Bifurcation Theory

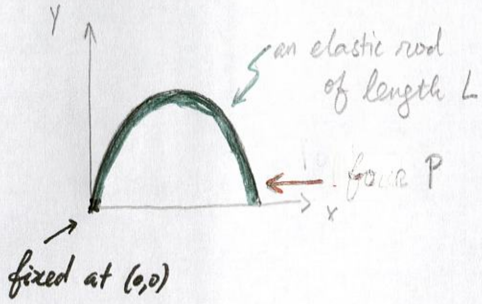
Chap 1. Introduction

$F(\lambda, x) = 0$, $F: \mathbb{R}^m \times X \to Y$, X, Y Banach-Räume (reelle-)

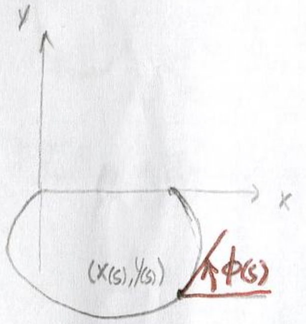
Goal: qualitative description of the solution set $S_\lambda := \{x \in X : F(\lambda, x) = 0\}$ for $\lambda \in \mathbb{R}^m$, especially around "singular" values of λ.

1.1 Example: Bending an elastic rod

(an example where all solutions can be explicitly computed.)



An elastic rod of length L is fixed at one end and placed in a plane. When a force is applied at the free end, the rod will bend in the plane.



For $s \in [0, L]$, let $(x, y) = (x(s), y(s))$ describe the shape of the rod. Then,

$$x(s) = \int_0^s \cos\phi(t)\, dt, \quad y(s) = \int_0^s \sin\phi(t)\, dt,$$

where $\phi(t)$ is the angle of the tangent and the horizontal.

Euler-Bernoulli $\Rightarrow$ $-\underbrace{k\phi'(s)}_{\text{curvature}} = \underbrace{P y(s)}_{\text{moment}}$ for a constant k (related to the material of the rod.)

$$\xrightarrow{y'(s) = \sin\phi(s)} \boxed{\phi''(s) + \lambda \sin\phi(s) = 0}^{(*)}, \quad s \in [0, L], \quad \phi'(0) = \phi'(L) = 0,$$

where $\lambda = P/k > 0$ is a parameter depending on $P > 0$.

Note: if ϕ is a solution of $(*)$, then $\phi + 2k\pi$ is also a solution, $\forall k \in \mathbb{Z}$.

Assume thus that $\phi(0) \in (-\pi, \pi)$ (if $\phi(0) = \pm\pi$, then $\phi \equiv$ constant)

Note : $(\lambda, \phi) = (\lambda, 0)$ is a solution of $(*)$, $\forall \lambda > 0$, called the trivial solutions.

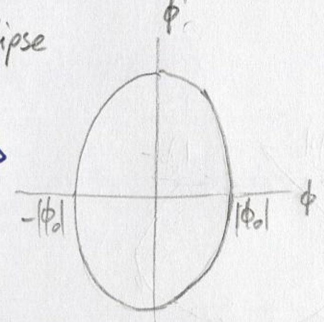
a straight rod, $\phi \equiv 0$.

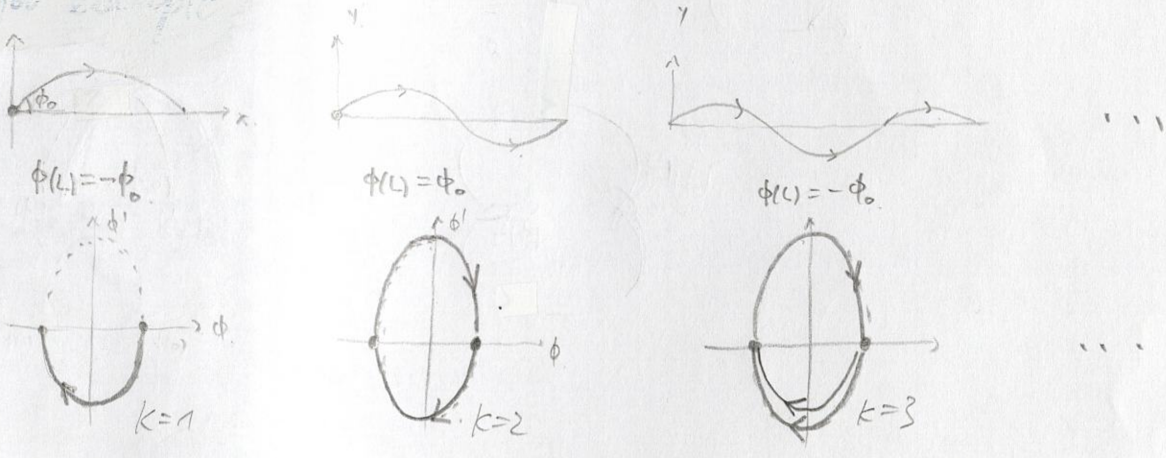
H One shows that any solution of $(*)$ satisfies

