Chap 3. Calculus in Banach Spaces

"Big O" and "small o"

Let f, g be functions defined from a neighborhood of a in X to Y.

We write

$$f(x) = g(x) + o(x-a) \quad \text{as } x \to a, \text{ if } \lim_{x\to a} \frac{\| f(x) - g(x) \|}{\| x-a \|} = 0$$

$$f(x) = g(x) + O(x-a) \quad \text{as } x \to a, \text{ if } \limsup_{x\to a} \frac{\| f(x) - g(x) \|}{\| x-a \|} < \infty.$$

3.1. Fréchet differentiation

Let X, Y be Banach, $U \subset X$ be open, $F : U \to Y$.

Def 3.1.1 The function F is Fréchet differentiable at $x_0 \in U$ if $\exists A \in \mathcal{L}(X,Y)$ s.t.

$$\lim_{0<\|h\|\to 0} \frac{\| F(x_0+h) - F(x_0) - Ah \|}{\| h \|} = 0.$$

or equivalently, $F(x_0+h) - F(x_0) - Ah = o(\|h\|)$ as $h \to 0$.

If such an operator A exists, it is unique #. and is called the Fréchet derivative of F at x_0, written as

$$A = dF[x_0].$$

The evaluation $dF[x_0]x \in Y$ for any $x \in X$, is called the directional derivative of F at x_0 in the direction x.

F is Fréchet differentiable on U if it is so at every $x_0 \in U$, in which case $x \mapsto dF[x]$ is a function from U to $\mathcal{L}(X,Y)$, which is denoted by dF.

H. Consider $f: \mathcal{L}(X,X) \to \mathcal{L}(X,X)$ given by $A \mapsto A^2 = A \circ A$.

Then, $Df[A](B) = A \circ B + B \circ A$ for all $A, B \in \mathcal{L}(X,X)$.

<u>Lemma 3.1.2</u>

addition
- $d(F+G)[x_0] = dF[x_0] + dG[x_0]$ for $F, G: \overset{\subset X}{U} \to Y$

chain rule
- $d(G \circ F)[x_0] = dG[F(x_0)] \circ dF[x_0]$ for $F: \overset{\subset X}{U} \to Y$, $G: \overset{\subset Y}{V} \to Z$

pf. H.

<u>Lemma 3.1.3</u> Let X, Y be Banach, $U \subset X$ be convex open, $F: U \to Y$ be Fréchet diff. at every point of U with

$$\sup\{\|dF[x]\| : x \in U\} = m < \infty.$$

Then, $\|F(x_1) - F(x_2)\| \le m\|x_1 - x_2\|$, $\forall x_1, x_2 \in U$.

<u>Def. 3.1.4</u> (Partial derivatives) Let X, Y and Z be Banach, $U \subset X \times Y$ be open and $F: U \to Z$. Suppose that $(x_0, y_0) \in U$.

Then $U_{y_0} := \{x \in X : (x, y_0) \in U\} \subset X$ is open. Consider

$$F(\cdot, y_0): U_{y_0} \to Z$$
$$x \mapsto F(x, y_0)$$

If $F(\cdot, y_0)$ has a Fréchet derivative at x_0, we denote it by $\partial_x F[(x_0, y_0)] \in \mathcal{L}(X, Z)$ and call it the partial Fréchet derivative of F w.r.t. x at (x_0, y_0). Similar for $\partial_y F[(x_0, y_0)]$.

Beispiel 3.1.5 $U = X \times Y = \mathbb{R}\times\mathbb{R}$,

$$F(x,y) = \begin{cases} 1 & \text{when } xy=0 \\ 0 & \text{otherwise} \end{cases}$$

- F is not continuous at $(0,0)$, so not Fréchet-diff. at $(0,0)$
- $\partial_x F[(0,0)]$, $\partial_y F[(0,0)]$ exist and $\partial_x F[(0,0)] = \partial_y F[(0,0)] = 0$.

Lemma 3.1.6 X, Y, Z are Banach spaces, $U \subset X\times Y$ is open.

(a) If $F: U \to Z$ is such that $dF[(x_0,y_0)]$ exists, then $\partial_x F[(x_0,y_0)]$ and $\partial_y F[(x_0,y_0)]$ exist with

$$dF[(x_0,y_0)](x,y) = \partial_x F[(x_0,y_0)]x + \partial_y F[(x_0,y_0)]y \qquad \forall (x,y)\in X\times Y$$

(b) Suppose that $\partial_x F[(x_0,y_0)]$ exists and $\partial_y F$, which is defined at every point in a nbhd of (x_0,y_0), is continuous at (x_0,y_0).

