Chap 4. Local Bifurcation Theory

Let $F: U \subset \mathbb{F} \times X \to Y$ be Fréchet-differentiable and

$F(\lambda, 0) = 0 \quad \forall_{\lambda \in \mathbb{F}}$.

Question: for which $\lambda_0 \in \mathbb{F}$ is there a sequence $\{(\lambda_n, x_n)\} \subset \mathbb{F} \times X \setminus \{0\}$ of solutions converging to $(\lambda_0, 0)$ in $\mathbb{F} \times X$?

Such $(\lambda_0, 0)$ are called bifurcation points on the line of trivial solutions $\{(\lambda, 0) : \lambda \in \mathbb{F}\}$; λ_0 is also referred as the bifurcation point.

4.1. A necessary condition

Assume that F is cont. diff. in a neighborhood of $(\lambda_0, 0)$

Implicit Function Theorem

I.F.T: $\partial_x F[\lambda_0, 0]$ is a homeomorphism $\Rightarrow$ in a nbhd of $(\lambda_0, 0)$, all solutions of $F(\lambda, x) = 0$ lie on a unique curve $\{(\lambda, x) : x = \phi(\lambda),\ \lambda \in (\lambda_0 - \varepsilon, \lambda_0 + \varepsilon)\}$.

$$\xRightarrow{F(\lambda,0)=0 \ \forall_{\lambda \in \mathbb{F}}} \quad F^{-1}(0) \cap \underset{\text{of } (\lambda_0,0)}{\text{a nbhd}} = \{(\lambda, 0) : \lambda_0 - \varepsilon < \lambda < \lambda_0 + \varepsilon\} \Rightarrow (\lambda_0, 0) \text{ is } \underline{\underline{\text{no}}}\text{. bf. point}$$

In other words,

$$(\lambda_0, 0) \text{ is a bifurcation point} \Rightarrow \partial_x F[\lambda_0, 0] \text{ is not a homeom.}$$

$$\underset{\partial_x F[\lambda_0,0] \in \mathcal{L}(X,Y)}{\overset{2.3.2}{\Longleftrightarrow}} \partial_x F[\lambda_0, 0] \text{ is not a bijection.}$$

$$\xRightarrow{\text{if } \partial_x F[\lambda_0,0] \text{ is Fredholm of index } 0} \{0\} \neq \ker \partial_x F[\lambda_0, 0] : X \to Y$$

Example 4.1.1 Let $X=Y$ and $F: \mathbb{F}\times X \to Y$ is a C^1-function (w.r.t. x) of form $F(\lambda,x)=x-G(\lambda,x)$, where $G(\lambda,\cdot): X\to Y$ is a compact nonlinear operator with $G(\lambda,0)=0$ $\forall_{\lambda\in\mathbb{F}}$. Then,

$$\partial_x F[\lambda_0,0] = I - \partial_x G[\lambda_0,0],$$

where $\partial_x G[\lambda_0,0]$ is a compact linear operator on X by Lemma 3.1.8. By Theorem 2.4.7, $\partial_x F[\lambda_0,0]$ is a Fredholm operator of index 0. Thus, $\partial_x F[\lambda_0,0]$ is a homeo. $\Leftrightarrow$ $\ker \partial_x F[\lambda_0,0]=\{0\}$. So

$$(\lambda_0,0) \text{ is a bf. point } \Longrightarrow \exists\, \xi\neq 0 \text{ s.t. } \partial_x F[\lambda_0,0]\xi = 0.$$

#4.1.1.

The condition "$\partial_x F[\lambda_0,0]$ is not a bijection" is however, not sufficient for $(\lambda_0,0)$ to be a bf. point:

Example 4.1.2 $X=Y=\mathbb{C}$, regarded as a Banach space over $\mathbb{R}$.

$$F: \mathbb{R}\times X \to Y$$
$$(\lambda, z) \mapsto z-\lambda z - i|z|^2 z.$$

Then, for $\lambda_0=1$, $\partial_z F[\lambda_0,0]$ is not a bijection, however, $(\lambda_0,0)=(1,0)$ is not a bifurcation point. H.

4.2. Lyapunov-Schmidt Reduction

existence of sol. of $F(\lambda,x)=0$ in a nbhd of $(\lambda_0,0)$ $\underset{(\lambda,x)\in\mathbb{F}\times X}{}$ ↖ infinit dim. $\xrightarrow{L\text{-}S}$ existence of sol. of $h(\lambda,\xi)=0$ in a nbhd of $(\lambda_0,0)$. $\underset{(\lambda,\xi)\in\mathbb{F}\times\mathbb{F}^p}{}$ ↑ finite dim.

Theorem 4.2.1 (Lyapunov-Schmidt Reduction) Suppose $F\in C^k(U,Y)$ for $U\subset\mathbb{F}\times X$ open and $F(\lambda_0,x_0)=0$ for some $(\lambda_0,x_0)\in U$. such that

$$L:=\partial_x F[\lambda_0,x_0]: X\to Y$$

is a Fredholm operator with $\ker(L)\neq\{0\}$, $q:=\operatorname{codim}\operatorname{range}(L)$.

Then, there exist $U_0\subset U$, $V\subset\mathbb{F}\times\ker(L)$, $\psi\in C^k(V,X)$, $h\in C^k(V,\mathbb{F}^q)$ s.t.