$$\phi'(s)^2 + 4\lambda \sin^2\left(\frac{\phi(s)}{2}\right) = 4\lambda \sin^2\left(\frac{\phi_0}{2}\right), \text{ where } \phi_0 := \phi(0), \text{ for } \forall s \in [0, L].$$

an ellipse

$\rightsquigarrow$ a solution is a segment on connecting $(-|\phi_0|, 0)$ to $(|\phi_0|, 0)$.



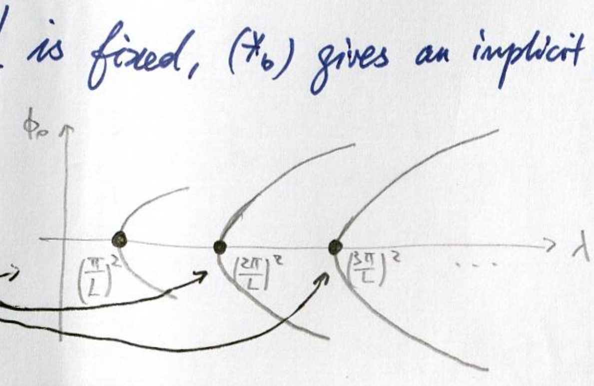


Thus, the length of the rod can be expressed as

$$L = \int_0^L \frac{|d\phi|}{|d\phi|/ds} = K \cdot \int_{-|\phi_0|}^{|\phi_0|} \frac{d\phi}{\sqrt{4\lambda \sin^2\left(\frac{\phi_0}{2}\right) - 4\lambda \sin^2\left(\frac{\phi}{2}\right)}} = \frac{K}{\sqrt{\lambda}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1 - \sin^2\left(\frac{|\phi_0|}{2}\right) \sin^2\theta}}, \quad (*_b)$$

where $K \in \mathbb{N}$, θ is given by $\sin\left(\frac{\phi}{2}\right) = \sin\frac{|\phi_0|}{2} \sin\theta$.

Since L is fixed, $(*_b)$ gives an implicit relation between ϕ_0 and λ.



"bifurcation points"

"bifurcation diagram" of $(*)$

$$\phi''(s) + \lambda \sin(\phi(s)) = 0 \qquad s \in [0,L], \quad \phi'(0) = \phi'(L) = 0$$

$$\phi''(s)\cdot\phi'(s) + \lambda \sin(\phi(s))\phi'(s) = 0$$

$$\underbrace{\int_0^t \phi''(s)\cdot\phi'(s)}_{\mathrm{I}} + \underbrace{\int_0^t \lambda \sin(\phi(s))\phi'(s)\,ds}_{\mathrm{II}} = 0$$

$$\frac{d}{dt}\left(4\lambda \sin^2\left(\tfrac{1}{2}\phi(t)\right)\right)$$

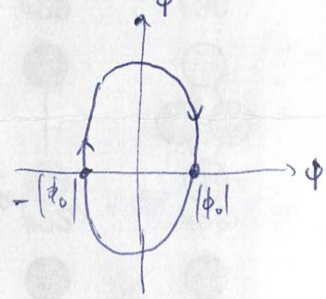
$$= 4\lambda\cdot\cancel{2}\sin\left(\tfrac{1}{2}\phi(t)\right)\cos\left(\tfrac{1}{2}\phi(t)\right)\cdot\tfrac{1}{\cancel{2}}\cdot\phi'(t)$$

