathbb{R}\times(-\delta,\delta);\mathbb{R})$ be 2π-per. in the 2nd. variable.

Consider the boundary-value problem on $[0,2\pi]$

$$(**)\quad \begin{cases} \dot{w}(t) + \lambda w(t) + w(t)^2 A(w(t),t,\lambda) + \lambda^2 w(t) B(w(t),t,\lambda) = 0 \\ w(0) = w(2\pi) \end{cases} \quad \text{for } w \in C^1[0,2\pi],$$

Note that $(\lambda, w) = (\lambda, 0)$ is a solution for all $\lambda \in \mathbb{R}$.

To find non-constant periodic solutions, define

$$\mathbb{F} = \mathbb{R}, \quad X = \{u \in C^1[0,2\pi] : u(0) = u(2\pi),\ u'(0) = u'(2\pi)\},$$

$$Y = \{v \in C[0,2\pi] : u(0) = v(2\pi)\}$$

and for $0 \le t \le 2\pi$, $u \in X$, define $F : \mathbb{R}\times X \to Y$ by

$$F(\lambda,u)(t) = \dot{u}(t) - \lambda u(t) + u(t)^2 A(u(t),t,\lambda) + \lambda^2 u(t) B(u(t),t,\lambda).$$

Then, $F \in C^2(U,Y)$, where

$$U = \{(\lambda,u) : |\lambda| < \delta,\ \max_{0\le t\le 2\pi} |u(t)| < \delta\} \subset \mathbb{R}\times X.$$

Moreover,

$$\text{H.}\ \begin{cases} \partial_u F[0,0]u = \dot{u} \quad \forall_{u\in X} \\ \ker(\partial_u F[0,0]) = \{u \in X : u \text{ is a constant}\}, \\ \operatorname{range}(\partial_u F[0,0]) = \{v \in Y : \int_0^{2\pi} v(t)\,dt = 0\} \end{cases}$$

$\Rightarrow$ $\partial_u F[0,0]$ is a Fredholm operator of index 0. Let $\xi_0 = 1$.

Then, $\partial^2_{\lambda,u} F[0,0](1,\xi_0) = \xi_0$ H. $\notin \operatorname{range} \partial_u F[0,0]$.

Consequently, by Theorem 4.3.1,

$\exists$ C^1-curve $\{(\Lambda(s), s\chi(s)) \in U : s \in (-\varepsilon, \varepsilon)\}$ s.t.

$$F(\Lambda(s), s\chi(s)) = 0, \quad \chi(0) = \xi_0 = 1, \quad \Lambda(0) = 0.$$

For s small, $\chi(s) = \xi_0 + \eta(s) = 1 + \eta(s)$, (where $\eta(0) = 0$) is positive.

Thus, for $s \in (0, \varepsilon)$, $w = s\chi(s)$ is a positive solution of $(**)$.

Moreover, under the additional assumption that

$$\partial_t A[0,t,0] \not\equiv 0,$$

$w = s\chi(s)$ is a $\underline{\underline{\text{non-constant}}}$ solution for all $|s| < \varepsilon$, for s sufficiently small.

$\ulcorner$

assume to the contrary, that $\exists \{(\lambda_n, c_n)\}$ of solutions, where $c_n \neq 0$ is constant,

s.t. $(\lambda_n, c_n) \xrightarrow{X} (0,0)$, then $\dot{c}_n = 0$ and from $(**)$, we have

$$(P) \qquad \lambda_n + c_n A(c_n, t, \lambda_n) + \lambda_n^2 B(c_n, t, \lambda_n) = 0 \quad \forall_n, \ \forall_{t \in [0, 2\pi]}$$

Since $c_n \neq 0$, let $(\bar{\lambda}_n, \bar{c}_n) = \dfrac{(\lambda_n, c_n)}{\|(\lambda_n, c_n)\|_{\mathbb{R} \times X}}$ then $\|(\bar{\lambda}_n, \bar{c}_n)\|_{\mathbb{R} \times X} = 1$.

$\mathbb{R} \times X$ is complete $\Rightarrow \exists (\lambda_*, c_*)$ s.t. $(\bar{\lambda}_n, \bar{c}_n) \to (\lambda_*, c_*) \in \mathbb{R} \times X$, so $\|(\lambda_*, c_*)\| = 1$.

It follows from (P) that

$$\lambda_* + c_* A(0, t, 0) = 0 \quad \forall_{t \in [0, 2\pi]}$$

Note $c_* \neq 0$ (if $c_* = 0$, then $\lambda_* = 0$, $\lightning \|(\lambda_*, c_*)\| = 1$), thus

$$A(0, t, 0) = -\frac{\lambda_*}{c_*} \quad \forall_{t \in [0, 2\pi]}$$

$$\Rightarrow \quad \partial_t A[0, t, 0] \equiv 0 \quad \forall_{t \in [0, 2\pi)} \quad \lightning$$

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