- $(\lambda_0,x_0)\in U_0$, $(\lambda_0,0)\in V$, $\psi(\lambda_0,0)=x_0$ and
- $F(\lambda,x)=0$ for some $(\lambda,x)\in U_0$ $\iff$ $\begin{cases}\psi(\lambda,\xi)=x\\ h(\lambda,\xi)=0\end{cases}$ for some $(\lambda,\xi)\in V$

Remark 4.2.2 The infinite-dimensional problem $F(\lambda,x)=0$ is "reduced" to the equivalent finite-dimensional problem "find $(\lambda,\xi)\in V\subset\mathbb{F}\times\ker(L)$ such that $h(\lambda,\xi)=0$".

Pf. 4.2.1 L is Fredholm $\Rightarrow$ $X=\ker(L)\oplus \underset{\text{closed}}{W}$, $Y=\underset{\cong\mathbb{F}^q}{Z}\oplus\operatorname{range}(L)$

$L|_W$ is a Bijection

Let $P: Y\to Y$ be a (bounded) projection s.t. $\begin{cases}\ker P=\operatorname{range}(L)\\ \operatorname{range}P=Z\end{cases}$.

$$\ulcorner\ P: Z\oplus\operatorname{range}(L)\to Z\oplus\operatorname{range}(L)$$
$$(x,\ y)\mapsto(x,\ 0)\ \lrcorner$$

Then, $(I-P)\underset{\in\operatorname{range}(L)=\ker(P)}{Lx}=Lx$ for all $x\in X$.

Thus, $(I-P)L$ is a bijection, thus a homeom. (since P, L are bounded) from W to range(L).

Consider $(\lambda, x) \in U$ and write $(\lambda, x) = (\lambda, x_0 + \xi + \eta)$ for $\xi \in \ker(L), \eta \in W$, since $U \subset \mathbb{F} \times X = \mathbb{F} \times \ker(L) \oplus W$. Define $G : \{(\lambda, \xi, \eta) \in \mathbb{F} \times \ker(L) \times W : (\lambda, x_0 + \xi + \eta) \in U\} \to Y$ by

$$G(\lambda, \xi, \eta) = (I-P)F(\lambda, x_0 + \xi + \eta) \in \mathrm{rang}(L)$$

$\mathrm{rang}(I-P) = \ker(P) = \mathrm{range}(L)$

Then, we have

- $G(\lambda_0, 0, 0) = (I-P)\underbrace{F(\lambda_0, x_0)}_{=0} = 0$
- $\partial_\eta G[\lambda_0, 0, 0]\eta = (I-P)\,\partial_x F[\lambda_0, x_0]\eta = (I-P)L\eta \quad \forall \eta \in W$

thus, $\partial_\eta G[\lambda_0, 0, 0]$ is a homeom. from W to range(L).

I.F.T. : $\exists\, U_0 \subset U,\ V \subset \mathbb{F} \times \ker(L),\ \phi \in C^k(V, W)$ s.t.

$(\lambda_0, 0) \in V,\ (\lambda_0, x_0) \in U_0,\ \phi(\lambda_0, 0) = 0$ and $G(\lambda, \xi, \phi(\lambda, \xi)) = 0 \quad \forall (\lambda, \xi) \in V$

and

$$\{(\lambda, x_0 + \xi + \eta) \in U_0 : (I-P)F(\lambda, x_0 + \xi + \eta) = 0\} = \{(\lambda, x_0 + \xi + \eta) : (\lambda, \xi) \in V,\ \eta = \phi(\lambda, \xi)\}.$$

Thus, let $\psi(\lambda, \xi) = x_0 + \xi + \phi(\lambda, \xi)$ and $h(\lambda, \xi) = PF(\lambda, \psi(\lambda, \xi)) \in Z$.

range $P = Z$

Then, for all $(\lambda, \xi) \in V$,

$$h(\lambda, \xi) = 0 \iff PF(\lambda, x_0 + \xi + \phi(\lambda, \xi)) = 0$$

$$\iff F(\lambda, x_0 + \xi + \phi(\lambda, \xi)) = 0$$

$\lceil\ G(\lambda, \xi, \phi(\lambda, \xi)) = 0 \quad \forall (\lambda, \xi) \in V \overset{\partial_\xi}{\Rightarrow} \underbrace{\partial_\xi G[\lambda_0, 0, 0]\cdot h}_{(I-P)\underbrace{Lh}_{=0 \text{ since } h \in \ker(L)} = 0} + \underbrace{\partial_\eta G[\lambda_0, 0, 0]\cdot \partial_\xi \phi[\lambda_0, 0]\cdot h}_{\underbrace{(I-P)L}_{\text{homeo. on } W}(\underbrace{\partial_\xi \phi[\lambda_0, 0]\cdot h}_{\in W})} = 0 \quad \forall h \in \ker(L)$

4.2.1.

$\Rightarrow \partial_\xi \phi[\lambda_0, 0]\cdot h = 0 \quad \forall h \in \ker$

$\Rightarrow \partial_\xi \phi[\lambda_0, 0] = 0. \quad (*)\ \rfloor$

4.3 Crandall-Rabinowitz transversality

We give a sufficient condition for $(\lambda_0,0)$ to be a b.f. point.