$$= 2\lambda \sin(\phi(t))\,\phi'(t)$$

$$\mathrm{I} = \int_0^t \phi'(s)\,d\phi'(s) = \tfrac{1}{2}\left(\phi'(s)\right)^2\Big|_0^t = \tfrac{1}{2}\left(\phi'(t)\right)^2 - \cancel{\tfrac{1}{2}\left(\phi'(0)\right)^2} = \tfrac{1}{2}\left(\phi'(t)\right)^2$$

$$\mathrm{II} = \int_0^t \tfrac{1}{2}\,\frac{d}{ds}\left(4\lambda\sin^2\tfrac{1}{2}(\phi(s))\right)ds = \tfrac{1}{2}\cdot 4\lambda\sin^2\left(\tfrac{1}{2}\phi(s)\right)\Big|_0^t = 2\lambda\sin^2\left(\tfrac{1}{2}\phi(t)\right) - 2\lambda\sin^2\left(\tfrac{1}{2}\phi_0\right)$$

$$\mathrm{I} + \mathrm{II} = 0 \Rightarrow \tfrac{1}{2}\left(\phi'(t)\right)^2 + 2\lambda\sin^2\left(\tfrac{1}{2}(\phi(t))\right) = 2\lambda\sin^2\left(\tfrac{1}{2}\phi_0\right)$$



Formally, the problem $^{(*)}$ of bending rod can be rewritten as:

$$X = \{ \phi \in C^2_{[0,L]} : \phi'(0) = \phi'(L) = 0 \},$$

$$Y = C_{[0,L]}$$

$$F : \mathbb{R} \times X \to Y, \quad F(\lambda, \phi) = \phi'' + \lambda \sin\phi \in Y,$$

then,

$$(*) \Leftrightarrow F(\lambda,\phi) = 0 \quad \text{for } (\lambda,\phi) \in \mathbb{R}\times X.$$

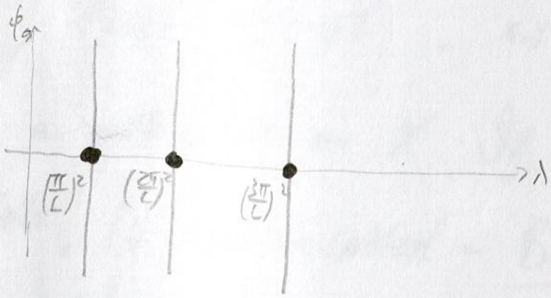
<u>Remark</u>: The linearization of $(*)$ at $(\lambda_0, 0)$ gives

$$\phi''(s) + \lambda_0 \phi(s) = 0, \quad s \in [0,L], \quad \phi'(0) = \phi'(L) = 0, \qquad (*_\ell)$$

which has nontrivial solutions iff

$$\lambda_0 = \left(\frac{K\pi}{L}\right)^2 \quad \text{with } \phi(s) = c \cdot \cos\left(\frac{K\pi s}{L}\right) \text{ for all constant } c \in \mathbb{R},$$

for $K \in \mathbb{N}$. That is, the bifurcation diagram of $(*_\ell)$ looks like:



Later, we will see that under certain transversality condition, the linearization gives a good description locally around a bifurcation point.

However, a global picture is often more difficult.

Chap 2. Linear Functional Analysis

2.1. Banach spaces

Let X be a linear space over $\mathbb{F}$ ($\mathbb{R}$ or $\mathbb{C}$). A norm on X is an $\mathbb{R}$-valued function $\|\cdot\|$ such that

- $\|x\| \geq 0 \quad \forall_{x \in X}$, and $\|x\| = 0 \Leftrightarrow x = 0$;
- $\|\alpha x\| = |\alpha| \cdot \|x\| \quad \forall_{x \in X}, \alpha \in \mathbb{F}$;
- $\|x+y\| \leq \|x\| + \|y\| \quad \forall_{x,y \in X}$.

