An equivariant degree theory for networked dynamical systems

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To my grandfather Zhenyin

Abstract

We extend the scope of equivariant degree theory and methods to the realm of networked dynamical systems. Focusing on two types of hidden symmetries of networked systems, we introduce two types of equivariant degrees: the *lattice equivariant degree* for quotient symmetries and the *interior equivariant degree* for interior symmetries. We show how these degrees can be applied to study synchrony-breaking bifurcations in general coupled cell networks, and provide topological classifications of bifurcating branches by their symmetry and synchrony properties.

Compared to the usual equivariant degrees, the lattice equivariant degree takes on an extensive versatile algebraic properties; and the interior equivariant degree shows an extreme elegance in treating bifurcations related to interior symmetries.

We also contribute to the theory of equivariant degrees by re-introducing equivariant degrees using their computational formulas which embody a geometric grace. In addition, we introduce a concept of *secondary dominating orbit types* in order to extend the classical equivariant classification results to bifurcating branches of secondary maximal isotropies.

Acknowledgment

It is difficult to express fully my gratitude to Herr Prof. Dr. Reiner Lauterbach at University of Hamburg, who has been hosting me as a postdoctoral researcher in his group and who has been constantly guiding me to grow as an independent scientist and a qualified college teacher. His determination in searching the truth and passion in conveying it have been a continuous inspiration for me.

I am deeply indebted to the Department of Mathematics at the University of Hamburg, which has offered me a wonderful opportunity in teaching mathematical lectures and developing my own curricula.

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Living in Hamburg is a blessing. Its open vital embracing vibes, its lush fertile verdant lands and its kind warm compassionate people, have been giving me immense amount of energy and hope to reach beyond.

Contents

Chapter 1

Introduction

A network is a set of interconnected elements that influence each other. They can be a group of people that share common social interests, a range of aquatic plants and animals that live in a wetland, or a cluster of neurons that transmit electric signals through dendrites and axons to each other.

Individual elements of a network operate not on their own, but in dependence on each other through connections. These connections can be social relations that influence the formation of an individual's mind, food chains that determine the existence of an individual species, or neural connections that excite or inhibit an individual neuron's action (cf. [77, 47, 46]).

More remarkably, individual elements, through connections with each other, constitute an entire entity that can operate in a global manner, giving rise to phenomena that cannot be accounted for by the individuals. Examples of such include conformity in social groups, biodiversity in ecosystems, and visual perception in cerebral cortices (cf. [54, 51, 78]).

One of the most prevailing collective dynamics on networks is synchronization, where two or more elements of the network start operating in a synchronous fashion. It is ubiquitous in nature, science and engineering (cf. [76, 61, 7]). It emerges from physical, biological, chemical, social and ecological systems. It occurs to networks composed of homogeneous or heterogeneous elements [30, 31], having strong or weak interactions [18], configured in a symmetric or non-symmetric, local or global structure [40].

The seemingly inevitable tendency of networks to synchronize in time has attracted a broad spectrum of research interest across a variety of scientific disciplines (cf. [7] and references therein). Yet, the general understanding of synchronization process on networks remains limited and rigorous analysis remains scarce.

In the field of complex networks, where networks, as often arise from social or biological systems, possess structures that are neither entirely regular nor completely random, many have reported a positive correlation between synchronization and some statistical property of the network. For example, it has been suggested that synchronization tends to emerge more easily in networks having a heterogeneous degree distribution which means networks that have a number of nodes, known as “hubs”, that reach out to a large number of nodes and act as pacemakers for the entire network (cf. [56, 30, 31]). It has also been suggested that synchronization can be promoted by reducing average path length on the graph which gives way to a quicker onset of fully synchronized behavior (cf. [53]).

However, observed correlations can be far from being conclusive. They can sometimes even point to the opposite directions. For example, the authors in [59] have found that networks with a homogeneous degree distribution can be easier to synchronize than the heterogeneous ones, even when the average path length on the graph becomes larger. The surprising behavior was explained using another statistical property of the network — the load distribution. A load of a given node is defined by the number of shortest paths between two nodes passing through it. Apparently, the advantage of having a few hubs acting as pacemakers can be forfeited by the overload of traffic communications passing through the hubs. But whether any single or combination of these statistical properties can be used as a causal explanation for synchronization remains unclear (cf. [8]).

Network structure is more than a mere graphic description of some interconnected chunk, but it lays down a structural frame within which network dynamics is to develop and unfold. Rigorous mathematical analysis is almost indispensable in analyzing synchronization as network dynamics.

Study of ensembles of coupled phase oscillators was motivated by spontaneous coordination of variety of biological rhythms into life processes (cf. [79]). Whether and when the oscillators are to synchronize typically relies on how far apart the frequencies of the oscillators are and how strong the coupling (interaction) strength is (cf. [7]). A mathematical tractable model for a rigorous description of this phenomenon was proposed by Kuramoto [52], who suggested a mean field approach and used a macroscopic complex order parameter to describe different scales of synchronization that lead to the final synchronization. Despite its simplicity and clarity, and despite a series of attempt [62, 63, 42, 43] extending it to more general scenarios such as finitely many oscillators or general couplings, no exact analytical results for Kuramoto models on general complex networks are available up to date (cf. [7]).

At the same time, some stability analysis of fully synchronized states of ensembles of identical oscillators has been initiated [60] and developed [17] using the master stability function, which is defined by the largest Lyapunov exponent transverse to the fully synchronized states. It has been widely used to assess and engineer synchronizability of networks of identical oscillators (cf. [57, 58]). It is a convenient tool since it separates the impact of network structure from particularities of oscillators and measures this by a single real-valued function. Despite its advantages, the stability acquired in this way is a *linear* and a *local* stability, which is a necessary, but not sufficient condition for synchronization.

On the other side of the coin lies the question of instability and bifurcations of the synchronized states. Rarely any real-world systems take on an unperturbed full synchrony from beginning to the end of their existence. Fully synchronized states, as ideal and appealing as they may appear, are frangible and evanescent. They emerge from progressively more synchronized pitches and readily give way to abundance of broken rhymes.

Formalism of coupled cell networks and coupled cell systems has been introduced [73, 25] and further developed [39, 37] for the study of synchronized dynamics in general networked systems, using coupled ordinary differential equations and their associated bifurcations. Here, a *cell* is a set of ordinary differential equations that describes an individual's dynamics; an incoming arrow to the cell indicates the dependence of its evolution on other cells. A finite set of cells together with the arrows connecting them constitutes a *coupled cell network*. A dynamical system that is consistent with the network structure is an admissible *coupled cell system*.

In analogue to other structures of dynamical systems such as the symmetry of the equivariant systems or the Hamiltonian of the Hamiltonian systems, network structure imposes strong restrictions on the dynamics of the associated admissible coupled cell systems.

One striking example is the existence of *synchrony subspaces*, which are subspaces defined by equivalence relations on cells such that they are flow-invariant for *every* coupled cell system admissible to the network. It has been shown that synchrony subspaces are intrinsic properties of the network structure, and they are in one-to-one correspondence with equivalence relations on cells that satisfy a combinatorial property called *balanced* (cf. [73, 39]). Coupled cell systems restricted to these synchrony subspaces are again coupled cell systems admissible to smaller networks, called the *quotient networks*. They characterize the network dynamics of the initial system subject to the cell equivalence defining the synchrony subspaces. The existence of synchrony subspaces and their associated quotient networks has strong implications on synchronized dynamics in coupled cell systems (cf. [73, 4, 1, 35]).

It is a scenario reminiscent of symmetry and equivariant systems. In case of equivariant systems, *fixed-point subspaces*, which are subspaces defined by equivalence relations induced by sub-symmetries (isotropy subgroups), are always flow-invariant for *every* equivariant system of the symmetry. The existence of fixed-point subspaces and their restricted systems has played a central and fundamental role in the study of equivariant dynamics in equivariant systems (cf. [38, 24, 65]).

One interesting dynamics arising from coupled cell networks is the *synchrony-breaking bifurcations*, where a fully synchronized state loses its stability and breaks into multiple clusters of synchronized cells, characterized by their synchrony patterns. A counterpart of synchrony-breaking bifurcations is the *symmetry-breaking bifurcations* in equivariant systems, where a completely symmetric state loses its stability and breaks into multiple states of less symmetry, characterized by their isotropy subgroups. This strong resemblance has motivated a series of scientific endeavors in adapting a parallel theory of studying synchronized dynamics on coupled cell networks by transferring essential ideas from equivariant bifurcation theory (cf. [73, 34, 5, 3, 4, 1, 72]).

Equivariant degree theory is a topological degree theory that concerns *equivariant maps*, which are maps commuting with the group symmetry in their domain and image. A topological degree is, in its simplest form, a generalization of the *winding number* of continuous maps that map the unit circle S^1 to a punched plane $\mathbb{C} \setminus \{p\}$ for some punched position p. It counts how many times the image of the map has traveled counterclockwise around the point p. The count remains unchanged, if the map is perturbed slightly or deformed largely without trespassing the point p. The addition of winding numbers corresponds to the conjunction of maps, and the negation of winding numbers is realized by rewinding the direction of maps. The topological degree is thus known as "an algebraic count of the zeros of a continuous map".

A main objective of the equivariant degree theory is to attain the topological structure of the zeros of an equivariant map and their algebraic properties induced by the equivariance. The past two decades have witnessed continuous progress in the development of equivariant degree, both in theory and in practice (cf. [21, 44, 29, 28, 68, 16] and the references therein). Among others, a *primary equivariant degree*, which is a truncated part of the full equivariant degree, turns out to be most "computable" and serves as an effective topological tool in the study of equivariant systems.

As the first part of the thesis, we review in Chapter 3 the two most common primary equivariant degrees in use: the *equivariant degree without parameter* and the *equivariant degree with one parameter*, which together have contributed to a systematic treatment of equivariant bifurcations (cf. [12, 13, 14, 15, 10] for example and [16] for further references). We position ourselves with a slightly different angle from the classical literature and introduce them by their geometric meanings and subsequently the so-called *recurrence formulas* (cf. (3.4), (3.11)). We show that usual properties of an equivariant degree follow naturally from this definition (cf. Theorem 3.1.4, Theorem 3.1.6). In the application part of these degrees in equivariant bifurcations, we focus on explaining how to use the one-line command `showdegree` in the "Equivariant Degree Maple© Library Package" (EDML) to produce an

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The Equivariant Degree Maple© Library Package was created by A. Biglands and W. Krawcewicz at the University of Alberta in 2006 supported by NSERC research grant. It is open source and can be freely downloaded, for example, from `http://www.math.uni-hamburg.de/home/ruan/download`.

instant topological classification of bifurcating branches by their symmetry types (cf. Subsection 3.2.3). A concept of *secondary dominating orbit types* will be introduced to sharpen the statement on the symmetry properties of the bifurcating branches (cf. Proposition 3.2.2).

The main advantage of using equivariant degree theory lies in both of its topological and algebraic properties. On one hand, equivariant degree is a topological invariant that remains unchanged against all adequate deformations of maps. This allows flexibility and freedom in their computations. As a result, equivariant degrees can be applied to equivariant bifurcations caused by multiple critical eigenvalues. In other words, as a topological invariant, equivariant degree sees no additional complications in treating critical eigenvalues with large eigenspaces. On the other hand, equivariant degree is algebraic and has an extended list of algebraic properties. For example, it is intrinsically compatible with homomorphisms of group symmetries. In application, this results in clarity and simplicity in treating changes of equivariance such as different equivariance (quotient symmetries) of a coupled cell system at different quotient levels.

As the second part of the thesis, we introduce in Chapter 4 a topological degree theory that is suitable for studying maps that are equivariant with respect to different symmetries on different subspaces of their domain. As shown in [70], the set of all synchrony subspaces of a coupled cell network forms a (complete) *lattice* under the inclusion relation. In a way, it gives a filtration of the total phase space by subspaces. To incorporate additional symmetry at each quotient level, we introduce the concept of *representation lattices*, which are lattices of representation spaces that are fitted together through connecting homomorphisms of groups alongside the inclusions of subspaces (cf. Definition 4.1.2). Following the geometric meaning of equivariant degrees, we define a *lattice equivariant degree* for maps that are equivariant at each level of the representation lattice (cf. Definition 4.2.2). We apply the degree for studying synchrony-breaking (Hopf) bifurcations in coupled cell systems (with or without quotient symmetries), which results in a topological classification of bifurcating branches of oscillating states by their synchrony and symmetry types (cf. Theorem 4.3.2).

Network structure can manifest simultaneously through different types of symmetry. Besides the quotient symmetries, there can also be interior symmetries. A *interior symmetry* is a permutational symmetry on a subset of cells so that the input structure of the subset is preserved. The concept of interior symmetry has been introduced in [34], as an intermediate between the general (groupoid) invariance of a coupled cell network and the stringent group symmetry of a symmetrically coupled cell network. It turns out that both the equivariant branching lemma and the equivariant Hopf theorem can be extended for networks with interior symmetries (cf. [34, 5])

As the third part of the thesis, we introduce in Chapter 5 a topological degree theory for studying maps with interior symmetries. Different from the definition of the lattice equivariant degree, which is of an agreeable algebraic nature, the construction of the *interior equivariant degree* here is of somewhat technical nature. We needed to establish an approximation scheme of *regular normal maps* following its equivariant counterpart (cf. Proposition 5.2.9). However, the resulting degree turns out to be an additionally merciful case when applied to the synchrony-breaking bifurcations. We show that bifurcating branches arising from breaking an interior symmetry are interior-equivariantly homotopic to bifurcating branches arising from an equivariant bifurcation problem (cf. Theorem 5.3.1). Thus, all the computational resources available for equivariant bifurcations are directly applicable for synchrony-breaking bifurcations caused by an interior symmetry. In analogue, interior symmetries of a quotient network, known as the *quotient interior symmetries*, can also be treated using quotient symmetries.

Throughout the thesis, we use the 5-cell regular network shown in Figure 1.1 as a running example to illustrate our results. The network arises

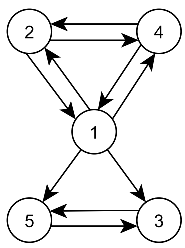


Figure 1.1: A regular 5-cell network frequently in use throughout the thesis.

from the discussions in [4] of possible lifts of 3-cell bidirectional rings to a regular 5-cell coupled cell network (cf. Network (6) of Figure 5 in [4]).

It possesses many interesting structures in one:

(a) Symmetry: $\mathbb{Z}_2 \times \mathbb{Z}_2 = \langle(2\ 4)\rangle \times \langle(3\ 5)\rangle$ is the permutational symmetry generated by switching cells 2 with 4 and cells 3 with 5.

(b) Quotient symmetry (cf. Example 2.1.18 and Example 2.1.8):

 (b1) The full permutational symmetry S_3 on the 3 cell quotient network obtained by identifying cells 2 with 5 and cells 3 with 4, which is related to the synchrony subspace Δ_4 (cf. [4] or Figure 2.6 for notations Δ_*'s).

 (b2) The full permutational symmetry S_3 on the 3 cell quotient network obtained by identifying cells 2 with 3 and cells 4 with 5, which is related to the synchrony subspace Δ_1.

 (b3) The symmetry $\mathbb{Z}_2 = \langle(3\ 5)\rangle$ on the 3 cell quotient network obtained by identifying cells 1, 2 and 4, which is related to the synchrony subspace Δ_2.

(b4) The symmetry $\mathbb{Z}_2 = \langle(3\ 5)\rangle$ on the 4 cell quotient network obtained by identifying cells 2 with 4, which is related to the synchrony subspace Δ_{00}.

(b5) The symmetry $\mathbb{Z}_2 = \langle(2\ 4)\rangle$ on the 4 cell quotient network obtained by identifying cells 3 with 5, which is related to the synchrony subspace Δ_{01}.

(b6) The symmetry $\mathbb{Z}_2 = \langle(3\ 5)\rangle$ on the 4 cell quotient network obtained by identifying cells 1 with 2, which is related to the synchrony subspace Δ_{02}.

(b7) The symmetry $\mathbb{Z}_2 = \langle(3\ 5)\rangle$ on the 4 cell quotient network obtained by identifying cells 1 with 4, which is related to the synchrony subspace Δ_{03}.

(c) Interior symmetry (cf. Example 2.1.12):

(c1) The full permutational symmetry S_3 on the subset $\{1, 2, 4\}$ with their input arrows.

(c2) The symmetry $\mathbb{Z}_2$ on the subset $\{3, 5\}$ with their input arrows.

(d) Quotient interior symmetry (cf. Example 2.1.18): the permutational symmetry S_3 on the cells $1, 2, 3$ of the 4 cell quotient network obtained by identifying cells 3 with 5, which is related to the synchrony subspace Δ_{01}.

We deal with the quotient symmetry part of this example in Subsection 4.3.3 and the interior symmetry part (including the quotient interior symmetry) in Subsection 5.3.3. The summary is given at the end of Chapter 5 (cf. Table 5.3).

Remarks and Open Questions

All the classification results obtained using a degree argument, regardless if it is equivariant, lattice equivariant or interior equivariant, are topological results. We refer especially to the statements presented in Subsection 3.2.2, Subsection 4.3.2 and Subsection 5.3.2. They provide a *least estimate* on number of bifurcating branches of solutions to the system, and by no means, exhaust all possible solutions. Depending on the type of systems and their nonlinear terms, additional bifurcating solutions can very well exist. We refer to an interesting case study on the network (6) in [4]. However, all bifurcating branches of solutions can be adequately deformed to those inferred by the bifurcation invariants. Additional branches correspond to zeros of homotopy-null class of maps.

It is interesting to compare the three different bifurcation invariants (4.28), (5.42) and (5.46), associated to the same problem of synchrony-breaking bifurcations in the coupled cell systems associated to our running example. They describe the *total* topological invariance of the *total* bifurcating branches of solutions from three different angles. The first one portraits the topological invariance in the light of lattice equivariant deformations; the second one produces a snapshot of the topological invariance through the lens of interior equivariant deformations; and the third one is a showcase of the topological invariance wearing costumes of quotient interior equivariant deformations. Each tells a story, a story of its own kind and from its own perspective.

It is natural to ask whether there is a way to achieve the wholeness and the entirety of the story, the story of a topological invariance in the full light of network structure, the story that is told by no more and no less than the network structure itself. Let this be our inspiration for the further pursuit and may it guide us through a wonderland of sparkling ideas, with an ever affirmed hope in seeing the beauty and truth of the whole story.

Chapter 2

Preliminaries

Important and necessary definitions from coupled cell networks, group representations and Euler rings of compact Lie groups are collected in this chapter, accompanied by examples. Most examples are based on the dihedral group D_3 and its permutational actions, which we will use later as running examples.

2.1 Coupled Cell Networks and Coupled Cell Systems

In this section, we give a brief account of the theory of coupled cell networks and coupled cell systems. We collect mainly definitions of important concept such as balanced equivalence relations and synchrony subspaces and present some basic examples to illustrate them. We also review several generalized forms of symmetry for networks. These include the interior symmetry and quotient symmetry, which will play an important role in our later discussions. Throughout, we restrict our attention mostly to the regular coupled cell networks, which give a simple organic setting for the study of synchrony-related bifurcations that we consider in later chapters. For details on coupled cell networks and coupled cell systems, we refer to Stewart *et al.* [73] and Golubitsky *et al.* [39, 37].

2.1.1 Coupled Cell Networks

A *cell* is a set of ordinary differential equations that describes the temporal evolution of an entity. A set of cells becomes a *coupled cell system* when evolution of individual cells is dependent on the evolution of others. In terms of dynamics, we mean that the change in state variable x_i of the cell i is subject to x_i and also possibly x_j for some $j \neq i$. That is, the evolution of x_i can be described possibly using

$$\dot{x}_i = f(x_i, x_{j_1}, \ldots, x_{j_k}),$$

for some function f and x_i being influenced by k other cells $j_1, \dots, j_k$.

A *coupled cell network* is a graph representation of a coupled cell system, where cells are represented by nodes and dependence is indicated by arrows.

The theory of coupled cell networks and coupled cell systems was mainly motivated by studying the phenomena of *synchronization* in networks, where different entities start operating in synchrony as a result of their interactions. Depending on the definition of synchrony, this can mean for example, $x_i(t) = x_j(t)$ for some $i \neq j$ and for all $t \in \mathbb{R}$. If synchrony occurs as a *structural* property of the network (that is, not as a result of some special form of the vector field), then this necessarily means x_i, x_j satisfy

$$\begin{cases} \dot{x}_i = f(x_i, x_{j_1}, \dots, x_{j_k}) \\ \dot{x}_j = f(x_j, x_{l_1}, \dots, x_{l_k}) \end{cases}$$

for some mutual function f and there is a one-to-one correspondence among the influencing cells, or *input sets*, $\{x_{j_1}, \dots, x_{j_k}\}$ and $\{x_{l_1}, \dots, x_{l_k}\}$.

Cells satisfying such conditions are called *input-equivalent* cells. Formally, on every coupled cell network there is a defined equivalence relation $\sim_C$ on the set C of cells and an equivalence relation $\sim_E$ on the set $\mathcal{E}$ of arrows such that they are compatible:

$$(c_1, d_1) \sim_E (c_2, d_2) \quad \Longrightarrow \quad c_1 \sim_C c_2, \quad d_1 \sim_C d_2,$$

where (c, d) denotes an arrow from c to d. Two equivalent cells c, d are *input equivalent*, written as $c \sim_I d$, if there exists a bijection $\beta : I(c) \to I(d)$ between their input sets $I(c), I(d)$ such that

$$(e, c) \sim_E (\beta(e), d), \quad \forall\, e \in I(c).$$

Note that the input equivalence is a refinement of the cell equivalence $\sim_C$.

On a coupled cell network, it is convenient to use the same node symbols for the input-equivalent cells and the same arrow symbols for the $\sim_E$-equivalent arrows. We use an example to explain.

Example 2.1.1. Consider the two graphs in Figure 2.1. The left graph has three types of nodes $\bigcirc$, $\square$, $\triangle$ and three types of arrows $\longrightarrow$, $\dashrightarrow$, $\cdots\!\!\blacktriangleright$. They indicate the cell equivalence:

$$\sim_C = \{\bigcirc = \{1\}, \square = \{2, 3\}, \triangle = \{4\}\} \tag{2.1}$$

and the edge equivalence

$$\sim_E = \{\longrightarrow = \{(1,2), (1,3), (2,4), (3,4)\}, \dashrightarrow = \{3,1\}, \cdots\!\!\blacktriangleright = \{4,1\}\}.$$

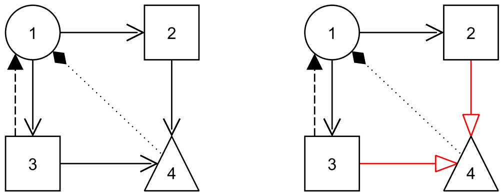


Figure 2.1: Directed graph that is not a coupled cell network (left) and that is a coupled cell network (right).

It is not a coupled cell network, since $(1,2) \sim_E (2,4)$ but $1 \nsim_C 2$. On the contrary, the right graph is a coupled cell network, where

$$\longrightarrow = \{(1,2),(1,3)\}, \; \longrightarrow = \{(2,4),(3,4)\}$$

and the compatibility condition is satisfied.

Next, we explain why it is convenient to use the same node symbols for input equivalent cells (instead of for cell equivalent cells). Consider the two graphs in Figure 2.2, where both are coupled cell networks. On the left, we

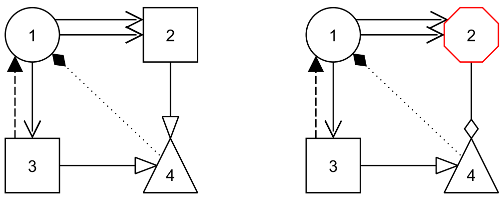


Figure 2.2: Coupled cell networks using same symbols for cell-equivalence (left) and using input-equivalence (right).

have

$$2 \sim_C 3, \quad 2 \nsim_I 3,$$

since $2, 3$ are given the same symbol $\square$, yet receive different number of input. In terms of dynamics, we have

$$\begin{cases} \dot{x}_2 = f(x_2, \overline{x_1, x_1}) \\ \dot{x}_3 = g(x_3, x_1) \end{cases}$$

for some functions f, g and the overhead bar means the interchangeability of the two last variables in f: $f(x,y,z) = f(x,z,y)$, to indicate that the two input arrows are of the same edge type. Thus, synchronization are generally not expected to happen between cells $2, 3$, since even starting with a synchronized initial condition, cells $2, 3$ will not evolve in a synchronized way.

Thus, we prefer to use different symbols for different input equivalent cells, for example as the right graph in Figure 2.2. In this case, only cells of the same symbol are relevant for discussion of synchronization. $\diamond$

In what follows, we use different node symbols ○, □, △, ⬠ . . . to indicate different classes of input equivalent cells, and different arrows ⟶, ⇢, ⋯◆, ─▷ . . . for different classes of edge equivalent arrows.

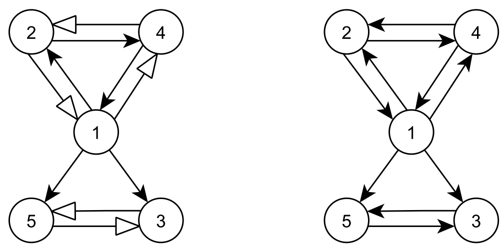
If we always associate the same vector field to input equivalent cells and require additional interchangeability for edge equivalent arrows, then we obtain a vector field that is *admissible* to the coupled cell network. For example, for the right graph in Figure 2.1, we have

$$\dot{\boldsymbol{x}} = F(\boldsymbol{x}), \quad F(\boldsymbol{x}) = \begin{pmatrix} f(x_1, x_3, x_4) \\ g(x_2, x_1) \\ g(x_3, x_1) \\ h(x_4, \overline{x_2, x_3}) \end{pmatrix},$$

for $\boldsymbol{x} = (x_1, x_2, x_3, x_4)^T$. The system of ODEs in this case is also called *admissible*.

A coupled cell network having only one input-equivalence class is called *homogeneous*; a homogeneous network having only one edge-equivalence class is called *regular*.

Example 2.1.2. (**Running Example**) Consider the two coupled cell networks in Figure 2.3. In the left one, every cell is input equivalent (indicated by ○)



(a) (b)

Figure 2.3: A coupled cell network of 5 cells that is (left) homogeneous (right) regular.

which receives two input arrows ⟶ and ─▷. Thus, it is homogeneous. The admissible vector field is of form

$$\begin{pmatrix} f(x_1, x_4, x_2) \\ f(x_2, x_1, x_4) \\ f(x_3, x_1, x_5) \\ f(x_4, x_2, x_1) \\ f(x_5, x_3, x_1) \end{pmatrix},$$

where the first variable is reserved for the internal dynamics, the second for $\longrightarrow$ and the last for $\longrightarrow\!\!\triangleright$.

The right graph gives a regular coupled cell network of *valency** 2. The admissible vector field is of form

$$\begin{pmatrix} f(x_1, \overline{x_4, x_2}) \\ f(x_2, \overline{x_1, x_4}) \\ f(x_3, \overline{x_1, x_5}) \\ f(x_4, \overline{x_2, x_1}) \\ f(x_5, \overline{x_3, x_1}) \end{pmatrix},$$

where the overhead bar means f satisfies $f(x, y, z) = f(x, z, y)$. ◇

The network (b) in Figure 2.3 arises from [4] by Aguiar *et al.*as a lift from a 3-cell bidirectional ring. It possesses especially many synchrony subspaces and hidden symmetries, as we shall see in the following subsections. We will use it as running example to illustrate our theoretical results in Chapter 4–5.

2.1.2 Synchrony Subspaces

Synchrony subspaces are linear subspaces in phase spaces of coupled cell systems that are defined by equalities of cell coordinates and are flow-invariant for *all* the coupled cell systems admissible to the underlying network structure. They describe synchrony patterns on networks that are independent of the specific form of vector fields, but induced by the network structure directly. They are an immediate showcase of the influence of the network structure on network dynamics.

Synchrony subspaces are properties of networks and can be determined by a graph combinatorial condition on networks, called *balanced colorings*. Given an equivalence relation $\bowtie$ on the set of cells of a coupled cell network, we can color the cells in the following way: two cells i, j receive the same color precisely when they belong to the same $\bowtie$-equivalence class. The coloring is called *balanced*, if any pair of cells of the same color receive the same number and type of input arrows from cells of color b, for every b.

More formally,

Definition 2.1.3. (cf. [39]) Given a coupled cell network $\mathcal{G} = (C, \mathcal{E}, \sim_C, \sim_E)$, an equivalence relation $\bowtie$ on the set C is called *balanced*, if for every $c, d \in C$ with $c \bowtie d$, there exists a bijection $\beta : I(c) \to I(d)$ between their input sets, which preserves the edge-equivalence relation $\sim_E$, and such that for all $i \in I(c)$, the tail cells of i and $\beta(i)$ are in the same $\bowtie$-class. ◇

* A *valency* of a cell is the total number of input arrows of this cell. Since every cell in a regular network has the same input arrows of the same type, one speaks of *valency* of the network.

Example 2.1.4. Let $\mathcal{G}$ be the regular network in Example 2.1.2 (cf. Figure 2.3(b)). Consider the equivalence relation

$$\bowtie_1 = \{\{1,2,4\},\{3\},\{5\}\}.$$

It induces a coloring on the network shown by Figure 2.4 (left). It is a

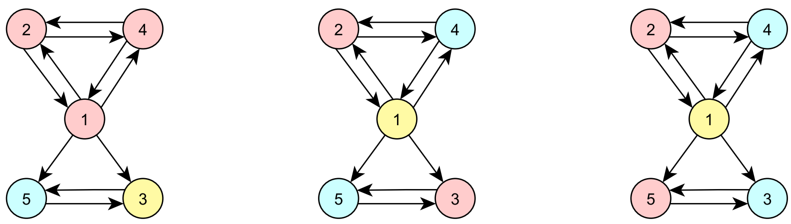


Figure 2.4: Balanced colorings on the network (b) in Figure 2.3.

balanced coloring, since every red node receives two inputs from red nodes. Similarly, one can verify that

$$\bowtie_2 = \{\{1\},\{2,3\},\{4,5\}\}, \quad \bowtie_3 = \{\{1\},\{2,5\},\{3,4\}\}$$

are also balanced equivalence relations on the network (cf. Figure 2.4).

For the homogeneous network in Example 2.1.2, it can be similarly verified that $\bowtie_1$ is balanced, but $\bowtie_2$ and $\bowtie_3$ are not balanced. $\diamond$

Given a coupled cell network $\mathcal{G}$, an equivalence relation $\bowtie$ on the cells and a choice of the total phase space P for the associated coupled cell systems, there is a *polydiagonal subspace*

$$\Delta_{\bowtie} = \{x : x_c = x_d, \quad \text{if } c \bowtie d\} \subset P$$

associated with $\bowtie$ defined by equating $\bowtie$-equivalent cell coordinates. A polydiagonal subspace is called a *synchrony subspace*, if it is flow-invariant for all the $\mathcal{G}$-admissible vector fields on P.

Example 2.1.5. Let $\bowtie_1, \bowtie_2, \bowtie_3$ be given by Example 2.1.4. Choose $\mathbb{R}^k$ be the phase space for each cell, so $P = (\mathbb{R}^k)^5$. The associated polydiagonal subspaces are

$$\begin{aligned}
\Delta_{\bowtie_1} &= \{(a,a,a,b,c) : a,b,c \in \mathbb{R}^k\} \\
\Delta_{\bowtie_2} &= \{(a,b,b,c,c) : a,b,c \in \mathbb{R}^k\} \\
\Delta_{\bowtie_3} &= \{(a,b,c,c,b) : a,b,c \in \mathbb{R}^k\}
\end{aligned}$$

Coupled cell systems admissible to the network with total phase space P are of form

$$\begin{cases} \dot{x}_1 = f(x_1, \overline{x_4, x_2}) \\ \dot{x}_2 = f(x_2, \overline{x_1, x_4}) \\ \dot{x}_3 = f(x_3, \overline{x_1, x_5}) \\ \dot{x}_4 = f(x_4, \overline{x_2, x_1}) \\ \dot{x}_5 = f(x_5, \overline{x_3, x_1}) \end{cases}, \quad x_i \in \mathbb{R}^k. \tag{2.2}$$

It can be verified that $\Delta_{\bowtie_1}$, $\Delta_{\bowtie_1}$, $\Delta_{\bowtie_1}$ are flow-invariant subspaces for (2.2). The system restricted to these flow-invariant subspaces are of form

$$\begin{cases} \dot{x}_1 = f(x_1, \overline{x_4, x_1}) \\ \dot{x}_3 = f(x_3, \overline{x_1, x_5}) \\ \dot{x}_5 = f(x_5, \overline{x_3, x_1}) \end{cases} \quad \begin{cases} \dot{x}_1 = f(x_1, \overline{x_4, x_2}) \\ \dot{x}_2 = f(x_2, \overline{x_1, x_4}) \\ \dot{x}_4 = f(x_4, \overline{x_2, x_1}) \end{cases} \quad \begin{cases} \dot{x}_1 = f(x_1, \overline{x_3, x_2}) \\ \dot{x}_2 = f(x_2, \overline{x_1, x_3}) \\ \dot{x}_3 = f(x_3, \overline{x_1, x_2}) \end{cases}. \tag{2.3}$$

◇

This is due to the fact that $\bowtie_1, \bowtie_2, \bowtie_3$ are balanced.