Theorem 4.3.1 Suppose that $F:\mathbb{F}\times X\to Y$ is of C^k, $k\geq 2$, and $F(\lambda,0)=0\ \forall_{\lambda\in\mathbb{F}}$. Assume that

- $L=\partial_x F[\lambda_0,0]$ is a Fredholm operator of index 0;
- $\ker L=\{\xi\in X:\ \xi=s\xi_0 \text{ for some } s\in\mathbb{F}\}$ for some $\xi_0\in X\setminus\{0\}$ is one-dimensional;
- the transversality condition holds: $\partial^2_{\lambda,x}F[\lambda_0,0](1,\xi_0)\notin \text{range}(L)$.

Then, $(\lambda_0,0)$ is a bifurcation point. More precisely, $\exists\,\varepsilon>0$ and a branch of solutions

$$\{(\lambda,x)=(\Lambda(s),s\chi(s)) : s\in\mathbb{F},\ |s|<\varepsilon\}\subset\mathbb{F}\times X$$

such that $\Lambda(0)=\lambda_0$, $\chi(0)=\xi_0$, $F(\Lambda(s),s\chi(s))=0\ \forall_s$ with $|s|<\varepsilon$, and

Λ, $s\mapsto s\chi(s)$ are of class C^{k-1}, χ is of class C^{k-2} on $(-\varepsilon,\varepsilon)$;

$\exists\, U_0\subset\mathbb{F}\times X$ s.t. $(\lambda_0,0)\in U_0$ and

$$\{(\lambda,x)\in U_0:\ F(\lambda,x)=0,\ x\neq 0\}=\{(\Lambda(s),s\chi(s)):\ 0<|s|<\varepsilon\}.$$

Remark 4.3.2 The notation $\partial^2_{\lambda,x}F[\lambda_0,0](1,\xi_0)$ means the following:

$$F: \underset{(\lambda_0,\,x_0)}{U}\subset\mathbb{F}\times X\to Y \rightsquigarrow \partial_x F[\lambda_0,x_0]\in\mathcal{L}(X,Y)$$

$$\rightsquigarrow G=\partial_x F:\mathbb{F}\times X\longrightarrow\mathcal{L}(X,Y)\rightsquigarrow \partial_\lambda G=\partial^2_{\lambda,x}F[\lambda_0,x_0]\in\mathcal{L}(\mathbb{F},\mathcal{L}(X,Y))$$

let $1\in\mathbb{F}$, $\xi_0\in X$, then $\partial^2_{\lambda,x}F[\lambda_0,x_0]\cdot 1\in\mathcal{L}(X,Y)$, $\partial^2_{\lambda,x}F[\lambda_0,x_0](1,\xi_0)\in Y$. So:

H. $$\partial^2_{\lambda,x}F[\lambda_0,0](1,\xi_0)=\lim_{t\to 0}\frac{\partial_x F[\lambda_0+t,0]\,\xi_0-\partial_x F[\lambda_0,0]\,\xi_0}{t}\in Y.$$

Pf 4.3.1: Let U_0, V, ϕ, ψ and h be given by the Lyapunov-Schmidt reduction of $F(\lambda,x)=0$ in a neighborhood of $(\lambda_0,0)\in\mathbb{F}\times X$. Then,

$$\begin{cases} U_0 \subset \mathbb{F}\times X,\ V\subset \mathbb{F}\times \ker(L),\ \phi: V\to W \text{ s.t.} \\ \{(\lambda, x_0+\xi+\eta)\in U_0 : (I-P)F(\lambda, x_0+\xi+\eta)=0\} = \{(\lambda, x_0+\xi+\eta) : (\lambda,\xi)\in V,\ \eta=\phi(\lambda,\xi)\} \text{ where } P: Y\to Y \text{ proj. } \ker P = \operatorname{range} L, \\ h(\lambda,\xi) := PF(\lambda, x_0+\xi+\phi(\lambda,\xi)), \quad \text{where } x_0=0. \end{cases}$$

Therefore, we have $\phi(\lambda,0)=0\ \forall_{\lambda\in\mathbb{F}}$ and H

- $h(\lambda,0)=0 \quad \forall_{(\lambda,0)\in V}$
- $\partial_\xi h[\lambda_0,0]=0$
- $\partial^2_{\lambda,\xi} h[\lambda_0,0](1,\xi_0)\neq 0$

Define $g: \underset{\subset \mathbb{F}\times\ker(L)=\mathbb{F}\times\langle\xi_0\rangle}{V}\to\mathbb{F}$ by $g(\lambda,\xi)=\int_0^1 \partial_\xi h[\lambda,t\xi]\xi_0\,dt$.

Then, g is of class C^{k-1} (since h is C^{k-1}) and

- $g(\lambda_0,0)=0$
- $\partial_\lambda g(\lambda_0,0)=\partial^2_{\lambda,\xi} h[\lambda_0,0](1,\xi_0)\neq 0$

$\partial_\lambda g[\lambda_0,0]=\int_0^1 \partial^2_{\lambda,\xi} h[\lambda_0,0](1,\xi_0)dt$
$=\partial^2_{\lambda,\xi} h[\lambda_0,0](1,\xi_0)\cdot 1$

I.F.T: $\exists\ \Lambda\in C^{k-1}(\{s\in\mathbb{F}: |s|<\varepsilon\},\mathbb{F})$ for some $\varepsilon>0$ s.t.