Two norms $\|\cdot\|_1$ and $\|\cdot\|_2$ on X are equivalent, if $\exists\, k, K > 0$ s.t.

$$k\|x\|_1 \leq \|x\|_2 \leq K\|x\|_1 \quad \forall_{x \in X}.$$

If $\|\cdot\|$ is a norm on X,

$$\rho(x,y) := \|x-y\| \quad , \quad x, y \in X,$$

defines a metric ρ on X. If the metric space (X, ρ) is complete, $(X, \|\cdot\|)$ is called a Banach space.

H. Every finite-dimensional linear space with any norm is a Banach space.

Example 2.1.1 $X = C^n([0,1]; \mathbb{F}^N) = \{ u = (u_1, \ldots u_N) : [0,1] \to \mathbb{F}^N$ is a continuous function, and the n-th derivatives exist on $(0,1)$ with cont. extensions on $[0,1] \}$ is a Banach space with respect to the norm

$$\|u\| = \sup\{ |u(x)| + |u^{(n)}(x)| \ : \ x \in (0,1) \},$$

where $u^{(n)} = \left(\frac{d^n u_1}{dx^n}, \dots, \frac{d^n u_N}{dx^n} \right)$. By convention, $C^0([0,1]; \mathbb{F}^N) = C([0,1]; \mathbb{F}^N)$.

An inner product $\langle \cdot , \cdot \rangle$ on a linear space X over $\mathbb{F}$ is an $\mathbb{F}$-valued function on $X \times X$ such that

- $y \mapsto \langle x, y \rangle$ is $\mathbb{F}$-linear for every fixed $x \in X$;
- $\langle x, y \rangle = \overline{\langle y, x \rangle}$ for all $(x,y) \in X \times X$;
- $\langle x, x \rangle \geq 0$ for all $x \in X$ and $\langle x, x \rangle = 0 \iff x = 0$.

If $\langle \cdot , \cdot \rangle$ is an inner product on X, then

$$\|x\|^2 = \langle x, x \rangle \ , \quad x \in X,$$

defines a norm on X and the Cauchy-Schwarz inequality,

$$|\langle x, y \rangle| \leq \|x\| \cdot \|y\| \quad \forall_{x,y \in X},$$

holds. A complete inner-product space is called a Hilbert space.

H. Every finite-dimensional linear space is a Hilbert space.

Lemma 2.1.1. A Banach space X is finite-dimensional $\iff$ the closed unit ball $\{x \in X : \|x\| \leq 1\}$ is compact.

2.2. Subspaces

Let X be a Banach space with norm $\|\cdot\|$ and $Y \subset X$ be a linear subspace of X. Then,

$$(Y, \|\cdot\|) \text{ is Banach} \iff Y \text{ is closed in } X.$$

If X_1, X_2 are closed linear subspaces of X such that

- $X_1 \cap X_2 = \{0\}$;
- $\forall_{x \in X} \ \exists_{x_1 \in X_1,\, x_2 \in X_2}$ s.t. $x = x_1 + x_2$,

then X is the direct sum of X_1 and X_2, written as $X = X_1 \oplus X_2$.

H. If $X = X_1 \oplus X_2$, then every $x \in X$ can be written in a unique way as $x = x_1 + x_2$ for $x_i \in X_i$, $i = 1, 2$.

Lemma 2.2.1 Let X be Banach and X_1 be a finite-dimensional subspace of X. Then, X_1 is closed and there exists a closed subspace X_2 of X such that $X = X_1 \oplus X_2$.

In this case, $\dim X_1$ is called the codimension of X_2.

2.3. Linear operators

Let X, Y be two Banach spaces over the same field $\mathbb{F}$.

A function $A: X \to Y$ is linear, if

$$A(\alpha \cdot x) = \alpha \cdot A(x) \quad \forall_{\alpha \in \mathbb{F}},\ x \in X.$$

Note Linearity is intimately connected with the field $\mathbb{F}$.

H. The operator $L: \mathbb{C} \to \mathbb{C}$, $z \mapsto \bar{z}$ is $\mathbb{R}$-linear, but not $\mathbb{C}$-linear.

A linear function $A: X \to Y$ is called a bounded linear operator, written $A \in \mathcal{L}(X, Y)$, if $\exists\, K > 0$ s.t.

$$\|Ax\| \leq K \cdot \|x\| \quad \forall_{x \in X}$$

Example. $\mathbb{1} \in \mathcal{L}(X, X)$, the identity operator, $x \mapsto x$;

$0 \in \mathcal{L}(X, Y)$, the zero operator, $x \mapsto 0 \in Y$.