In general, it is shown in [39] that given a coupled cell network with a choice of total phase space and an equivalence relation $\bowtie$ on the cells, the polydiagonal subspace $\Delta_{\bowtie}$ is a synchrony subspace if and only if $\bowtie$ is balanced. That is, there is a one-to-one correspondence between balanced equivalence relations and synchrony subspaces of a coupled cell network.

Lattice of synchrony subspaces

The set of balanced equivalence relations moreover forms a complete lattice under the refinement of equivalence relations; or equivalently, the set of all synchrony subspaces (for a choice of total phase space) forms a complete lattice under the set inclusion (cf. [70], Theorem 5.7).

Definition 2.1.6. (cf. [70]) A *lattice* is a partially ordered set such that any two elements have a unique greatest lower bound or *meet*, and a unique least upper bound or *join*. A lattice is *complete* if every subset has a unique greatest lower bound or *meet*, and a unique least upper bound or *join*. ◇

The set of equivalence relations on a finite set C has a partial order

$$\bowtie_1 \preceq \bowtie_2, \quad \text{if } [c]_{\bowtie_1} \subseteq [c]_{\bowtie_2},\ \forall\, c \in C,$$

given by the refinement "$\preceq$". There are meet and join operations on the set such that it becomes a complete lattice. The meet of $\bowtie_1, \bowtie_2$ is an equivalence relation $\bowtie$ on C such that

$$c \bowtie d \quad \Longleftrightarrow \quad c \bowtie_1 d \quad \text{and} \quad c \bowtie_2 d$$

which is usually denoted by $\bowtie_1 \wedge \bowtie_2$. The join of $\bowtie_1, \bowtie_2$ is an equivalence relation $\bowtie$ on C such that

$$c \bowtie d \quad \Longleftrightarrow \quad c \bowtie_1 d \quad \text{or} \quad c \bowtie_2 d$$

which is denoted by $\bowtie_1 \vee \bowtie_2$. Thus, one can speak of the *lattice* of balanced equivalence relations.

The set of balanced equivalence relations on C is a partially ordered set with respect to "$\preceq$", which is a *subset* of the lattice of equivalence relations. But it is generally not a *sublattice*. While the join of two balanced equivalence relations is again balanced, the meet may not be balanced.

Example 2.1.7. (cf. [70]) Consider the graphs in Figure 2.5. They are three

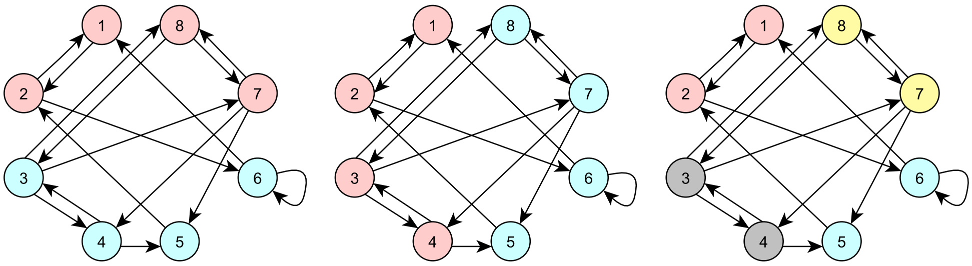


Figure 2.5: Balanced colorings.

different colorings on a regular network of 8 cells that are induced by

$$\begin{aligned}
\bowtie_1 &= \{\{1,2,7,8\},\{3,4,5,6\}\},\\
\bowtie_2 &= \{\{1,2,3,4\},\{5,6,7,8\}\},\\
\bowtie_3 &= \{\{1,2\},\{3,4\},\{5,6\},\{7,8\}\}.
\end{aligned}$$

Note that $\bowtie_3 = \bowtie_1 \wedge \bowtie_2$. The equivalence relations $\bowtie_1, \bowtie_2$ are balanced on the network. However, $\bowtie_3$ is not balanced, since the blue cell 5 receives one input from yellow and one input from gray cells, while the blue cell 6 receives one input from blue and one input from red cells. $\diamond$

Nevertheless, it is possible to define the meet for balanced equivalence relations in terms of join operation and to show that the set of balanced equivalence relations has a natural structure of a complete lattice. This structure carries over to the set of all synchrony subspaces through a order-reversing lattice isomorphism (cf. [70], Theorem 5.7).

Example 2.1.8. Let $\mathcal{G}$ be the regular network in Example 2.1.2. It can be shown that $\mathcal{G}$ admits 18 balanced equivalence relations, or equivalently, 18 synchrony subspaces (cf. [4]). See Figure 2.6, where synchrony subspaces are defined using symbols a, b, c, d, e from a phase space of choice and spaces are connected if one includes the other.

$\diamond$

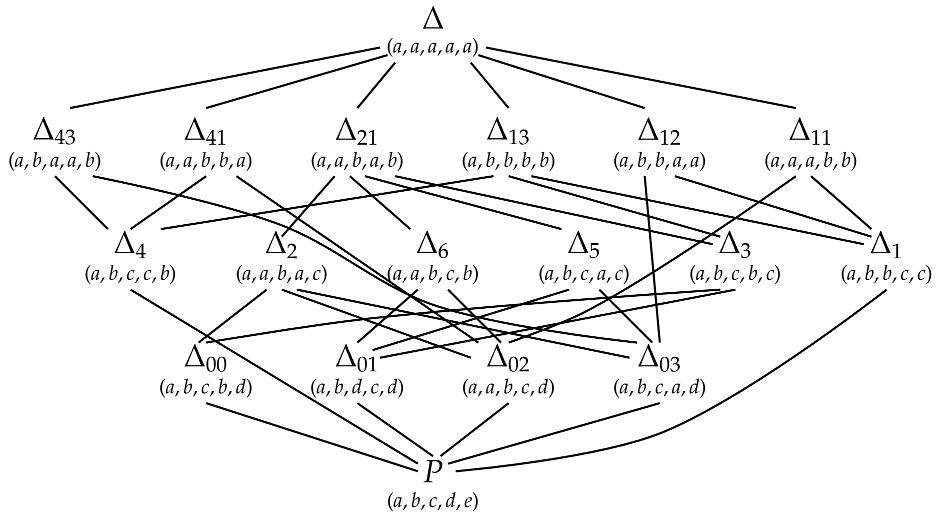


Figure 2.6: The lattice of synchrony subspaces for network (b) in Figure 2.3.

Synchrony subspaces can also be determined using *adjacency matrices*. For a coupled cell network of n cells, these are $n \times n$-matrices, one for each edge type such that the (i, j)-th entry is equal to the number of edges (of the given edge type) directing from the j-th to the i-th cell. Algorithms of finding synchrony subspaces exist in the literature such as an algorithm in [49] that uses symbolic adjacency matrices, an algorithm in [2] that uses eigenvalue structure of network adjacency matrices, or more recently, an algorithm in [55] based on the usage of special Jordan subspaces of adjacency matrices.

2.1.3 Network Structure and Hidden Symmetries

Coupled cell networks are typically non-symmetrically coupled, so symmetry is not in general, a character of networks. Yet, networks may possess various kinds of symmetry that can influence the network dynamics the same way as symmetry does to the equivariant systems. These include the *interior symmetry* and the *quotient symmetry*.

Symmetry of networks

By a *symmetry* of a coupled cell network, we mean a permutation σ on the cells such that both cell equivalence and edge equivalence are preserved. In case of homogeneous networks, this reduces to requiring

$$(c, d) \sim_E (\sigma(c), \sigma(d)), \quad \forall\, (c, d) \in \mathcal{E}.$$

Lemma 2.1.9. *A symmetry of a homogeneous network is an equivariance of the admissible system.*

Proof. Let σ be a symmetry of a homogeneous network $\mathcal{G}$ and F be an admissible vector field to the network $\mathcal{G}$. For any $\boldsymbol{x}$, we show that

$$\sigma\dot{\boldsymbol{x}} = F(\sigma\boldsymbol{x}). \tag{2.4}$$

Since F is admissible to a homogeneous network, F is generated by a single function f which takes $(v + 1)$ variables, where the first variable is reserved for the internal dynamics and the rest for the inputting cells, and the symbol v stands for the valency.

Comparing the left hand and right hand sides of (2.4) and considering their i-th components, we have

$$\mathrm{LHS}(2.4)_i = \dot{x}_{\sigma(i)} = f(x_{\sigma(i)}, x_{j_1}, \ldots, x_{j_v}),$$

where $j_1, \ldots, j_v$ are the inputs of cell $\sigma(i)$; and we have

$$\mathrm{RHS}(2.4)_i = f(x_{\sigma(i)}, x_{\sigma(i_1)}, \ldots, x_{\sigma(i_v)}),$$

where $i_1, \ldots, i_v$ are the inputs of cell i.

Since σ is a symmetry of $\mathcal{G}$, there is an edge equivalence preserving bijection between the input sets of cells i and $\sigma(i)$. Thus, $j_s = \sigma(i_s)$ for $1 \leq s \leq v$, up to a re-indexing, which concludes the statement. □

In what follows, we use the standard notion $(i_1 \ldots i_k)$ to denote the permutation that maps i_j to i_{j+1} for $1 \leq j \leq k-1$ and maps i_k to i_1, where i_j's are distinct integers.

Example 2.1.10. Consider the networks in Example 2.1.2. The homogeneous network on the left is symmetric under the permutation (3 5). The symmetry group in this case is

$$\mathbb{Z}_2 = \langle (3\ 5) \rangle.$$

The regular network on the right is symmetric under the permutations (2 4) and (3 5). Thus, the symmetry of the network is

$$\mathbb{Z}_2 \times \mathbb{Z}_2 = \langle (2\ 4) \rangle \times \langle (3\ 5) \rangle.$$

◇

Interior symmetry

The concept of interior symmetry of a coupled cell network is a generalized notion of a symmetry of a coupled cell network. Roughly speaking, it is a permutation of the cells that preserves certain amount of input structure. The notion of interior symmetry was first introduced by Golubitsky *et al.* [34]. We adapt and simplify the definition in [34] to define an interior symmetry of a homogeneous network as follows.

Definition 2.1.11. Let $\mathcal{G} = (C, \mathcal{E}, \sim_C, \sim_E)$ be a homogeneous network. Let $\mathcal{S} \subseteq C$ be a subset. An *interior symmetry* of $\mathcal{G}$ on $\mathcal{S}$ is a permutation σ on C such that σ fixes every element in $C \setminus \mathcal{S}$, and there is a bijection between edges $(\sigma(a), \sigma(b))$ and (a, b), which preserves edge-equivalence relation $\sim_E$, for $a \in \mathcal{S}$, $b \in C$. $\diamond$

Note that in the case $\mathcal{S} = C$, an interior symmetry on C is precisely a symmetry of $\mathcal{G}$. In what follows, when referring to interior symmetry, we also include the case of symmetry.

Example 2.1.12. Consider the networks in Example 2.1.2 again. Let $\mathcal{S} = \{1, 2, 4\}$. Then, both networks possess some interior symmetry. The homogeneous network has an interior symmetry of $\mathbb{Z}_3$ and the regular network has an interior symmetry of D_3. The different arrow types in the homogeneous network prevent the reflection symmetry. $\diamond$

The next proposition states that every interior symmetry permutation determines a balanced equivalence relation.

Proposition 2.1.13. *Let $\mathcal{G}$ be an n-cell homogeneous network and σ be an interior symmetry of $\mathcal{G}$ on a subset $\mathcal{S} \subseteq C$. If $\bowtie$ is an equivalence relation on the cells C of $\mathcal{G}$ such that*

$$c \bowtie d \ \Leftrightarrow\ c, d \text{ belong to the same orbit under } \sigma,$$

then $\bowtie$ is balanced.

Proof. Let c, d be such that $c \bowtie d$. Then, $\sigma^m(c) = d$ for some $m \in \mathbb{N}$. Note that σ^m is an interior symmetry of $\mathcal{G}$ on $\mathcal{S}$, for all $m \in \mathbb{N}$. Thus, by the definition of interior symmetry, there exists an edge-equivalence preserving bijection between the edges $(\sigma^m(c), \sigma^m(x))$ and (c, x), for every input arrows (c, x). Thus, there exists a bijection between the input sets of $d = \sigma^m(c)$ and c, which preserves the edge-equivalence relation. On the other hand, the tail cells x and $\sigma^m(x)$ are in the same orbit by σ, thus are in the same $\bowtie$-class. Therefore, $\bowtie$ is a balanced equivalence relation. $\square$

In fact, the set of all these equivalence relations forms a sublattice of the total lattice of balanced equivalence relations on $\mathcal{G}$ (cf. Stewart [70]).

Example 2.1.14. Let $\mathcal{G}$ be the regular network in Example 2.1.2. As shown in Example 2.1.12, $\mathcal{G}$ has an interior symmetry of D_3 on $\{1, 2, 4\}$. Consider the permutation $\sigma = (1\ 2\ 4)$ and its induced equivalence relation

$$\bowtie = \{\{1, 2, 4\}, \{3\}, \{5\}\}.$$

Then, $\bowtie$ is balanced, which is confirmed in Example 2.1.8. $\diamond$

Quotient networks and quotient symmetry

Definition 2.1.15. *(cf. [39])*Given a balanced equivalence relation $\bowtie$ on a coupled cell network $\mathcal{G}$, a *quotient network* $\mathcal{G}_{\bowtie} = (\mathcal{C}_{\bowtie}, \mathcal{E}_{\bowtie}, \sim_{C_{\bowtie}}, \sim_{E_{\bowtie}})$ can be defined naturally as follows: the cells in $\mathcal{C}_{\bowtie}$ are the $\bowtie$-equivalence classes of the cells of $\mathcal{G}$ and the edges in $\mathcal{E}_{\bowtie}$ from quotient cell $[c]_{\bowtie}$ to quotient cell $[d]_{\bowtie}$, where $[c]_{\bowtie}$ denotes the $\bowtie$-equivalence class of c, are in correspondence with the edges (c', d') of $\mathcal{G}$, for all $c' \bowtie c$, $d' \bowtie d$. The cell-equivalence $\sim_{C_{\bowtie}}$ and edge-equivalence $\sim_{E_{\bowtie}}$ relations for $\mathcal{G}_{\bowtie}$ are induced from those of $\mathcal{G}$. $\diamond$

Let $\mathcal{G}$ be a homogeneous network of n-cells with s edge-equivalence classes whose adjacency matrices are $A_1, A_2, \ldots, A_s$. Let $\bowtie$ be a balanced equivalence relation, which divides the cells of $\mathcal{G}$ into p equivalence-classes. Then, $\mathcal{G}_{\bowtie}$ is a homogeneous network of p-cells with s edge-equivalence classes. Denote the adjacency matrices of $\mathcal{G}_{\bowtie}$ by $A_{1_{\bowtie}}, A_{2_{\bowtie}}, \ldots, A_{s_{\bowtie}}$. Let $A_{l_{\bowtie}} = [\bar{a}^{(l)}_{\alpha\beta}]_{p\times p}$. Then, for $\alpha = [i]_{\bowtie}$, $\beta = [j]_{\bowtie}$ in $\mathcal{C}_{\bowtie}$, we have

$$\bar{a}^{(l)}_{\alpha\beta} = \sum_{k\in[j]_{\bowtie}} a^{(l)}_{ik}. \tag{2.5}$$

Example 2.1.16. Let $\mathcal{G}$ be the regular network in Figure 2.3(b) again. Consider the balanced equivalence relations $\bowtie_2, \bowtie_3$ in Example 2.1.4. The quotient networks are shown in Figure 2.7, which correspond precisely to the middle and right coupled cell systems in (2.3). Note that they both have a non-trivial symmetry. $\diamond$

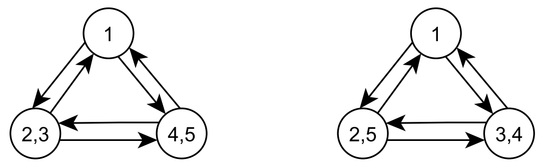


Figure 2.7: Quotient networks for $\mathcal{G}$ in Figure 2.3(b) given by $\bowtie_1 = \{\{1\}, \{2, 3\}, \{4, 5\}\}$ (left) and $\bowtie_4 = \{\{1\}, \{2, 5\}, \{3, 4\}\}$ (right).

Similar for networks, one can also define symmetry and interior symmetry for quotient networks.

Definition 2.1.17. Let $\mathcal{G}$ be a coupled cell network. We say that a permutation σ is a *quotient (interior) symmetry* of $\mathcal{G}$, if $\mathcal{G}$ has a quotient network $\mathcal{G}_{\bowtie}$ which has σ as an (interior) symmetry, for some balanced equivalence relation $\bowtie$. $\diamond$

Example 2.1.18. Continuing from Example 2.1.16, we see that both quotient networks $\mathcal{G}_{\bowtie_1}$ and $\mathcal{G}_{\bowtie_4}$ have symmetry D_3, thus $\mathcal{G}$ has a quotient symmetry D_3.

Moreover, $\mathcal{G}$ has a quotient interior symmetry. It is related to the quotient network induced by $\bowtie = \{\{1\}, \{2\}, \{3,5\}, \{4\}\}$ (cf. Figure 2.8). The quotient

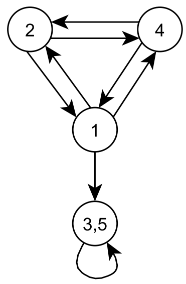


Figure 2.8: A quotient network for $\mathcal{G}$ in Figure 2.3(b) given by $\bowtie = \{\{1\}, \{2\}, \{3,5\}, \{4\}\}$, which has an interior symmetry D_3.

network possesses an interior symmetry on the subset $\{1, 2, 4\}$. Thus, $\mathcal{G}$ has a quotient interior symmetry D_3.

$\square$

2.2 Groups and Group Representations

Basic concepts from theory of transformation groups and theory of group representations will be reviewed. They provide a foundation for understanding compact Lie groups and their representations, which we use later in our theory.

2.2.1 Groups and Actions

Groups

A *group* G is a set of elements together with a binary operation "$\cdot$" such that the following axioms are fulfilled:

- CLOSURE If $a, b \in G$, then $a \cdot b \in G$
- ASSOCIATIVITY $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ for all $a, b, c \in G$
- IDENTITY An element $e \in G$ exists such that $e \cdot a = a \cdot e = e$ for all $a \in G$
- INVERSE There exists $b \in G$ for each $a \in G$ such that $a \cdot b = b \cdot a = e$

If $a \cdot b = b \cdot a$ for all $a, b \in G$, then G is called an *abelian* group.

A *subgroup* $H \subset G$ is a non-empty subset that is closed under the binary and the inverse operations. Two subgroups H, H' of G are *conjugate* in G, if $H' = gHg^{-1}$ for some $g \in G$. The *conjugacy class*

$$(H) := \{gHg^{-1} \ : \ g \in G\}$$

of a subgroup H is the set of all its conjugate subgroups. The *(left) co-set* of H is defined by

$$G/H := \{gH \,:\, g \in G\}.$$

There is a canonical identification between (H) and G/H.

Groups can be used to describe symmetry of objects. Indeed, the set of all transformations under which an object is left invariant forms a group with the binary operation given by the composition of transformations. Groups in this case are also referred as *symmetry groups*.

Example 2.2.1. Consider an equilateral triangle in $\mathbb{R}^2$ and all the transformations that leave it invariant (cf. Figure 2.9). These are rotations of 0°,

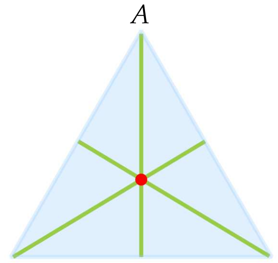


Figure 2.9: An equilateral triangle and its transformation symmetry.

120°, 240° around the center point and reflections with respect to the three hight lines. See Figure 2.9, where the red dot stands for the center and green lines are the reflection axes.

Together they form a group. Let r_0, r_1, r_2 denote the rotations of 0°, 120°, 240° respectively; and k_A, k_B, k_C denote the reflection with respect to the hight lines going through A, B, C respectively. Then,

$$G = \{r_0, r_1, r_2, k_A, k_B, k_C\}$$

gives the set of all transformations that keep the triangle invariant. It forms a group with r_0 being the identity element e. The elements r_1, r_2 are of order* 3 being inverse to each other. The elements k_A, k_B, k_C are of order 2 being self-inverse.

Moreover, every element of G can be written as a product of r_1 and k_A. For example, the reflection k_B can be written as the product $k_A \cdot r_1$ of first rotating 120° around the center and then reflecting around the hight line through A (cf. Figure 2.10). Similarly, one shows that $k_C = k_A \cdot r_1^2$. We say

*An element $a \in G$ is of *order* m if m is a smallest positive integer such that $a^m = e$.

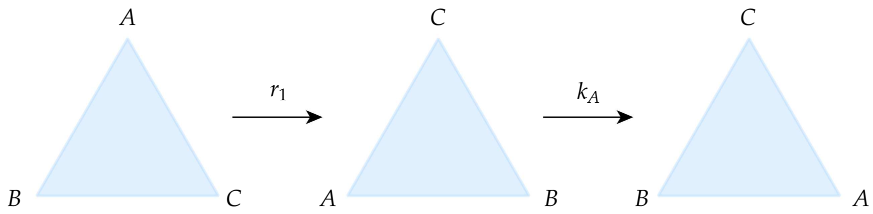


Figure 2.10: Illustration for showing $k_B = k_A \cdot r_1$.

that G is a group *generated* by r_1 and k_A, denoted as

$$G = \langle r_1, k_A \rangle,$$

where $r_1 \cdot k_A = k_A \cdot r_1^{-1}$.

◇

The group G in Example 2.2.1 is known as the *dihedral group* of order 6, which is commonly denoted by D_3. More generally, the dihedral group D_n which is defined as the group of order $2n$ generated by an element r of order n and an element k of order 2 such that $rk = kr^{-1}$, describes the transformation symmetry of n-sided regular polygons in $\mathbb{R}^2$.

Group Actions

Formally, groups can be used to describe symmetry of mathematical objects through *group actions*. By *an action* of a group G on an object X, we mean a homomorphism

$$\varphi : G \to \operatorname{Aut}(X)$$

from the group G into the group of automorphisms of X. For simplicity, we write gx instead of $(\varphi(g))(x)$ for the result of applying g on x, for $g \in G$, $x \in X$.

A subset $Y \subset X$ is called G-*invariant* if $gx \in Y$ for all $x \in Y$ and $g \in G$. An action on an invariant subset $Y \subset X$ is called *free* if $gx = x$ for some $x \in Y$ implies $g = e$ is the identity element.

Depending on structure of X, one can endow the group G with additional structure and set further restrictions on φ accordingly. For example, one can use *topological groups* to describe symmetry of topological spaces through *continuous* actions; or *Lie groups* to describe symmetry of differentiable manifolds through *differentiable* actions; or one can use groups to describe symmetry of vector spaces through *linear* actions. We give precise definitions.

Definition 2.2.2. (cf. [22]) A *topological group* G is a group and a topological space such that the binary operation and the inverse operation on G are continuous functions. An *action* of a topological group G on a topological space X is a continuous group homomorphism $\varphi : G \to \mathrm{Aut}(X)$, in which case X is called a *G-space*. $\diamond$

Example 2.2.3. Let $\mathbb{Q} \subset \mathbb{R}$ be the set of rational numbers on the real line. It forms a group under the addition operation with 0 being the identity element (cf. Figure 2.11). Consider the inherited topology from $\mathbb{R}$ on $\mathbb{Q}$,

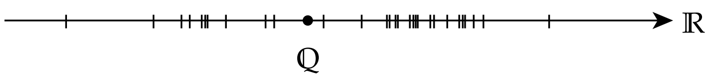


Figure 2.11: The topological group $\mathbb{Q} \subset \mathbb{R}$ under addition.

where a subset of $\mathbb{Q}$ is open if and only if it can be written as an intersection of an open subset of $\mathbb{R}$ with $\mathbb{Q}$. Both addition and inverse operations are continuous functions under this topology, since they are continuous on $\mathbb{R}$. Thus, $\mathbb{Q}$ endowed with the inherited topology is a topological group. $\diamond$

Definition 2.2.4. (cf. [22, 50]) A *Lie group* G is a group and a differentiable manifold such that the binary operation and the inverse operation on G are differentiable maps. An *action* of a Lie group G on a differentiable manifold M is a differentiable group homomorphism $\varphi : G \to \mathrm{Aut}(M)$, in which case M is called a *G-manifold*. $\diamond$

Lie groups are automatically topological groups using the same topology. An example of topological group that is not a Lie group is given by Example 2.2.3, as $\mathbb{Q}$ fails to be a manifold under the inherited topology.

Important examples of Lie groups that we use later are finite groups Γ, the circle group S^1 of complex number of unit length and the product group $\Gamma \times S^1$. See Section 2.3.3.

In case the differentiable manifold on which a Lie group G acts is indeed a finite-dimensional vector space, then the group of automorphisms coincides with the general linear group of the vector space. The G-manifold in this case becomes a *representation*.

Definition 2.2.5. Let G be a Lie group and V be a finite-dimensional vector space. An *action* of G on V is a differentiable group homomorphism

$$\varphi : G \to \mathrm{GL}(V)$$

from the group G to the general linear group $\mathrm{GL}(V)$ of V. The vector space V together with the action is called a *representation* of G.

Example 2.2.6. Let D_3 be the dihedral group discussed in Example 2.2.1. It is a Lie group under the discrete topology. Let it act on $\mathbb{R}^3$ by permuting the vector components according to

$$\begin{aligned} r_1: &\quad (x_1, x_2, x_3) \mapsto (x_3, x_1, x_2) \\ k_A: &\quad (x_1, x_2, x_3) \mapsto (x_1, x_3, x_2) \end{aligned}$$

(cf. Figure 2.12). Then, $\mathbb{R}^3$ becomes a representation of D_3. This action can

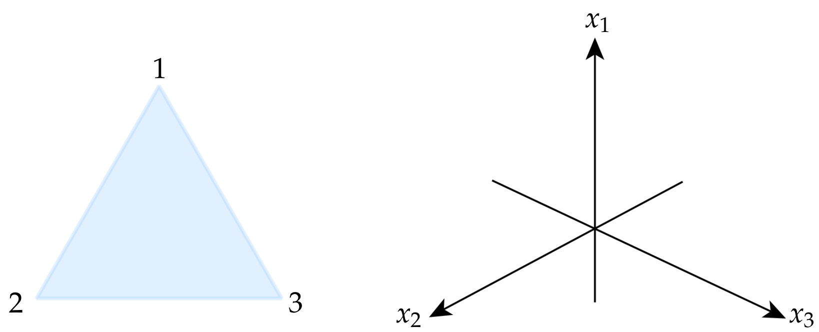


Figure 2.12: A representation of D_3 in $\mathbb{R}^3$.

be extended to P^3 for any vector space P. ◇

Further representations of D_3 can be found in Subsection 2.2.3.

2.2.2 Orbits and Orbit Structure

Let G be a topological group and X be a G-space.

Consider the set

$$G(x) = \{gx : g \in G\} \subset X$$

of all elements in X that can be reached by applying a group element to x and call it the *orbit* of x. An orbit of an element plays a similar role in a G-space, as an element in a topological space (without group actions). It represents an elementary unit of the space that is to be treated as a whole.

One sees that orbits of different elements $x, y \in X$ are either disjoint or identical. That is, X is a disjoint union of orbits. Moreover, $G(x) = G(y)$ if and only if $x = gy$ for some $g \in G$. If we define an equivalence relation $\sim$ on X by: $x \sim y$ if and only if $G(x) = G(y)$, then the quotient space X/G together with the quotient topology is called the *orbit space*.

Elements from the same orbit share the same symmetric property in space. If we use the *isotropy subgroup* of x defined by

$$G_x = \{g \in G : gx = x\}$$

to measure how symmetric the element x is situated in space, then elements from the same orbit have conjugate symmetries:

$$G_{gx} = gG_x g^{-1}, \tag{2.6}$$

for $g \in G$. Thus, one speaks of the *orbit type* of an orbit given by

$$(G_x) := \{gG_x g^{-1} \, : \, g \in G\} \tag{2.7}$$

which can be used to measure the symmetry of the orbit of x. Note that the obit type is independent of the choice of x and indeed is a property of orbit.

Elements having isotropy subgroups at least H form a closed subspace:

$$X^H = \{x \in X \, : \, hx = x, \, \forall \, h \in H\} = \operatorname{Fix}(H), \tag{2.8}$$

for any subgroup $H \subset G$, which is called the *fixed-point subspace* of H. Fixed-point subspaces of conjugate subgroups are isomorphic, since

$$gX^H = X^{gHg^{-1}}, \quad \forall \, g \in G. \tag{2.9}$$

In particular, (2.9) implies that fixed-point subspaces are generally not G-invariant. But if $g \in G$ is such that $gHg^{-1} = H$, then $X^H = gX^H$. Thus, X^H is invariant under the *normalizer* $N(H) := \{g \in G \, : \, gHg^{-1} = H\}$ of H. This action is redundant, since H acts trivially on X^H. Denote by

$$W(H) = N(H)/H = \{g \in G \, : \, gHg^{-1} = H\}/H, \tag{2.10}$$

which is called *Weyl group* of H.

Example 2.2.7. Let D_3 act on $\mathbb{R}^3$ as in Example 2.2.6. Consider the three points from the three axes: $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$ (cf. Figure 2.13). They form an orbit together. The isotropy subgroups of the three points are

$$\begin{aligned} D_{3\,(1,0,0)} &= \{1, k_A\} := D_1 \\ D_{3\,(0,1,0)} &= \{1, k_A r_1\} := D_1' \\ D_{3\,(0,0,1)} &= \{1, k_A r_1^2\} := D_1'' \end{aligned}$$

which are conjugate to each other through

$$D_1' = r_1 D_1 r_1^{-1}, \quad D_1'' = r_1^2 D_1 r_1^{-2}.$$

Thus, the orbit type of the orbit $\{(1,0,0),(0,1,0),(0,0,1)\}$ is (D_1).

Denote by $X = \mathbb{R}^3$. Then, the fixed-point subspaces of D_1, D_1', D_1'' are

$$\begin{aligned} X^{D_1} &= \{(a,b,b) \, : \, a,b \in \mathbb{R}\} \\ X^{D_1'} &= \{(b,a,b) \, : \, a,b \in \mathbb{R}\} \\ X^{D_1''} &= \{(b,b,a) \, : \, a,b \in \mathbb{R}\}, \end{aligned}$$

which are isomorphic closed subspaces in $\mathbb{R}^3$. ◇

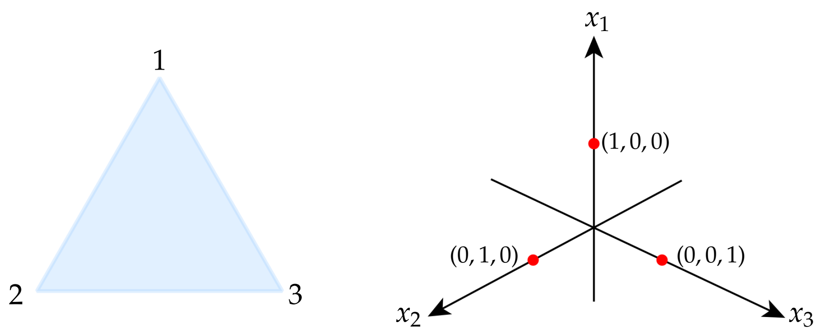


Figure 2.13: An orbit of three points under the D_3-action in $\mathbb{R}^3$.

Observe that in Example 2.2.7, the orbit of $(1,0,0)$ consists of three elements, while the left co-set D_3/D_1 of the isotropy of $(1,0,0)$ as well consists of three elements: $D_1, r_1D_1, r_1^2D_1$. This one-to-one correspondence holds in general. Indeed, one can show directly that

$$G/G_x \simeq G(x)$$

are isomorphic as sets (without topology), using the map

$$\begin{aligned} f : G/G_x &\to G(x) \\ gG_x &\mapsto gx \end{aligned} \tag{2.11}$$

However, f may not be a homeomorphism, with respect to the quotient topology on G/G_x and the inherited topology on $G(x)$ from X.

Example 2.2.8. (cf. [50]) Let S^1 be the set of complex number of unit length (with inherited topology from $\mathbb{R}^2$) and T^2 be the torus $S^1 \times S^1$. Given an irrational number a, define

$$\begin{array}{rccc} \varphi : & \mathbb{R} \times T^2 & \to & T^2 \\ & (r,(z_1,z_2)) & \mapsto & (z_1 \exp(2\pi i r), z_2 \exp(2\pi i a r)), \end{array}$$

for $r \in \mathbb{R}$ and $z_1, z_2 \in S^1$. Then, φ gives a continuous action of the topological group $\mathbb{R}$ (with the standard topology) on T^2. The isotropy of $(1,1) \in T^2$ is $\mathbb{Z}_1$ so that $\mathbb{R}/\mathbb{R}_{(1,1)} = \mathbb{R}/\mathbb{Z}_1 = \mathbb{R}$, which is a locally connected set. But the orbit $\mathbb{R}((1,1))$ of $(1,1)$ is a dense curve that goes by any small neighborhoods of $(1,1)$ infinitely many times, which is not a locally connected set. Thus, $\mathbb{R}/\mathbb{R}_{(1,1)}$ and $\mathbb{R}((1,1))$ are not homeomorphic. $\diamond$

Example 2.2.8 is due to the lack of compactness of the group $\mathbb{R}$.