$\Lambda(0)=\lambda_0,\ g(\Lambda(s), s\xi_0)=0$ if $|s|<\varepsilon$.

$g(\lambda,s\xi_0)=\int_0^1\partial_\xi h[\lambda,ts\xi_0]\xi_0\,dt$
(if $s\neq0$) $=\frac{1}{s}\int_0^{s\xi_0}\partial_\xi h[\lambda,ts\xi_0]\,d(ts\xi_0)$
$=\frac{1}{s}\Big(h(\lambda,s\xi_0)-\underbrace{h(\lambda,0)}_{=0}\Big)=\frac{1}{s}h(\lambda,$

Moreover, since $h(\lambda,0)=0\ \forall_\lambda$,

$$(*)\quad g(\lambda,\xi)=\begin{cases}\dfrac{h(\lambda,\xi)}{s} & \text{if } s\neq0\\ \partial_\xi h[\lambda,0]\xi_0 & \text{if } s=0\end{cases}\quad \text{for } (\lambda,\xi)=(\lambda,s\xi_0)\in V$$

Now let $\chi(s) = s^{-1}\psi(\Delta(s), s\xi_0)$ for $0<|s|<\varepsilon$ and $\chi(0)=\xi_0$

then $\lim\limits_{0\neq s\to 0} \chi(s) = \lim\limits_{s\to 0} \dfrac{\frac{d}{ds}\psi(\Delta(s), s\xi_0)}{\frac{d}{ds}(s)} = \lim\limits_{s\to 0}\left(\partial_\lambda \psi[\Delta(s), s\xi_0]\Delta'(s) + \partial_\xi \psi[\Delta(s), s\xi_0]\xi_0\right)$

$$= \underbrace{\partial_\lambda \psi[\lambda_0, 0]}_{=0}\Delta'(0) + \underbrace{\partial_\xi \psi[\lambda_0, 0]\xi_0}_{=\xi_0} = \xi_0$$

$\psi(\lambda,0) = \phi(\lambda,0) = 0 \ \forall \lambda$

$\psi(\lambda,\xi) = \xi + \phi(\lambda,\xi)$

$\Rightarrow \partial_\xi \psi[\lambda_0, 0] = I_\xi + \underbrace{\partial_\xi \phi[\lambda_0,0]}_{\overset{(*)}{=}0} = I_\xi$

$\Rightarrow \partial_\xi \psi[\lambda_0, 0]\xi_0 = \xi_0.$

Also, for all $|s|<\varepsilon$,

$$g(\Delta(s), s\xi_0) = 0 \overset{(*)}{\Rightarrow} h(\Delta(s), s\xi_0) = 0 \overset{L-S}{\Rightarrow} F(\Delta(s), \psi(\Delta(s), s\xi_0)) = 0$$

$$\Rightarrow F(\Delta(s), s\chi(s)) = 0 \quad \forall\, |s|<\varepsilon.$$

\# 4.3.1.

Example 4.3.2 (Concerning transversality) H.

(a) $\mathbb{F}=\mathbb{R}$, $X=Y=\mathbb{R}$, $F(\lambda,x) := x(\lambda^2+x^2)$, then at $(\lambda_0,0)=(0,0)$, we have

$\partial^2_{\lambda,x}F[0,0](1,\xi_0) \in \mathrm{range}(L)$, and $(0,0)$ is not a bifurcation point.

(b) $\mathbb{F}=\mathbb{R}$, $X=Y=\mathbb{R}$, $F(\lambda,x) = x(\lambda+x^2)$, then at $(0,0)$,

$\partial^2_{\lambda,x}F[0,0](1,\xi_0) \notin \mathrm{range}(L)$ and $(0,0)$ is a bifurcation point.

(c) $\mathbb{F}=\mathbb{R}$, $X=Y=\mathbb{R}$, $F(\lambda,x) = x(\lambda^3+x^3)$, then at $(0,0)$,

$\partial^2_{\lambda,x}F[0,0](1,\xi_0) \in \mathrm{range}(L)$ but $(0,0)$ is a bifurcation point.

Chap 3. Calculus in Banach Spaces

"Big O" and "small o"

let f, g be functions defined from a neighborhood of a in X to Y.

We write

$$f(x) = g(x) + o(x-a) \quad \text{as } x \to a, \text{ if } \lim_{x\to a} \frac{\| f(x) - g(x) \|}{\| x-a \|} = 0$$

$$f(x) = g(x) + O(x-a) \quad \text{as } x \to a, \text{ if } \limsup_{x\to a} \frac{\|f(x)-g(x)\|}{\|x-a\|} < \infty.$$

3.1. Fréchet differentiation

Let X, Y be Banach, $U \subset X$ be open, $F: U \to Y$.

Def 3.1.1 The function F is Fréchet differentiable at $x_0 \in U$ if $\exists A \in \mathcal{L}(X,Y)$ s.t.

$$\lim_{0<\|h\|\to 0} \frac{\| F(x_0+h) - F(x_0) - Ah \|}{\| h\|} = 0.$$

or equivalently, $F(x_0+h) - F(x_0) - Ah = o(\|h\|)$ as $h \to 0$.