Then, $dF[(x_0,y_0)]$ exists.

Pf: (a) H.

(b)
$$\begin{aligned}
&\| F(x_0+x, y_0+y) - F(x_0,y_0) - \partial_x F[(x_0,y_0)]x - \partial_y F[(x_0,y_0)]y \| \\
&\le \underbrace{\| F(x_0+x, y_0+y) - F(x_0+x, y) - \partial_y F[(x_0,y_0)]y \|}_{I_1} \\
&\quad + \underbrace{\| F(x_0+x, y) - F(x_0,y_0) - \partial_x F[(x_0,y_0)]x \|}_{I_2}
\end{aligned}$$

$\partial_x F[(x_0,y_0)]$ exists $\Rightarrow I_2 = o(\|x\|)$ as $x\to 0$

$\Rightarrow I_2 = o(\|(x,y)\|)$ as $\|(x,y)\|\to 0$

$= o(\|x\|+\|y\|)$

To estimate I_1, let $\varepsilon > 0$ and choose $\delta > 0$ s.t. (Since $\partial_y F$ is cont. at (x_0, y_0))

$$\| \partial_y F[(x,y)] - \partial_y F[(x_0,y_0)] \| < \varepsilon \quad \forall_{(x,y)} \text{ with } \|(x-x_0, y-y_0)\| < \delta .$$

Define

$$u(t) = F(x_0+x, y_0+ty) - t\,\partial_y F[(x_0,y_0)]\, y \quad \text{for } (x,y) \text{ with } \|(x,y)\| < \delta .$$

By chain Rule,

$$\begin{aligned} \| du[t] \| &= \| \partial_y F[(x_0+x, y_0+ty)] \cdot y - \partial_y F[(x_0,y_0)]\, y \| \\ &\le \| \partial_y F[(x_0+x, y_0+ty)] - \partial_y F[(x_0,y_0)] \| \cdot \|y\| \\ &< \varepsilon \cdot \|y\| \end{aligned}$$

⌈ Meanvalue theorem in Banach spaces:

$U \subset X$ convex open, X, Y Banach, $F: U \to Y$ Fréchet-diff everywhere in U s.t.

$$m := \sup \{ \|dF[x]\| : x \in U \} < \infty .$$

Then, $\| F(x_1) - F(x_2) \| \le m \, \|x_1 - x_2\| \quad \forall_{x_1, x_2 \in U}$. ⌋

$$\Rightarrow \quad I_1 = \| u(1) - u(0) \| \le \varepsilon \|y\| \le \varepsilon (\|x\| + \|y\|) \quad \text{for } \|(x,y)\| < \delta$$

i.e. $I_1 = o(\|x\| + \|y\|) = o(\|(x,y)\|)$ as $\|(x,y)\| \to 0$.

Thus, $dF[(x_0,y_0)]$ exists and $dF[(x_0,y_0)](x,y) = \partial_x F[(x_0,y_0)]\, x + \partial_y F[(x_0,y_0)]\, y$.

\# 3.1.6

<u>Definition 3.1.7</u> A (nonlinear) function $F: U \subset X \to Y$ is compact, if $\overline{F(W)}$ is <u>compact</u> in Y when $W \subset U$ is bounded in X.

<u>Lemma 3.1.8</u> (Differentiation of compact operators) Suppose U is open in X, $F: U \to Y$ is compact and $dF[x_0]$ exists at $x_0 \in U$.

Then, $dF[x_0] \in \mathcal{L}(X,Y)$ is compact.

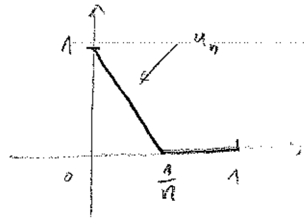
pf. H.

Example 3.1.9 The converse of 3.1.8 is false.

$X = Y = C[0,1]$, $F : X \to Y$ defined by $F(u) = u^2$. Then,

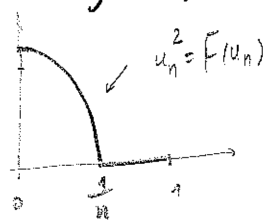
- $dF[0] = 0 \in \mathcal{L}(X, Y)$ is a compact operator,
- let $\{u_n\} \subset X$ be given by

$$u_n(x) = \begin{cases} 0 & \frac{1}{n} \le x \le 1 \\ 1 - nx & 0 \le x \le \frac{1}{n} \end{cases}$$



$\{u_n\}$ is bounded in X, but $\{F(u_n)\}$ is not relatively compact in X,

so F is not compact.