H. A linear mapping is continuous $\Leftrightarrow$ it is bounded.

Lemma 2.3.1. If X, Y are Banach spaces over $\mathbb{F}$, then $\mathcal{L}(X, Y)$ is a Banach space with respect to the norm

$$\|B\| = \inf\{K \geq 0 : \|Bx\| \leq K\|x\| \ \forall_{x \in X}\}$$

$$= \sup\{\|Bx\| : x \in X,\ \|x\| \leq 1\}.$$

For $A \in \mathcal{L}(X,Y)$, denote by

$$\ker(A) = \{ x \in X : Ax = 0 \},$$ closed linear subspace of X

$$\mathrm{range}(A) = \{ y \in Y : y = Ax \text{ for some } x \in X \}.$$ linear subspace of Y.

A is injective, if $\ker(A) = \{0\}$; surjective, if $\mathrm{range}(A) = Y$; and bijective, if both hold. An operator $A \in \mathcal{L}(X,Y)$ is a homeomorphism if it is a bijection and $A^{-1} \in \mathcal{L}(Y,X)$.

Lemma 2.3.2 (Corollary of Open Mapping Theorem)

If X and Y are Banach spaces and $A \in \mathcal{L}(X,Y)$ is a bijection, then A is a homeomorphism.

2.4. Compact and Fredholm operators

Let (M, ρ) be a compact metric space. Then, the linear space $C(M, \mathbb{F})$ of continuous functions $u : M \to \mathbb{F}$ with norm

$$\|u\| = \max \{ |u(x)| : x \in M \}$$

is a Banach space. The following characterizes its compact sets:

Theorem 2.4.1 (Ascoli-Arzelà Theorem) Suppose that M is a compact metric space. A set $B \subset C(M, \mathbb{F})$ has compact closure if and only if

(i) $\exists\, k$ s.t. $\|u\| \leq k \quad \forall_{u \in B}$; and

(ii) $\forall_{\varepsilon > 0} \; \exists_{\delta > 0}$ (independent of u) s.t. $|u(x) - u(y)| < \varepsilon \quad \forall_{u \in B}$, x, y with $\rho(x,y) < \delta$.

Def. 2.4.2 Let X and Y be Banach spaces. An operator $K \in \mathcal{L}(X, Y)$ is called compact if every bounded sequence $\{x_n\} \subset X$ has a subsequence $\{x_{n_k}\}$ such that $\{K(x_{n_k})\}$ converges in Y.

H. If X and Y are finite-dimensional, then all linear operators are bounded and thus compact; if Y is finite-dimensional, X is Banach, then any linear operator is compact. cf. 2.1.1.

H. If W, X, Y, Z are Banach spaces, and $K \in \mathcal{L}(X, Y)$ is compact, $B \in \mathcal{L}(Z, X)$, $C \in \mathcal{L}(Y, W)$, then $C \circ K \circ B \in \mathcal{L}(Z, W)$ is compact.

Def. 2.4.3 Let X and Y be linear spaces over $\mathbb{F}$ and $X \subset Y$. Suppose that $(X, |\cdot|)$ and $(Y, \|\cdot\|)$ are Banach spaces for which the mapping $\iota: X \hookrightarrow Y$ given by $\iota(x) = x \in Y$ for all $x \in X$, is bounded. We say that the embedding of X in Y is continuous and ι is called the embedding operator. If ι is compact, then X is compactly embedded in Y.

Example 2.4.4 Let $X = C^1([0,1], \mathbb{F})$, $Y = C([0,1], \mathbb{F})$ be Banach spaces defined in Example 2.1.1. Then:

$$\iota: X \hookrightarrow Y \qquad f \mapsto f$$

is compact, by Thm. 2.4.1 and the mean-value theorem. H.

Lemma 2.4.5 Let X and Y be Banach spaces. Then, the compact operators form a closed linear subspace in $\mathcal{L}(X,Y)$. In particular, if there exists a sequence $\{A_n\} \subset \mathcal{L}(X,Y)$ such that $\|A_n - A\| \to 0$ as $n \to \infty$ and A_n has finite-dimensional range for each n, then A is a compact operator.