If such an operator A exists, it is unique H. and is called the Fréchet derivative of F at x_0, written as

$$A = dF[x_0].$$

The evaluation $dF[x_0]x \in Y$ for any $x \in X$, is called the directional derivative of F at x_0 in the direction x.

F is Fréchet differentiable on U if it is so at every $x_0 \in U$, in which case $x \mapsto dF[x]$ is a function from U to $\mathcal{L}(X,Y)$, which is denoted by dF.

H. Consider $f: \mathcal{L}(X,X) \to \mathcal{L}(X,X)$ given by $A \mapsto A^2 = A \circ A$.

Then, $Df[A](B) = A \circ B + B \circ A$ for all $A, B \in \mathcal{L}(X,X)$.

<u>Lemma 3.1.2</u>

- addition: $d(F+G)[x_0] = dF[x_0] + dG[x_0]$ for $F, G: U \to Y$, $U \subset X$
- chain rule: $d(G \circ F)[x_0] = dG[F(x_0)] \circ dF[x_0]$ for $F: U \to Y$, $U \subset X$; $G: V \to Z$, $V \subset Y$

pf. H.

<u>Lemma 3.1.3</u> Let X, Y be Banach, $U \subset X$ be convex open, $F: U \to Y$ be Fréchet diff. at every point of U with

$$\sup \{ \|dF[x]\| : x \in U \} = m < \infty .$$

Then, $\|F(x_1) - F(x_2)\| \leq m \|x_1 - x_2\|$, $\forall x_1, x_2 \in U$.

<u>Def. 3.1.4</u> (Partial derivatives) Let X, Y and Z be Banach, $U \subset X \times Y$ be open and $F: U \to Z$. Suppose that $(x_0, y_0) \in U$. Then $U_{y_0} := \{ x \in X : (x, y_0) \in U \} \subset X$ is open. Consider

$$\begin{aligned} F(\cdot, y_0) : \quad & U_{y_0} \to Z \\ & x \mapsto F(x, y_0) \end{aligned}$$

If $F(\cdot, y_0)$ has a Fréchet derivative at x_0, we denote it by $\partial_x F[(x_0, y_0)] \in \mathcal{L}(X, Z)$ and call it the partial Fréchet derivative of F w.r.t. x at (x_0, y_0). Similar for $\partial_y F[(x_0, y_0)]$.

Beispiel 3.1.5 $U = X \times Y = \mathbb{R} \times \mathbb{R}$,

$$F(x,y) = \begin{cases} 1 & \text{when } xy = 0 \\ 0 & \text{otherwise} \end{cases}$$

- F is not continuous at $(0,0)$, so not Fréchet-diff. at $(0,0)$
- $\partial_x F[(0,0)]$, $\partial_y F[(0,0)]$ exist and $\partial_x F[(0,0)] = \partial_y F[(0,0)] = 0$.

Lemma 3.1.6 X, Y, Z are Banach spaces, $U \subset X \times Y$ is open.

(a) If $F: U \to Z$ is such that $dF[(x_0,y_0)]$ exists, then $\partial_x F[(x_0,y_0)]$ and $\partial_y F[(x_0,y_0)]$ exist with

$$dF[(x_0,y_0)](x,y) = \partial_x F[(x_0,y_0)]x + \partial_y F[(x_0,y_0)]y \qquad \forall (x,y) \in X \times Y$$

(b) Suppose that $\partial_x F[(x_0,y_0)]$ exists and $\partial_y F$, which is defined at every point in a nbhd of (x_0,y_0), is continuous at (x_0,y_0).

Then, $dF[(x_0,y_0)]$ exists.

pf: (a) H.

(b) $\| F(x_0+x, y_0+y) - F(x_0,y_0) - \partial_x F[(x_0,y_0)]x - \partial_y F[(x_0,y_0)]y \|$

$$\le \underbrace{\| F(x_0+x, y_0+y) - F(x_0+x, y_0) - \partial_y F[(x_0,y_0)]y \|}_{I_1}$$

$$+ \underbrace{\| F(x_0+x, y_0) - F(x_0,y_0) - \partial_x F[(x_0,y_0)]x \|}_{I_2}$$

$\partial_x F[(x_0,y_0)]$ exists $\Rightarrow$ $I_2 = o(\|x\|)$ as $x \to 0$

$\Rightarrow$ $I_2 = o(\|(x,y)\|)$ as $\|(x,y)\| \to 0$

$= o(\|x\| + \|y\|)$

To estimate I_1, let $\varepsilon > 0$ and choose $\delta > 0$ s.t. (Since $\partial_y F$ is cont. at (x_0, y_0))

$$\| \partial_y F[(x,y)] - \partial_y F[(x_0,y_0)] \| < \varepsilon \quad \forall_{(x,y)} \text{ with } \|(x-x_0, y-y_0)\| < \delta .$$

Define

$$u(t) = F(x_0+x, y_0+ty) - t\,\partial_y F[(x_0,y_0)]\, y \qquad \text{for } (x,y) \text{ with } \|(x,y)\| < \delta .$$