Theorem 2.2.9. *(cf. [50], Proposition 1.53) If G is a compact topological group and X is a Hausdorff G-space, then f in (2.11) is a G-homeomorphism.*

A parallel statement holds for compact Lie groups and G-manifolds.

Theorem 2.2.10. *(cf. [50], Corollary 4.4) If G is a compact Lie group and M is a G-manifold, then the orbit $G(x)$ of every point x of M is a G-invariant submanifold of M and f in (2.11) is a G-diffeomorphism.*

The compactness assumption is also important in Theorem 2.2.10, which can be shown by Example 2.2.8. The group $\mathbb{R}$ is in fact a Lie group and T^2 is a differentiable manifold on which $\mathbb{R}$ acts smoothly through φ. The orbit $\mathbb{R}((1,1))$ of $(1,1)$ is clearly not a submanifold of T^2 since any small neighborhood of $(1,1)$ looks locally like a union of infinitely many disjoint copies of $\mathbb{R}$.

Further structure of orbits can be described for compact Lie groups. Let $H \subset G$ be a closed subgroup and M be a G-manifold. Denote by

$$M_{(H)} = \{x \in M \,:\, (G_x) = (H)\} \tag{2.12}$$

$$M_H = \{x \in M \,:\, G_x = H\} \subset M^H, . \tag{2.13}$$

Then, they are both submanifolds of M.

Theorem 2.2.11. *(cf. [50], Theorem 4.19) Let G be a compact Lie group and M be a G-manifold. For every $H \subset G$, let $W(H)$ be the Weyl group of H (cf. (2.10)). Then, we have*

(i) $M_{(H)}$ is a G-invariant submanifold of M;

(ii) M_H is a $W(H)$-invariant submanifold of M, and the action is free.

2.2.3 Representations of Compact Lie Groups

Representations

Let G be a Lie group and V be a finite-dimensional vector space. Recall that an *action* of G on V is a differentiable group homomorphism

$$\varphi : G \to \mathrm{GL}\,(V)$$

from the group G to the general linear group $\mathrm{GL}\,(V)$ of V. The vector space V together with the action is called a *representation* of G (cf. Definition 2.2.5).

If V is a real vector space and the action is $\varphi : G \to \mathrm{GL}_{\mathbb{R}}(V)$, then we call V a *real representation* of G. Similarly, if V is a complex vector space with an action $\varphi : G \to \mathrm{GL}_{\mathbb{C}}(V)$, then we call V a *complex representation* of G.

A linear map $A : V_1 \to V_2$ between two representations of G is called *G-equivariant*, if $A(gv) = g(Av)$ for all $v \in V_1$ and $g \in G$. Two representations V_1, V_2 of G are called *equivalent*, if there exists a G-equivariant linear isomorphism between them.

A representation V of G is called *irreducible* if $\{0\}$ and V are the only G-invariant linear subspaces in V. A representation that is not irreducible is called *reducible*. Further, a representation is called *completely reducible*, if it is equivalent to a direct sum of irreducible representations.

Example 2.2.12. Let $G = \{\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} : n \in \mathbb{Z}\}$ which is a group under matrix multiplication and forms a topological (and also a Lie) group with respect to the standard topology on matrices. Let it act on $\mathbb{C}^2$ by the usual matrix multiplication:

$$\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} z_1 + nz_2 \\ z_2 \end{pmatrix},$$

for $z_1, z_2 \in \mathbb{C}$. It gives a continuous (and also differentiable) action of G on $\mathbb{C}^2$ and $\mathbb{C}^2$ is a complex representation of G.

The complex representation $\mathbb{C}^2$ is reducible, since $\mathbb{C} := \{(z_1, 0) : z_1 \in \mathbb{C}\}$ is a G-invariant linear subspace of $\mathbb{C}^2$. It is however, not completely reducible, since $\mathbb{C}$ is the only G-invariant linear subspace in $\mathbb{C}^2$ besides $\{0\}$ and $\mathbb{C}^2$. $\diamond$

Example 2.2.12 is due to the lack of compactness of the group G.

Theorem 2.2.13. *(cf. Proposition (1.9) in [20] or Corollary 2.42 in [50]) Every finite-dimensional representation of a compact topological group is completely reducible.*

Example 2.2.14. Let D_3 act on $\mathbb{R}^3$ as in Example 2.2.6. Then,

$$\begin{aligned} U &:= \{(x, y, z) : x = y = z \in \mathbb{R}\} \\ V &:= \{(x, y, z) : x + y + z = 0,\ x, y, z \in \mathbb{R}\} \end{aligned}$$

are both D_3-invariant linear subspaces of $\mathbb{R}^3$ and $\mathbb{R}^3 = U \oplus V$ as space. The group D_3 acts on U trivially, that is, every group element acts like the identity map. The group D_3 acts on V by

$$r_1 : (x, y, -x-y) \mapsto (-x-y, x, y), \quad k_A : (x, y, -x-y) \mapsto (x, -x-y, y).$$

It is convenient to identify V with

$$V' := \{(x, y) : x, y \in \mathbb{R}\}$$

using the projection $(x, y, -x-y) \mapsto (x, y)$ to the first two coordinates. The D_3-action on V can then be expressed as a D_3-action on V' by

$$r_1 = \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}, \quad k_A = \begin{pmatrix} 1 & 0 \\ -1 & -1 \end{pmatrix}.$$

It follows that V is an irreducible representation of D_3 and $\mathbb{R}^3 = U \oplus V$ as G-representation is a direct sum of irreducible representations of D_3.

$\diamond$

It is interesting to list all irreducible representations of D_3.

Example 2.2.15. (cf. [16]) It can be shown, using the character theory of group representations for example, that there are exactly three distinct irreducible real representations of the dihedral group D_3:

(i) The trivial representation $\mathcal{V}_0 \simeq \mathbb{R}$, where every group element acts as the identity map on $\mathbb{R}$.

(ii) The representation $\mathcal{V}_1 \simeq \mathbb{R}$ given by r_1 as the identity map and k_A as the antipodal map on $\mathbb{R}$, i.e. $k_A x = -x$.

(iii) The natural representation $\mathcal{V}_2 \simeq \mathbb{R}^2$ given by r_1 as a rotation of 120° of the plane and k_A as the reflection with respect to the first axis of the plane.

There are also exactly three distinct irreducible complex representations of the dihedral group D_3:

(i) The trivial representation $\mathcal{U}_0 \simeq \mathbb{C}$, where every group element acts as the identity map on $\mathbb{C}$.

(ii) The representation $\mathcal{U}_1 \simeq \mathbb{R}$ given by r_1 as the identity map and k_A as the antipodal map on $\mathbb{C}$.

(iii) The natural representation $\mathcal{U}_2 \simeq \mathbb{C}^2$ given by r_1 as the map $(z_1, z_2) \mapsto (r_1 z_1, r_1^{-1} z_2)$ and k_A as the interchange of coordinates: $(z_1, z_2) \mapsto (z_2, z_1)$.

◇

The representation $\mathbb{R}^3$ from Example 2.2.14 is equivalent to $\mathcal{V}_0 \oplus \mathcal{V}_1$.

Banach representations

Definition 2.2.16. Let G be a Lie group and W be a Banach space. An *action* of G on W is a differentiable group homomorphism

$$\varphi : G \to \mathrm{GL}_B(W),$$

from the group G to the group of invertible bounded linear operators on W. The Banach space together with this action is called a *Banach representation* of G. A Banach representation W is called *isometric*, if $\|gw\| = \|w\|$, for all $g \in G, w \in W$. ◇

Using the Haar measure, one can show that every Banach representation of a compact Lie group is equivalent to an isometric Banach representation.

Example 2.2.17. Let $n \in \mathbb{N}$ be a positive integer and $C([0,T];\mathbb{R}^n)$ be the set of all continuous T-periodic functions defined on $[0,T]$ and valued in $\mathbb{R}^n$. Then, $C([0,T];\mathbb{R}^n)$ is a vector space over reals. Moreover, it is a real Banach space with respect to the supremum norm $\|\cdot\|$, which is defined by

$$\|f\| := \sup_{x\in[0,T]} |f(x)|, \quad \forall\ f \in C([0,T];\mathbb{R}^n).$$

Let $G = S^1$ and define an action of S^1 on $C([0,T];\mathbb{R}^n)$ by

$$(e^{i\theta} f)(t) := f(t + \frac{\theta T}{2\pi}), \quad \forall\ f \in C([0,T];\mathbb{R}^n),\ e^{i\theta} \in S^1,\ t \in [0,T], \tag{2.14}$$

which is clearly differentiable. Also, $\|e^{i\theta} f\| = \|f\|$ for all $f \in C([0,T];\mathbb{R}^n)$. Thus, $C([0,T];\mathbb{R}^n)$ is a real isometric Banach representation of S^1 with respect to the action (2.14). ◇

2.3 Euler Ring and Ring Homomorphisms

Let G be a compact Lie group. In Subsection 2.2.2, we have seen that the conjugacy classes of subgroups can be used to measure symmetry of orbits in G-spaces (cf. (2.7)). Later, we will use their finite sums to define degrees for equivariant maps. It is interesting to explore the ring structure of the set of all finite sums of conjugacy classes of subgroups, known as the *Euler ring*, which will be very useful in computations of degrees.

In what follows, we consider only *closed* subgroups of G.

2.3.1 Euler Ring

Let $\Phi(G)$ be the set of all conjugacy classes of subgroups of G. Consider the set $A(G)$ of all finite sums of elements of $\Phi(G)$, that is, the set of elements that look like

$$(H_1) + (H_2) + \cdots + (H_m)$$

for $(H_i) \in \Phi(G)$. If $(H_i) = (H_j)$, then we will group them together and write

$$(H_i) + (H_j) = 2(H_i).$$

The set $A(G)$ forms an abelian group under the addition "+" given by

$$\sum_{i=1}^{N} n_i(H_i) + \sum_{i=1}^{N} m_i(H_i) = \sum_{i=1}^{N} (n_i + m_i)(H_i) \tag{2.15}$$

for $(H_i) \in \Phi(G)$, with the identity element given by $\sum_{i=1}^{N} n_i(H_i)$ for all $n_i = 0$.

Technically speaking, $A(G)$ is a free (left) $\mathbb{Z}$-module generated by $\Phi(G)$.

Moreover, one can define a multiplication operation

$$* : A(G) \times A(G) \to A(G)$$

on $A(G)$, with respect to which the abelian group $A(G)$ becomes a ring.

Let $(H) \in \Phi(G)$. The left co-set G/H is a G-manifold under the action

$$G \times G/H \to G/H, \quad (g', gH) \mapsto (g'g)H. \tag{2.16}$$

The isotropy subgroup of gH is then given by gHg^{-1}. This coincides with the conjugate symmetry (2.6) of elements from the same orbit that we observe earlier. In fact, (2.16) resembles the G-action on the G-invariant submanifold $G(x)$ under the identification (2.11).

Now consider $(H), (K) \in \Phi(G)$ and the product space $G/H \times G/K$ (which is in resemblance of product of orbits) under the same action

$$\begin{aligned} G \times \Big(G/H \times G/K\Big) &\to G/H \times G/K, \\ \Big(g', (g_1H, g_2K)\Big) &\mapsto \Big((g'g_1)H, (g'g_2K)\Big). \end{aligned} \tag{2.17}$$

We are interested in classifying orbit types in $G/H \times G/K$ under (2.17). Note that the isotropy subgroup of $(g_1H, g_2K) \in G/H \times G/K$ is

$$L := g_1Hg_1^{-1} \cap g_2Kg_2^{-1}.$$

The set of all elements of orbit type (L) forms a G-invariant submanifold of $G/H \times G/K$ (cf. Theorem 2.2.11). The product

$$(H) * (K) = \sum n_L(L) \tag{2.18}$$

expresses a topological count of orbit types in $G/H \times G/K$, where n_L is the count associated with the orbit type (L). The sum in (2.18) is finite, since $G/H \times G/K$ is a *compact* G-manifold.

Formally, we can define the *Euler ring* as follows.

Definition 2.3.1. (cf. [22]) Let G be a compact Lie group and $\Phi(G)$ be the set of conjugacy classes of closed subgroups of G. Let $A(G)$ be the set of all finite sums of elements of $\Phi(G)$ equipped with "+" by (2.15) and "$*$" by (2.18), where

$$n_L := \chi_c((G/H \times G/K)_{(L)}/G), \tag{2.19}$$

is defined by the Euler characteristics χ_c of the orbit space of the G-invariant submanifold $(G/H \times G/K)_{(L)}$ in $G/H \times G/K$ (cf. [69]).

◇

We explain the geometric meaning of (2.19) using an example.

Example 2.3.2. Consider the dihedral group D_3 in Example 2.2.1, which is generated by the rotation r_1 and the reflection k_A. The group D_3 has the following subgroups

$$\begin{aligned}&D_1 = \{1, k_A\}, \quad D_1' = \{1, k_B\}, \quad D_1'' = \{1, k_C\}\\ &\mathbb{Z}_3 = \{1, r_1, r_2\}, \quad \mathbb{Z}_1 = \{1\}\end{aligned}$$

besides D_3 itself. Since D_1, D_1', D_1'' are conjugate in D_3, we have

$$\Phi(D_3) = \{(D_3), (D_1), (\mathbb{Z}_3), (\mathbb{Z}_1)\}$$

consists of four elements. The Euler ring multiplication is listed in Table 2.1 (cf. [16]).

$*$	(D_3)	(D_1)	$(\mathbb{Z}_3)$	$(\mathbb{Z}_1)$
(D_3)	(D_3)	(D_1)	$(\mathbb{Z}_3)$	$(\mathbb{Z}_1)$
(D_1)	(D_1)	$(D_1) + (\mathbb{Z}_1)$	$(\mathbb{Z}_1)$	$3(\mathbb{Z}_1)$
$(\mathbb{Z}_3)$	$(\mathbb{Z}_3)$	$(\mathbb{Z}_1)$	$2(\mathbb{Z}_3)$	$2(\mathbb{Z}_1)$
$(\mathbb{Z}_1)$	$(\mathbb{Z}_1)$	$3(\mathbb{Z}_1)$	$2(\mathbb{Z}_1)$	$6(\mathbb{Z}_1)$

Table 2.1: Multiplication table for the Euler ring $A(D_3)$

The geometric meaning of $(H) * (K)$ is that it counts the G-orbits in the product G-space $G/H \times G/K$ according to their orbit types. For example, consider

$$(D_1) * (D_1) = (D_1) + (\mathbb{Z}_1).$$

Note that $D_3/D_1 = \{D_1, r_1D_1, r_1^2D_1\}$. Thus, the product space $D_3/D_1 \times D_3/D_1$ consists of 9 elements, which are represented by hollow squares and triangles in Figure 2.14. The isotropy of the element $(r_1^aD_1, r_1^bD_1) \in D_3/D_1 \times D_3/D_1$ is given by $r_1^aD_1r_1^{-a} \cap r_1^bD_1r_1^{-b}$, for $a, b \in \{0, 1, 2\}$, as indicated in Figure 2.14. These isotropies give rise to two orbit types in $D_3/D_1 \times D_3/D_1$:

$$\begin{aligned}&(D_1) = \{D_1, r_1D_1r_1^{-1}, r_1^2D_1r_1^{-2}\},\\ &(\mathbb{Z}_1) = \{\mathbb{Z}_1\},\end{aligned}$$

corresponding to the hollow triangles and squares in Figure 2.14, respectively. Moreover, all the hollow triangles (resp. all the hollow squares) consist of 1 orbit under the D_3-action on $D_3/D_1 \times D_3/D_1$. Therefore, $D_3/D_1 \times D_3/D_1$ consists of 1 orbit of orbit type (D_1) and 1 orbit of orbit type $(\mathbb{Z}_1)$, or equivalently written as

$$(D_1) * (D_1) = (D_1) + (\mathbb{Z}_1).$$

◇

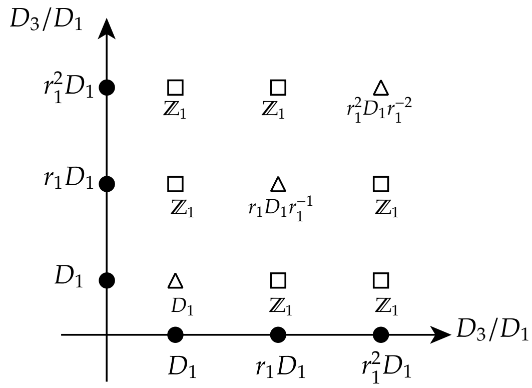


Figure 2.14: Geometric meaning of $(D_1) * (D_1)$ in the Euler ring $A(D_3)$.

2.3.2 Ring Homomorphisms

The Euler ring defined in Subsection 2.3.1 provides a natural environment for multiplying group orbits. In this subsection, we discuss how the Euler ring structure of compact Lie groups behaves with respect to group homomorphisms. The resulting ring homomorphism will play an important role in Section 4, when we deal with different (quotient) symmetries of a network at the same time.

Definition 2.3.3. Let G_1, G_2 be compact Lie groups and $\mathsf{h} : G_2 \to G_1$ be a group homomorphism. Let X be a G_1-space. Define a G_2-action on X by

$$g_2 x := \mathsf{h}(g_2)x, \quad \text{for } g_2 \in G_2,\ x \in X, \tag{2.20}$$

and call it the *induced action* of G_2 on X through h. Then, X is also a G_2-space.

Let $K \subset G_1$ be a closed subgroup. Then, the left co-set G_1/K is a G_1-space under the natural action (2.16). Consider the induced action of G_2 on G_1/K through h. Then, the isotropy of g_1K under this G_2-action is given by

$$\tilde{K} = \mathsf{h}^{-1}(g_1 K g_1^{-1}).$$

The set of all elements of orbit type $(\tilde{K})$ forms a G_2-invariant submanifold of the G_2-space G_1/K by Theorem 2.2.11. Based on the topological nature of this submanifold $(G_1/K)_{(\tilde{K})}$, one can define a map

$$\begin{aligned} \mathsf{H} : A(G_1) &\to A(G_2) \\ (K) &\mapsto \sum n_{\tilde{K}} (\tilde{K}), \end{aligned} \tag{2.21}$$

between the Euler rings $A(G_1)$ and $A(G_2)$, where

$$n_{\tilde{K}} = \chi_c((G_1/K)_{(\tilde{K})}/G_2), \tag{2.22}$$

is defined by the Euler characteristics of the orbit space of $(G_1/K)_{(\tilde{K})}$ and expresses the topological count of orbits of orbit type $(\tilde{K})$.

In a sense, the map H enables us to view G_1-orbits as G_2-orbits under the induced action (2.20). It "lifts" a G_1-orbit of orbit type (K) to several G_2-orbits of orbit type $(\tilde{K})$ for $\tilde{K} = \mathsf{h}^{-1}(g_1 K g_1^{-1})$, $g_1 \in G_1$.

One can show that the map H defined by (2.21)-(2.22) is indeed a ring homomorphism, which we call the *Euler ring homomorphism induced by* h.

Theorem 2.3.4. *(cf. [22, 15]) Let G_i be a compact Lie group for $i = 1, 2, 3$ and $\mathsf{h}_i : G_i \to G_{i+1}$ a group homomorphism for $i = 1, 2$. Let H_i be defined by (2.21) for $i = 1, 2$. Then, we have*

(i) H_i *is an Euler ring homomorphism, for $i = 1, 2$.*

(ii) $\mathsf{H}_2 \circ \mathsf{H}_1$ *is precisely the Euler ring homomorphism induced by* $\mathsf{h}_2 \circ \mathsf{h}_1$.

We explain the geometric meaning of the ring homomorphism using an example.

Example 2.3.5. Let $G_1 = D_3, G_2 = D_1$ and $\mathsf{h} : D_1 \hookrightarrow D_3$ be the inclusion homomorphism. Let $\mathsf{H} : A(D_3) \to A(D_1)$ be the Euler ring homomorphism induced by h. Then, we have

$$\begin{aligned} \mathsf{H} : \quad & (D_3) \mapsto (D_1), && (\mathbb{Z}_3) \mapsto (\mathbb{Z}_1) \\ & (D_1) \mapsto (D_1) + (\mathbb{Z}_1), && (\mathbb{Z}_1) \mapsto 3(\mathbb{Z}_1). \end{aligned}$$

The geometric meaning of $\mathsf{H}((K))$ is that it counts the G_2-orbits in the G_2-space G_1/K according to their orbit types. For example, consider $K = D_1$. Then, the space D_3/D_1 consists of 3 elements: D_1, r_1D_1, $r_1^2D_1$. Consider the

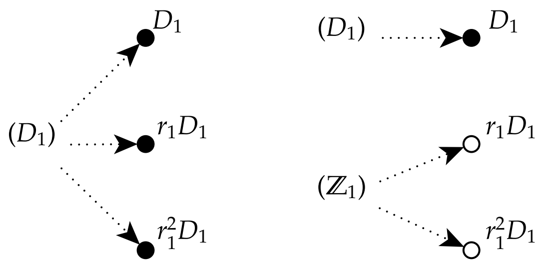


Figure 2.15: (left) The space D_3/D_1 considered as D_3-space; (right) the space D_3/D_1 considered as D_1-space.

D_1-action on D_3/D_1. Then, the isotropies are

$$\mathsf{h}^{-1}(D_1) = D_1, \quad \mathsf{h}^{-1}(r_1 D_1 r_1^{-1}) = \mathbb{Z}_1, \quad \mathsf{h}^{-1}(r_1^2 D_1 r_1^{-2}) = \mathbb{Z}_1,$$

respectively (cf. Figure 2.15). Moreover, the elements $r_1 D_1, r_1^2 D_1$ belong to the same orbit under the D_1-action, since

$$\mathsf{h}(k_A) r_1 D_1 = k_A r_1 D_1 = r_1^2 k_A D_1 = r_1^2 D_1.$$

Therefore, the space D_3/D_1 (with respect to the induced D_1-action) consists of 1 orbit of orbit type (D_1) and 1 orbit of orbit type $(\mathbb{Z}_1)$, i.e.

$$\mathsf{H}((D_1)) = (D_1) + (\mathbb{Z}_1).$$

2.3.3 Euler Ring of $\Gamma \times S^1$

Important groups that we use later are compact Lie groups of form

$$\Gamma \times S^1$$

where Γ is a finite group and S^1 is the group of complex numbers of unit length with standard topology. In the setting of (equivariant) bifurcations, the group Γ is usually used to describe the symmetry of the system in phase space while S^1 describes the temporal symmetry of possible periodic states.

Twisted subgroups of $\Gamma \times S^1$

There are two kinds of (closed) subgroups in $\Gamma \times S^1$: those that are finite and those that are of dimension one. The latter one is always of form $K \times S^1$ for subgroups $K \subset \Gamma$, while the finite ones are what we call the *twisted subgroups*. Twisted subgroups are of particular importance, since they are precisely the symmetries of *non-constant* periodic states in Hopf bifurcations.

Definition 2.3.6. *(cf. [16])*A subgroup $H \subset \Gamma \times S^1$ is called a *twisted l-folded subgroup*, if there exists a subgroup $K \subset \Gamma$, an integer $l \geq 0$ and a group homomorphism $\varphi : K \to S^1$ such that

$$H = K^{\varphi,l} := \{(\gamma, z) \, : \, \varphi(\gamma) = z^l\}.$$

◇

It can be verified that every finite subgroup $H \subset \Gamma \times S^1$ is twisted.

Example 2.3.7. Let $\Gamma = D_3$, where $D_3 = \mathbb{Z}_3 \cup \kappa\mathbb{Z}_3$ and $\mathbb{Z}_3 = \langle \xi \rangle$. Then, up to conjugacy, $D_3 \times S^1$ has the following twisted 1-folded subgroups: $\mathbb{Z}_1$, $\mathbb{Z}_3$, D_1, D_3 and (cf. [16])

$$\mathbb{Z}_3^t = \{(1,1), (\xi, \xi), (\xi^2, \xi^2)\}, \quad D_1^z = \{(1,1), (\kappa, -1)\},$$
$$D_3^z = \{(1,1), (\xi, 1), (\xi^2, 1), (\kappa, -1), (\kappa\xi, -1), (\kappa\xi^2, -1)\}.$$

Let D_3 act on $\mathbb{R}^3$ as the permutation group $S_3 \simeq D_3$. Let $C([0,T];\mathbb{R}^3)$ be given by Example 2.2.17. Define a $D_3 \times S^1$-action on $C([0,T];\mathbb{R}^3)$ by

$$\Big((\gamma, e^{i\theta})f\Big)(t) := \gamma_{\bullet} f(t + \frac{\theta T}{2\pi}),$$

where "$\bullet$" stands for the D_3-action on $\mathbb{R}^3$. Then, a function $u \in C([0,T];\mathbb{R}^3)$ has an isotropy subgroup $\mathbb{Z}_3^t$ under this action if and only if

$$(\xi,\xi)u(t) := (\xi,\xi)\begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix} = \begin{pmatrix} z(t+\frac{T}{3}) \\ x(t+\frac{T}{3}) \\ y(t+\frac{T}{3}) \end{pmatrix} = \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix}, \quad \forall\, t \in [0,T].$$

Thus, $x(t) = y(t+\frac{2T}{3}) = z(t+\frac{T}{3})$ and u is of form

$$u(t) = (x(t), x(t+\frac{T}{3}), x(t+\frac{2T}{3})), \quad \forall\, t \in [0,T].$$

That is, knowing the (twisted) isotropy subgroups of a periodic function u helps determine the form of u. More examples of this kind can be found in Table 4.1. ◇

Euler ring of $\Gamma \times S^1$

We describe the Euler ring of $\Gamma \times S^1$ for a finite group Γ. For a more detailed discussion, we refer to [67], Section 2.3.

In this case, the set $\Phi(\Gamma \times S^1)$ of all conjugacy classes of closed subgroups in $\Gamma \times S^1$, splits into two subsets

$$\Phi_0(\Gamma \times S^1) = \{(H) : H = K \times S^1, \text{ for } K \subset \Gamma\}, \tag{2.23}$$

$$\Phi_1(\Gamma \times S^1) = \{(H) : H = K^{\varphi,l}, \text{ for } K \subset \Gamma, \varphi : K \to S^1, l = 0,1,2,\dots\}, \tag{2.24}$$

Let

$$A_k(G) := \mathbb{Z}\Big[\Phi_k(G)\Big], \quad \text{for } k = 0,1, \tag{2.25}$$

be the free $\mathbb{Z}$-module generated by $\Phi_k(G)$. Then,

$$A(\Gamma \times S^1) = A_0(\Gamma \times S^1) \times A_1(\Gamma \times S^1).$$

The multiplication can be summarized using Table 2.2, where the parts **(I)** and **(II)** are related to the identification $(K \times S^1) \mapsto (K)$ from $A_0(\Gamma \times S^1)$ to the Euler ring $A(\Gamma)$ of Γ. The part **(III)** is zero (cf. [67], Proposition 2.3.3.1)

Example 2.3.8. Let $\Gamma = D_3$. Consider the Euler ring $A(D_3 \times S^1)$ of $D_3 \times S^1$. The multiplication table of the Euler ring $A(D_3 \times S^1)$ is given in Table 2.3 (cf. [64], Appendix A3.2).

◇

$*$	$A_0(\Gamma \times S^1)$	$A_1(\Gamma \times S^1)$
$A_0(\Gamma \times S^1)$	**(I):** $A(\Gamma)$-multiplication	**(II):** $A(\Gamma)$-module multiplication
$A_1(\Gamma \times S^1)$	**(II):** $A(\Gamma)$-module multiplication	**(III):** 0

Table 2.2: The Euler ring multiplication table for $G = \Gamma \times S^1$, where Γ is a finite group.

$*$	$(D_3 \times S^1)$	$(D_1 \times S^1)$	$(\mathbb{Z}_3 \times S^1)$	$(\mathbb{Z}_1 \times S^1)$
$(D_3 \times S^1)$	$(D_3 \times S^1)$	$(D_1 \times S^1)$	$(\mathbb{Z}_3 \times S^1)$	$(\mathbb{Z}_1 \times S^1)$
$(D_1 \times S^1)$	$(D_1 \times S^1)$	$(D_1 \times S^1) + (\mathbb{Z}_1 \times S^1)$	$(\mathbb{Z}_1 \times S^1)$	$3(\mathbb{Z}_1)$
$(\mathbb{Z}_3 \times S^1)$	$(\mathbb{Z}_3 \times S^1)$	$(\mathbb{Z}_1 \times S^1)$	$2(\mathbb{Z}_3 \times S^1)$	$2(\mathbb{Z}_1 \times S^1)$
$(\mathbb{Z}_1 \times S^1)$	$(\mathbb{Z}_1 \times S^1)$	$3(\mathbb{Z}_1 \times S^1)$	$2(\mathbb{Z}_1 \times S^1)$	$6(\mathbb{Z}_1 \times S^1)$
(D_3^z)	(D_1^z)	$(\mathbb{Z}_3)$	$(\mathbb{Z}_1)$	(D_3^z)
(D_1^z)	$(D_1^z) + (\mathbb{Z}_1)$	$(\mathbb{Z}_1)$	$3(\mathbb{Z}_1)$	(D_1^z)
$(\mathbb{Z}_3^t)$	$(\mathbb{Z}_1)$	$2(\mathbb{Z}_3^t)$	$2(\mathbb{Z}_1)$	$(\mathbb{Z}_3^t)$

Table 2.3: Multiplication table for the Euler ring $A(D_3 \times S^1)$, where the upper half is essentially Table 2.1.

Chapter 3

Equivariant Degree and Equivariant Bifurcations

Equivariant degree theory is a mapping degree theory for *equivariant maps*, that is, maps that commute with a group of symmetries. Zeros of equivariant maps form orbits under the group action, so one speaks from *zero orbits* of equivariant maps. Different orbits can exhibit different symmetries: some situate more symmetrically in space than others. These can be conveniently measured by the orbit type, which is defined as the conjugacy class of the isotropy of any point on the orbit. One aim of the equivariant degree theory is to provide a "count" of these zero orbits by their orbit types, using possibly integer numbers, like in the case of a usual mapping degree. However, this turns out to be not always possible, as one deals with different orbit types of potentially different dimension. Indeed, the original definition of an equivariant degree for abelian groups in [45] introduced it as an element in some group of equivariant homotopy classes of maps between spheres, which cannot generally be expressed as an integer count. Nevertheless, as shown in [29], part of the equivariant degree is "expressible" as integer count of zero orbits by their orbit types. This is the so-called *primary equivariant degree* and takes the form

$$n_1 \cdot (H_1) + n_2 \cdot (H_2) + \cdots + n_k \cdot (H_k), \quad n_i \in \mathbb{Z}.$$

In this chapter, we review two most common (primary) equivariant degrees: the *equivariant degree without parameter* and the *equivariant degree with one parameter*. They provide together, foundation to effective (topological) treatment of equivariant bifurcation problems in various context (cf. [12, 13, 14, 15, 10] for example and [16] for further references). Here, we introduce them in a slightly different fashion as in the literature (where equivariant topology, representation theory and equivariant approximations stand in the foreground) in that we define these degrees based on their geometric meanings and using directly the so-called *recurrence for-*

mulas (cf. (3.4), (3.11)). We show that degrees defined in this way satisfy the usual properties of a degree theory including the *existence*, *homotopy*, *suspension* (cf. Theorem 3.1.4, Theorem 3.1.6).

In Section 3.2, we review the standard degree-approach for treating equivariant bifurcations which reformulates the bifurcation problem into an (equivariant) fixed point problem in an appropriate functional setting, and then defines a *bifurcation invariant* for the locally bifurcating equilibium. Based on the value of the bifurcation invariant, one can make various statement about the bifurcating branches including their existence and symmetry properties. Here, we introduce the concept of *secondary dominating orbit types*, to sharpen the symmetry statement about the bifurcating solutions. The main result is stated in Proposition 3.2.2.

In Subsection 3.2.3, we explain how to use the command `showdegree` from the "Equivariant Degree Maple© Library Package" (EDML) to obtain exact values of the bifurcating invariant. As we will see, it only takes input from the spectrum of the Jacobian operator. Thus, topological existence of bifurcating branches of solutions is a linear property of the system, even though bifurcations are non-linear phenomena.

3.1 Equivariant Maps and Equivariant Degrees

Let X, Y be two isometric Banach representations of a compact Lie group G.

Definition 3.1.1. A continuous map $f : X \to Y$ is called *equivariant* if $f(g_\circ x) = g_* f(x)$, for all $x \in X$ and $g \in G$, where $\circ$ and $*$ stand for the G-actions on X and Y, respectively. A homotopy $h : [0,1] \times X \to Y$ is called *equivariant*, if $h(t, \cdot)$ is equivariant for all $t \in [0,1]$. ◇

Remark 3.1.2. Note that by equivariance, if $f(x) = 0$ for some $x \in X$, then $f(g_\circ x) = 0$ for all $g \in G$. Thus, the zero set of f is composed of (disjoint) group orbits. An equivariant degree counts zero orbits of equivariant maps in bounded domains according to their orbit types. ◇

To define an adequate equivariant degree, one needs a no-zero boundary condition on the domain, since any meaningful topological count should remain stable against homotopies including small perturbations on the map.