3.2. Gradient Operators

Let X be a Hilbert space over $\mathbb{F}$ with $\langle \cdot, \cdot \rangle$ and $U \subset X$ open.

If $g : U \to \mathbb{F}$ is Fréchet diff at $x_0 \in U$, then

$$dg[x_0] \in \mathcal{L}(X, \mathbb{F}) =: X^* \leftarrow \text{the dual space of } X$$

By the Riesz representation theorem, $\exists$ a vector $v \in X$ s.t.

$$dg[x_0]\, y = \langle \underset{\substack{\uparrow \\ \text{written as } \nabla g(x_0)}}{v}, y \rangle \quad \text{for all } y \in X.$$

The element $v =: \nabla g(x_0)$ is the gradient of g at x_0.

In general,

Def. 3.2.1 Suppose $Y \hookrightarrow (X, \langle\cdot,\cdot\rangle)$ is continuously embedded. (Banach Y, Hilbert X)

Let $U \subset Y$ open and $g: U \to \mathbb{F}$ be Fréchet diff at $y_0 \in U$.

Then, $dg[y_0] \in \mathcal{L}(Y, \mathbb{F}) = Y^*$. Let $x_g \in X$ be such that

$$dg[y_0]\, y = \langle x_g, y \rangle \quad \forall y \in Y .$$

then x_g is called the gradient of g in X at y_0, write $x_g = \nabla_X g(y_0)$.

If $G: U \to X$, $y_0 \mapsto x_g$ coincides with $\nabla_X g$ on U, then G is called the gradient operator of g.

H. Let $(X, \langle\cdot,\cdot\rangle)$ be Hilbert over $\mathbb{R}$. Show that

(a) $f: X \to \mathbb{R}$, $x \mapsto \|x\|^2$ is conts. diff and $\nabla f(x) = 2x$.

(b) $\tilde{f}: X \to \mathbb{R}$, $x \mapsto \|x\|$ is " " on $X \setminus \{0\}$ and $\nabla \tilde{f}(x) = \frac{x}{\|x\|}$ for $\forall x \neq 0, x \in X$.

3.3 Perturbation of a simple eigenvalue

Consider a family of bounded linear operators:

$$s \mapsto L(s) \in \mathcal{L}(X, Y), \quad \text{for } s \in (-1, 1)$$

Such that for $s = 0$, $L(0)$ has a simple eigenvalue μ_0 with eigenvec. ξ_0.

We want to know about the spectrum of $L(s)$, for s small.

Proposition 3.3.1 Let $X \subset Y$ be Banach spaces, where $\iota : X \hookrightarrow Y$ is a continuous embedding. Let $s \mapsto L(s)$ be a mapping of C^k, for some $k \geq 1$, from $(-1,1)$ into $\mathcal{L}(X,Y)$. If μ_0 is a simple eigenvalue of $L(0)$ with eigenvector $\xi_0 \in X$ s.t. $\|\iota\xi_0\|_Y = 1$, then

$\exists\, \varepsilon > 0$ and a C^k-curve $\{(\mu(s), \xi(s)) : s \in (-\varepsilon, \varepsilon)\} \subset \mathbb{R} \times X$ s.t.

(a) $(\mu(0), \xi(0)) = (\mu_0, \xi_0)$

(b) $L(s)\xi(s) = \mu(s)\iota\xi(s)$

(c) $\xi(s) = \xi_0 + \eta(s)$ for some $\eta(s)$

(d) $\mu(s)$ is a simple eigenvalue of $L(s)$ and

if μ is an eigenvalue of $L(s)$ with $|\mu_0 - \mu| < \varepsilon$, $|s| < \varepsilon$, then $\mu = \mu(s)$.