By chain Rule,

$$\begin{aligned} \| du[t] \| &= \| \partial_y F[(x_0+x, y_0+ty)] \cdot y - \partial_y F[(x_0,y_0)]\, y \| \\ &\le \| \partial_y F[(x_0+x, y_0+ty)] - \partial_y F[(x_0,y_0)] \| \cdot \|y\| \\ &< \varepsilon \cdot \|y\| \end{aligned}$$

⌜ Meanvalue theorem in Banach spaces:

$U \subset X$ convex open, X, Y Banach, $F: U \to Y$ Fréchet-diff everywhere in U s.t.

$$m := \sup \{ \|dF[x]\| : x \in U \} < \infty .$$

Then, $\|F(x_1) - F(x_2)\| \le m \, \|x_1 - x_2\| \quad \forall_{x_1, x_2 \in U}$. ⌟

$\Rightarrow \quad I_1 = \| u(1) - u(0) \| \le \varepsilon \|y\| \le \varepsilon (\|x\| + \|y\|) \quad$ for $\|(x,y)\| < \delta$

i.e. $I_1 = o(\|x\| + \|y\|) = o(\|(x,y)\|)$ as $\|(x,y)\| \to 0$.

Thus, $dF[(x_0,y_0)]$ exists and $dF[(x_0,y_0)](x,y) = \partial_x F[(x_0,y_0)]\, x + \partial_y F[(x_0,y_0)]\, y$.

\# 3.1.6.

Definition 3.1.7 A (nonlinear) function $F: U \subset X \to Y$ is compact, if $\overline{F(W)}$ is compact in Y when $W \subset U$ is bounded in X.

Lemma 3.1.8 (Differentiation of compact operators) Suppose U is open in X, $F: U \to Y$ is compact and $dF[x_0]$ exists at $x_0 \in U$.

Then, $dF[x_0] \in \mathcal{L}(X,Y)$ is compact.

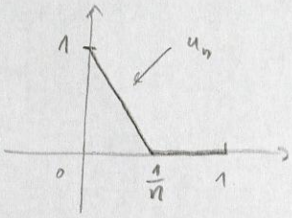
pf. H.

Example 3.1.9 The converse of 3.1.8 is false.

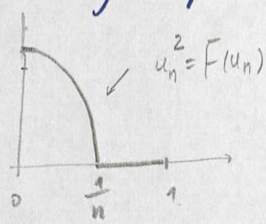
$X = Y = C[0,1]$, $F: X \to Y$ defined by $F(u) = u^2$. Then,

- $dF[0] = 0 \in \mathcal{L}(X,Y)$ is a compact operator.
- let $\{u_n\} \subset X$ be given by

$$u_n(x) = \begin{cases} 0 & \frac{1}{n} \le x \le 1 \\ 1 - nx & 0 \le x \le \frac{1}{n} \end{cases}$$



$\{u_n\}$ is bounded in X, but $\{F(u_n)\}$ is not relatively compact in X, so F is not compact.



3.2. Gradient Operators

Let X be a Hilbert space over $\mathbb{F}$ with $\langle \cdot, \cdot \rangle$ and $U \subset X$ open.

If $g: U \to \mathbb{F}$ is Fréchet diff at $x_0 \in U$, then

$$dg[x_0] \in \mathcal{L}(X, \mathbb{F}) =: X^* \leftarrow \text{the dual space of } X$$

By the Riesz representation theorem, $\exists$ a vector $v \in X$ s.t.

$$dg[x_0]\, y = \langle v, y \rangle \quad \text{for all } y \in X.$$

written as $\nabla g(x_0)$

The element $v =: \nabla g(x_0)$ is the gradient of g at x_0.

In general,

Def. 3.2.1 Suppose $\underset{\text{Banach}}{Y} \hookrightarrow (\underset{\text{Hilbert}}{X}, \langle \cdot,\cdot \rangle)$ is continuously embedded.

Let $U \subset Y$ open and $g : U \to \mathbb{F}$ be Fréchet diff at $y_0 \in U$.

Then, $dg[y_0] \in \mathcal{L}(Y, \mathbb{F}) = Y^*$. Let $x_g \in X$ be such that

$$dg[y_0]\, y = \langle x_g, y \rangle \quad \forall_{y \in Y} .$$

then x_g is called the gradient of g in X at y_0, write $x_g = \nabla_X g(y_0)$.

If $G : U \to X$ coincides with $\nabla_X g$ on U, then G is called the
$y_0 \mapsto x_g$

gradient operator of g.

H. Let $(X, \langle \cdot,\cdot \rangle)$ be Hilbert over $\mathbb{R}$. Show that

(a) $f : X \to \mathbb{R}$, $x \mapsto \|x\|^2$ is conts. diff and $\nabla f(x) = 2x$.

(b) $\tilde{f} : X \to \mathbb{R}$, $x \mapsto \|x\|$ is " " on $X \setminus \{0\}$ and $\nabla \tilde{f}(x) = \frac{x}{\|x\|}$ for $\forall_{x \neq 0,\, x \in X}$.

3.3 Perturbation of a simple eigenvalue

Consider a family of bounded linear operators:

$$s \mapsto L(s) \in \mathcal{L}(X, Y), \quad \text{for } s \in (-1, 1)$$

Such that for $s = 0$, $L(0)$ has a simple eigenvalue μ_0 with eigenvec. $\xi_0 \in X$

We want to know about the spectrum of $L(s)$, for s small.