Definition 3.1.3. Let $\Omega \subset X$ be an open bounded invariant subset. A map $f : X \to Y$ is called *admissible* on Ω if $f(x) \neq 0$ for all $x \in \partial\Omega$. A pair (f, Ω) is called *admissible* if f is admissible on Ω. A homotopy $h : [0,1] \times X \to Y$ is called *admissible* if $h(t, \cdot)$ is admissible for all $t \in [0,1]$. ◇

The condition of being admissible is necessary for defining any degree theory, since a non-admissible map would have zeros on the boundary of Ω, which may fall inside or outside of Ω depending on the kind of perturbations involved. Then, it would be hard, if not impossible, to count zeros of f in Ω that stays the same for any perturbations.

In the next two subsections we review from [12, 16] two types of equivariant degrees defined for equivariant maps with one parameter or without parameters. They provide a basis for defining degrees for networks in later chapters. In both cases, we focus on the simplest case of group forms and discuss the computational properties of the equivariant degrees.

3.1.1 Equivariant Degree without Parameters

Let $G = \Gamma$ be a finite group and Φ collect all conjugacy classes of subgroups of Γ. In particular, all possible orbit types of zero orbits of Γ-equivariant maps are elements of Φ. Let $\geq$ be the partial order on Φ defined by

$$(K_1) \geq (K_2) \quad \Longleftrightarrow \quad \exists\, \gamma \in \Gamma\, s.t.\, K_1 \supseteq \gamma K_2 \gamma^{-1}. \tag{3.1}$$

Let X be a finite-dimensional Γ-representation and $\Omega \subset X$ be an open bounded Γ-invariant subset. Consider an equivariant map $f : \overline{\Omega} \subset X \to X$ that is admissible on Ω. For every $(K) \in \Phi$, consider the restriction of f on the fixed point set $\Omega^K = \operatorname{Fix}(K) \cap \Omega$. One is allowed to take any representative from conjugacy class (K), due to the equivariance of f. Then, the restricted map

$$f|_{\Omega^K} : \Omega^K \to X^K$$

is admissible on Ω^K, so one can define the usual Brouwer degree to the pair $(f|_{\Omega^K}, \Omega^K)$, which we will denote by $\deg(f|_{\Omega^K}, \Omega^K) \in \mathbb{Z}$. This gives a topological count of zeros of f lying in Ω^K, i.e. having isotropy *at least* K.

We use these integers $\{\deg(f|_{\Omega^K}, \Omega^K) : (K) \in \Phi\}$ to define an equivariant degree for (f, Ω), which gives an integer n_K to every orbit type $(K) \in \Phi$. We explain in details.

Since we are interested in counting zero orbits instead of zeros, we need to "quotient out" the repetition of counting zeros from the same orbit. The set Ω^K is generally not invariant under the whole group Γ-action (see dihedral example), but is invariant under the normalizer group $N(K)$-action. Indeed, if $\gamma \in N(K)$, then $\gamma K = K\gamma$, so for all $x \in \Omega^K$ and $k \in K$, we have

$$k\gamma x = \gamma \tilde{k} x = \gamma x,$$

that is, $\gamma x \in \Omega^K$. The subgroup $K \subset N(K)$ acts trivially on Ω^K, so we effectively have a Weyl group $W(K) = N(K)/K$-action on Ω^K. Even better, this action is free on the open dense subset $\Omega_K \subset \Omega^K$, which is composed of $x \in \Omega^K$ that has isotropy *precisely* K.

Therefore, if $n_K \in \mathbb{Z}$ counts the zero orbits of f of orbit type (K), then $n_K \cdot |W(K)|$ counts the zeros of f of isotropy *precisely* K. It follows that

$$\sum_{\tilde{K} \geq K} n_{\tilde{K}} \cdot |W(\tilde{K})| = \deg(f|_{\Omega^K}, \Omega^K). \tag{3.2}$$

It is convenient to rewrite the above in terms of $(K) \in \Phi$ as

$$\sum_{(\tilde{K})\geq(K)} n_{\tilde{K}} \cdot n(K,\tilde{K}) \cdot |W(K)| = \deg\,(f|_{\Omega^K}, \Omega^K), \tag{3.3}$$

where $n(K,\tilde{K})$ is the number of distinct subgroups $\gamma\tilde{K}\gamma^{-1}$ that contain K.

Following the order (3.1) on Φ and using (3.3), we can define n_K's as follows. For a maximal orbit type $(K_{\max}) \in \Phi$, define

$$n_{K_{\max}} = \frac{\deg\,(f|_{\Omega^{K_{\max}}}, \Omega^{K_{\max}})}{|W(K_{\max})|}.$$

Assume that $n_{\tilde{K}}$'s are defined for all $(\tilde{K}) > (K)$. Then,

$$n_K = \frac{\deg\,(f|_{\Omega^K}, \Omega^K) - \sum_{(\tilde{K})>(K)} n_{\tilde{K}} \cdot n(K,\tilde{K}) \cdot |W(\tilde{K})|}{|W(K)|}, \tag{3.4}$$

which is known as the *recurrence formula* in the literature.

Define the *equivariant degree (without parameter)* of f in Ω by a finite sum of integer-indexed orbit types:

$$\Gamma\text{-Deg}\,(f,\Omega) = \sum_{(K)\in\Phi} n_K \cdot (K), \tag{3.5}$$

where $n_K \in \mathbb{Z}$ is defined by (3.4).

We show that the degree defined in this way satisfies all classical properties of a degree theory. It should be pointed out that the Theorem 3.1.4 below has been proven for equivariant degrees in a much broader sense using regular normal approximations (cf. [12, 16]). But the proof using recurrence formula (3.4) can be much easier and present a direct connection to the classical Brouwer degree.

Theorem 3.1.4. *The degree defined by (3.4)–(3.5) satisfies the following properties:*

(P1) (Existence) *If* $\Gamma\text{-Deg}\,(f,\Omega) = \sum_{(K)\in\Phi} n_K \cdot (K)$ *is such that* $n_{K_o} \neq 0$ *for some* $(K_o) \in \Phi$*, then there exists* $x \in f^{-1}(0) \cap \Omega^{K_o}$.

(P2) (Additivity) *Assume that* Ω_1 *and* Ω_2 *are two open invariant subsets of* Ω *such that* $f^{-1}(0) \cap \Omega \subset \Omega_1 \cup \Omega_2$*. Then,*

$$\Gamma\text{-Deg}\,(f,\Omega) = \Gamma\text{-Deg}\,(f,\Omega_1) + \Gamma\text{-Deg}\,(f,\Omega_2).$$

(P3) (Homotopy) *Suppose* $h : [0,1] \times X \to X$ *is an* Ω*-admissible equivariant homotopy. Then,*

$$\Gamma\text{-Deg}\,(h_t,\Omega) = \text{constant}$$

(here $h_t := h(t,\cdot,\cdot)$*,* $t \in [0,1]$*).*

(P4) (Suspension) *Suppose that Y is another orthogonal Γ-representation and let U be an open bounded invariant neighborhood of* 0 *in Y. Then,*

$$\Gamma\text{-Deg}\,(f \times \mathrm{Id}, \Omega \times U) = \Gamma\text{-Deg}\,(f, \Omega).$$

Proof. (P1) Consider the set $\{(K) : (K) \geq (K_o), n_K \neq 0\}$ and let (M) be a maximal element of the set. Then, $n_K = 0$ for all $(K) > (M)$. It follows from (3.4) that

$$n_M = \frac{\deg\,(f|_{\Omega^M}, \Omega^M)}{|W(M)|}.$$

Since $n_M \neq 0$, we have $\deg\,(f|_{\Omega_M}, \Omega^M) \neq 0$. By existence property of $\deg$, we have $f^{-1}(0) \cap \Omega^M \neq \emptyset$. By equivariance of f, we have

$$f^{-1}(0) \cap \Omega^{\gamma M \gamma^{-1}} \neq \emptyset, \quad \forall\, \gamma \in \Gamma.$$

On the other hand, $(M) \geq (K_o)$ means $\gamma M \gamma^{-1} \supseteq K_o$ for some $\gamma \in \Gamma$, so $\Omega^{\gamma M \gamma^{-1}} \subset \Omega^{K_o}$. Thus, $f^{-1}(0) \cap \Omega^{K_o} \neq \emptyset$.
(P2) The statement follows from the additivity property of $\deg$:

$$\deg\,(f|_{\Omega^K}, \Omega^K) = \deg\,(f|_{\Omega_1^K}, \Omega_1^K) + \deg\,(f|_{\Omega_2^K}, \Omega_2^K),$$

applied inductively to (3.4). (P3) It is sufficient to notice that $h|_{[0,1]\times\Omega^K}$ is an Ω^K-admissible homotopy. By applying the homotopy property of $\deg$, we obtain (P3).
(P4) It follows from the suspension property of $\deg$:

$$\deg\,(f \times \mathrm{Id}|_{\Omega^K \times U}, \Omega^K \times U) = \deg\,(f|_{\Omega^K}, \Omega^K).$$

□

Example 3.1.5. Let $\Gamma = D_3$ be the dihedral group of order 6:

$$D_3 = \{1, \xi, \xi^2, \kappa, \kappa\xi, \kappa\xi^2\},$$

where $\xi^3 = 1$, $\kappa^2 = 1$ and $\kappa\xi = \xi^{-1}\kappa$. Consider the natural action of D_3 on the complex plane $\mathbb{C}$ where ξ rotates by 120° and κ reflects around the x-axis. Denote by

$$D_1 = \{1, \kappa\}, \quad \mathbb{Z}_1 = \{1\}.$$

It can be directly verified that points of $\mathbb{C}$, depending on their location in space, can have isotropy of D_3, D_1, $\xi D_1 \xi^{-1}$, $\xi^2 D_1 \xi^{-2}$ or $\mathbb{Z}_1$ (cf. Figure 3.1). These isotropy subgroups form three classes under group conjugation: (D_3), (D_1) and $(\mathbb{Z}_1)$. Thus, equivariant degree of any D_3-equivariant map $f : \Omega \subset \mathbb{C} \to \mathbb{C}$ on an admissible domain Ω is of form

$$\Gamma\text{-Deg}\,(f, \Omega) = n_1(D_3) + n_2(D_1) + n_3(\mathbb{Z}_1),$$

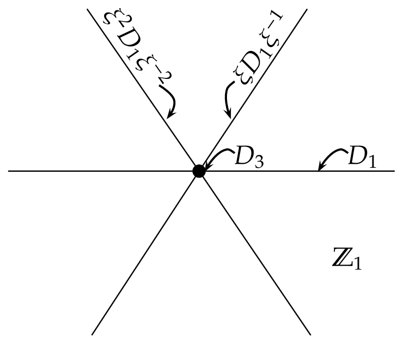


Figure 3.1: Orbit types in $\mathbb{C}$ under D_3-action: (D_3), (D_1) and $(\mathbb{Z}_1)$.

for some integers $n_1, n_2, n_3 \in \mathbb{Z}$.

As an example, consider $f = -\mathrm{Id} : B \subset \mathbb{C} \to \mathbb{C}$ defined by

$$f(x) = -x, \quad \forall\, x \in B,$$

on a disc B of positive radius $r > 0$. It is admissible since $f(x) = 0$ if and only if $x = 0$, which is not on the boundary of B. We use (3.4) to compute Γ-Deg$(-\mathrm{Id}, B)$. As illustrated in Figure 3.1, we have $B^{D_3} = \{0\}$, $B^{D_1} = x$-axis and $B^{\mathbb{Z}_1} = B$. It follows that

$$\deg(-\mathrm{Id}|_{B^{D_3}}, B^{D_3}) = 1,\ \deg(-\mathrm{Id}|_{B^{D_1}}, B^{D_1}) = -1,\ \deg(-\mathrm{Id}|_{B^{\mathbb{Z}_1}}, B^{\mathbb{Z}_1}) = 1, \tag{3.6}$$

since $\deg(-\mathrm{Id}, B_o) = (-1)^d$ for the dimension d of B_o. On the other hand, we have

$$N(D_3) = D_3,\ N(D_1) = D_1,\ N(\mathbb{Z}_1) = D_3. \tag{3.7}$$

Moreover, $n(D_1, D_3) = 1$ since $\{\xi D_3 \xi^{-1} \supset D_1 : \xi \in D_3\}$ contains only one element D_3; $n(\mathbb{Z}_1, D_1) = 3$ since $\{\xi D_1 \xi^{-1} \supset \mathbb{Z}_1 : \xi \in D_3\}$ contains three elements D_1, $\kappa D_1 \kappa^{-1}$, $\kappa^2 D_1 \kappa^{-2}$. Therefore, we have

$$n_1 = \frac{\deg(-\mathrm{Id}|_{B^{D_3}}, B^{D_3})}{|N(D_3)/D_3|} = \frac{1}{1} = 1$$

$$n_2 = \frac{\deg(-\mathrm{Id}|_{B^{D_1}}, B^{D_1}) - 1 \cdot 1 \cdot 1}{|N(D_1)/D_1|} = \frac{-2}{1} = -2$$

$$n_3 = \frac{\deg(-\mathrm{Id}|_{B^{\mathbb{Z}_1}}, B^{\mathbb{Z}_1}) - 1 \cdot 1 \cdot 1 - (-2) \cdot 1 \cdot 3}{|N(\mathbb{Z}_1)/\mathbb{Z}_1|} = \frac{6}{6} = 1.$$

The above amounts to a Maple© command:

$$\texttt{showdegree}[D3](0,0,1,1,0,0)$$

3.1.2 Equivariant Degree with One Parameter

In the same spirit, one can define an equivariant degree for maps with one parameter using a recurrence formula. In this case, we consider groups of form $G = \Gamma \times S^1$ for a finite group Γ and the circle group S^1, and maps of form $f : \mathbb{R} \times X \to X$. The choice of the product form of G is motivated by the study of equivariant Hopf bifurcations in equivariant systems, where Γ describes the spatial symmetry of the system and S^1 describes the temporal symmetry of potential periodic solutions. The basic topological count of zeros is given by the S^1-equivariant degree as counterpart of the Brouwer degree in the Subsection 3.1.1.

There are two types of subgroups in $\Gamma \times S^1$: those of form $K \times S^1$ for subgroup $K \subset \Gamma$, and those that are twisted (cf. Definition 2.3.6). Subgroups of form $K \times S^1$ and twisted subgroups can also be distinguished by the dimension of their Weyl groups. Indeed, we have

$$H = K \times S^1 \quad \Longleftrightarrow \quad \dim W(H) = 0$$
$$H = K^{\phi,l} \quad \Longleftrightarrow \quad \dim W(H) = 1.$$

Let Φ_1 be the set of all conjugacy classes of twisted subgroups. It permits a partial order given by

$$(H_1) > (H_2) \quad \Longleftrightarrow \quad \exists\, g \in G\, s.t.\, H_1 \supset gH_2g^{-1}. \tag{3.8}$$

Let X be a finite-dimensional G-representation and $O \subset \mathbb{R} \times X$ be an open bounded G-invariant subset. Consider an equivariant map $F : \overline{O} \subset \mathbb{R} \times X \to X$ that is admissible on O. We define an equivariant degree for (F, O), which associates an integer n_H to every twisted orbit type $(H) \in \Phi_1$ as follows.

For every $(H) \in \Phi_1$, consider the fixed point subspace O^H, which is invariant under S^1-action, since S^1 is abelian. The restricted map

$$F|_{O^H} : O^H \subset \mathbb{R} \times X^H \to X^H$$

is a map with one parameter and admissible on O^H. Moreover, it is S^1-equivariant, since S^1 acts on O^H. Thus, for the admissible pair $(F|_{O^H}, O^H)$, the classical S^1-degree is defined (cf. [45]) and of form

$$S^1\text{-Deg}\,(F|_{O^H}, O^H) = s_1 \cdot (\mathbb{Z}_1) + \cdots + s_m \cdot (\mathbb{Z}_m), \quad s_i \in \mathbb{Z}. \tag{3.9}$$

The integer s_i gives a topological count of circles in the zero set of $f|_{O^H}$ that have isotropy precisely $\mathbb{Z}_i$. So the total count of circles in the zero set of $f|_{O^H}$ will be $\sum_i s_i$. On the other hand, as explained in Subsection 3.1.1, the total equivariance in O^H is given by the Weyl group $W(H) = N(H)/H$. Thus, besides S^1-equivariance, the additional equivariance on O^H is $W(H)/S^1$, which is a finite group, since $\dim(W(H)) = 1$.

Therefore, if n_H gives a count of zero orbits of f of orbit type (H), then $n_H \cdot |W(H)/S^1|$ counts the zero orbits of f of isotropy precisely H. It follows that

$$\sum_{\tilde{H} \geq H} n_{\tilde{H}} \cdot |W(\tilde{H})/S^1| = \sum_i s_i.$$

The above can be conveniently rewritten in terms of orbit types by

$$\sum_{(\tilde{H}) \geq (H)} n_{\tilde{H}} \cdot |W(\tilde{H})/S^1| \cdot n(H, \tilde{H}) = \sum_i s_i, \tag{3.10}$$

where $n(H, \tilde{H})$ is the number of distinct subgroups $g\tilde{H}g^{-1}$ that contain H for $g \in G$.

Based on (3.10) and following the order (3.8), we can define n_H inductively. For a maximal orbit type $(H_{\max}) \in \Phi_1$, define

$$n_{H_{\max}} = \frac{\sum_i s_i^{H_{\max}}}{|W(H_{\max})/S^1|},$$

where we use the superscript ${}^{H_{\max}}$ to indicate the s_i's are from the map $f|_{O^{H_{\max}}}$. Assume that $n_{\tilde{H}}$'s are defined for all $(\tilde{H}) > (H)$. Then, define

$$n_H = \frac{\sum\limits_i s_i^H - \sum\limits_{(\tilde{H}) > (H)} n_{\tilde{H}} \cdot |W(\tilde{H})/S^1| \cdot n(H, \tilde{H})}{|W(H)/S^1|}, \tag{3.11}$$

which is known as the *recurrence formula with one parameter*.

Define an *equivariant degree (with one parameter)* of f in Ω by a finite sum of integer-indexed twisted orbit types:

$$\Gamma \times S^1\text{-Deg}\,(f, \Omega) = \sum_{(H) \in \Phi_1} n_H \cdot (H), \tag{3.12}$$

where $n_H \in \mathbb{Z}$ is defined by (3.11).

In the same way as for the equivariant degree without parameter, one can show that $\Gamma \times S^1$-Deg defined using the recurrence formula (3.11) satisfies the *existence, additivity, homotopy, suspension* properties in a similar way as shown in Theorem 3.1.4. We state this result without direct proof. A proof of a much more general result using regular normal approximations can be found in [16].

Theorem 3.1.6. *The degree defined by (3.11)–(3.12) satisfies the following properties:*

(P1) (Existence) *If* $\Gamma \times S^1\text{-Deg}\,(F, O) = \sum_{(H) \in \Phi_1} n_H \cdot (H)$ *is such that* $n_{H_o} \neq 0$ *for some* $(H_o) \in \Phi_1$*, then there exists* $x \in F^{-1}(0) \cap O^{H_o}$.

(P2) (Additivity) *Assume that O_1 and O_2 are two open invariant subsets of O such that $F^{-1}(0) \cap O \subset O_1 \cup O_2$. Then,*

$$\Gamma \times S^1\text{-Deg}\,(F, O) = \Gamma \times S^1\text{-Deg}\,(F, O_1) + \Gamma \times S^1\text{-Deg}\,(F, O_2).$$

(P3) (Homotopy) *Suppose $h : [0,1] \times O \to X$ is an O-admissible equivariant homotopy. Then,*

$$\Gamma \times S^1\text{-Deg}\,(h_t, O) = \text{constant}$$

for $h_t := h(t, \cdot, \cdot)$.

(P4) (Suspension) *Suppose that Y is another orthogonal G-representation and let U be an open bounded invariant neighborhood of 0 in Y. Then,*

$$\Gamma \times S^1\text{-Deg}\,(F \times \mathrm{Id}, O \times U) = \Gamma \times S^1\text{-Deg}\,(F, O).$$

3.1.3 Multiplication of Equivariant Degrees

Recall that the $\Gamma \times S^1$-equivariant degree with one parameter has a multiplication property corresponding to the $A(\Gamma)$-module structure on the set $A_1(\Gamma \times S^1)$, which coincides with the Euler ring multiplication in $A(\Gamma \times S^1)$ (cf. [16, 67]).

Proposition 3.1.7. *Let $F : \overline{O} \subset \mathbb{R} \times X \to X$ be an admissible $\Gamma \times S^1$-equivariant map in $O \subset \mathbb{R} \times X$. Let $f : \overline{\Omega} \subset Y \to Y$ be an admissible Γ-equivariant map in $\Omega \subset Y$. Then, we have*

(P5) (Multiplication) *The product map $F \times f : \mathbb{R} \times X \times Y \to X \times Y$ is $O \times \Omega$-admissible and*

$$\Gamma \times S^1\text{-Deg}\,(F \times f, O \times \Omega) = \Gamma\text{-Deg}\,(f, \Omega) * G\text{-Deg}\,(f_1, \Omega_1),$$

where "$*$" stands for the Euler ring multiplication (II) in Table 2.2.

3.2 Equivariant Bifurcations

In this section, we review the standard procedure of treating equivariant bifurcations using equivariant degrees. It starts with reformulating the bifurcation problem as a fixed point problem in an equivariant functional setting, and then it defines a *bifurcation invariant* for the bifurcating equilibrium using equivariant degrees of appropriate maps. The exact value of bifurcation invariant then gives a complete topological classification of the bifurcating branches by their symmetry properties.

Before we review this procedure, it is worthwhile pointing out that bifurcation invariants can be exactly calculated using the EDML (Equivariant Degree Maple Library) Package, by calling

$$\texttt{showdegree}[\Gamma](n_0, n_1, \dots, n_r, m_0, m_1, \dots, m_s), \quad \text{for } n_i, m_j \in \mathbb{Z}, \tag{3.13}$$

where Γ stands for the symmetry group, n_i's and m_j's are integers to be determined by the critical spectrum of the linearization at the equilibrium. The integers r and s in (3.13) are predetermined by Γ and are equal to the number of all distinct (nontrivial) irreducible representations of Γ over reals and over complex numbers, respectively.

3.2.1 Definition of a Bifurcation Invariant

Let Γ be a finite group and $\mathbb{R}^n$ be an (orthogonal) Γ-representation.

Consider

$$\dot{x} = f(\lambda, x), \quad (\lambda, x) \in \mathbb{R} \times \mathbb{R}^n, \tag{3.14}$$

where $f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ is an equivariant map of class C^1 and $\lambda \in \mathbb{R}$ is the bifurcation parameter. Here, Γ acts trivially on the parameter space $\mathbb{R}$.

Assume that

(E1) $x_o \in \mathbb{R}^n$ is an equilibrium of (3.14), i.e. $f(\lambda, x_o) = 0, \quad \forall \lambda \in \mathbb{R}$.

Let $J(\lambda) := Df_x(\lambda, x_o)$ be the Jacobian of f at x_o. We call (λ_o, x_o) a *bifurcation center* of (3.14), if $J(\lambda_o)$ has a purely imaginary eigenvalue $i\beta_o$. Assume that

(B1) (λ_o, x_o) is an *isolated* bifurcation center, i.e. (λ_o, x_o) is the only bifurcation center in a neighborhood of (λ_o, x_o) in $\mathbb{R} \times \mathbb{R}^n$.

Moreover, to avoid steady-state bifurcations, we assume

(B2) $J(\lambda_o) : \mathbb{R}^n \to \mathbb{R}^n$ is an isomorphism.

We are interested in describing possible oscillating states arising from x_o as λ crosses λ_o.

Functional Reformulation Let $p > 0$ be the unknown period of the bifurcating solution x of (3.14). Let $\beta := \frac{2\pi}{p}$ and $u(t) := x(\frac{1}{\beta}t)$. Then, finding a p-periodic solution x of (3.14) is equivalent to solving

$$\begin{cases} \dot{u} = \frac{1}{\beta} f(\lambda, u), \\ u(0) = u(2\pi). \end{cases} \tag{3.15}$$

Let $W := H^1(S^1; \mathbb{R}^n)$ be the first Sobolev space of $\mathbb{R}^n$-valued functions defined on S^1. Then, W admits a natural $\Gamma \times S^1$-action by

$$((\gamma, e^{i\theta})u)(t) := \gamma u(t + \theta), \quad \gamma \in \Gamma,\ e^{i\theta} \in S^1.$$

Define

$$L : W \to L^2(S^1; \mathbb{R}^n), \qquad L(u) = \dot{u} \tag{3.16}$$

$$j : W \to C(S^1; \mathbb{R}^n), \qquad j(u) = \tilde{u} \tag{3.17}$$

$$N_f : \mathbb{R} \times C(S^1; \mathbb{R}^n) \to L^2(S^1; \mathbb{R}^n), \quad (N_f(\lambda, v))(t) = f(\lambda, v(t)). \tag{3.18}$$

Then, (3.15) is equivalent to

$$Lu = \frac{1}{\beta} N_f(\lambda, j(u)). \tag{3.19}$$

Define $K : W \to L^2(S^1; \mathbb{R}^n)$ by $Ku := \frac{1}{2\pi} \int_0^{2\pi} u(t)dt$. Then, $(L + K)$ is invertible and (3.19) is equivalent to

$$u = (L + K)^{-1}\Big[\frac{1}{\beta} N_f(\lambda, j(u)) + Ku\Big] := F_1(\lambda, \beta, u), \tag{3.20}$$

where F_1 is a $\Gamma \times S^1$-equivariant compact map, due to the compactness of j.

Bifurcation Invariant Let (λ_o, x_o) be the isolated bifurcation center given by (B1) and $i\beta_o$ be the purely imaginary eigenvalue of $J(\lambda_o)$. Define a neighborhood $O \subset \mathbb{R}^2 \times W$ of $(\lambda_o, \beta_o, u_o)$ by (cf. Figure 3.2)

$$O := \{(\lambda, \beta, u) \,:\, \sqrt{(\lambda - \lambda_o)^2 + (\beta - \beta_o)^2} < \varepsilon,\ \|u\| < r\} \subset \mathbb{R}^2 \times W, \tag{3.21}$$

where $\mathbb{R}^2$ is considered as a parameter space (on which $\Gamma \times S^1$ act trivially). Note that O is $\Gamma \times S^1$-invariant, since $\Gamma \times S^1$ acts isometrically on W.

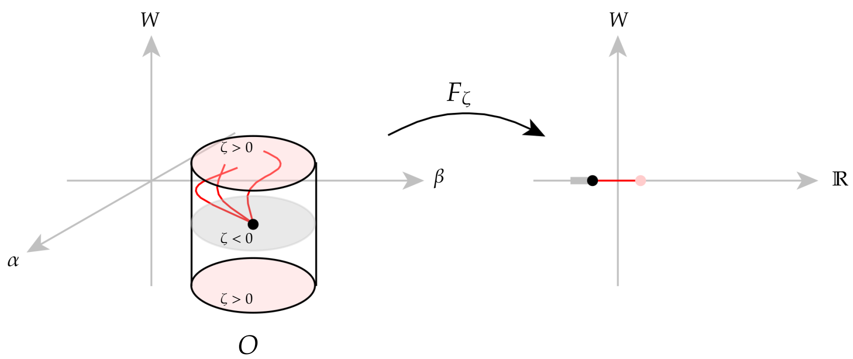


Figure 3.2: Bifurcation invariant as equivariant degree of F_ζ in O: non-trivial bifurcating solutions (in red) are "separated" from the trivial solutions using an auxiliary function ζ.

To define an admissible map on O and consider only the non-trivial solutions for (3.20), we introduce an auxiliary function $\zeta : \overline{O} \to \mathbb{R}$ such that

$$\begin{cases} \zeta > 0, & \text{for } \|u\| = r \\ \zeta < 0, & \text{for } \|u\| = 0 \end{cases}. \tag{3.22}$$

Such auxiliary function can be defined for example, by

$$\zeta(\lambda, \beta, u) := \sqrt{(\lambda - \lambda_o)^2 + (\beta - \beta_o)^2}(\|u\| - r) + \|u\| - \frac{r}{2}.$$

For the definition of bifurcation invariant, it is not essential how the auxiliary function is precisely defined, but the property (3.22) that it satisfies.

Now, we are ready to define an admissible map on O:

$$\begin{aligned} F_\zeta : \quad & \overline{O} \to \mathbb{R} \times W \\ F_\zeta(\lambda, \beta, u) &= (\zeta(\lambda, \beta, u), u - F_1(\lambda, \beta, u)). \end{aligned} \tag{3.23}$$

By construction, non-trivial solutions of (3.15) near (λ_o, x_o) are precisely zeros of F_ζ in O for some auxiliary function ζ. Also, F_ζ is $\Gamma \times S^1$-equivariant and amounts to a compact perturbation of identity up to parameters. Therefore, we can define

$$\omega(\lambda_o, \beta_o, x_o) := \Gamma \times S^1\text{-Deg}\,(F_\zeta, O)$$

and call it the *bifurcation invariant* around (λ_o, x_o).

3.2.2 Classification Results

Based on the existence property of the equivariant degree, a non-zero coefficient n_H in $\omega(\lambda_o, \beta_o, x_o)$ indicates the existence of a zero of F_ζ in O having isotropy at least H. As we will see in the next subsection, the value of $\omega(\lambda_o, \beta_o, x_o)$ is independent of choice of ζ, thus a non-zero coefficient n_H implies indeed the existence of a *branch* of bifurcating zeros of F_ζ in O having isotropy at least H, since these bifurcating zeros can be "traced" along by varying the choice of auxiliary functions. For a rigorous proof in this perspective, we refer to Lemma 9.19 in [16].

Theorem 3.2.1. *(cf. [16]) If $\omega(\lambda_o, \beta_o, x_o) = \sum_{(H)} n_H \cdot (H)$ contains a non-zero coefficient $n_H \neq 0$ for some (H), then there exists a bifurcating branch of oscillating states of isotropy at least (H).*

Theorem 3.2.1 gives an existence result of bifurcating branches with their *least* symmetry. To sharpen to the *precise* symmetry, one can work with orbit types satisfying certain maximality condition. We recall the concept of *dominating orbit types* from [16] and introduce a complementing definition of *secondary dominating orbit types* (cf. [9]).

There is a natural way of "converting" a complex Γ-representation into a real $\Gamma \times S^1$-representation. Let U be a complex Γ-representation. Define a $\Gamma \times S^1$-action on U by

$$(\gamma, z)u = z \cdot (\gamma u), \quad \text{for } (\gamma, z) \in \Gamma \times S^1,\ u \in U, \tag{3.24}$$

where $\cdot$ stands for the complex multiplication. The obtained representation is denoted by $\bar{U}$ and called the $\Gamma \times S^1$-representation induced from U. Note that

$\bar{U}$ is irreducible as a real $\Gamma \times S^1$-representation if U is irreducible as a complex Γ-representation.

Definition 3.2.1. Let $\{\mathcal{U}_1, \mathcal{U}_2, \dots, \mathcal{U}_m\}$ be the set of irreducible Γ-representations that occur in $\mathbb{C}^n$, where $\mathbb{C}^n$ is the complexification of the phase space $\mathbb{R}^n$ of the system (3.14). Let $\bar{\mathcal{U}}_j$ be the $\Gamma \times S^1$-representation induced from $\mathcal{U}_j$, for $j = 1, 2, \dots, m$ (cf. (3.24)). Collect maximal orbit types from $\bar{\mathcal{U}}_j$, for $j = 1, 2, \dots, m$, and denote this collection by $\mathcal{M}$. An orbit type $(H) \in \mathcal{M}$ is called *dominating* if (H) is maximal in $\mathcal{M}$. A non-dominating orbit type $(L) \in \mathcal{M}$ is called *secondary dominating* if all orbit types $(H) \in \mathcal{M}$ satisfying $(L) < (H)$ are dominating. ◇

Proposition 3.2.2. *Under the assumptions of Theorem 3.2.1, we have*

(i) If $(H) = (K^{\phi,l})$ is a dominating orbit type, then there exist at least $|\Gamma/K|$ different bifurcating branches of non-constant oscillating states of (3.14), which have isotropy subgroups $\gamma H \gamma^{-1}$, as γ run through Γ/K.

(ii) If (H) is a secondary dominating orbit type and for every dominating orbit type $(\tilde{H}) > (H)$ there exists a flow-invariant subspace $S \subset \mathbb{R}^n$ such that

(a) S contains every state of isotropy $\tilde{H}$; and

(b) $J(\lambda_o)|_S$ has no purely imaginary eigenvalues.

Then there exists a bifurcating branch of oscillating states of symmetry precisely (H). Moreover, if $(H) = (K^{\phi,l})$, then the conclusion of (i) also holds for (H).