⌜ Implicit Function theorem:

$(x_0, y_0) \in U \subset X \times Y$ open, $F \in C^k(U, Z)$ for some k, $F(x_0, y_0) = z_0$, $\partial_x F[(x_0,y_0)] \in \mathcal{L}(X, Z)$

Banach $\quad F : U \subset X \times Y \to Z$

is a homeomorphism. Then, $\exists$ open ball $V \subset Y$ centered at y_0, a connected open set $W \subset U$ and $\phi \in C^k(V, X)$ s.t.

$$(x_0, y_0) \in W \quad \text{and} \quad F^{-1}(z_0) \cap W = \{(\phi(y), y) : y \in V\}.$$

⌟

Pf 3.3.1: Define $G : \mathbb{R} \times X \times (-1,1) \to Y \times \mathbb{R}$ by

$$(\mu, x, s) \longmapsto (\mu\iota x - L(s)x,\ y^*(\iota x) - 1),$$

where $y^* \in Y^*$ such that $y^*(\iota\xi_0) = 1$ and $\ker y^* = \text{range}(\mu_0\iota - L(0))$

Lemma 3.4.11 $\rightsquigarrow$ $\mu_0\iota - L(0) : \langle \xi_0 \rangle \oplus X_1 \to \langle \iota\xi_0 \rangle \oplus Y_1$

$\cong$ homeo.

0

range$(\mu_0\iota - L(0))$

Then:

(i) $G(\mu_0, \xi_0, 0) = (0,0)$

(ii) $\partial_{(\mu,x)} G[(\mu_0,\xi_0,0)](\mu,x) = (\mu\iota\xi_0 + \mu_0\iota x - L(0)x,\ y^*(\iota x)) \in Y \times \mathbb{R} \quad \forall_{(\mu,x)\in\mathbb{R}\times X}$

(iii) $\partial_{(\mu,x)} G[(\mu_0,\xi_0,0)] : \mathbb{R}\times X \to Y \times \mathbb{R}$ is a bijection,

thus by Lemma 2.3.2, is a homeomorphism.

⌜ (i) $G(\mu_0,\xi_0,0) = (\underbrace{\mu_0\iota\xi_0 - L(0)\xi_0}_{\overset{\shortparallel}{0}},\ y^*(\iota\xi_0)-1) = (0,0)$. since ξ_0 is the eigenvalue of $L(0)$

(ii) #

(iii) Assume $\partial_{(\mu,x)} G[(\mu_0,\xi_0,0)](\mu,x) = (0,0)$ for some $(\mu,x)\in\mathbb{R}\times X$

then $\left\{\begin{array}{l} \mu\iota\xi_0 + \mu_0\iota x - L(0)x = 0 \Rightarrow \mu\iota\xi_0 \in \text{range}(\mu_0\iota - L(0)) \\ \qquad\qquad \langle\iota\xi_0\rangle \cap \text{range}(\mu_0\iota - L(0)) = \{0\} \\ y^*(\iota x) = 0 \end{array}\right\} \Rightarrow \mu = 0$

$\left.\begin{array}{l} \mu = 0 \Rightarrow (\mu_0\iota - L(0))x = 0 \xRightarrow[\text{simple eigenvalue}]{\mu_0 \text{ is a}} x \in \langle\xi_0\rangle \\ \qquad\qquad y^*(\iota\xi_0) = 1 \end{array}\right\} \Rightarrow x = 0 \qquad \Rightarrow \text{injective.}$

$\left.\begin{array}{l} Y = \langle\iota\xi_0\rangle \oplus \text{range}(\mu_0\iota - L(0)) \\ \partial_{(\mu,x)} G[(\mu_0,\xi_0,0)] : (\mu,x) \mapsto (\underbrace{\mu\iota\xi_0}_{\in\langle\iota\xi_0\rangle} + \underbrace{\mu_0\iota x - L(0)x}_{\in\text{range}(\mu_0\iota - L(0))},\ y^*(\iota x)) \\ y^*(\iota\xi_0) = 1 \end{array}\right\} \Rightarrow \text{surjective.}$

$\Rightarrow \partial_{(\mu,x)} G[(\mu_0,\xi_0,0)]$ is a bijection.

Thus, it follows from (i)–(iii) and the implicite function theorem

that $\exists$ a C^k-curve $\{(\mu(s), \xi(s), s) : s \in (-\varepsilon,\varepsilon)\} = G^{-1}(0,0) \cap \underset{\text{a nbhd of } (\mu_0, x_0)}{W}$ $\rightsquigarrow$ (b) ✓

with $(\mu(0), \xi(0)) = (\mu_0, \xi_0)$ $\rightsquigarrow$ (a) ✓

This implies that $\gamma^*(\iota\xi(s))=1 \xrightarrow[\ker\gamma^* = \mathrm{range}(..)]{\gamma^*(\iota\xi_0)=0} \xi(s)=\xi_0+\underbrace{\eta(s)}_{\eta(s)\in\mathrm{range}(\mu_0\iota-L(0))}.$

$\rightsquigarrow$ (c) ✓

To show (d):

$\overset{\text{Prop. 2.4.9}}{\Longrightarrow}$ for s small, $\mu(s)\iota - L(s)$ is a Fredholm operator of index 0.