Proposition 3.3.1 Let $X \subset Y$ be Banach spaces, where $\iota : X \hookrightarrow Y$ is a continuous embedding. Let $s \mapsto L(s)$ be a mapping of C^k, for some $k \geq 1$, from $(-1,1)$ into $\mathcal{L}(X,Y)$. If μ_0 is a simple eigenvalue of $L(0)$ with eigenvector $\xi_0 \in X$ s.t. $\|\iota\xi_0\|_Y = 1$, then

$\exists\, \varepsilon > 0$ and a C^k-curve $\{(\mu(s), \xi(s)) : s \in (-\varepsilon, \varepsilon)\} \subset \mathbb{R} \times X$ s.t.

(a) $(\mu(0), \xi(0)) = (\mu_0, \xi_0)$

(b) $L(s)\xi(s) = \mu(s)\iota\xi(s)$

(c) $\xi(s) = \xi_0 + \eta(s)$ for some $\eta(s)$

(d) $\mu(s)$ is a simple eigenvalue of $L(s)$ and

if μ is an eigenvalue of $L(s)$ with $|\mu_0 - \mu| < \varepsilon$, $|s| < \varepsilon$, then $\mu = \mu(s)$.

⌜ Implicit Function theorem:

$(x_0, y_0) \in U \subset X \times Y$ open, $F \in C^k(U, Z)$ for some k, $F(x_0,y_0) = z_0$, $\partial_x F[(x_0,y_0)] \in \mathcal{L}(X,Z)$

Banach $\quad F : U \subset X \times Y \to Z$

is a homeomorphism. Then, $\exists$ open ball $V \subset Y$ centered at y_0, a connected open set $W \subset U$ and $\phi \in C^k(V, X)$ s.t.

$$(x_0, y_0) \in W \text{ and } F^{-1}(z_0) \cap W = \{(\phi(y), y) : y \in V\}.$$

⌟

Pf. 3.3.1: Define $G : \mathbb{R} \times X \times (-1,1) \to Y \times \mathbb{R}$ by

$$(\mu, x, s) \longmapsto (\mu\iota x - L(s)x,\ y^*(\iota x) - 1),$$

where $y^* \in Y^*$ such that $y^*(\iota\xi_0) = 1$ and $\ker y^* = \operatorname{range}(\mu_0\iota - L(0))$

Lemma 2.4.11 $\to$ $\mu_0\iota - L(0) : \underbrace{\langle\xi_0\rangle}_{\to\, 0} \oplus \overbrace{X_1 \to \langle\iota\xi_0\rangle \oplus Y_1}^{\cong\ \text{homeo}}$

$Y_1 = \operatorname{range}(\mu_0\iota - L(0))$

Then:

(i) $G(\mu_0, \xi_0, 0) = (0,0)$

(ii) $\partial_{(\mu,x)} G[(\mu_0, \xi_0, 0)](\mu, x) = (\mu \iota \xi_0 + \mu_0 \iota x - L(0)x, \, y^*(\iota x)) \in Y \times \mathbb{R} \quad \forall_{(\mu,x) \in \mathbb{R} \times X}$

(iii) $\partial_{(\mu,x)} G[(\mu_0, \xi_0, 0)] : \mathbb{R} \times X \to Y \times \mathbb{R}$ is a bijection,

thus by Lemma 2.3.2, is a homeomorphism.

Γ (i) $G(\mu_0, \xi_0, 0) = (\underbrace{\mu_0 \iota \xi_0 - L(0)\xi_0}_{\overset{\shortparallel}{0}}, \, y^*(\iota \xi_0) - 1) = (0, 0)$.

since ξ_0 is the eigenvector of $L(0)$

(ii) #

(iii) Assume $\partial_{(\mu,x)} G[(\mu_0, \xi_0, 0)](\mu, x) = (0,0)$ for some $(\mu, x) \in \mathbb{R} \times X$

then $\left\{ \begin{array}{l} \mu \iota \xi_0 + \mu_0 \iota x - L(0)x = 0 \Rightarrow \mu \iota \xi_0 \in \text{range}(\mu_0 \iota - L(0)) \\ \qquad \langle \iota \xi_0 \rangle \cap \text{range}(\mu_0 \iota - L(0)) = \{0\} \\ y^*(\iota x) = 0 \end{array} \right\} \Rightarrow \mu = 0$

$\left. \begin{array}{l} \mu = 0 \Rightarrow (\mu_0 \iota - L(0))x = 0 \xRightarrow[\text{simple eigenvalue}]{\mu_0 \text{ is a}} x \in \langle \xi_0 \rangle \\ y^*(\iota \xi_0) = 1 \end{array} \right\} \Rightarrow x = 0 \qquad \Rightarrow$ injective.

$\left. \begin{array}{l} Y = \langle \iota \xi_0 \rangle \oplus \text{range}(\mu_0 \iota - L(0)) \\ \partial_{(\mu,x)} G[(\mu_0, \xi_0, 0)] : (\mu, x) \mapsto (\underbrace{\mu \iota \xi_0}_{\in \langle \iota \xi_0 \rangle} + \underbrace{\mu_0 \iota x - L(0)x}_{\in \text{range}(\mu_0 \iota - L(0))}, \, y^*(\iota x)) \\ y^*(\iota \xi_0) = 1 \end{array} \right\} \Rightarrow$ surjective.