Proof. Statement (i) follows from [16], and (ii) follows from the implicit function theorem. More precisely, let (H) be a secondary dominating orbit type with a nonzero coefficient in $\omega(\lambda_o, \beta_o, x_o)$. By Theorem 3.2.1, there exists a a bifurcating branch of oscillating states of symmetry at least (H). Let $(\tilde{H})$ be the precise symmetry of this branch and suppose that $(H) < (\tilde{H})$. By definition of secondary dominating orbit types, the only orbit types that are strictly larger than (H) are dominating orbit types. Thus $(\tilde{H})$ is dominating, and so there exists a flow-invariant subspace S in $\mathbb{R}^n$ satisfying (a)–(b). Consider the restricted flow on S. The bifurcating branch of oscillating states, by condition (a), is contained in S. However, by condition (b) and the implicit function theorem, there can be no bifurcation taking place in S. This leads to a contradiction.

□

3.2.3 The Command `showdegree`

Exact value of bifurcation invariants can be obtained by calling

$$\texttt{showdegree}[\Gamma](n_0, n_1, \dots, n_r, m_0, m_1, \dots, m_s), \quad \text{for } n_i, m_j \in \mathbb{Z}, \qquad (3.25)$$

from the EDML (Equivariant Degree Maple Library) package.

We explain below how to use `showdegree` to obtain value of the bifurcation invariant.

The Symmetry Γ The entry Γ of `showdegree` refers to the phase symmetry of (3.14). Currently available groups in EDML are the quaternion group Q_8, the dihedral groups D_n for $3 \leq n \leq 13$, the symmetric group S_4, the alternating groups A_4, A_5, the orthogonal group $O(2)$ and the special orthogonal group $SO(3)$.

The integers r and s in (3.25) are predetermined by Γ and equal to the total number of its distinct (nontrivial) irreducible representations over reals and over complex numbers, respectively.

We usually use $\mathcal{V}_0, \mathcal{V}_1, \ldots, \mathcal{V}_r$ for the distinct real irreducible representations and $\mathcal{U}_0, \mathcal{U}_1, \ldots, \mathcal{U}_s$ for the complex ones, where $\mathcal{V}_0$ and $\mathcal{U}_0$ are reserved for the trivial representations.

The Integers $n_0, n_1, \ldots, n_r$ Consider $J(\lambda_o) : \mathbb{R}^n \to \mathbb{R}^n$ as a real linear transformation. Denote by σ_+ the set of all its positive eigenvalues.

For $\mu \in \sigma_+$, let $E(\mu)$ be the generalized eigenspace of μ. We decompose $E(\mu)$ into pieces of $\mathcal{V}_i$'s as follows. Since $\mathbb{R}^n$ is a Γ-representation ovre reals, we can decompose $\mathbb{R}^n$ as

$$\mathbb{R}^n = V_0 \times V_1 \times \cdots \times V_r,$$

where every V_i

$$V_i = \underbrace{\mathcal{V}_i \times \cdots \times \mathcal{V}_i}_{c_i \text{ times}} \tag{3.26}$$

is a product of c_i copies of $\mathcal{V}_i$ for some integer $c_i \in \mathbb{N} \cup \{0\}$. Now, since $E(\mu) \subset \mathbb{R}^n$ is Γ-invariant, this leads to a decomposition of $E(\mu)$ as

$$E(\mu) = E_0 \times E_1 \times \cdots \times E_r,$$

for $E_i = E(\mu) \cap V_i$ which can be then written as

$$E_i = \underbrace{\mathcal{V}_i \times \cdots \times \mathcal{V}_i}_{n_i(\mu) \text{ times}} \tag{3.27}$$

for some integer $n_i(\mu)$ with $0 \leq n_i(\mu) \leq c_i$. Then,

$$n_i := \sum_{\mu \in \sigma_+} n_i(\mu) \tag{3.28}$$

are the integer input $n_0, n_1, \ldots, n_r$ to `showdegree`.

The Integers $m_0, m_1, \ldots, m_s$ Consider $J(\lambda_o) : \mathbb{C}^n \to \mathbb{C}^n$ as a complex linear transformation. Recall that it has $i\beta_o$ as a purely imaginary eigenvalue. Denote by $E^c(i\beta_o)$ the eigenspace. We decompose $E^c(i\beta_o)$ into pieces of $\mathcal{U}_j$'s.

There is a natural extension of the Γ-action on $\mathbb{R}^n$ to $\mathbb{C}^n$, which is defined by

$$\gamma \cdot (z \otimes x) = z \otimes (\gamma \cdot x),$$

for $\gamma \in \Gamma$, $z \in \mathbb{C}$ and $x \in \mathbb{R}^n$. Then, $\mathbb{C}^n$, as a Γ-representation over complex numbers, can be decomposed as

$$\mathbb{C}^n = U_0 \times U_1 \times \cdots \times U_r,$$

where every U_j

$$U_j = \underbrace{\mathcal{U}_j \times \cdots \times \mathcal{U}_j}_{d_j \text{ times}} \tag{3.29}$$

is a product of d_j copies of $\mathcal{U}_i$ for some integer $d_j \in \mathbb{N} \cup \{0\}$.

Note that $E^c(i\beta_o)$ is invariant under this induced action, thus it admits a decomposition as

$$E^c(i\beta_o) = C_0 \times C_1 \times \cdots \times C_s,$$

where every $C_j = E^c(i\beta_o) \cap U_j$ can be written as

$$C_j = \underbrace{\mathcal{U}_j \times \cdots \times \mathcal{U}_j}_{m_j \text{ times}} \tag{3.30}$$

a product of m_j copies of $\mathcal{U}_j$ for some integer m_j with $0 \leq m_j \leq d_j$. These are the integer input $m_0, m_1, \ldots, m_s$ to `showdegree`.

Remark 3.2.3. The integers $m_0, m_1, \ldots, m_s$ defined above differs from the usual definition in the literature by a sign.Indeed, these integers are usually given by the *crossing numbers t_j's*, which count the net escape of eigenvalues through $i\beta_o$ (with respect to $\mathcal{U}_j$), as the bifurcation parameter λ moves across λ_o (cf. Figure 3.3).

The relation between m_j's and t_j's are (cf. Remark 9.33 in [16])

$$m_j = \pm t_j,$$

depending on whether there are more eigenvalues escaping or more eigenvalues entering through $i\beta_o$. In the case illustrated by Figure 3.3 for example, these quantities coincide.

◇

Thus, we have

$$\omega(\lambda_o, \beta_o, x_o) = \pm\texttt{showdegree}[\Gamma](n_0, n_1, \ldots, n_r, m_0, m_1, \ldots, m_s),$$

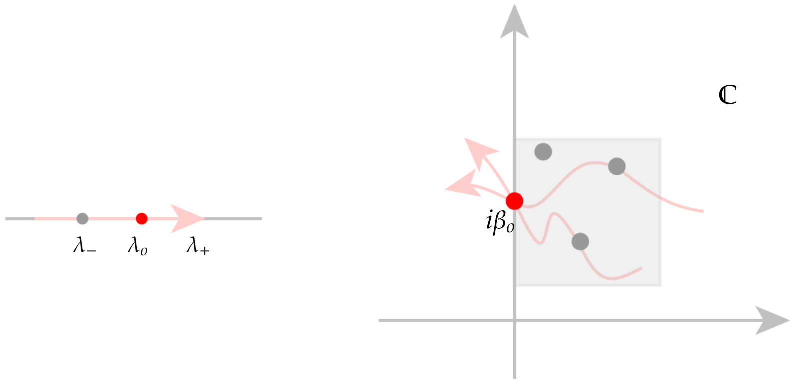


Figure 3.3: Geometric meaning of crossing numbers: it measures the net escape of eigenvalues through $i\beta_o$, as the bifurcation parameter λ moves across λ_o. The eigenvalues are counted by their algebraic multiplicities.

where Γ stands for the phase symmetry of (3.14), the integers n_i's are given by (3.28) and m_j's are indicated in (3.30).

Note that

$$\omega(\lambda_o, \beta_o, x_o) = \sum_{(H)\in\Phi_1} n_H \cdot (H) \quad \Leftrightarrow \quad -\omega(\lambda_o, \beta_o, x_o) = \sum_{(H)\in\Phi_1} -n_H \cdot (H).$$

Also, $n_H \neq 0$ if and only if $-n_H \neq 0$. Therefore, Theorem 3.2.1 and Proposition 3.2.2 can be applied to `showdegree`$[\Gamma](n_0, n_1, \ldots, n_r, m_0, m_1, \ldots, m_s)$ directly.

Chapter 4

Quotient Symmetry and Equivariant Degree

Genuine similarity exists between equivariant bifurcations in equivariant systems and synchrony-related bifurcations in networked systems. Symmetry, as the main feature of equivariant systems, places strong restrictions on possible bifurcating states of the equivariant systems in their occurrence and pattern. Network structure, on the other hand, as the global structure obeyed by the networked systems, prescribes balanced equivalence relations, robust synchronies and their associated bifurcations. The key idea is to replace symmetry with network structure, isotropy subgroups with equivalence relations, fixed-point subspaces with synchrony subspaces, and orbit types with synchrony patterns (cf. [73]).

Network structure is more than an extended version of symmetry. Systems that lie in the intersection of equivariant systems and networked systems are symmetrically coupled systems. Examples of such systems show that even in this case symmetry alone cannot always reveal all synchrony patterns dictated by network structure (cf. [33]).

Network structure also often induces different kinds of symmetry at the same time. These include those that occur at quotient level, the *quotient symmetry*, and those that occur at subnetwork level, the *interior symmetry*. As we have seen in Subsection 2.1.1, the network (b) in Figure 2.3 has quotient symmetry at different quotient levels and also interior symmetry for some subnetwork.

In this chapter, we focus on the aspect of quotient symmetry and introduce an equivariant degree theory that is suitable for studying synchrony-related bifurcations in networks, with or without quotient symmetry. The key idea is to explore algebraic structure of lattices of synchrony subspaces and to use Euler ring homomorphisms to bridge zero orbits at different quotient level. The resulting equivariant degree theory will be called the *lattice equivariant degree*.

It is worthwhile mentioning that in case of symmetrically coupled networks, the lattice equivariant degree coincides with the usual equivariant degree. In case of coupled networks with adjacency matrices having only simple eigenvalues, the networks are free of symmetry and the lattice equivariant degree reduces to the lattice indices introduced by Kamei in [48].

The content of this chapter is based on [66].

4.1 Representation Lattices

In this section, we give the definition of representation lattices and discuss some of their properties, which include basic properties as lattices and algebraic properties related to the lattice inclusion and product operations.

Definition 4.1.1. Let X be a real (resp. complex) Banach space and $\mathcal{L}$ be a finite set of closed linear subspaces of X. $\mathcal{L}$ is called a *lattice* in X, if $X \in \mathcal{L}$ and

$$U_1 \cap U_2 \in \mathcal{L}, \quad \forall\, U_1, U_2 \in \mathcal{L}.$$

We write $U_1 \leq U_2$ (resp. $U_1 < U_2$), if $U_1 \subset U_2$ (resp. $U_1 \subsetneq U_2$). A subset $\mathcal{S} \subset \mathcal{L}$ is called a *sublattice* of $\mathcal{L}$, if it is a lattice on its own right.

A representation lattice is a lattice of compatible representations, where different representations are "connected" by group homomorphisms.

Definition 4.1.2. Let X be a real (resp. complex) Banach space and $\mathcal{L}$ be a lattice in X. Assume that

(i) (REPRESENTATION) U is a real (resp. complex) isometric Banach representation of a compact Lie group G_U, for every $U \in \mathcal{L}$;

(ii) (COMPATIBILITY) there exists a group homomorphism

$$\mathsf{h}_{U_1,U_2} : G_{U_2} \to G_{U_1},$$

for every U_1, U_2 with $U_1 \leq U_2$ such that

$$g_{2\clubsuit} x = \mathsf{h}_{1,2}(g_2)_{\diamond} x, \quad \forall\, g_2 \in G_{U_2},\ x \in U_1,$$

where "$\clubsuit$" and "$\diamond$" stand for the G_2-action on U_2 and the G_1-action on U_1, respectively.

(iii) (CONSISTENCE) $\mathsf{h}_{U_1,U_2} \circ \mathsf{h}_{U_2,U_3} = \mathsf{h}_{U_1,U_3}$ for every $U_1 \leq U_2 \leq U_3$.

Then, $\mathcal{L}$ is called a *real (resp. complex) representation lattice* in X. The collection

$$\{(U_i, G_{U_i}, \mathsf{h}_{U_i,U_j}) \,:\, U_i, U_j \in \mathcal{L},\, U_i \leq U_j\}$$

is called the *structure of representation lattice* of $\mathcal{L}$. $\diamond$

It is instantly clear that a sublattice of a representation lattice $\mathcal{L}$ is again a representation lattice, which we call a *representation sublattice* of $\mathcal{L}$.

We illustrate the definition with the following example, which will serve as a running example throughout the chapter.

Example 4.1.3. Let $X = \mathbb{R}^5$ and $\mathcal{L}$ be a lattice given in Figure 4.1, where the pair (Δ_*, Γ_*) indicates that Δ_* is a representation of Γ_* and Δ_* is the linear subspace composed of vectors of form indicated below the pair (Δ_*, Γ_*). The arrows give the direction of homomorphisms between Γ_*'s.

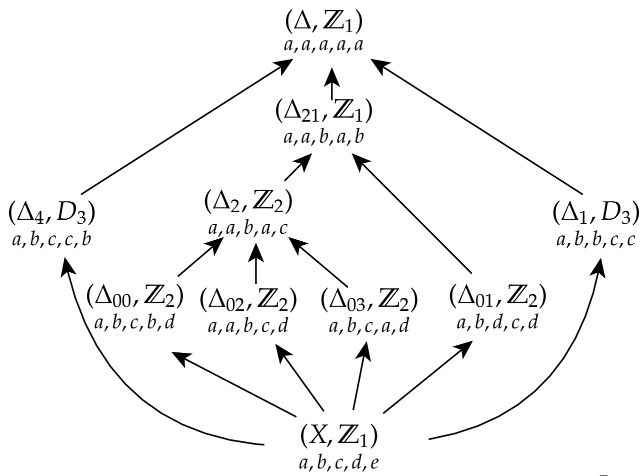


Figure 4.1: A representation lattice $\mathcal{L}$ in $\mathbb{R}^5$.

The structure of the representation lattice $\mathcal{L}$ is specified as follows:
Representations: $\mathbb{Z}_1$ acts on $\Delta, \Delta_{21}, \mathbb{R}^5$ trivially; $\mathbb{Z}_2 = \langle \kappa \rangle$ acts on $\Delta_{00}, \Delta_{02}, \Delta_{03}, \Delta_2$ by the permutation $\kappa = (3\ 5)$; $\mathbb{Z}_2 = \langle \tilde{\kappa} \rangle$ acts on Δ_{01} by the permutation $\tilde{\kappa} = (2\ 4)$; and $D_3 \simeq S_3$ acts on Δ_4, Δ_1 by the natural action of S_3 on symbols a, b, c.
Homomorphisms: $\mathbb{Z}_1 \to \Gamma_x$ are given by the inclusion; homomorphisms $\Gamma_x \to \mathbb{Z}_1$ are given by the projection; and homomorphisms $\mathbb{Z}_2 \to \mathbb{Z}_2$ are given by the identity homomorphism. Under the above structure, $\mathcal{L}$ becomes a real representation lattice in X. $\diamond$

4.1.1 Basic Properties of Lattices

We discuss some basic properties of representation lattices viewed as lattices (without the representation structure).

Let $\mathcal{L}$ be a lattice in a Banach space X and $U_1, U_2 \in \mathcal{L}$. If $U_1 < U_2$, then U_2 is called a *descendant* of U_1. A minimal descendant is called an *immediate descendant*. Denote by

$$\mathcal{L}^\top := \{U \in \mathcal{L} \,:\, U \text{ has a unique immediate descendant in } \mathcal{L}\}.$$

Lemma 4.1.4. *Let X be a Banach space and $\mathcal{L}$ be a lattice in X. Then,*

(i) $\mathcal{L}$ has a unique minimal element;

(ii) for $U \in \mathcal{L}$, the set $\mathcal{L} \setminus \{U\}$ is a sublattice of $\mathcal{L}$ if and only if $U \in \mathcal{L}^\top$;

(iii) Let $\mathcal{S} \subset \mathcal{L}$ be a sublattice and set $k := |\mathcal{L} \setminus \mathcal{S}|$, where $|\cdot|$ is the count of elements. Then there exists a flag of lattices of length k

$$\mathcal{L} = \mathcal{L}_0 \supset \mathcal{L}_1 \supset \cdots \supset \mathcal{L}_k = \mathcal{S}$$

such that $\mathcal{L}_{i+1} = \mathcal{L}_i \setminus \{U_i\}$ for certain $U_i \in \mathcal{L}_i$, $i = 0, 1, \ldots, k-1$;

(iv) Let X' be another Banach space and $\mathcal{M}$ be a lattice in X'. Then,

$$\mathcal{L} \times \mathcal{M} := \{U \times M \,:\, U \in \mathcal{L},\, M \in \mathcal{M}\}$$

is a lattice in $X \times X'$.

Proof. (i) Since $\mathcal{L}$ is closed under set intersections, the minimal element is given by the intersection of all elements of $\mathcal{L}$.
(ii) Let $\mathcal{S} = \mathcal{L} \setminus \{U\}$. If $U \in \mathcal{L}^\top$, then $U \neq X$ and $U = U_1 \cap U_2$ for some $U_1, U_2 \in \mathcal{L}$ only if $U \in \{U_1, U_2\}$. It follows that $X \in \mathcal{S}$ and $U \neq U_1 \cap U_2$ for any $U_1, U_2 \in \mathcal{S}$. Thus, $\mathcal{S}$ is a sublattice. If $U \notin \mathcal{L}^\top$, then U has more than one immediate descendants. Let U_1, U_2 be two distinct immediate descendants of U in $\mathcal{L}$. Then, $U = U_1 \cap U_2$ for $U_1, U_2 \in \mathcal{S}$. But $U \notin \mathcal{S}$, which implies that $\mathcal{S}$ is not a sublattice.
(iii) We claim that

$$\mathcal{P}^\top \setminus \mathcal{S} \neq \emptyset, \quad \text{for every lattice } \mathcal{P} \text{ s.t. } \mathcal{L} \supset \mathcal{P} \supsetneq \mathcal{S}. \tag{4.1}$$

Assume to the contrary and let U be a maximal element of $\mathcal{P} \setminus \mathcal{S}$. In particular, since $U \neq X$, U has descendants. By assumption, U has at least two distinct immediate descendants in $\mathcal{P}$, say U_1, U_2. Then, $U = U_1 \cap U_2$. Moreover, since U is a maximal element of $\mathcal{P} \setminus \mathcal{S}$, we have $U_1, U_2 \in \mathcal{S}$. It follows that $U = U_1 \cap U_2 \in \mathcal{S}$, which is a contradiction to the fact that $U \notin \mathcal{S}$. Thus, (4.1) holds.

It follows from (4.1) that there exists $U_0 \in \mathcal{L}^\top \setminus \mathcal{S}$. By (ii), $\mathcal{L}_1 := \mathcal{L} \setminus \{U_0\}$ is a sublattice. By applying (4.1) inductively to $\mathcal{L}_{i+1} = \mathcal{L}_i \setminus \{U_i\}$, for $U_i \in \mathcal{L}_i^\top \setminus \mathcal{S}$, $i = 1, \ldots, k-1$, we obtain the desired flag of lattices.
(iv) It follows from the fact that

$$(U_1 \times Q_1) \cap (U_2 \times Q_2) = (U_1 \cap U_2) \times (Q_1 \cap Q_2)$$

for $U_i \in \mathcal{L}$, $Q_i \in \mathcal{M}$, $i = 1, 2$. □

In analogue, we have

Corollary 4.1.5. *The properties (i)–(iv) in Lemma 4.1.4 hold for representation lattices.*

Proof. Let $\mathcal{L}$ be a representation lattice. Then, (i) clearly holds. Moreover, since representation sublattices are precisely sublattices of representation lattices, (ii) and (iii) also hold.

Let X' be another Banach space and $\mathcal{M}$ be a lattice in X'. Then,

$$\{(U \times Q, G_U \times G_Q, \mathsf{h}_{U,V} \times \mathsf{h}_{Q,P}) \,:\, U, V \in \mathcal{L},\, Q, P \in \mathcal{M}, U \subset V, Q \subset P\} \tag{4.2}$$

gives $\mathcal{L} \times \mathcal{M}$ a structure of representation lattice. □

Definition 4.1.6. Let $\mathcal{L}$ be a representation lattice with structure $\{U, G_U, \mathsf{h}_{U,V}\}$ and $\mathcal{M}$ be a representation lattice with structure $\{Q, G_Q, \mathsf{h}_{Q,P}\}$. Then, the lattice $\mathcal{L} \times \mathcal{M}$ with the structure (4.2) is called *the product representation lattice* of $\mathcal{L}$ and $\mathcal{M}$. ◇

4.1.2 Algebraic Properties of Representation Lattices

Using Euler rings of compact Lie groups, we associate to a representation lattice an algebraic structure that serves to be the range of the lattice equivariant degree that we introduce later. We show that this algebraic structure is compatible with the usual lattice operations such as the inclusion and the product operation.

Definition 4.1.7. Let $\mathcal{L}$ be a representation lattice with structure $\{U, G_U, \mathsf{h}_{U,V}\}$. For $U \in \mathcal{L}$, let $A(G_U)$ be the Euler ring of the compact Lie group G_U (cf. Definition 2.3.1). Let

$$R(\mathcal{L}) := \Big\{ \sum_{U \in \mathcal{L}} (U, a_U) \,:\, a_U \in A(G_U) \Big\}, \tag{4.3}$$

which is a free (left) $\mathbb{Z}$-module with respect to

$$\begin{aligned} &\sum_{U \in \mathcal{L}} (U, a_U) + \sum_{U \in \mathcal{L}} (U, b_U) := \sum_{U \in \mathcal{L}} (U, a_U + b_U), \quad a_U, b_U \in A(G_U), \\ &k \sum_{U \in \mathcal{L}} (U, a_U) := \sum_{U \in \mathcal{L}} (U, k a_U), \quad k \in \mathbb{Z}. \end{aligned}$$

Define a ring multiplication on $R(\mathcal{L})$ by

$$\sum_{U \in \mathcal{L}} (U, a_U) \cdot \sum_{U \in \mathcal{L}} (U, b_U) := \sum_{U \in \mathcal{L}} (U, a_U * b_U), \quad a_U, b_U \in A(G_U), \tag{4.4}$$

where '$*$' stands for the Euler ring multiplication in $A(G_U)$. The free (left) $\mathbb{Z}$-module $R(\mathcal{L})$ together with the ring multiplication (4.4) is called the *associated ring* of $\mathcal{L}$. ◇

Reduction Map

Let $\mathcal{L}$ be a representation lattice and $\mathcal{S} \subset \mathcal{L}$ be a representation sublattice. Then, every $U \in \mathcal{L} \setminus \mathcal{S}$ has a unique minimal descendant in $\mathcal{S}$, which is given by the intersection of all the descendants of U in $\mathcal{S}$.

Definition 4.1.8. Let $\mathcal{L}$ be a representation lattice with structure $\{U, G_U, h_{U,V}\}$ and $\mathcal{S} \subset \mathcal{L}$ be a representation sublattice. Define the *reduction map* from $R(\mathcal{L})$ to $R(\mathcal{S})$ by

$$\begin{aligned} \Phi^{\mathcal{L}}_{\mathcal{S}} &: R(\mathcal{L}) \to R(\mathcal{S}) \\ (U, a) &\mapsto \begin{cases} (U_d, \mathsf{H}_{U,U_d}(a)), & \text{if } U \in \mathcal{L} \setminus \mathcal{S}, \\ (U, a), & \text{if } U \in \mathcal{S}, \end{cases} \end{aligned} \tag{4.5}$$

where $U \in \mathcal{L}$, $a \in A(G_U)$, U_d stands for the unique minimal descendant of U in $\mathcal{S}$ and H_{U,U_d} is the Euler ring homomorphism induced by h_{U,U_d} (cf. (2.21)). $\diamond$

The reduction map is compatible with the lattice inclusion.

Proposition 4.1.9. *Let $\mathcal{L}$ be a representation lattice, $\mathcal{S}, \mathcal{P} \subset \mathcal{L}$ be representation sublattices such that $\mathcal{L} \supset \mathcal{P} \supset \mathcal{S}$. Then, we have $\Phi^{\mathcal{L}}_{\mathcal{S}} = \Phi^{\mathcal{P}}_{\mathcal{S}} \circ \Phi^{\mathcal{L}}_{\mathcal{P}}$.*

Proof. Let $U \in \mathcal{L}$ and $a \in A(G_U)$. If $U \in \mathcal{S}$, then (U, a) is a fixed point of $\Phi^{\mathcal{L}}_{\mathcal{S}}$, $\Phi^{\mathcal{P}}_{\mathcal{S}}$ and $\Phi^{\mathcal{L}}_{\mathcal{P}}$. Thus, the statement holds.

Let $U \in \mathcal{L} \setminus \mathcal{S}$ and U_d be the unique minimal descendant of U in $\mathcal{S}$. Then,

$$\Phi^{\mathcal{L}}_{\mathcal{S}}\big((U, a)\big) = \big(U_d, \mathsf{H}_{U,U_d}(a)\big).$$

If $U \in \mathcal{P}$, then (U, a) is a fixed point of $\Phi^{\mathcal{L}}_{\mathcal{P}}$ and $\Phi^{\mathcal{P}}_{\mathcal{S}}((U, a)) = (U_d, \mathsf{H}_{U,U_d}(a))$. So $\Phi^{\mathcal{P}}_{\mathcal{S}} \circ \Phi^{\mathcal{L}}_{\mathcal{P}}\big(U, a\big)$ agrees with $\Phi^{\mathcal{L}}_{\mathcal{S}}\big((U, a)\big)$. Otherwise if $U \in \mathcal{L} \setminus \mathcal{P}$, then $\Phi^{\mathcal{L}}_{\mathcal{P}}((U, a)) = (U_c, \mathsf{H}_{U,U_c}(a))$, where U_c is the unique minimal descendant of U in $\mathcal{P}$. In the case $U_c \in \mathcal{S}$, we have $U_c = U_d$, by the uniqueness of minimal descendant. In the case $U_c \in \mathcal{P} \setminus \mathcal{S}$, U_d is the unique minimal descendant of U_c in $\mathcal{S}$. Consequently, in both cases, we have

$$\Phi^{\mathcal{P}}_{\mathcal{S}} \circ \Phi^{\mathcal{L}}_{\mathcal{P}}\big((U, a)\big) = \Phi^{\mathcal{P}}_{\mathcal{S}}\big((U_c, \mathsf{H}_{U,U_c}(a))\big) = \big(U_d, \mathsf{H}_{U,U_d}(a)\big).$$

$\square$

Example 4.1.10. Let $\mathcal{L}$ be the representation lattice given in Example 4.1.3 and $\mathcal{S}$ be a representation sublattice given by $\mathcal{S} = \mathcal{L} \setminus \{\Delta_{01}\}$ (cf. Figure 4.2). Let $\Phi^{\mathcal{L}}_{\mathcal{S}}$ be the reduction map defined by (4.5). Let $\mathsf{H}_{\Delta_{01},\mathbb{R}_5}$ be the ring homomorphism induced by the inclusion homomorphism $\mathsf{h}_{\Delta_{01},\mathbb{R}_5} : \mathbb{Z}_1 \to \mathbb{Z}_2$.

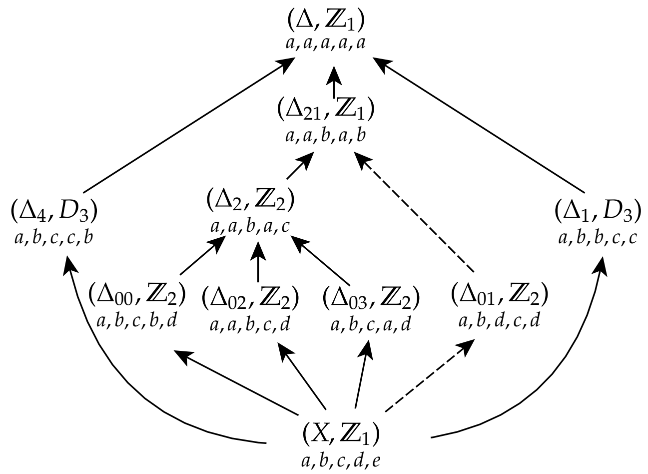


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Figure 4.2: A representation sublattice $\mathcal{S} \subset \mathcal{L}$, where $\mathcal{S} = \mathcal{L} \setminus \{\Delta_{01}\}$ and the dashed arrows are removed.

Then, $\Phi^{\mathcal{L}}_{\mathcal{S}}$ fixes all generators of $R(\mathcal{L})$ except

$$\Phi^{\mathcal{L}}_{\mathcal{S}}\big(\Delta_{01}, (\mathbb{Z}_2)\big) = \big(\mathbb{R}^5, \mathsf{H}_{\Delta_{01},\mathbb{R}_5}\big((\mathbb{Z}_2)\big)\big) = \big(\mathbb{R}^5, (\mathbb{Z}_1)\big),$$
$$\Phi^{\mathcal{L}}_{\mathcal{S}}\big(\Delta_{01}, (\mathbb{Z}_1)\big) = \big(\mathbb{R}^5, \mathsf{H}_{\Delta_{01},\mathbb{R}_5}\big((\mathbb{Z}_1)\big)\big) = \big(\mathbb{R}^5, 2(\mathbb{Z}_1)\big).$$

◇

Product Map

Let $\mathcal{L}$ be a representation lattice with structure $\{U, G_U, \mathsf{h}_{U,V}\}$ and $\mathcal{M}$ be a representation lattice with structure $\{P, G_P, \mathsf{h}_{P,Q}\}$. The projection homomorphisms on groups

$$\mathrm{proj}_U : G_U \times G_P \to G_U, \quad \mathrm{proj}_P : G_U \times G_P \to G_P$$

induce the inclusion homomorphisms on Euler rings (cf. (2.21))

$$\mathrm{inc}_U : A(G_U) \hookrightarrow A(G_U \times G_P),$$
$$\mathrm{inc}_P : A(G_P) \hookrightarrow A(G_U \times G_P).$$

Thus, we can define a product of $a \in A(G_U)$ and $b \in A(G_P)$ through

$$A(G_U) \times A(G_P) \hookrightarrow A(G_U \times G_P) \times A(G_U \times G_P) \xrightarrow{*} A(G_U \times G_P),$$

where $*$ is the ring multiplication in $A(G_U \times G_P)$, i.e.

$$a \star b := \mathrm{inc}_U(a) * \mathrm{inc}_P(b). \tag{4.6}$$

The product '$\star$' is compatible with representation lattice structure.

Proposition 4.1.11. *Let $\mathcal{L}$ be a representation lattice with structure $\{U, G_U, \mathsf{h}_{U,V}\}$ and $\mathcal{M}$ be a representation lattice with structure $\{P, G_P, \mathsf{h}_{P,Q}\}$. Then, the diagram in Figure 4.3 commutes, where '$\star$' is defined by (4.6) and H_* is the induced*

$$\begin{tikzcd}
A(G_U) \times A(G_P) \arrow[r, "\star"] \arrow[d, "\mathsf{H}_{U,V} \times \mathsf{H}_{P,Q}"] & A(G_U \times G_P) \arrow[d, "\mathsf{H}_{U\times P, V\times Q}"] \\
A(G_V) \times A(G_Q) \arrow[r, "\star"] & A(G_V \times G_Q)
\end{tikzcd}$$

Figure 4.3: A commutative diagram for Proposition 4.1.11.

homomorphism through h_* (cf. (2.21)).