Consider $\underline{(\mu(s)\iota - L(s))x(s)=0}$ for some $\|x(s)\|_X = 1$, $x(s)\in X$,

$X = \langle\xi_0\rangle \oplus X_1 \Leftarrow \iota^{-1}(\mathrm{range}(\mu_0\iota - L(0)))$

Write $x(s) = \alpha(s)\xi_0 + z(s)$ for $\alpha(s)\in\mathbb{R}$, $z(s)\in X_1$.

we assume $\alpha(s)\geq 0$.

$$\Rightarrow (\mu_0\iota - L(0))z(s) = (\mu_0\iota - L(0))x(s) = \Big(\mu_0\iota - \mu(s)\iota - (L(0)-L(s))\Big)x(s) \longrightarrow 0 \text{ in } Y$$

as $s\to 0$

$$\xrightarrow[X_1\cap\ker(\mu_0\iota - L(0))=\{0\}]{z(s)\in X_1} z(s)\to 0 \text{ in } X \text{ as } s\to 0$$

$$\overset{\|x(s)\|=1}{\Rightarrow} \alpha(s)\to\|\xi_0\|_X^{-1} \text{ as } s\to 0.$$ (in particular, $\alpha(s)\neq 0$ for small s)

Let $\hat{x}(s) = \dfrac{x(s)}{\alpha(s)}$, then $(\mu(s),\hat{x}(s))\to(\mu_0,\xi_0)$ in $\mathbb{R}\times X$

and $G(\mu(s),\hat{x}(s),s)=0$

$\xrightarrow[\text{theorem}]{\text{the implicit function}}$ for s small, $\hat{x}(s)=\xi(s) \Rightarrow x(s)\in\langle\xi(s)\rangle$

Thus $\ker(\mu(s)\iota - L(s)) = \langle\xi(s)\rangle$. (*1)

Also, $\left.\begin{array}{l}\xi(s)\to\xi_0 \text{ in } X\\ \xi_0\notin X_1\\ X_1 \text{ is closed}\end{array}\right\} \Rightarrow \xi(s)\notin X_1$ for s small $\Rightarrow X = X_1\oplus\langle\xi(s)\rangle$ (*2)

for small s.

Moreover,

if $(\mu(s)\iota - L(s))\, p(s) = \iota\,\xi(s)$ for some $p(s) \in X$,

(w.l.o.G. we can assume $p(s) \in X_1$)

then $\|p(s)\|_X$ is bounded for s small

⌈ Otherwise, $\exists$ a sequence $\{s_n\}$ s.t. $s_n \to 0$ and $\|p(s_n)\|_X \to \infty$

$$\Rightarrow \quad (\mu_0\iota - L(0)) \frac{p(s_n)}{\|p(s_n)\|_X} \to 0 \quad \text{in } Y \quad \text{as } n \to \infty$$

$p(s_n) \in X_1$

$$\Rightarrow \quad \frac{p(s_n)}{\|p(s_n)\|_X} \to 0 \quad ↯ \quad \text{since } x_n = \frac{p(s_n)}{\|p(s_n)\|_X} \text{ is s.t. } \|x_n\|_X = 1 \ \forall n.$$ ⌋

and thus

$$(\mu_0\iota - L(0))\, p(s) = \Big((\mu_0 - \mu(s))\iota - (L(0) - L(s))\Big) p(s) + \iota\,\xi(s) \longrightarrow \iota\,\xi(0) \quad \text{as } s \to 0 \text{ in } Y$$

$Y_1 = \text{range}\,(\mu_0\iota - L(0))$ is closed in Y $\Rightarrow$ $\iota\,\xi_0 \in Y_1$ ↯

$Y_1 \cap \langle \xi_0 \rangle = \{0\}$
$\|\xi_0\|_Y = 1$.

$$\Rightarrow \quad \langle \iota\,\xi(s) \rangle \cap \text{range}\,(\mu(s)\iota - L(s)) = \{0\} \qquad (*3)$$

By $(*1)$–$(*3)$, $\mu(s)$ is a simple eigenvalue of $L(s)$. $\rightsquigarrow$ (d)

$\#$ 3.3.1.