$\Rightarrow \partial_{(\mu,x)} G[(\mu_0, \xi_0, 0)]$ is a bijection.

Thus, it follows from (i)-(iii) and the implicite function theorem

that $\exists$ a C^k-curve $\{(\mu(s), \xi(s), s) : s \in (-\varepsilon, \varepsilon)\} = G^{-1}(0,0) \cap W$ ⤳ (b) ✓

a nbhd of (μ_0, x_0)

with $(\mu(0), \xi(0)) = (\mu_0, \xi_0)$

⤳ (a) ✓

This implies that $\psi^*(\iota\xi(s)) = 1 \xrightarrow[\ker\psi^* = \text{range}(..)]{\psi^*(\iota\xi_0)=0} \xi(s) = \xi_0 + \underbrace{\eta(s)}_{(\eta(s)\in \text{range}(\mu_0\iota - L(0)))}.$

$\rightsquigarrow$ (c) ✓

To show (d):

$\xrightarrow{\text{Prop. 2.4.9}}$ for s small, $\mu(s)\iota - L(s)$ is a Fredholm operator of index 0.

Consider $(\mu(s)\iota - L(s))\,x(s) = 0$ for some $\|x(s)\|_X = 1$, $x(s)\in X$,

$X = \langle \xi_0\rangle \oplus X_1 \Leftarrow \iota^{-1}(\text{range}(\mu_0\iota - L(0)))$

Write $x(s) = \alpha(s)\xi_0 + z(s)$ for $\alpha(s)\in\mathbb{R}$, $z(s)\in X_1$.

we assume $\alpha(s) \geq 0$.

$\Rightarrow (\mu_0\iota - L(0))\,z(s) = (\mu_0\iota - L(0))\,x(s) = \Big(\mu_0\iota - \mu(s)\iota - (L(0) - L(s))\Big)\,x(s) \to 0$ in Y as $s\to 0$

$\xrightarrow[X_1\cap\ker(\mu_0\iota - L(0)) = \{0\}]{z(s)\in X_1}$ $z(s)\to 0$ in X as $s\to 0$

$\overset{\|x(s)\|=1}{\Rightarrow}$ $\alpha(s) \to \|\xi_0\|_X^{-1}$ as $s\to 0$. (in particular, $\alpha(s)\neq 0$ for small s)

Let $\hat{x}(s) = \dfrac{x(s)}{\alpha(s)}$, then $(\mu(s), \hat{x}(s)) \to (\mu_0, \xi_0)$ in $\mathbb{R}\times X$

and $G(\mu(s), \hat{x}(s), s) = 0$

$\xrightarrow[\text{theorem}]{\text{the implicit function}}$ for s small, $\hat{x}(s) = \xi(s) \Rightarrow x(s)\in\langle\xi(s)\rangle$

thus $\ker(\mu(s)\iota - L(s)) = \langle \xi(s)\rangle$. (*1)

Also, $\left.\begin{array}{l}\xi(s)\to\xi_0 \text{ in } X\\ \xi_0\notin X_1\\ X_1 \text{ is closed}\end{array}\right\}$ $\Rightarrow \xi(s)\notin X_1$ for s small $\Rightarrow X = X_1\oplus\langle\xi(s)\rangle$ (*2) for small s.

Moreover,

Veranstaltung

Name

Datum

if $(\mu(s)\iota - L(s))\,p(s) = \iota\,\xi(s)$ for some $p(s) \in X$,

(w.l.o.g. we can assume $p(s) \in X_1$)

then $\|p(s)\|_X$ is bounded for s small

Otherwise, $\exists$ a sequence $\{s_n\}$ s.t. $s_n \to 0$ and $\|p(s_n)\|_X \to \infty$

$$\Rightarrow \quad (\mu_0\iota - L(0))\frac{p(s_n)}{\|p(s_n)\|_X} \to 0 \text{ in } Y \text{ as } n \to \infty$$

$$\overset{p(s_n)\in X_1}{\Rightarrow} \quad \frac{p(s_n)}{\|p(s_n)\|_X} \to 0 \quad ↯ \text{ since } x_n = \frac{p(s_n)}{\|p(s_n)\|_X} \text{ is s.t. } \|x_n\|_X = 1 \ \forall n.$$

and thus

$$(\mu_0\iota - L(0))\,p(s) = \Big((\mu_0 - \mu(s))\iota - (L(0) - L(s))\Big)p(s) + \iota\,\xi(s) \to \iota\,\xi(0) \text{ as } s \to 0 \text{ in } Y$$

$Y_1 = \text{range}(\mu_0\iota - L(0))$ is closed in Y $\Rightarrow$ $\iota\,\xi_0 \in Y_1$ ↯ $Y_1 \cap \langle \xi_0 \rangle = \{0\}$, $\|\xi_0\|_Y = 1$.

$$\Rightarrow \quad \langle \iota\,\xi(s) \rangle \cap \text{range}(\mu(s)\iota - L(s)) = \{0\} \qquad (*3)$$

By $(*1)-(*3)$, $\mu(s)$ is a simple eigenvalue of $L(s)$. $\rightsquigarrow$ (d)

$\#$ 3.3.1.