Proof. By Theorem 2.3.4, $\mathsf{H}_{U\times P, V\times Q}$ is an Euler ring homomorphism. Thus, it suffices to show that the following diagram commutes.

$$\begin{tikzcd}
A(G_U) \arrow[r, hook, "\mathrm{inc}_U"] \arrow[d, "\mathsf{H}_{U,V}"] & A(G_U \times G_P) \arrow[d, "\mathsf{H}_{U\times P, V\times Q}"] \\
A(G_V) \arrow[r, hook, "\mathrm{inc}_V"] & A(G_V \times G_Q)
\end{tikzcd}$$

Let $(K) \in A(G_U)$. It follows from the definition of H_* that (cf. (2.21))

$$\mathsf{H}_{U,V}\big((K)\big) = \sum_{(\tilde{K})\in\Phi(G_V)} \chi_c((G_U/K)_{(\tilde{K})}/G_V)(\tilde{K}),$$
$$\mathsf{H}_{U\times P, V\times Q}\big((K\times G_P)\big) = \sum_{(K')\in\Phi(G_V\times G_Q)} \chi_c((G_U\times G_P/K\times G_P)_{(K')}/G_V\times G_Q)(K').$$

Note that G_P acts trivially on $G_U \times G_P/K \times G_P$, which implies that G_Q also acts trivially on $G_U \times G_P/K \times G_P$ through $\mathsf{h}_{U\times P, V\times Q}$. Therefore, $(G_U \times G_P/K \times G_P)_{(K')} \neq \emptyset$ if and only if $(K') = (\tilde{K} \times G_P)$ for some $\tilde{K}$ such that $(G_U/K)_{(\tilde{K})} \neq \emptyset$. Moreover, $(G_U\times G_P/K\times G_P)_{(K')}/G_V\times G_Q$ is G_V-homeomorphic to $(G_U/K)_{(\tilde{K})}/G_V$. Thus, we have

$$\mathsf{H}_{U\times P, V\times Q}\big((K\times G_P)\big) = \mathsf{H}_{U,V}\big((K)\big), \quad \forall\, (K) \in A(G_U).$$

□

Definition 4.1.12. Let $\mathcal{L}$ be a representation lattice with structure $\{U, G_U, \mathsf{h}_{U,V}\}$ and $\mathcal{M}$ be a representation lattice with structure $\{P, G_P, \mathsf{h}_{P,Q}\}$. Consider the product lattice $\mathcal{L} \times \mathcal{M}$ with the product structure (cf. (4.2)). Let $R(\mathcal{L})$, $R(\mathcal{M})$

and $R(\mathcal{L} \times \mathcal{M})$ be the associated rings defined by (4.3). Define a *product map* by

$$\begin{aligned} \bullet : R(\mathcal{L}) \times R(\mathcal{M}) &\to R(\mathcal{L} \times \mathcal{M}) \\ ((U, a), (P, b)) &\mapsto (U \times P, a \star b), \end{aligned} \tag{4.7}$$

where $U \in \mathcal{L}$, $P \in \mathcal{M}$, $a \in A(G_U)$, $b \in A(G_P)$ and $a \star b$ is defined by (4.6). $\diamond$

4.2 Lattice Equivariant Degree

In this section, we give the definition of lattice equivariant maps and formulate a degree theory for these maps, which we call the *lattice equivariant degree*. We show that this degree satisfies the usual properties of a topological degree, and moreover, it has algebraic properties compatible with the inclusion and the product of representation lattices.

In what follows, $\mathbb{R}$ stands for a parameter space, in which all groups act trivially.

Definition 4.2.1. Let $\mathcal{L}$ be a representation lattice in $\mathbb{R}^n$ with structure $\{U, G_U, \mathsf{h}_{U,V}\}$. An open bounded subset $\Omega \subset \mathbb{R} \times \mathbb{R}^n$ is called *$\mathcal{L}$-invariant*, if $\Omega \cap (\mathbb{R} \times U)$ is G_U-invariant, for every $U \in \mathcal{L}$. A continuous map $f : \overline{\Omega} \to \mathbb{R}^n$ is called *$\mathcal{L}$-equivariant*, if $f(\overline{\Omega} \cap (\mathbb{R} \times U)) \subset U$ and $f|_{\overline{\Omega} \cap (\mathbb{R} \times U)}$ is G_U-equivariant for every $U \in \mathcal{L}$. The map f is called *Ω-admissible*, if $f^{-1}(0) \cap \partial\Omega = \emptyset$. In this case, we say that the pair (f, Ω) is an *admissible pair*. Similarly, one defines *Ω-admissible* and *$\mathcal{L}$-equivariant* homotopies. $\diamond$

A degree theory can be defined for lattice equivariant maps on their admissible domains, using equivariant degrees and Euler ring homomorphisms. For simplicity, we consider only representation lattices $\mathcal{L}$ with a structure $\{U, G_U, \mathsf{h}_{U,V}\}$ such that

$$G_U = \Gamma_U \times S^1, \quad \text{for a finite group } \Gamma_U,\ U \in \mathcal{L}.$$

The resulting degree will be used for studying synchrony-related bifurcations in coupled cell networks, where Γ_U describes the quotient symmetry associated with a synchrony subspace U.

4.2.1 Definition and Basic Properties

Let $\mathcal{L}$ be a representation lattice with structure $\{U, G_U, \mathsf{h}_{U,V}\}$, where $G_U = \Gamma_U \times S^1$ for a finite group Γ_u. Recall that (cf. (4.3))

$$R(\mathcal{L}) := \Big\{ \sum_{U \in \mathcal{L}} (U, a_U) \ : \ a_U \in A(G_U) \Big\},$$

where $A(G_U)$ is the Euler ring of G_U. Denote by

$$R_k(\mathcal{L}) := \Big\{ \sum_{U\in\mathcal{L}} (U, a_U) \in R(\mathcal{L}) \, : \, a_U \in A_k(G_U) \Big\}, \quad \text{for } k = 0, 1,$$

where $A_k(G)$ is defined by (2.25).

The lattice equivariant degree we define below takes value in $R(\mathcal{L})$ and is directly related to the equivariant degree with one parameter $\Gamma_U \times S^1$-Deg for $U \in \mathcal{L}$ discussed in Subsection 3.1.2.

Definition 4.2.2. Let $\mathcal{L}$ be a representation lattice in $\mathbb{R}^n$ with the structure given by $\{U, G_U, \mathsf{h}_{U,V}\}$, where G_U is of form $\Gamma_U \times S^1$ for a finite group Γ_U. Let $\Omega \subset \mathbb{R} \times \mathbb{R}^n$ be an open bounded $\mathcal{L}$-invariant subset and $f : \overline{\Omega} \to \mathbb{R}^n$ be an Ω-admissible $\mathcal{L}$-equivariant map.

For the minimal element $U_{\min} \in \mathcal{L}$, define

$$a_{U_{\min}} := \Gamma_{U_{\min}} \times S^1\text{-Deg}\,(f|_{\Omega\cap(\mathbb{R}\times U_{\min})}, \Omega \cap (\mathbb{R} \times U_{\min})). \tag{4.8}$$

Suppose that $a_{U'}$ is defined for all $U' < U$. Then, define

$$a_U := \Gamma_U \times S^1\text{-Deg}\,(f|_{\Omega\cap(\mathbb{R}\times U)}, \Omega \cap (\mathbb{R} \times U)) - \sum_{U'<U} \mathsf{H}_{U',U}(a_{U'}), \tag{4.9}$$

where $\mathsf{H}_{U',U}$ is the Euler ring homomorphism induced by $\mathsf{h}_{U',U}$ (cf. (2.21)). The *lattice equivariant degree* of f in Ω is then defined by

$$\mathcal{L}\text{-Deg}\,(f, \Omega) := \sum_{U\in\mathcal{L}} (U, a_U) \in R_1(\mathcal{L}). \tag{4.10}$$

$\diamond$

It follows from Definition 4.2.2 that in the case of a trivial lattice structure: $\mathcal{L}$ is composed of a single element $\mathbb{R}^n$ as a representation of $\Gamma \times S^1$, the lattice equivariant degree $\mathcal{L}$-Deg coincides with the equivariant degree $\Gamma \times S^1$-Deg with one parameter.

We show that the degree defined by (4.8)–(4.10) satisfies basic properties of a degree theory.

Theorem 4.2.3. *Let $\mathcal{L}$ be a representation lattice in $\mathbb{R}^n$ with the structure given by $\{U, G_U, \mathsf{h}_{U,V}\}$, where $G_U = \Gamma_U \times S^1$ for a finite group Γ_U. Then, the function* $\mathcal{L}$-Deg *defined by (4.8)–(4.10) satisfies:*

(i) (Existence) *Suppose that $\mathcal{L}$-Deg$\,(f, \Omega) = \sum (U, a_U)$ and $a_U \neq 0$ for some $U \in \mathcal{L}$. Write $a_U = \sum n_H(H)$. If (H) is such that $n_H \neq 0$, then*

$$f^{-1}(0) \cap \big(\Omega^H \cap (\mathbb{R} \times U)\big) \neq \emptyset,$$

where Ω^H is the fixed-point subspace in Ω (cf. (2.8)).

(ii) (Homotopy Invariance) *If $H : [0,1] \times \overline{\Omega} \to \mathbb{R}^n$ is an Ω-admissible $\mathcal{L}$-equivariant homotopy, then*

$$\mathcal{L}\text{-Deg}\,(H(t,\cdot),\Omega) = constant, \quad \forall t \in [0,1].$$

(iii) (Additivity) *If $\Omega_1, \Omega_2 \subset \Omega$ are disjoint open bounded $\mathcal{L}$-invariant subsets such that $f^{-1}(0) \cap \Omega \subset \Omega_1 \cup \Omega_2$ and f is Ω_i-admissible for $i = 1,2$, then*

$$\mathcal{L}\text{-Deg}\,(f,\Omega) = \mathcal{L}\text{-Deg}\,(f,\Omega_1) + \mathcal{L}\text{-Deg}\,(f,\Omega_2).$$

(iv) (Suspension) *Let $\mathcal{M}$ be a representation lattice in $\mathbb{R}^m$, $\mathrm{Id} : V' \to V'$ be the identity map and $\Omega' \subset \mathbb{R}^m$ be an open bounded $\mathcal{M}$-invariant neighborhood of 0. Then, $\mathcal{L} \times \mathcal{M}$-$Deg^t\,(f \times \mathrm{Id}, \Omega \times \Omega')$ is well-defined. Moreover,*

$$\mathcal{L} \times \mathcal{M}\text{-}Deg^t\,(f \times \mathrm{Id}, \Omega \times \Omega') = \mathcal{L}\text{-Deg}\,(f,\Omega),$$

under the identification: $U \mapsto U \times P_{\min}$ and $(H) \mapsto (H \times \Gamma_{P_{\min}})$ for every $U \in \mathcal{L}$ and $(H) \in \Phi_1(\Gamma_U \times S^1)$, where $P_{\min} \in \mathcal{M}$ is the minimal element.

Proof. (ii) and (iii) follow immediately from the corresponding properties of the equivariant degree with one parameter (cf. Theorem 3.1.6).

To show (i), assume that $f^{-1}(0) \cap \left(\Omega^H \cap (\mathbb{R} \times U)\right) = \emptyset$. If $U = U_{\min}$, then by the existence property of the equivariant degree, we have that $a_{U_{\min}} = \Gamma_U \times S^1\text{-Deg}\,(f|_{\Omega\cap(\mathbb{R}\times U_{\min})}, \Omega \cap (\mathbb{R} \times U_{\min}))$ has a zero (H)-coefficient, which is a contradiction. Assume that the statement holds for all $U' < U$. By assumption, a_U has a nonzero (H)-coefficient. By the existence property of the equivariant degree, $\Gamma_U \times S^1\text{-Deg}\,(f|_{\Omega\cap(\mathbb{R}\times U)}, \Omega \cap (\mathbb{R} \times U))$ has a zero (H)-coefficient. It follows from the definition of a_U that there exists $U' < U$ such that $\mathsf{H}_{U',U}(a_{U'})$ has a nonzero (H)-coefficient. Thus, there exists an $H' \subset \Gamma_{U'} \times S^1$ such that $H = \mathsf{h}^{-1}_{U',U}(H')$ and the (H')-coefficient in $a_{U'}$ is nonzero. By the induction assumption, we have

$$f^{-1}(0) \cap \left(\Omega^{H'} \cap (\mathbb{R} \times U')\right) \neq \emptyset.$$

Let $x \in f^{-1}(0) \cap \left(\Omega^{H'} \cap (\mathbb{R} \times U')\right)$ and $g \in H$. Then, $\mathsf{h}_{U',U}(g) \in H'$, so $\mathsf{h}_{U',U}(g)x = x$. By the definition of representation lattice (cf. Definition 4.1.2(ii)), we have then

$$gx = \mathsf{h}_{U',U}(g)x = x.$$

It follows that $x \in f^{-1}(0) \cap \left(\Omega^H \cap (\mathbb{R} \times U)\right)$. In particular, $f^{-1}(0) \cap \left(\Omega^H \cap (\mathbb{R} \times U)\right) \neq \emptyset$, which is a contradiction to our initial assumption.

To show (iv), let

$$\mathcal{L} \times \mathcal{M}\text{-Deg}^t\,(f \times \mathrm{Id}, \Omega \times \Omega') = \sum (U \times P, b_{U\times P})$$
$$\mathcal{L}\text{-Deg}\,(f,\Omega) = \sum (U, a_U).$$

Let $U_{\min} \in \mathcal{L}$ and $P_{\min} \in \mathcal{M}$ be the minimal element, respectively. Let $\Gamma_{U_{\min}}$ act trivially on $P_{\min}$ and $\Gamma_{P_{\min}}$ act trivially on $\mathbb{R} \times U_{\min}$. Then, $\Omega \cap (\mathbb{R} \times U_{\min})$ becomes $\Gamma_{U_{\min}} \times \Gamma_{P_{\min}} \times S^1$-invariant and the restricted f is a $\Gamma_{U_{\min}} \times \Gamma_{P_{\min}} \times S^1$-equivariant map. Similarly, $\Omega' \cap P_{\min}$ becomes $\Gamma_{U_{\min}} \times \Gamma_{P_{\min}}$-invariant and the Id is $\Gamma_{U_{\min}} \times \Gamma_{P_{\min}}$-equivariant. By the suspension property of the equivariant degree, we have

$$\begin{aligned} b_{U_{\min} \times P_{\min}} &= \Gamma_{U_{\min}} \times \Gamma_{P_{\min}} \times S^1\text{-Deg}^t\,(f \times \mathrm{Id}, (\Omega \cap (\mathbb{R} \times U_{\min})) \times (\Omega' \cap P_{\min})) \\ &= \Gamma_{U_{\min}} \times \Gamma_{P_{\min}} \times S^1\text{-Deg}^t\,(f, \Omega \cap (\mathbb{R} \times U_{\min})) \\ &\simeq a_{U_{\min}}, \end{aligned}$$

where '$\simeq$' means identifying $(H \times \Gamma_{P_{\min}})$ with (H), for $H \subset \Gamma_{U_{\min}} \times S^1$. Using the suspension property of the equivariant degree inductively, one shows

$$b_{U \times P_{\min}} \simeq a_U, \quad b_{U_{\min} \times P} = 0, \quad \text{for } P > P_{\min}. \tag{4.11}$$

Let $U > U_{\min}$ and $P > P_{\min}$. We show that $b_{U \times P} = 0$. Assume that $b_{U' \times P'} = 0$ for all $U' \times P' < U \times P$ and $P' > P_{\min}$. Then, we have

$$\begin{aligned} b_{U \times P} &= \Gamma_U \times \Gamma_P \times S^1\text{-Deg}^t\,(f, (\Omega \cap (\mathbb{R} \times U)) - \sum_{U' \times P' < U \times P} \mathsf{H}_{U' \times P', U \times P}(b_{U' \times P'}) \\ &\overset{(4.11)}{=} \Gamma_U \times \Gamma_P \times S^1\text{-Deg}^t\,(f, (\Omega \cap (\mathbb{R} \times U)) - \sum_{U' \leq U} \mathsf{H}_{U' \times P_{\min}, U \times P}(b_{U' \times P_{\min}}) \\ &\overset{(4.11)}{\simeq} \Gamma_U \times S^1\text{-Deg}^t\,(f, (\Omega \cap (\mathbb{R} \times U)) - \sum_{U' \leq U} \mathsf{H}_{U', U}(a_{U'}) = 0. \end{aligned}$$

Thus, (iv) holds.

□

4.2.2 Algebraic Properties

We discuss further algebraic properties of lattice equivariant degrees, with respect to the reduction map (cf. Definition 4.1.8) and the product map (cf. Definition 4.1.12) on representation lattices.

Proposition 4.2.4. (Reduction Homomorphism) *Let $\mathcal{L}$ be a representation lattice in $\mathbb{R}^n$ and $\mathcal{S} \subset \mathcal{L}$ be a representation sublattice. Let $\Phi^{\mathcal{L}}_{\mathcal{S}}$ be the reduction map from $\mathcal{L}$ to $\mathcal{S}$ defined by (4.5). Then, we have*

$$\Phi^{\mathcal{L}}_{\mathcal{S}}(\mathcal{L}\text{-Deg}\,(f, \Omega)) = \mathcal{S}\text{-}Deg(f, \Omega), \tag{4.12}$$

for every admissible $\mathcal{L}$-equivariant pair (f, Ω).

Proof. Let (f,Ω) be an admissible $\mathcal{L}$-equivariant pair. By Proposition 4.1.9 and Lemma 4.1.4(iii), we can assume without loss of generality, that $\mathcal{S} = \mathcal{L}\setminus\{U_o\}$ for some $U_o \in \mathcal{L}^\top$. Let

$$\mathcal{L}\text{-Deg}\,(f,\Omega) = \sum_{U\in\mathcal{L}}(U,a_U), \quad \mathcal{S}\text{-Deg}(f,\Omega) = \sum_{U\in\mathcal{S}}(U,b_U).$$

By the definition of a_U, we have $b_U = a_U$ if $U_o \nless U$. Let $U \in \mathcal{L}$ be such that $U_o < U$ and U_+ be the unique immediate descendant of U_o. Then, $U_+ \le U$. In case $U = U_+$, we have

$$\begin{aligned}
b_{U_+} &= \Gamma_{U_+}\times S^1\text{-Deg}(f,\Omega\cap(\mathbb{R}\times U_+)) - \sum_{\substack{U'<U_+\\ U'\in\mathcal{S}}}\mathsf{H}_{U',U_+}(b_{U'})\\
&= a_{U_+} + \sum_{\substack{U'<U_+\\ U'\in\mathcal{L}}}\mathsf{H}_{U',U_+}(a_{U'}) - \sum_{\substack{U'<U_+\\ U'\in\mathcal{S}}}\mathsf{H}_{U',U_+}(b_{U'})\\
&= a_{U_+} + \mathsf{H}_{U_o,U_+}(a_{U_o}) + \sum_{\substack{U'<U_+\\ U'\in\mathcal{S}}}\mathsf{H}_{U',U_+}(a_{U'}) - \sum_{\substack{U'<U_+\\ U'\in\mathcal{S}}}\mathsf{H}_{U',U_+}(b_{U'})\\
&= a_{U_+} + \mathsf{H}_{U_o,U_+}(a_{U_o})
\end{aligned}\tag{4.13}$$

For $U > U_+$, suppose that $a_{U'} = b_{U'}$ for all $U_+ < U' < U$, then we have

$$\begin{aligned}
b_U &= \Gamma_U\times S^1\text{-Deg}(f,\Omega\cap(\mathbb{R}\times U)) - \sum_{\substack{U'<U\\ U'\in\mathcal{S}}}\mathsf{H}_{U',U}(b_{U'})\\
&= a_U + \sum_{\substack{U'<U\\ U'\in\mathcal{L}}}\mathsf{H}_{U',U}(a_{U'}) - \sum_{\substack{U'<U\\ U'\in\mathcal{S}}}\mathsf{H}_{U',U}(b_{U'})\\
&= a_U + \mathsf{H}_{U_+,U}(a_{U_+}) + \mathsf{H}_{U_o,U}(a_{U_o}) - \mathsf{H}_{U_+,U}(b_{U_+})\\
&\overset{(4.13)}{=} a_U + \mathsf{H}_{U_o,U}(a_{U_o}) - \mathsf{H}_{U_o,U_+}\mathsf{H}_{U_+,U}(a_{U_o}),
\end{aligned}$$

which implies that $b_U = a_U$ by Theorem 2.3.4(ii). □

Recall that the $\Gamma\times S^1$-equivariant degree with one parameter has a multiplication property corresponding to the $A(\Gamma)$-module structure on $A_1(\Gamma\times S^1)$, which coincides with the Euler ring multiplication in $A(\Gamma\times S^1)$ (cf. Subsection 3.1.3). We show that this multiplication can be naturally extended to the lattice equivariant degree.

Proposition 4.2.5. (Product Property) *Let $\mathcal{L}$ be a representation lattice with structure $\{U,\Gamma_U\times S^1,\mathsf{h}_{U,V}\}$ in $\mathbb{R}^n$ and $\mathcal{M}$ be a representation lattice with structure $\{P,\Gamma_P,\mathsf{h}_{P,Q}\}$, where Γ_* are finite groups. Suppose that $\Omega\subset\mathbb{R}\times\mathbb{R}^n$ (resp. $\Omega'\subset\mathbb{R}^m$) is an open bounded $\mathcal{L}$-invariant (resp. $\mathcal{M}$-invariant) subset and $f:\overline{\Omega}\to\mathbb{R}^n$*

(resp. $g : \overline{\Omega}' \to \mathbb{R}^m$) is an Ω-admissible $\mathcal{L}$-equivariant (resp. Ω'-admissible $\mathcal{M}$-equivariant) map. Then, we have

$$\mathcal{L} \times \mathcal{M}\text{-}Deg^t(f \times g, \Omega \times \Omega') = \mathcal{L}\text{-Deg}\,(f, \Omega) \bullet \mathcal{M}\text{-}Deg(g, \Omega'), \tag{4.14}$$

where "$\bullet$" is defined by (4.7).

Proof. Let $\mathcal{L} \times \mathcal{M}\text{-Deg}^t(f \times g, \Omega \times \Omega') = \sum(U \times P, a_{U\times P})$, $\mathcal{L}\text{-Deg}\,(f, \Omega) = \sum_{U\in\mathcal{L}}(U, b_U)$ and $\mathcal{M}\text{-Deg}(g, \Omega') = \sum_{P\in\mathcal{M}}(P, c_P)$. It sufficies to show

$$a_{U\times P} = b_U \star c_P.$$

Denote by $U_{\min}$ and $P_{\min}$ the minimal element of $\mathcal{L}$ and $\mathcal{M}$ respectively. Then,

$$\begin{aligned}
&a_{U_{\min}\times P_{\min}} = \Gamma_{U_{\min}} \times \Gamma_{P_{\min}} \times S^1\text{-Deg}^t\,(f \times g, (\Omega \cap (\mathbb{R} \times U_{\min})) \times (\Omega' \cap P_{\min}))\\
&= \Gamma_{U_{\min}} \times \Gamma_{P_{\min}} \times S^1\text{-Deg}^t\,(f, \Omega \cap (\mathbb{R} \times U_{\min}))) * \Gamma_{U_{\min}} \times \Gamma_{P_{\min}}\text{-Deg}(g, \Omega' \cap P_{\min})\\
&= \Gamma_{U_{\min}} \times S^1\text{-Deg}^t\,(f, \Omega \cap (\mathbb{R} \times U_{\min}))) \star \Gamma_{P_{\min}}\text{-Deg}(g, \Omega' \cap P_{\min})\\
&= b_{U_{\min}} \star c_{P_{\min}}.
\end{aligned}$$

Assume that $a_{U'\times P'} = b_{U'} \star c_{P'}$ for all U', P' such that $U' \times P' < U \times P$. Then, we have

$$\begin{aligned}
a_{U\times P} &= \sum_{U'\le U} b_{U'} \star \sum_{P'\le P} c_{P'} - \sum_{U'\times P'<U\times P} \mathsf{H}_{U'\times P', U\times P}(a_{U'\times P'})\\
&= \sum_{U'\le U} b_{U'} \star \sum_{P'\le P} c_{P'} - \sum_{U'\times P'<U\times P} \mathsf{H}_{U'\times P', U\times P}(b_{U'} \star c_{P'})\\
&= \sum_{U'\le U} b_{U'} \star \sum_{P'\le P} c_{P'} - \sum_{U'\times P'<U\times P} \mathsf{H}_{U', U}(b_{U'}) \star \mathsf{H}_{P', P}(c_{P'}),\\
&= b_U \star c_P,
\end{aligned}$$

where the second last equality is based on Proposition 4.1.11. □

4.2.3 Extension to Infinite-dimensional Vector Spaces

In this subsection, we extend the lattice equivariant degree to infinite-dimensional lattice representations for compact lattice equivariant vector fields. The desired approximation of compact maps by finite-dimensional maps is based on an equivariant version of the Schauder projection.

In what follows, W is an infinite-dimensional real Banach space. Recall that for a bounded subset $X \subset \mathbb{R} \times W$, a continuous map $F : X \to W$ is called *compact*, if $\overline{F(X)}$ is compact in W; and F is called *finite dimensional*, if $F(X)$ is contained in a finite dimensional subspace of W. Let $\pi : \mathbb{R} \times W \to W$ be the projection on W and $F : X \to W$ be a compact map, then $\pi - F$ is called a *compact vector field*.

Definition 4.2.6. Let G be a compact Lie group and W be a Banach representation of G. Let $N = \{c_1, c_2, \dots, c_n\} \subset W$ be a finite set. For any fixed $\varepsilon > 0$, let

$$U(N,\varepsilon) = \bigcup_{i=1}^{n} \bigcup_{g \in G} gB(c_i,\varepsilon), \tag{4.15}$$

where the symbol gA means the union of all elements gx for $x \in A$ and $B(c_i,\varepsilon)$ stands for the open ε-disk around c_i in W. For $x, y \in W$, define $\rho_\varepsilon(x,y) = \max\{0, \varepsilon - \|x-y\|\}$. We call the map $p_{N,\varepsilon} : U(N,\varepsilon) \to W$ defined by

$$p_{N,\varepsilon}(x) = \frac{\sum_{i=1}^{n} \int_G \rho_\varepsilon(g^{-1}x, c_i) gc_i \, d\mu(g)}{\sum_{i=1}^{n} \int_G \rho_\varepsilon(g^{-1}x, c_i) \, d\mu(g)}, \tag{4.16}$$

for μ being the Haar measure of G, the *equivariant Schauder projection*.

Note that the denominator of $p_{N,\varepsilon}$ is never zero. For every $x \in U(N,\varepsilon)$, there exists $i \in \{1, 2, \dots, n\}$, $g \in G$ such that $x = gy$ for some $y \in B(c_i,\varepsilon)$. That is, $x \in U(N,\varepsilon)$ if and only if $\|g^{-1}x - c_i\| < \varepsilon$, which implies that $\rho_\varepsilon(g^{-1}x, c_i) > 0$.

The equivariant Schauder projection has the following properties.

Lemma 4.2.7. *Let G be a compact Lie group and W be an isometric Banach representation of G. Let $N = \{c_1, c_2, \dots, c_n\} \subset W$ be a finite set and $\varepsilon > 0$. Let $U(N,\varepsilon)$ be given by (4.15) and $p_{N,\varepsilon}$ be given by (4.16). Then,*

(i) $p_{N,\varepsilon}$ is G-equivariant;

(ii) $p_{N,\varepsilon}$ is a finite-dimensional map;

(iii) $\|x - p_{N,\varepsilon}(x)\| < \varepsilon$, for all $x \in U(N,\varepsilon)$.

Proof. (i) Let $a(x)$ be the numerator of $p_{N,\varepsilon}(x)$ and $b(x)$ be the denominator of $p_{N,\varepsilon}(x)$. We show that the map a is G-equivariant and b is G-invariant. Let $g_o \in G$. Then,

$$\begin{aligned} a(g_o^{-1}x) &= \sum_{i=1}^{n} \int_G \rho_\varepsilon(g^{-1}g_o^{-1}x, c_i) gc_i \, d\mu(g) = \sum_{i=1}^{n} \int_G \rho_\varepsilon((g_o g)^{-1}x, c_i) g_o^{-1} g_o g c_i \, d\mu(g) \\ &= g_o^{-1} \sum_{i=1}^{n} \int_G \rho_\varepsilon((g_o g)^{-1}x, c_i)(g_o g)c_i \, d\mu(g) = g_o^{-1} a(x), \end{aligned}$$

and

$$b(g_o^{-1}x) = \sum_{i=1}^{n} \int_G \rho_\varepsilon(g^{-1}g_o^{-1}x, c_i) \, d\mu(g) = \sum_{i=1}^{n} \int_G \rho_\varepsilon((g_o g)^{-1}x, c_i) \, d\mu(g) = b(x).$$

Thus, $p_{N,\varepsilon}$ is G-equivariant. (ii) Note that the G-orbit of c_i is a finite-dimensional smooth manifold of W, thus is contained in a subspace $W_i \subset W$ with $\dim W_i < \infty$, for $i = 1, 2, \dots, n$. It follows that $\int_G \rho_\varepsilon(g^{-1}x, c_i) g c_i \, d\mu(g) \in W_i$ and $p_{N,\varepsilon}(x) \in \operatorname{span}\{W_1, W_2, \dots, W_n\}$ for all $x \in W$.
(iii) Let $x \in U(N, \varepsilon)$. Assume that $\rho_\varepsilon(g^{-1}x, c_i) \neq 0$. Then, $\|g^{-1}x - c_i\| < \varepsilon$. Thus, $\|x - gc_i\| < \varepsilon$, since G acts isometrically on W. Therefore,

$$\|x - p_{N,\varepsilon}(x)\| = \left\| \frac{\sum_{i=1}^{n} \int_G \rho_\varepsilon(g^{-1}x, c_i)(x - gc_i) \, d\mu(g)}{\sum_{i=1}^{n} \int_G \rho_\varepsilon(g^{-1}x, c_i) \, d\mu(g)} \right\| \leq \frac{\sum_{i=1}^{n} \int_G \rho_\varepsilon(g^{-1}x, c_i) \, \|x - gc_i\| \, d\mu(g)}{\sum_{i=1}^{n} \int_G \rho_\varepsilon(g^{-1}x, c_i) \, d\mu(g)} < \varepsilon.$$

□

We have the following approximation theorem.

Proposition 4.2.8. *Let W be an infinite-dimensional real Banach space and $\mathcal{T}$ be representation lattice with structure $\{Y, G_Y, \mathsf{h}_{Y,Y'}\}$ in W. Let $X \subset \mathbb{R} \times W$ be a bounded $\mathcal{T}$-invariant subset and $F : X \to W$ be a $\mathcal{T}$-equivariant compact map. Then, for every $\varepsilon > 0$, there exists a $\mathcal{T}$-equivariant finite-dimensional map $F_\varepsilon : X \to W$ such that*

$$\|F(x) - F_\varepsilon(x)\| < \varepsilon, \quad \textit{for all } x \in X.$$

Proof. For convenience, we numerate the elements of $\mathcal{T}$ as $Y_1, Y_2, \dots, Y_m$ such that

$$Y_i \subset Y_j \implies i \leq j.$$

Based on Lemma 4.2.7, we define F_ε inductively on $Y = Y_i$ using the equivariant Schauder projection. Set $\varepsilon = \varepsilon_1$.

For $Y = Y_1$, since F is a compact map, $\overline{F(X \cap (\mathbb{R} \times Y_1))}$ is a compact set in W. Thus, there exists a finite set $N_1 = \{c_1, c_2, \dots, c_{n_1}\} \subset Y_1$ such that the set $U(N_1, \varepsilon_1)$ defined by (4.15) covers $\overline{F(X \cap (\mathbb{R} \times Y_1))}$. Let p_{N_1,ε_1} be given by (4.16) and define

$$F_{\varepsilon_1}(x) = p_{N_1,\varepsilon_1}(F(x)), \quad \forall\, x \in X \cap (\mathbb{R} \times Y_1).$$

For $Y = Y_2$, choose $\varepsilon_2 > 0$ such that $\varepsilon_2 < \varepsilon_1$ and

$$\{y \in Y_2 \,:\, \operatorname{dist}(y, F(X \cap (\mathbb{R} \times Y_1)) < \varepsilon_2\} \subset U(N_1, \varepsilon_1).$$

Since $\overline{F(X \cap (\mathbb{R} \times Y_2))}$ is compact, there exists a finite set

$$N_2 = \{c_{n_1+1}, c_{n_1+2}, \dots, c_{n_1+n_2}\} \subset Y_2 \setminus U(N_1, \varepsilon_1)$$

such that $U(N_2, \varepsilon_2)$ defined by (4.15) covers $\overline{F(X \cap (\mathbb{R} \times Y_2))} \setminus U(N_1, \varepsilon_1)$. Note that by the choice of ε_2, we have

$$\operatorname{dist}(c_{n_1+j}, F(X \cap (\mathbb{R} \times Y_1)) \geq \varepsilon_2, \quad \forall\, j = 1, 2, \dots, n_2. \tag{4.17}$$

Define $p_{N_2,\varepsilon_2} : U(N_1, \varepsilon_1) \cup U(N_2, \varepsilon_2) \to W$ by

$$p_{N_2,\varepsilon_2}(x) = \frac{\sum_{i=1}^{n_1} \int_{G_{Y_1}} \rho_{\varepsilon_1}(g^{-1}x, c_i) g c_i \, d\mu(g) + \sum_{j=1}^{n_2} \int_{G_{Y_2}} \rho_{\varepsilon_2}(g^{-1}x, c_{n_1+j}) g c_{n_1+j} \, d\mu(g)}{\sum_{i=1}^{n_1} \int_{G_{Y_1}} \rho_{\varepsilon_1}(g^{-1}x, c_i) \, d\mu(g) + \sum_{j=1}^{n_2} \int_{G_{Y_2}} \rho_{\varepsilon_2}(g^{-1}x, c_{n_1+j}) \, d\mu(g)}.$$

It can be verified that p_{N_2,ε_2} is G_{Y_2}-equivariant in Y_2 (noting the compatibility condition (ii) of Definition 4.1.2), finite-dimensional and satisfies $\|x - p_{N_2,\varepsilon_2}(x)\| < \varepsilon_1 = \varepsilon$. Let

$$F_{\varepsilon_2}(x) = p_{N_2,\varepsilon_2}(F(x)), \quad \forall\, x \in X \cap (\mathbb{R} \times Y_2).$$

It should be noted that by (4.17), F_{ε_2} coincides with F_{ε_1} on $X \cap (\mathbb{R} \times Y_1)$. Thus, F_{ε_2} is a finite dimensional ε-approximation of F such that F_{ε_2} is lattice-equivariant with respect to the representation sublattice $\{Y_1, Y_2\}$ of $\mathcal{T}$.