Def. 2.4.6 An operator $A \in \mathcal{L}(X,Y)$ is a Fredholm operator of index p, if

- $\ker(A)$ has finite dimension n;
- $\mathrm{range}(A)$ is closed and has finite codimension r;
- $p = n - r$.

Clearly, any homeomorphism from X to Y is a Fredholm operator of index 0. Moreover,

Theorem 2.4.7 (Fredholm Alternative) Let $K \in \mathcal{L}(X,X)$ be compact. Then, $I - K \in \mathcal{L}(X,X)$ is a Fredholm operator of index zero.

Theorem 2.4.8 Let X and Y be Banach spaces and $K \in \mathcal{L}(X,Y)$ be compact and $T \in \mathcal{L}(X,Y)$ be a homeomorphism.

Then, $B = T + K$ is Fredholm with index zero.

Pf: $B = T + K = T(I + T^{-1}K)$, so

B is Fredholm of index 0 $\Leftrightarrow$ $I + T^{-1}K$ is Fredholm of index 0,

since T is a homeomorphism.

$\#$.

Proposition 2.4.9 The set of Fredholm operators is an open set in $\mathcal{L}(X,Y)$ and the Fredholm index of operators is constant on the components of this set.

Def. 2.4.10 Let $\iota \in \mathcal{L}(X,Y)$ be the continuous embedding of X in Y. Consider $\lambda_0 \in \mathbb{F}$ and $A \in \mathcal{L}(X,Y)$. We say that λ_0 is a simple eigenvalue of A, if

(S1) $(\lambda_0 \iota - A)$ is Fredholm of index 0 ;

(S2) $\ker(\lambda_0 \iota - A)$ is one-dimensional over $\mathbb{F}$;

(S3) $\mathrm{range}\,(\lambda_0 \iota - A) \cap \iota(\ker(\lambda_0\iota - A)) = \{0\}$.

An element $\xi_0 \in X \setminus \{0\}$ with $A\xi_0 = \lambda \iota \xi_0$ is called an eigenvector of A corresponding to the eigenvalue λ_0.

Lemma 2.4.11 Suppose λ_0 is a simple eigenvalue of A with eigenvector ξ_0. Let

$$Y_1 = \mathrm{range}\,(\lambda_0 \iota - A) \quad \text{and} \quad X_1 = \iota^{-1}(Y_1) \left(= X \cap Y_1\right)$$

Then, $X = X_1 \oplus \mathrm{span}\{\xi_0\}$ and $(\lambda_0 \iota - A)$ is a homeomorphism from X_1 to Y_1 with the norms inherited from X and Y.

pf: $(S1)-(S2) \Rightarrow \mathrm{codim}\, Y_1 = 1 \overset{(S3)}{\Rightarrow} Y = Y_1 + \mathrm{Span}\{\xi_0\}$ i.e. $\forall y \in Y,\ y = y_1 + \alpha \iota \xi_0$ for some $y_1 \in Y_1,\ \alpha \in \mathbb{F}$.

So $\forall x \in X \subset Y,\ x = y_1 + \alpha \iota \xi_0 \Rightarrow x - \alpha \xi_0 = \iota^{-1}(y_1) \in X_1$

$\Rightarrow x = x_1 + \alpha \xi_0$ for some $x_1 \in X_1,\ \alpha \in \mathbb{F}$.

Also, $X_1 \cap \operatorname{span}\{\xi_0\} = \iota^{-1}(Y \cap \operatorname{span}\{\iota\xi_0\}) \overset{(53)}{=} \{0\}$

Thus, $X = X_1 \oplus \operatorname{span}\{\xi_0\}$, so $(Y = Y_1 \oplus \operatorname{span}\{\iota\xi_0\})$

$$(\lambda_0\iota - A): X_1 \oplus \operatorname{span}\{\xi_0\} \longrightarrow Y_1 \oplus \operatorname{span}\{\iota\xi_0\}$$

is a bijection from X_1 to Y_1. Hence, by Lemma 2.3.2, it is a homeomorphism from X_1 to Y_1.