By iterating the above procedure until $Y = Y_m$, we obtain the desired map F_ε given by F_{ε_m}.

□

Let $\mathcal{T}$ be a representation lattice in W. Let $O \subset \mathbb{R} \times W$ be a $\mathcal{T}$-invariant open bounded subset and $F : \overline{O} \to W$ be a $\mathcal{T}$-equivariant compact map. By Proposition 4.2.8, for given $\varepsilon > 0$, F has a $\mathcal{T}$-equivariant finite-dimensional approximation $F_\varepsilon : \overline{O} \to W$ such that $\|F_\varepsilon(x) - F_1(x)\| < \varepsilon$, for $x \in \overline{O}$. Suppose that $F_\varepsilon(\overline{O}) \subset W_*$ for a finite-dimensional subspace $W_* \subset W$. Set

$$\mathcal{T}_* := \{Y \cap W_* \; : \; Y \in \mathcal{T}\}.$$

We define the *lattice equivariant degree* of $\pi - F$ in O by

$$\mathcal{T}\text{-Deg}^t(\pi - F, O) := \mathcal{T}_*\text{-Deg}^t(\pi - F_\varepsilon|_{O\cap(\mathbb{R}\times W_*)}, O \cap (\mathbb{R} \times W_*)), \tag{4.18}$$

where the function $\mathcal{T}_*$-Deg^t on the right hand side is defined by (4.10).

By a standard argument, one shows that the definition is independent of the choice of approximation F_ε and W_*. Moreover, the defined lattice equivariant degree by (4.18) satisfies similar properties as listed in Theorem 4.2.3 with f replaced by compact vector fields.

4.3 Bifurcations in Coupled Cell Systems

In the same way how equivariant degrees can be used for equivariant bifurcations, lattice equivariant degrees are suitable for studying synchrony-related bifurcations in coupled dynamical systems.

Homogeneous coupled cell networks are coupled cell networks whose cells are all input-equivalent. They provide basic models for studying fully synchronized states in networks and their breaking through bifurcations.

Recall that a homogeneous coupled cell system is of form

$$\begin{aligned}
\dot{x}_1 &= f_o(\lambda; x_1; x_{i_1}, \dots, x_{i_s}) \\
\dot{x}_2 &= f_o(\lambda; x_2; x_{j_1}, \dots, x_{j_s}), \\
&\cdots \\
\dot{x}_n &= f_o(\lambda; x_n; x_{k_1}, \dots, x_{k_s}),
\end{aligned} \tag{4.19}$$

where $\lambda \in \mathbb{R}$ is a bifurcation parameter, $x_i \in \mathbb{R}^k$ are cell variables of internal dimension k and $f_o : \mathbb{R} \times \mathbb{R}^k \times (\mathbb{R}^k)^s \to \mathbb{R}^k$ is some function of class C^1 determining the internal cell dynamics.

It is convenient to write (4.19) in a more compact form. Let

$$x = \begin{pmatrix} x_1 \\ x_2 \\ \cdots \\ x_n \end{pmatrix} \in (\mathbb{R}^k)^n, \quad f(\lambda, x) = \begin{pmatrix} f_o(\lambda; x_1; x_{i_1}, \dots, x_{i_s}) \\ f_o(\lambda; x_2; x_{j_1}, \dots, x_{j_s}) \\ \cdots \\ f_o(\lambda; x_n; x_{k_1}, \dots, x_{k_s}) \end{pmatrix}$$

Then, (4.19) is equivalent to

$$\dot{x} = f(\lambda, x). \tag{4.20}$$

Assume that $x = x_o \in (\mathbb{R}^k)^n$ is an equilibrium of (4.20). We are interested in studying the Hopf bifurcations of x_o as λ varies. The basic assumptions are (E1), (B1), (B2) from Section 3.2 which assume an isolated bifurcation center (λ_o, x_o) that cannot bifurcate to steady states.

As shown in Subsection 2.1.2, depending on how the cells $x_1, x_2, \dots, x_n$ are coupled together, (4.20) may admit various flow-invariant subspaces given by equalities of cell coordinates, i.e. the synchrony subspaces. For example, a homogeneous coupled cell system always has

$$\Delta_0 = \{x : x_1 = x_2 = \cdots = x_n\}$$

as a synchrony subspace, which is composed of states of *full synchrony*.

In general, the set of all synchrony subspaces admitted by a coupled cell system forms a lattice under set inclusion. We assume

(L1) (4.20) admits a lattice $\mathcal{L}$ of synchrony subspaces as flow-invariant subspaces, for all $\lambda \in \mathbb{R}$;

(L2) $\mathcal{L}$ has a structure $\{\Delta, \Gamma_\Delta, \mathsf{h}_{\Delta,\Delta'}\}$ of representation lattice for finite groups Γ_Δ such that f is $\mathcal{L}$-equivariant.

Here, the group Γ_Δ is reserved to express the quotient symmetry of (4.20) when restricted to the synchrony subspace Δ. In case there is no quotient symmetry on Δ, we set $\Gamma_\Delta = \mathbb{Z}_1$.

We also assume that

(E2) $x_o = 0 \in \Delta_0$.

We are interested in studying the *synchrony-breaking Hopf bifurcations* of x_o, where the fully synchronized equilibrium x_o losses its stability and breaks into oscillating states of multiple number of synchronized clusters.

Definition 4.3.1. Let $x \in (\mathbb{R}^k)^n$ and $\mathcal{L}$ be a lattice of synchrony subspaces of (4.20). If $x \in \Delta$ for some $\Delta \in \mathcal{L}$, then we say that x is *of synchrony type* Δ. If moreover, Δ is the smallest element in $\mathcal{L}$ that contains x, then we say that x is of *proper* synchrony type Δ. Similarly, a function $x : \mathbb{R} \to (\mathbb{R}^k)^n$ is *of (proper) synchrony type* Δ, if $x(t)$ is of (proper) synchrony type Δ, for all $t \in \mathbb{R}$. $\diamond$

4.3.1 Degree Approach

Following the same lines of Subsection 3.2.1, a bifurcation invariant can be defined for (λ_o, x_o) using lattice equivariant degrees. The functional reformulation follows from (3.15)–(3.20), which leads to a fixed-point problem in functional spaces. The bifurcation invariant can be defined as the degree of the map F_ζ in (3.23) on the admissible domain O in (3.21).

Additional care is needed only for the representation lattice structure.

Let $W := H^1(S^1; (\mathbb{R}^k)^n)$ be the first Sobolev space of $(\mathbb{R}^k)^n$-valued functions defined on S^1. The lattice $\mathcal{L}$ induces a representation lattice $\mathcal{T}$ in W as follows. Let

$$\breve{\Delta} := H^1(S^1; \Delta), \tag{4.21}$$

be the first Sobolev space of Δ-valued functions defined on S^1, for $\Delta \in \mathcal{L}$. Let Γ_Δ be the group of action on Δ (cf. (L2)). Define an (isometric) $\Gamma_\Delta \times S^1$-action on $\breve{\Delta}$ by

$$((\gamma, e^{i\theta})u)(t) := \gamma u(t + \theta), \quad \gamma \in \Gamma_\Delta,\ e^{i\theta} \in S^1.$$

Let

$$\mathcal{T} = \{\breve{\Delta} \,:\, \Delta \in \mathcal{L}\}, \tag{4.22}$$

where $\breve{\Delta}$ is defined by (4.21). Then, with respect to the structure

$$\{\breve{\Delta}, \Gamma_\Delta \times S^1, \mathfrak{h}_{\Delta,\Delta'} \times \mathrm{Id}_{S^1}\},$$

$\mathcal{T}$ is a representation lattice, which will be called the *induced lattice* from $\mathcal{L}$.

It can be verified that F_ζ is a $\mathcal{T}$-equivariant compact vector field and (F_ζ, O) is an admissible pair. The lattice equivariant degree of F_ζ in O

$$\omega(\lambda_o, \beta_o, x_o) := \mathcal{T}\text{-Deg}^t(F_\zeta, O)$$

is well-defined, which we call the *bifurcation invariant* around (λ_o, x_o).

4.3.2 Classification Result

Using bifurcation invariants defined by lattice equivariant degree, *finer* classification results are possible for the bifurcating branches of solutions. Here, the structure of representation lattices enables us to track down both the synchrony type and the isotropy type of the bifurcating states.

Recall that dominating and secondary dominating orbit types are defined in Definition 3.2.1.

Theorem 4.3.2. *Let $\mathcal{L}$ be a representation lattice of synchrony subspaces of (4.20) satisfying (L1)–(L2). Consider an equilibrium $x_o \in (\mathbb{R}^k)^n$ of (4.20) satisfying (E1)–(E2) and a bifurcation center (λ_o, x_o) satisfying (B1)–(B2). Let $\mathcal{T}$ be the induced lattice from $\mathcal{L}$ given by (4.22) and O, F_ζ be defined by (3.21) and (3.23). Assume that*

$$\mathcal{T}\text{-}Deg^t(F_\zeta, O) = \sum (\check{\Delta}, a_{\check{\Delta}}), \quad \text{for some } a_{\check{\Delta}} \neq 0,$$

and

$$a_{\check{\Delta}} = \sum_{(H) \in \Phi_1(\Gamma_\Delta \times S^1)} n_H \cdot (H), \quad \text{for some } n_H \neq 0.$$

Then,

- (Existence) *There exists a branch of non-constant periodic solutions of synchrony type Δ and having isotropy subgroup at least H that bifurcates from x_o.*
- (Multiplicity) *Write $(H) = (K^{\phi,l})$. There exist at least $|\Gamma_\Delta/K|$ distinct bifurcating branches of non-constant periodic solutions of proper synchrony type Δ and having precise isotropy subgroup $\gamma H \gamma^{-1}$ for $\gamma \in \Gamma_\Delta/K$, if one of the following is satisfied:*
  - *(i)* $i\beta_o \notin \sigma(J(\lambda_o)|_{\Delta'})$ $\forall\, \Delta' < \Delta$ *and (H) is dominating;*
  - *(ii)* $i\beta_o \notin \sigma(J(\lambda_o)|_{\Delta'})$ $\forall\, \Delta' < \Delta$ *and (H) is secondary dominating satisfying Proposition 3.2.2(ii) for a flow-invariant subspace $S \subset \Delta$;*
  - *(iii)* (H) *is dominating and for every $\Delta' < \Delta$ with $i\beta_o \in \sigma(J(\lambda_o)|_{\Delta'})$, Δ' does not contain states of isotropy H;*
  - *(iv)* (H) *is secondary dominating satisfying Proposition 3.2.2(ii) for a flow-invariant subspace $S \subset \Delta$ and for every $\Delta' < \Delta$ with $i\beta_o \in \sigma(J(\lambda_o)|_{\Delta'})$, Δ' does not contain states of isotropy H.*

Proof. The existence follows from Theorem 4.2.3(i), combined with a standard argument using parametrized auxiliary functions (cf. [16]).

The multiplicity is based on Proposition 3.2.2. It suffices to show that Δ is a proper synchrony type under any of the conditions (i)–(iv). Let $\Delta' \in \mathcal{L}$ be such that $\Delta' < \Delta$.

In case of (i) or (ii), $i\beta_o$ is not an eigenvalue of $J(\lambda_o)|_{\Delta'}$, then

$$\mathrm{Id} - D_u F_1(\lambda_o, \beta_o, \cdot) : \breve{\Delta}' \to \breve{\Delta}'$$

is an isomorphism. By the implicit function theorem, u_o is the unique zero of F_ζ in $O \cap (\mathbb{R}^2 \times \breve{\Delta}')$. Thus, the bifurcating solutions cannot belong to $\breve{\Delta}'$.

In case of (iii) or (iv), if $i\beta_o$ is an eigenvalue of $J(\lambda_o)|_{\Delta'}$, then the predicted branch could have synchrony type Δ'. However, since Δ' is assumed not to contain any states of isotropy H, the branch cannot belong to $\breve{\Delta}'$.

Therefore, in either of the cases (i)–(iv), the synchrony type Δ is proper and the isotropy subgroup H is precise.

□

4.3.3 An Example

Consider the regular 5-cell network (b) in Example 2.1.2. The network is of configuration Figure 4.4. The adjacency matrix is

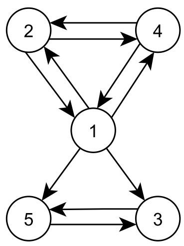


Figure 4.4: The running example of the 5-cell network $\mathcal{N}$ in Figure 2.3(b).

$$A = \begin{pmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \end{pmatrix}, \tag{4.23}$$

whose (i, j)-th element is equal to the number of arrows from the j-th cell to the i-th cell. The matrix A has the following spectrum

$$\sigma(A) = \Big\{\mu_1 = 2, \mu_2 = 1, \mu_3 = \mu_4 = \mu_5 = -1\Big\}.$$

We discuss the synchrony-breaking Hopf bifurcations related to -1. Define $f_o : \mathbb{R} \times \mathbb{R}^2 \times (\mathbb{R}^2)^2 \to \mathbb{R}^2$ by

$$f_o(\lambda, x, y, z) := \alpha(\lambda)x + \beta y + \beta z + xyz, \tag{4.24}$$

where "xyz" stands for the entry-wise multiplication of x, y, z and

$$\alpha(\lambda) = \begin{pmatrix} 1+\lambda & -2 \\ 2 & 1+\lambda \end{pmatrix}, \quad \beta = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}. \tag{4.25}$$

Consider the coupled cell system on $\mathcal{N}$ given by

$$\begin{cases} \dot{x}_1 &= f_o(\lambda, x_1, x_2, x_4) \\ \dot{x}_2 &= f_o(\lambda, x_2, x_1, x_4) \\ \dot{x}_3 &= f_o(\lambda, x_3, x_1, x_5) \\ \dot{x}_4 &= f_o(\lambda, x_4, x_1, x_2) \\ \dot{x}_5 &= f_o(\lambda, x_5, x_1, x_3), \end{cases} \tag{4.26}$$

where $x_i \in \mathbb{R}^2$, $\lambda \in \mathbb{R}$ and f_o is defined by (4.24). Then, $x = 0$ is an equilibrium.

The spectrum of Jacobian

Let $f : \mathbb{R} \times (\mathbb{R}^2)^5 \to (\mathbb{R}^2)^5$ be the right hand side of (4.26). It was shown in [32] that the linearisation $J(\lambda) = Df_x(\lambda, 0)$ of f at $(\lambda, 0)$ has the form

$$J(\lambda) = \alpha(\lambda) \otimes I_5 + \beta \otimes A,$$

where $I_5 : \mathbb{R}^5 \to \mathbb{R}^5$ is the identity matrix. Also, the eigenvalues of $J(\lambda)$ are the union of the eigenvalues of the 2×2-matrices $M_\mu := \alpha(\lambda) + \mu\beta$, for all $\mu \in \sigma(A)$. Moreover, if $v \in \mathbb{C}^5$ is an eigenvector of A and $u \in \mathbb{C}^2$ is an eigenvector of M_μ, then $u \otimes v$ is an eigenvector of $J(\lambda)$ (cf. [32]). More precisely, $J(\lambda)$ has the following eigenvalues and eigenvectors

$$\begin{array}{lll} M_2, & \sigma_{1,2} = 3 + \lambda \pm 4i, & \begin{pmatrix} -i \\ 1 \end{pmatrix} \otimes v_1, \begin{pmatrix} i \\ 1 \end{pmatrix} \otimes v_1 \\ M_1, & \sigma_{3,4} = 2 + \lambda \pm 3i, & \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes v_2, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes v_2 \\ M_{-1}, & \sigma_{5,6,7,8,9,10} = \lambda \pm i, & \begin{pmatrix} -i \\ 1 \end{pmatrix} \otimes v_j, \begin{pmatrix} i \\ 1 \end{pmatrix} \otimes v_j, \end{array} \tag{4.27}$$

where v_1, v_2, v_j for $3 \le j \le 5$ are eigenvectors of A corresponding to $2, 1, -1$. Consequently, (4.26) has three isolated bifurcation centers $(-3, 0)$, $(-2, 0)$ and $(0, 0)$. We describe the synchrony-breaking bifurcation around $(0, 0)$, i.e.

$$(\lambda_o, \beta_o, x_o) = (0, 1, 0).$$

The synchrony subspaces

The lattice of all synchrony subspaces is given by Figure 2.6 for our choice of phase space $a, b, c, d, e \in \mathbb{R}^2$. Denote by

$$\breve{U} := H^1(S^1; U),$$

the first Sobolev space of 2π-periodic functions valued in U, for $U \in \mathcal{L}$. Let

$$\mathcal{T} := \{\breve{U} \,:\, U \in \mathcal{L}\}.$$

Then, $\mathcal{T}$ is a representation lattice with structure $\{\breve{U}, \Gamma_U \times S^1, \mathsf{h}_{U,U''} \times \mathrm{Id}\}$, where $\mathrm{Id} : S^1 \to S^1$ is the identity homomorphism.

The representation lattice

Let $\tilde{\mathcal{L}}$ be the lattice given by Figure 2.6. For every $U \in \tilde{\mathcal{L}}$, there is a corresponding quotient network whose network structure is given by $A|_U$. The symmetry of the quotient network is called a quotient symmetry, which we denote by Γ_U. It can be verified that (4.26) has the following (non-trivial) quotient symmetries:

$$\Gamma_{\Delta_4} = \Gamma_{\Delta_1} = S_3 \simeq D_3, \qquad \Gamma_{\Delta_2} = \Gamma_{\Delta_{00}} = \Gamma_{\Delta_{01}} = \Gamma_{\Delta_{02}} = \Gamma_{\Delta_{03}} = \mathbb{Z}_2,$$

where D_3 acts as permutations on symbols a, b, c; $\mathbb{Z}_2 = \langle \kappa \rangle$ acts on $\Delta_{00}, \Delta_{02}, \Delta_{03}$ by $\kappa : (a, b, c, d, e) \mapsto (a, b, e, d, c)$ and acts on Δ_{01} by $\kappa : (a, b, c, d, e) \mapsto (a, d, c, b, e)$.

However, $\{U, \Gamma_U, \mathsf{h}_{U,U'}\}_{U \in \tilde{\mathcal{L}}}$ does not give a valid structure of representation lattice to $\tilde{\mathcal{L}}$. Indeed, a necessary condition for $\tilde{\mathcal{L}}$ to be a representation lattice is that U is $\Gamma_{U'}$-invariant subspace of U', for all $U \subset U'$ (cf. Definition 4.1.2 (ii)). But we have that $\Delta_{43}, \Delta_{41}, \Delta_{13}$ are not D_3-invariant in Δ_4; $\Delta_{13}, \Delta_{12}, \Delta_{11}$ are not D_3-invariant in Δ_1; and Δ_5, Δ_6 are not $\mathbb{Z}_2$-invariant in Δ_{01}. In fact, in each of these cases, the non-invariant subspaces belong to one orbit under the group action.

Let $\mathcal{L} = \tilde{\mathcal{L}} \setminus \{\Delta_{43}, \Delta_{41}, \Delta_{13}, \Delta_{12}, \Delta_{11}, \Delta_5, \Delta_6\}$ and Γ_U be the quotient symmetry related to U, for $U \in \mathcal{L}$ (cf. Figure 4.5). The arrows in Figure 4.5 stand for homomorphisms, where $\mathsf{h}_{*,*} : \mathbb{Z}_1 \to \Gamma_x$ are given by the inclusion, $\mathsf{h}_{*,*} : \Gamma_x \to \mathbb{Z}_1$ are given by the projection and $\mathsf{h}_{*,*} : \mathbb{Z}_2 \to \mathbb{Z}_2$ are the identity homomorphism. As shown in Example 4.1.3, $\mathcal{L}$ is a real representation lattice in $V = \mathbb{R}^2 \otimes \mathbb{R}^5$, with respect to this structure.

The bifurcation invariant

It can be verified that the assumptions (E1)-(E2), (B1)-(B2), (L1)-(L2) are satisfied by (4.26) with f_o given by (4.24) and $x_o = 0$. Thus, the bifurcation invariant

$$\omega(\lambda_o, \beta_o, x_o) = \mathcal{T}\text{-}\mathrm{Deg}^t(F_\zeta, O)$$

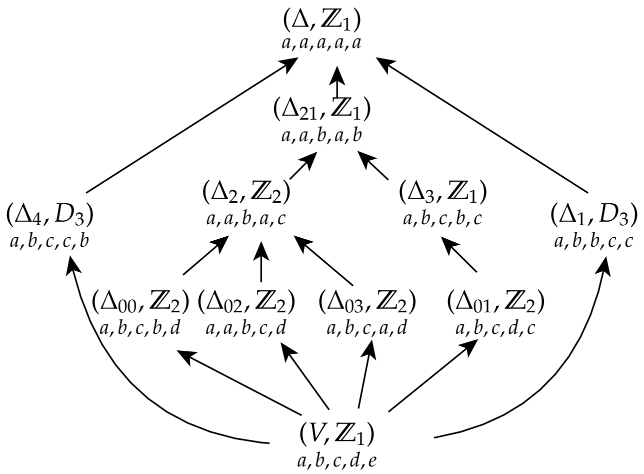


Figure 4.5: A representation lattice $\mathcal{L}$ in $\mathbb{R}^2 \otimes \mathbb{R}^5$.

is well-defined.

It can be verified that the dominating orbit types in $\check{\Delta} \in \mathcal{T}$ are

$$\check{\Delta}_1 : (\mathbb{Z}_3^t), (D_1), (D_1^z); \quad \check{\Delta}_2 : (\mathbb{Z}_2^-); \quad \check{\Delta}_3 : (\mathbb{Z}_1); \quad \check{\Delta}_4 : (\mathbb{Z}_3^t), (D_1), (D_1^z)$$
$$\check{\Delta}_{00} : (\mathbb{Z}_2^-); \quad \check{\Delta}_{01} : (\mathbb{Z}_2^-); \quad \check{\Delta}_{02} : (\mathbb{Z}_2^-); \quad \check{\Delta}_{03} : (\mathbb{Z}_2^-); \quad W : (\mathbb{Z}_1),$$

where $\mathbb{Z}_2^- := \{(1,1), (-1,-1)\} \subset \mathbb{Z}_2 \times S^1$ and $\mathbb{Z}_3^t, D_1, D_1^z$ are defined in Example 2.3.7.

The value of the bifurcation invariant can be computed systematically (cf. Appendix 5.3.3 for details), which is equal to

$$\begin{aligned}\mathcal{T}\text{-Deg}^t(F_\zeta, O) &= \Big(\boldsymbol{\check{\Delta}_1}, -2(\mathbf{Z_3^t}) - 2(\mathbf{D}_1) - 2(\mathbf{D}_1^z) + 2(\mathbb{Z}_1)\Big) + \Big(\boldsymbol{\check{\Delta}_2}, -(\mathbf{Z_2^-})\Big)\\ &+ \Big(\boldsymbol{\check{\Delta}_3}, -(\mathbf{Z_1})\Big) + \Big(\boldsymbol{\check{\Delta}_4}, -2(\mathbf{Z_3^t}) - 2(\mathbf{D}_1) - 2(\mathbf{D}_1^z) + 2(\mathbb{Z}_1)\Big) + \Big(\check{\Delta}_{00}, -(\mathbf{Z_2^-})\Big)\\ &+ \Big(\boldsymbol{\check{\Delta}_{01}}, -2(\mathbf{Z_2^-}) + (\mathbb{Z}_2)\Big) + \Big(\check{\Delta}_{02}, -(\mathbf{Z_2^-})\Big) + \Big(\check{\Delta}_{03}, -(\mathbf{Z_2^-})\Big) + (W, 11(\mathbf{Z_1})).\end{aligned} \tag{4.28}$$

We explain the meaning and implications of (4.28).

Consider the first entry in (4.28). Let $\Delta = \boldsymbol{\Delta_1} \simeq \mathbb{R}^3$. Then, $\Gamma_\Delta = D_3$ and

$$a_{\check{\Delta}_1} = -2(\mathbf{Z_3^t}) - 2(\mathbf{D}_1) - 2(\mathbf{D}_1^z) + 2(\mathbb{Z}_1),$$

where $(\mathbb{Z}_3^t), (D_1), (D_1^z)$ are dominating orbit types. Applying Theorem 4.3.2(i) to $\Delta = \Delta_1$ and $(H) = (\mathbb{Z}_3^t)$, we obtain at least $|D_3/\mathbb{Z}_3| = 2$ different branches of non-constant periodic solutions of (4.26), whose isotropy types are given by $\mathbb{Z}_3^t$ and $\kappa\mathbb{Z}_3^t\kappa^{-1}$, respectively. More precisely, the branch with the isotropy type $\mathbb{Z}_3^t$ has the form (cf. Example 2.3.7)

$$u(t) = (x(t), x(t + \frac{T}{3}), x(t + \frac{2T}{3})),$$

and the branch with the isotropy type $\kappa\mathbb{Z}_3^t\kappa^{-1}$ is of form

$$v(t) = (x(t), x(t + \frac{2T}{3}), x(t + \frac{T}{3})).$$

These branches of solutions are of proper synchrony type Δ_1. Similarly, one can apply Theorem 4.3.2(i) to $(H) = (\mathbf{D}_1)$. Then, there exist at least $|D_3/D_1| = 3$ branches of non-constant periodic solutions of (4.26), whose isotropy types are D_1, $\xi D_1 \xi^{-1}$, $\xi^2 D_1 \xi^{-2}$, respectively, and they have a proper synchrony type Δ_1. Also, the nontrivial $(\mathbf{D}_1^z)$-term indicates the existence of $|D_3/D_1| = 3$ branches of non-constant periodic solutions of (4.26), whose isotropy types are D_1^z, $\xi D_1^z \xi^{-1}$, $\xi^2 D_1^z \xi^{-2}$, respectively (cf. Table 4.1), and they are of a proper synchrony type Δ_1.

An analogous analysis can be applied to $\boldsymbol{\Delta_2}$, $\boldsymbol{\Delta_3}$, $\boldsymbol{\Delta_4}$ and their dominating orbit types. An additional branch can be obtained in Δ_{01} for the dominating orbit type $(\mathbb{Z}_2^-)$, since they satisfy the condition of Theorem 4.3.2(iii).

A summary of these bifurcating solutions can be found in Table 4.1. In brief, we predict **8** branches of non-constant periodic solutions of proper synchrony type Δ_1; **1** branch of non-constant periodic solutions of proper synchrony type Δ_2; **1** branch of non-constant periodic solutions of proper synchrony type Δ_3; **8** branches of non-constant periodic solutions of proper synchrony type Δ_4 (one of which coincides with one from Δ_1); and **1** branch of non-constant periodic solutions of proper synchrony type Δ_{01}.

Note that we do not exclude the possibility of additional periodic solutions bifurcating from $x_o = 0$, besides those listed in Table 4.1, since as a topological invariant, the lattice degree gives only a lower estimate of the number of solutions. In other words, other non-constant periodic solutions may also bifurcate from $x_o = 0$. These additional branches of solutions indeed amount to a homotopy-zero class.

Synchrony	**Symmetry**	**Form of Periodic Solutions** (for some period T)
Δ_1 (a,b,b,c,c)	$\mathbb{Z}_3^t$	$\big(x(t), x(t+\frac{T}{3}), x(t+\frac{T}{3}), x(t+\frac{2T}{3}), x(t+\frac{2T}{3})\big)$
	$\kappa\mathbb{Z}_3^t\kappa^{-1}$	$\big(x(t), x(t+\frac{2T}{3}), x(t+\frac{2T}{3}), x(t+\frac{T}{3}), x(t+\frac{T}{3})\big)$
	D_1	$\big(x(t), y(t), y(t), x(t), x(t)\big) \in \Delta_{12}$
	$\xi D_1\xi^{-1}$	$\big(x(t), x(t), x(t), y(t), y(t)\big) \in \Delta_{11}$
	$\xi^2 D_1\xi^{-2}$	$\big(x(t), y(t), y(t), y(t), y(t)\big) \in \Delta_{13}$
	D_1^z	$\big(x(t), y(t), y(t), x(t+\frac{T}{2}), x(t+\frac{T}{2})\big)$, for $y(t) = y(t+\frac{T}{2})$
	$\xi D_1^z\xi^{-1}$	$\big(x(t), x(t+\frac{T}{2}), x(t+\frac{T}{2}), y(t), y(t)\big)$, for $y(t) = y(t+\frac{T}{2})$
	$\xi^2 D_1^z\xi^{-2}$	$\big(x(t), y(t), y(t), y(t+\frac{T}{2}), y(t+\frac{T}{2})\big)$, for $x(t) = x(t+\frac{T}{2})$
Δ_2 (a,a,b,a,c)	$\mathbb{Z}_2^-$	$\big(x(t), x(t), y(t), x(t), y(t+\frac{T}{2})\big)$
Δ_3 (a,b,c,b,c)	$\mathbb{Z}_1$	$\big(x(t), y(t), z(t), y(t), z(t)\big)$
Δ_4 (a,b,c,c,b)	$\mathbb{Z}_3^t$	$\big(x(t), x(t+\frac{T}{3}), x(t+\frac{2T}{3}), x(t+\frac{2T}{3}), x(t+\frac{T}{3})\big)$
	$\kappa\mathbb{Z}_3^t\kappa^{-1}$	$\big(x(t), x(t+\frac{2T}{3}), x(t+\frac{T}{3}), x(t+\frac{T}{3}), x(t+\frac{2T}{3})\big)$
	D_1	$\big(x(t), y(t), x(t), x(t), y(t)\big) \in \Delta_{43}$
	$\xi D_1\xi^{-1}$	$\big(x(t), x(t), y(t), y(t), x(t)\big) \in \Delta_{41}$
	$\xi^2 D_1\xi^{-2}$	$\big(x(t), y(t), y(t), y(t), y(t)\big) \in \Delta_{13}$
	D_1^z	$\big(x(t), y(t), x(t+\frac{T}{2}), x(t+\frac{T}{2}), y(t)\big)$, for $y(t) = y(t+\frac{T}{2})$
	$\xi D_1^z\xi^{-1}$	$\big(x(t), x(t+\frac{T}{2}), y(t), y(t), x(t+\frac{T}{2})\big)$, for $y(t) = y(t+\frac{T}{2})$
	$\xi^2 D_1^z\xi^{-2}$	$\big(x(t), y(t), y(t+\frac{T}{2}), y(t+\frac{T}{2}), y(t)\big)$, for $x(t) = x(t+\frac{T}{2})$
Δ_{01} (a,b,c,d,c)	$\mathbb{Z}_2^-$	$\big(x(t), y(t), z(t), y(t+\frac{T}{2}), z(t)\big)$

Table 4.1: The summary of synchrony type and symmetric properties of bifurcating branches of solutions from $x_o = 0$ of system (4.26) (Part I), using quotient symmetries.

Chapter 5

Interior Symmetry and Equivariant Degree

Interior symmetry is a symmetry of networks that concerns subsets of cells and their inputs. It is a permutational symmetry on a subset of cells while fixing every cell outside the subset, so that the input structure of the subset is preserved.

If the subset is the total set of cells, then interior symmetry coincides with the usual (global) symmetry and the coupled cell systems admissible to the network structure are simply equivariant systems. Otherwise the systems are only equivariant in some cell coordinates and theory of equivariant bifurcations cannot be applied directly.

However, essential ideas from equivariant bifurcation theory can be transfered to extend the key statements. In [34], both the equivariant branching lemma and the equivariant Hopf theorem were extended for networks with interior symmetry, based on an adapted Lyapunov-Schmidt reduction. This was later followed up and complemented by [5] which presents a complete parallel of the equivariant Hopf theorem.

In this chapter, we extend the equivariant degree theory and introduce a degree theory that is suitable for studying maps that are equivariant for an interior symmetry. The resulting degree, called the *interior equivariant degree* is used to study interior-symmetry breaking bifurcations in coupled cell networks. A somehow surprising outcome is the homotopy equivalence between the bifurcation invariant associated to the interior-symmetry breaking bifurcation problem and the bifurcation invariant associated to a related equivariant bifurcation problem (cf. Theorem 5.3.1).

The main step in defining an adequate degree for interior symmetry is to establish a similar approximation scheme using regular normal maps (cf. Theorem 5.2.9), which is of somewhat technical nature. The approximation scheme in case of equivariant degree can be found in detail in [16].

5.1 Coupled Cell Systems with Interior Symmetry

We review important properties of coupled cell systems with interior symmetry. More details can be found in [5, 34].

5.1.1 Phase Space Decomposition

Let $\mathcal{G} = (\mathcal{C}, \mathcal{E}, \sim_C, \sim_E)$ be a coupled cell network that has an interior symmetry on a subset $\mathcal{S} \subset \mathcal{C}$. Denote by Σ the group of all interior symmetries on $\mathcal{S}$. We also write $\mathcal{N} = \mathcal{C} \setminus \mathcal{S}$.

Let P be the total phase space of coupled cell systems admissible to $\mathcal{G}$. There is a natural decomposition

$$P = P_{\mathcal{S}} \oplus P_{\mathcal{N}} \tag{5.1}$$

of the phase space induced by $\mathcal{C} = \mathcal{S} \sqcup \mathcal{N}$. Note that $P_{\mathcal{S}}$ and $P_{\mathcal{N}}$ may not not flow-invariant in general.

Example 5.1.1. Consider the coupled cell network in Figure 5.1(Left). It

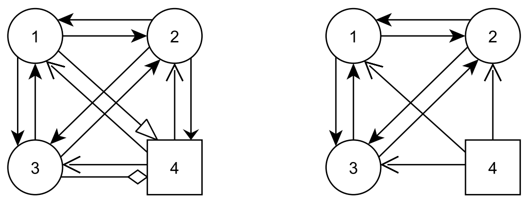


Figure 5.1: (Left) A coupled cell network with interior symmetry $\Sigma = D_3$. (Right) The preserved input structure of the subset $\mathcal{S} = \{1, 2, 3\}$.

admits an interior symmetry $\Sigma = S_3 \simeq D_3$ on $\mathcal{S} = \{1, 2, 3\} \subset \{1, 2, 3, 4\}$, since the input edges of cells of $\mathcal{S}$ are preserved under the permutations of $1, 2, 3$ (cf. Figure 5.1(Right)).

Coupled cell systems admissible to the network are of form

$$\begin{cases} \dot{x}_1 = f(x_1, \overline{x_2, x_3}, x_4) \\ \dot{x}_2 = f(x_2, \overline{x_1, x_3}, x_4) \\ \dot{x}_3 = f(x_3, \overline{x_1, x_2}, x_4) \\ \dot{x}_4 = g(x_4, x_1, x_2, x_3) \end{cases}, \quad \text{for } x_1, x_2, x_3 \in P_1, x_4 \in P_2, \tag{5.2}$$

where P_1, P_2 are some phase spaces of choice. Consider the decomposition (5.1). Then, we have

$$P_{\mathcal{S}} = \{(x_1, x_2, x_3, 0) : x_1, x_2, x_3 \in P_1\}, \quad P_{\mathcal{N}} = \{(0, 0, 0, x_4) : x_4 \in P_2\}.$$

For general form of f, g, the space $P_{\mathcal{S}}$ is not flow-invariant, since an initial condition x^o with $x^o_4 = 0$ does not guarantee $x_4(t) = 0$ for all positive $t > 0$. Similarly, $P_{\mathcal{N}}$ is not flow-invariant, since an initial condition x^o with $x^o_1 = x^o_2 = x^o_3 = 0$ does not guarantee $x_1(t) = x_2(t) = x_3(t) = 0$ for all positive $t > 0$. ◇

There is, however, a phase space decomposition that respects flows, which is induced by the interior symmetry group Σ. Recall that in case of symmetry and equivariant systems, fixed point subspaces of any (closed) subgroups of the symmetry are flow-invariant for the equivariant systems. This turns out to be true also for interior symmetry.

Proposition 5.1.2. *(cf. [34]) Let $\mathcal{G}$ be a network admitting a non-trivial interior symmetry group $\Sigma_{\mathcal{S}}$ and fix a phase space P. Let K be any subgroup of $\Sigma_{\mathcal{S}}$. Then, the equivalence relation $\bowtie_K$ defined by*

$$x \bowtie_K y \Leftrightarrow \exists\, \sigma \in K : y = \sigma(x)$$

is a balanced relation on $\mathcal{G}$. In particular, the fixed-point subspace

$$Fix_{\,P}(K) = \Delta_{\bowtie_K} = \{x \in P : x_c = x_{\sigma(c)}\, \forall\, \sigma \in K\}$$

is a flow invariant subspace for all $\mathcal{G}$-admissible vector fields.

Let Σ be the interior symmetry group of $\mathcal{G}$ on $\mathcal{S}$. Then,

$$\mathrm{Fix}_{\,P}(\Sigma) = \{x = (x_{\mathcal{S}}, x_{\mathcal{N}}) \in P : x_{\mathcal{S}} \in \mathrm{Fix}_{\,\mathcal{S}}(\Sigma)\}, \tag{5.3}$$

where $x_{\mathcal{S}}$ is of full synchrony, $x_{\mathcal{N}}$ is a free coordinate. By Proposition 5.1.2, $\mathrm{Fix}_{\,P}(\Sigma)$ is flow-invariant for all admissible vector fields. Let

$$U = \mathrm{Fix}_{\,P}(\Sigma) \tag{5.4}$$

and define a complement W of U so that

$$P = W \oplus U \tag{5.5}$$

in the following way. First write $\mathcal{S}$ as a disjoint union of Σ-orbits

$$\mathcal{S} = \mathcal{S}_1 \sqcup \mathcal{S}_2 \sqcup \cdots \sqcup \mathcal{S}_l.$$

Then, define

$$W = \{x = (x_{\mathcal{S}}, x_{\mathcal{N}}) \in P : x_{\mathcal{N}} = 0 \,\wedge\, \sum_{s \in \mathcal{S}_i} x_s = 0\, \forall\, i \in \{1, \ldots, l\}\}, \tag{5.6}$$

which is a Σ-invariant subspace and $W \cap U = \{0\}$. Thus, (5.5) holds.

Consider the system (5.2) in Example 5.1.1 again. Then, the decomposition holds for

$$W = \{(w_1, w_2, -w_1 - w_2, 0) : w_1, w_2 \in P_1\}$$
$$U = \{(u_1, u_1, u_1, u_2) : u_1, u_2 \in P_2\}.$$

The relation between the decompositions (5.1) and (5.5) is

$$P = W \oplus U = \underbrace{W \oplus U_1}_{P_{\mathcal{S}}} \oplus P_{\mathcal{N}}, \tag{5.7}$$

where $U_1 = U \cap P_{\mathcal{S}}$. That is, the Σ-fixed point subspace U is composed of the phase space $P_{\mathcal{N}}$ of the cells outside $\mathcal{S}$ (where Σ acts trivially) and the fixed point subspace for cells inside $\mathcal{S}$.

As we will see in Subsection 5.1.2, the advantage of using decomposition (5.1) is that it indicates the "equivariant" part of admissible vector fields (cf. (5.8)), while the decomposition (5.5) is based on a flow-invariant subspace U, which leads to a triangular form of the linearization of admissible vector fields (cf. (5.9)).

5.1.2 Admissible Vector Fields

Let f be a vector field on P such that it is admissible to $\mathcal{G}$. Write $f = (f_{\mathcal{S}}, f_{\mathcal{N}})$, where $f_{\mathcal{S}} : P \to P_{\mathcal{S}}$ and $f_{\mathcal{N}} : P \to P_{\mathcal{N}}$ are projections of f onto $P_{\mathcal{S}}$ and $P_{\mathcal{N}}$, respectively. The interior symmetry of $\mathcal{G}$ implies that f must satisfy

$$\begin{cases} \sigma f_{\mathcal{S}}(x_{\mathcal{S}}, x_{\mathcal{N}}) = f_{\mathcal{S}}(\sigma x_{\mathcal{S}}, \sigma x_{\mathcal{N}}) = f_{\mathcal{S}}(\sigma x_{\mathcal{S}}, x_{\mathcal{N}}), \\ \sigma f_{\mathcal{N}}(x_{\mathcal{S}}, x_{\mathcal{N}}) = f_{\mathcal{N}}(x_{\mathcal{S}}, x_{\mathcal{N}}), \end{cases} \quad \forall\, \sigma \in \Sigma, \tag{5.8}$$

where $x = (x_{\mathcal{S}}, x_{\mathcal{N}})$ is expressed with respect to (5.1) and we used the fact that Σ acts trivially on $\mathcal{N}$.

Consider the decomposition (5.5) and write $f = (f_W, f_U)$ for the projections $f_W : P \to W$ and $f_U : P \to U$. Since U is flow-invariant, we have

$$f(U) \subseteq U \;\Rightarrow\; f(0, u) \in U,\ \forall\, u \in U \;\Rightarrow\; f_W(0, u) = 0,\ \forall\, u \in U.$$

It follows that every linear admissible vector field must have the form

$$L = \begin{pmatrix} A & 0 \\ C & B \end{pmatrix} \tag{5.9}$$

with respect to the decomposition (5.5). As a special case, linearization of an admissible vector field at an equilibrium is of form (5.9). The equivariance (5.8) of f is translated to

$$\begin{cases} \sigma f_W(w, u) = f_W(\sigma w, u), \\ \sigma f_{U_1}(w, u) = f_{U_1}(w, u), \\ \sigma f_{\mathcal{N}}(x_{\mathcal{S}}, x_{\mathcal{N}}) = f_{\mathcal{N}}(x_{\mathcal{S}}, x_{\mathcal{N}}), \end{cases} \quad \forall\, \sigma \in \Sigma,$$

where $f_{U_1} : P \to U_1$ is the projection of f onto U_1.

Example 5.1.3. Consider the coupled cell network in Example 5.1.1 again. Then, the basis transformation between the decompositions (5.1) and (5.5) is given by Figure 5.2. Using the basis change, we obtain directly

$$
\begin{tikzcd}
P_{\mathcal{S}} \oplus P_{\mathcal{N}} \arrow[rr, "(f_{\mathcal{S}}, f_{\mathcal{N}})"] & & P_{\mathcal{S}} \oplus P_{\mathcal{N}} \\
W \oplus U \arrow[u, "\begin{smallmatrix} x_1 = w_1 + u_1 \\ x_2 = w_2 + u_1 \\ x_3 = -w_1 - w_2 + u_1 \\ x_4 = u_2 \end{smallmatrix}"'] \arrow[rr, "(f_W, f_U)"] & & W \oplus U \arrow[u, "\begin{smallmatrix} w_1 = x_1 - \frac{x_1+x_2+x_3}{3} \\ w_2 = x_2 - \frac{x_1+x_2+x_3}{3} \\ u_1 = \frac{x_1+x_2+x_3}{3} \\ u_2 = x_4 \end{smallmatrix}"']
\end{tikzcd}
$$

Figure 5.2: Basis transformation for system (5.2).

$$
f_W = \begin{pmatrix} \frac{1}{3}(2f(w_1 + u_1, \overline{w_2 + u_1, w_3 + u_1}, u_2) - f(w_2 + u_1, \overline{w_1 + u_1, w_3 + u_1}, u_2) - f(w_3 + u_1, \overline{w_1 + u_1, w_2 + u_1}, u_2)) \\ \frac{1}{3}(2f(w_2 + u_1, \overline{w_1 + u_1, w_3 + u_1}, u_2) - f(w_1 + u_1, \overline{w_2 + u_1, w_3 + u_1}, u_2) - f(w_3 + u_1, \overline{w_1 + u_1, w_2 + u_1}, u_2)) \end{pmatrix}
$$
$$
f_U = \begin{pmatrix} \frac{1}{3}(f(w_1 + u_1, \overline{w_2 + u_1, w_3 + u_1}, u_2) + f(w_2 + u_1, \overline{w_1 + u_1, w_3 + u_1}, u_2) + f(w_3 + u_1, \overline{w_1 + u_1, w_2 + u_1}, u_2)) \\ g(u_2, w_1 + u_1, w_2 + u_1, w_3 + u_1) \end{pmatrix}
$$

where $w_3 = -w_1 - w_2$.

The action of $S_3 = \langle (1\,2\,3), (1\,2) \rangle$ is given by

$$
\begin{aligned}
(1\,2\,3)(w_1, w_2, u_1, u_2) &= (w_2, -w_1 - w_2, u_1, u_2) \\
(1\,2)(w_1, w_2, u_1, u_2) &= (w_2, w_1, u_1, u_2). \qquad (5.10)
\end{aligned}
$$

That is, S_3 acts by permuting w_1, w_2, w_3 (with $w_3 = -w_1 - w_2$) while fixing u_1, u_2. The fixed-point subspaces of subgroups $K \subset S_3$ satisfy

$$
\mathrm{Fix}_P(K) = \mathrm{Fix}_W(K) \oplus U,
$$

for

$$
\begin{aligned}
\mathrm{Fix}_W(S_3) = \mathrm{Fix}_W(\mathbb{Z}_3) &= \{0\} \\
\mathrm{Fix}_W(D_1) &= \{(w_1, w_2) : w_1 = w_2\} \\
\mathrm{Fix}_W(D_1') &= \{(w_1, w_2) : w_2 = -w_1 - w_2\} \\
\mathrm{Fix}_W(D_1'') &= \{(w_1, w_2) : w_1 = -w_1 - w_2\} \\
\mathrm{Fix}_W(\mathbb{Z}_1) &= W,
\end{aligned}
$$

where $\mathbb{Z}_3 = \langle (1\,2\,3) \rangle$, $D_1 = \langle (1\,2) \rangle$, $D_1' = \langle (2\,3) \rangle$, $D_1'' = \langle (1\,3) \rangle$ and $\mathbb{Z}_1 = \langle (1) \rangle$ (cf. Figure 5.3) It can be verified directly that (f_W, f_{U_1}) is S_3-equivariant and all fixed-point subspaces $\mathrm{Fix}_P(K)$ are flow-invariant for admissible vector fields of (5.2).

$\diamond$

In what follows, we mainly work with the decomposition (5.5).

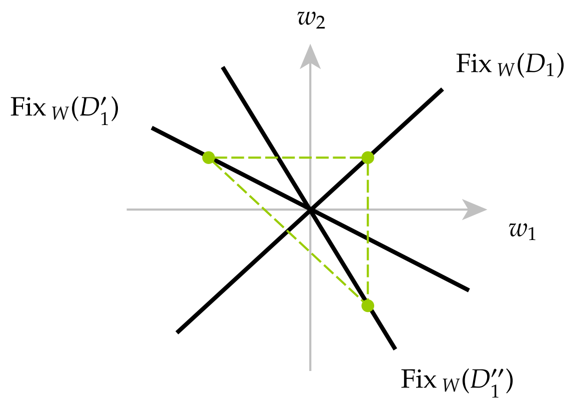


Figure 5.3: Fixed-point subspaces of S_3-action (5.10) in W, where $D_1 = \{(1), (1\ 2)\}$, $D_1' = \{(1), (2\ 3)\}$, $D_1'' = \{(1), (1\ 3)\}$ are conjugate subgroups in S_3 and they give isomorphic fixed-point subspaces $\text{Fix}_W(D_1)$, $\text{Fix}_W(D_1')$, $\text{Fix}_W(D_1'')$. The points connected by dashed lines form an orbit under S_3-action.

5.2 Interior Equivariant Degree

Motivated by coupled cell systems with interior symmetry, we introduce a notion of interior equivariant maps and their related homotopies. We prove a regular normal approximation scheme for these maps, based on which we define an *interior equivariant degree* for interior equivariant maps.

5.2.1 Interior Equivariant Maps

Let P be a finite-dimensional Euclidean space over reals. Let $W, U \subset P$ be two subspaces such that the decomposition (5.5) holds.

Let Σ be a finite group that acts on W orthogonally[‡] and on U trivially.

Definition 5.2.1. A map $f : P \to P$ is called Σ-*interior equivariant on* W, if $f = (f_W, g)$ for a Σ-equivariant map $f_W : P \to W$ and a continuous map $g : P \to U$. Similarly, a homotopy $h : [0,1] \times P \to P$ is called Σ-*interior equivariant on* W, if $h(t, \cdot) : P \to P$ is a Σ-interior equivariant map on W, for all $t \in [0,1]$. $\diamond$

Remark 5.2.2. An important property of Σ-interior equivariant maps f is

$$f(\text{Fix}(K)) \subseteq \text{Fix}(K), \quad \forall K \subset \Sigma, \tag{5.11}$$

[‡]Recall that every representation of a compact Lie group is equivalent to an orthogonal representation using the Haar measure.

which is an extension of the same property of equivariant maps. To see (5.11), let $x \in \text{Fix}\,(K)$, $\sigma \in K$, then we have

$$\sigma f(x) = (\sigma f_W(x), \sigma g(x)) = (f_W(\sigma x), g(x)) = (f_W(x), g(x)) = f(x).$$

This key property plays an important role in constructing a degree for interior equivariant maps. $\diamond$

We review the definition of a regular normal map for equivariant maps and then extend it to the case of interior symmetry. Let X be a Σ-manifold. For $K \subset \Sigma$, recall the notations $X_{(K)}$, X_K and X^K from (2.12), (2.13) and (2.8), where $X_{(K)}$, X_K are submanifolds of X and X^K is a closed subspace of X containing X_K (cf. Theorem 2.2.11).

Definition 5.2.3. (cf. [16]) Let V be an orthogonal representation of Σ and let $\Omega \subset V$ be an open bounded invariant subset[§]. A Σ-equivariant map $f : V \to V$ is called *normal* in Ω, if

$$\begin{aligned} &\forall\, K \subset \Sigma,\ \forall\, x \in f^{-1}(0) \cap \Omega_K,\ \exists\, \delta_x > 0\ s.t.\ \forall\, v \in \nu_x(\Omega_{(K)}) \text{ with } \|v\| < \delta_x : \\ &\qquad f(x+v) = f(x) + v = v, \end{aligned} \tag{5.12}$$

where $\nu(\Omega_{(K)})$ denotes the normal bundle of the submainfold $\Omega_{(K)}$ in Ω.

A normal map is called *regular normal*, if it is additionally of class C^1 and zero is a regular value of map $f|_{\Omega_K} : \Omega_K \to V^K$ for every $K \subset \Sigma$. $\diamond$

Example 5.2.4. Consider the S_3-action (5.10) in Example 5.1.3 again. Then, an invariant subset $\Omega \subset W$ is a disjoint union of different Ω_K's (cf. Figure 5.4(a)). The normality condition (5.12) is as illustrated in Figure 5.4(b). $\diamond$

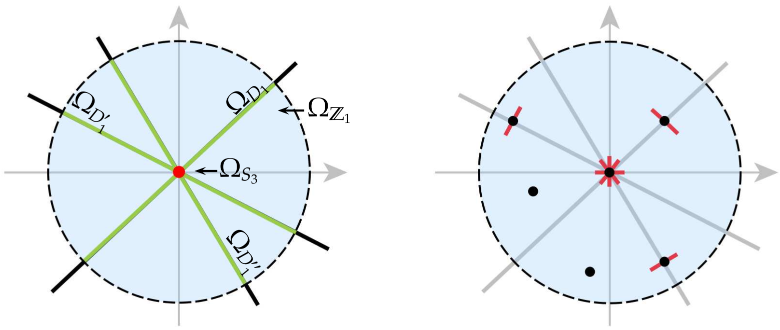
We adopt the same definition for interior equivariant maps.

Definition 5.2.5. Let V be an orthogonal representation of Σ and let $\Omega \subset V$ be an open bounded invariant subset. A continuous map $f : V \to V$ satisfying (5.11) is called *normal* in Ω, if (5.12) holds for every $K \subset \Sigma$. A normal map is called *regular normal*, if it is additionally of class C^1 and zero is a regular value of map $f|_{\Omega_K} : \Omega_K \to V^K$ for every $K \subset \Sigma$.

$\diamond$

Since interior equivariant maps always satisfy (5.11) by Remark 5.2.2, Definition 5.2.5 also defines regular normal maps for interior equivariant maps, thus is an extension of Definition 5.2.3.

[§]In particular, Ω is a Σ-manifold.



(*a*) (*b*)

Figure 5.4: (a) Decomposition of Ω into disjoint proper fixed-point sets: Ω_{S_3}, Ω_{D_1}, $\Omega_{D_1'}$, $\Omega_{D_1''}$, $\Omega_{\mathbb{Z}_1}$ for the S_3-action (5.10) in W. (b) Zeros of a normal S_3-equivariant map on Ω, where the short red lines indicate the normal direction to $\Omega_{(K)}$ (cf. Definition 5.2.3).

5.2.2 Regular Normal Approximations

Let $\Phi(\Sigma)$ be the set of all conjugacy classes of subgroups of Σ. Define a partial order "$\leq$" on $\Phi(\Sigma)$ by

$$(K_1) \leq (K_2) \quad \Leftrightarrow \quad \exists\, \sigma \in \Sigma \quad s.t. \quad \sigma K_1 \sigma^{-1} \subseteq K_2. \tag{5.13}$$

One can extend this partial order to a complete order so that all $\leq$-related pairs are preserved. The proof presented below is based on induction of orbit types in Ω from bigger to smaller orbit types.

Proposition 5.2.6. (Normal Approximation) *Let $f = (f_W, g) : W \oplus U \to W \oplus U$ be a Σ-interior equivariant map on W for a Σ-equivariant map $f_W : W \oplus U \to W$ and a continuous map $g : W \oplus U \to U$. Let $\Omega \subset W \oplus U$ be open, bounded and Σ-invariant. For every $\varepsilon > 0$, there exists a normal Σ-interior equivariant map $\tilde{f}$ on W such that $\|\tilde{f} - f\| < \varepsilon$ in Ω.*

Proof. Without loss of generality, we can assume that f_W is a normal map, based on the normal approximation theorem for equivariant maps (cf. [16], Theorem 3.17).

Then, the normality condition (5.12) implies that

$$f(x+v) = \begin{pmatrix} f_W(x+v) \\ g(x+v) \end{pmatrix} \overset{(*)}{=} \begin{pmatrix} f_W(x) + v \\ g(x) \end{pmatrix} = f(x) + v, \tag{5.14}$$

where (*) follows from the fact that $\Omega_{(K)} = (\Omega \cap W)_{(K)} \times (\Omega \cap U)$, thus $v \in W$ whenever $v \perp \Omega_{(K)}$.

The equality (*) of (5.14) leads to the condition

$$g(x+v) = g(x) \tag{5.15}$$

near zeros of f so that f is normal. It is thus sufficient to define $\tilde{f} = (f_W, \tilde{g})$ for a map $\tilde{g}$ that fulfills (5.15).

Let $\leq$ be the partial order defined by (5.13). Starting from the maximal orbit type $(K) = (\Sigma)$, we use local Urysohn functions to define $\tilde{g}$.

Consider the set $Z := g^{-1}(0) \cap \Omega_{(K)}$ for $K = \Sigma$. Since $g^{-1}(0) \cap \partial\Omega = \emptyset$, there exists a compact neighborhood N of Z in Ω such that $Z \subset N \subset \Omega$. Also, we choose an "intermediate" neighborhood A so that $Z \subset A \subset N$ to define a Urysohn function around Z (cf. Figure 5.5 (a)). Moreover, we assume that

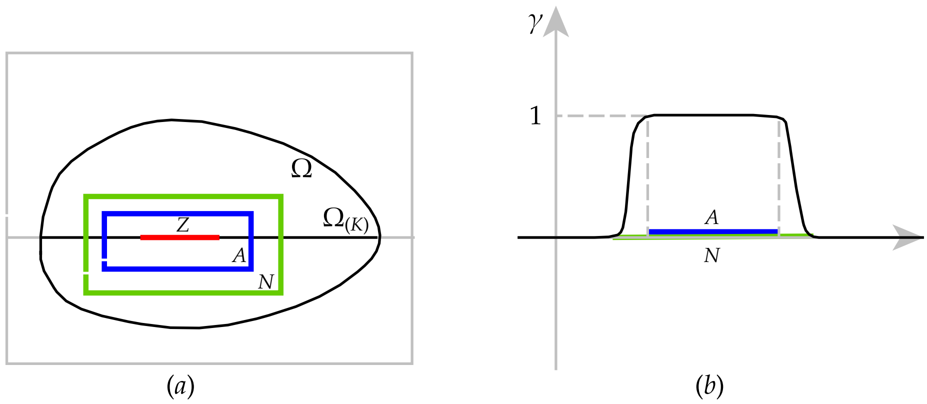


Figure 5.5: (a) Compact neighborhoods A, N around the zero set Z of orbit type (K) for $Z \subset A \subset N$; (b) A Urysohn function γ locally around the zero set such that $\gamma \equiv 1$ in A and $\gamma \equiv 0$ outside N.

N is so small that for every $x \in N$ we can write

$$x = y + v, \quad y \in N \cap \Omega_{(K)},\ v \in \nu_y(\Omega_{(K)})$$

Let $\gamma : W \oplus U \to [0,1]$ be a smooth Σ-invariant Urysohn such that

$$\gamma(x) = \begin{cases} 1 & \text{for} \quad x \in A \\ 0 & \text{for} \quad x \notin N \end{cases} \tag{5.16}$$

(cf. Figure 5.5 (b)). Define $\tilde{g} : W \oplus U \to U$ by

$$\tilde{g}(x) = \begin{cases} \gamma(x)g(y) + (1-\gamma(x))g(x) & \text{for} \quad x = y + v \in N \\ g(x) & \text{for} \quad x \notin N \end{cases} . \tag{5.17}$$

Note that $\tilde{g} \equiv g$ on A and outside of N. Thus, by appropriate choice of N, we can make $\tilde{g}$ and g as close as possible. Also, note that the new zeros of $\tilde{g}$ will have smaller orbit types, thus one can repeat this procedure by following the existing orbit types in decreasing order.

In this way, we obtain a map $\tilde{g} : W \oplus U \to U$ such that $\tilde{g}(x + v) = \tilde{g}(x)$ for all $x \in \tilde{g}^{-1}(0) \cap \Omega_{(K)}$ and $v \in \nu_x\Omega_{(K)}$ with sufficiently small $\|v\|$. It follows that the map $\tilde{f} = (f_W, \tilde{g})$ fulfills the normality condition and is as desired. □

Recall the following important result of the Sard-Brown Theorem.

Theorem 5.2.7. *Let $\Omega \subset \mathbb{R}^n$ be an open set, $f : \Omega \to \mathbb{R}^k$ be a smooth map and $K \subset \Omega$ be a compact set. For any $y \in \mathbb{R}^k$ and $\varepsilon > 0$, there exists a smooth map $g : \Omega \to \mathbb{R}^k$ such that y is a regular value of g and* $\sup\{|f(x) - g(x)| : x \in K\} < \varepsilon$.

Lemma 5.2.8. *Let $\Omega \subset \mathbb{R}^n$ be an open bounded set and $f : \overline{\Omega} \to \mathbb{R}^k$ be a smooth map such that $f^{-1}(0) \cap \partial\Omega = \emptyset$. For any compact neighborhood*[¶] *$C \subset \Omega$ of $f^{-1}(0)$ and $\varepsilon > 0$, there exists a smooth map $g : \Omega \to \mathbb{R}^k$ such that 0 is a regular value of g,* $\sup\{|f(x) - g(x)| : x \in C\} < \varepsilon$ *and $f(x) = g(x)$ for all $x \in \Omega \setminus C$.*

Proof. Let $C \subset \Omega$ be a compact neighborhood of $f^{-1}(0)$ and $\varepsilon > 0$. Choose another compact neighborhood C_1 of $f^{-1}(0)$ such that $C_1 \subset \text{int}(C)$ (cf. Figure 5.6). Clearly, the value of f outside C_1 is bounded away from zero. Thus,

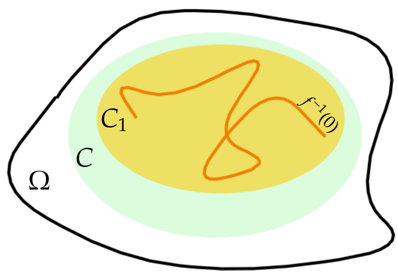


Figure 5.6: Compact neighborhoods of zeros of f (cf. Proof of Lemma 5.2.8).

we can set $\rho := \min\{|f(x)| : x \notin C_1\} > 0$ and assume that $\rho < \varepsilon$. Applying Theorem 5.2.7 to f for C and $\frac{\rho}{2}$, we obtain a smooth map $g_1 : \Omega \to \mathbb{R}^k$ such that 0 is a regular value of g_1 and $\sup\{|f(x) - g_1(x)| : x \in C\} < \frac{\rho}{2}$.

Consider now a Urysohn function $\gamma : \mathbb{R}^n \to [0, 1]$ such that

$$\gamma(x) = \begin{cases} 1 & \forall\, x \in C_1 \\ 0 & \forall\, x \notin C \end{cases} .$$

Define $g : \Omega \to \mathbb{R}^k$ by

$$g(x) = \gamma(x)g_1(x) + (1 - \gamma(x))f(x).$$

[¶]A set N is a *compact neighborhood* of a set S, if $S \subset \text{int}(N)$.

Then, $f(x) = g(x)$ for all $x \notin C$. For $x \in C$, we have

$$|f(x) - g(x)| = |\gamma(x)(f(x) - g_1(x))| \leq |f(x) - g_1(x)| < \frac{\rho}{2} < \varepsilon.$$

We show that 0 is also a regular value of g. Note that $f = g$ outside C and f has no zeros outside C, thus $g^{-1}(0) \subset C$. Further, for $x \in C \setminus C_1$, we have

$$|g(x)| = |f(x) + \gamma(x)(g_1(x) - f(x))| \geq |f(x)| - |g_1(x) - f(x)| \geq \rho - \frac{\rho}{2} > 0.$$

Thus, $g^{-1}(0) \subset C_1$. But $g = g_1$ on C_1. It follows that 0 is a regular value of g.
□

Proposition 5.2.9. (Regular Normal Approximation) *Let $f = (f_W, g) : W \oplus U \to W \oplus U$ be a Σ-interior equivariant map on W for a Σ-equivariant map $f_W : W \oplus U \to W$ and a continuous map $g : W \oplus U \to U$. Let $\Omega \subset W \oplus U$ be open, bounded and Σ-invariant. For every $\varepsilon > 0$, there exists a regular normal (Σ, W)-equivariant map $\tilde{f}$ such that $\|\tilde{f} - f\| < \varepsilon$ in Ω.*

Proof. The idea of the proof follows that of Proposition 5.2.6, where at every K-fixed point subspace we "correct" g by a regular map using Lemma 5.2.8.

Again, we can assume that f_W is a regular normal Σ-equivariant map in Ω, by the regular normal approximation theorem for equivariant maps (cf. Theorem 3.23 in [16]). Also, by Weierstrass Theorem, we can assume that g is a smooth map.

Let $\leq$ be the partial order defined by (5.13).

Consider the maximal orbit type $(K) = (\Sigma)$ and $Z = g^{-1}(0) \cap \Omega_{(K)}$. Choose two compact neighborhoods A, N of Z in Ω, just as shown in Figure 5.5 (a). Applying Lemma 5.2.8 to g and N, we obtain a smooth map $\hat{g} : \Omega \to U$ such that

(i) $\sup\{|g(x) - \hat{g}(x)| : x \in N\} < \frac{\varepsilon}{2}$;

(ii) $g(x) = \hat{g}(x)$ for all $x \notin N$; and

(iii) 0 is a regular value of $\hat{g}$.

Continue as in the proof of Proposition 5.2.6 by a Urysohn function γ satisfying (5.16) and define, in the spirit of (5.17),

$$g_1(x) = \begin{cases} \gamma(x)\hat{g}(y) + (1 - \gamma(x))\hat{g}(x) & \text{for} \quad x = y + v \in N \\ g(x) & \text{for} \quad x \notin N \end{cases}.$$

Then, $f_1 = (f_W, g_1)$ satisfies that $f(x) = f_1(x)$ for all $x \notin N$ and

$$\begin{aligned} \|f - f_1\| &= \sup\{|f(x) - f_1(x)| : x \in \Omega\} = \sup\{|g(x) - g_1(x)| : x \in N\} \\ &\leq \sup\{|\hat{g}(x) - \hat{g}(y)| + |g(x) - \hat{g}(x)| : x = y + v \in N\} \\ &\overset{(i)}{\leq} \sup\{|\hat{g}(x) - \hat{g}(y)| : x = y + v \in N\} + \frac{\varepsilon}{2}. \end{aligned}$$

Now, we can choose N so small that the first summand in the last line is less than $\frac{\varepsilon}{2}$ and consequently, $\|f - f_1\| < \varepsilon$. Also, for any $K' \in (K)$, 0 is a regular value of $g_1|_{\Omega_{K'}} : \Omega_{K'} \to U$. In fact, since $g_1 = g$ for $x \notin N$ and g has no zeros outside N, we have $g_1^{-1}(0) \subset N$, which implies that $(g_1|_{\Omega_{K'}})^{-1}(0) \subset N \cap \Omega_{K'}$. But $g_1|_{N \cap \Omega_{K'}} = \hat{g}$ by definition of g_1. It follows that zero is a regular value of $g_1|_{\Omega_{K'}}$. Therefore, $f_1|_{\Omega_{K'}} = (f_W|_{\Omega_{K'}}, g_1|_{\Omega_{K'}})$ has zero as a regular value, for any $K' \in (K)$.

Clearly, by definition of γ and g_1, f_1 is normal for orbit type (K), i.e. $f_1(x + v) = f_1(x) + v = v$ for all $x \in f_1^{-1}(0) \cap \Omega_{(K)}$ and $v \perp x$ with $(x, v) \in A$. Thus, zeros of f_1 of orbit type (K) are separated (by neighborhoods) away from zeros of other orbit types. Moreover, zeros $(x, v) \in N$ of f_1 for $v \neq 0$ will have smaller orbit types as (K), since $\operatorname{Iso}((x, v)) = \operatorname{Iso}(x) \cap \operatorname{Iso}(v)$. Thus, one can repeat this procedure to smaller orbit types by deceasing order and obtain a map of desired properties. □

Similarly, one can show the regular normal approximation for homotopies.

Proposition 5.2.10. *Let $h = (h_W, g) : [0,1] \times W \oplus U \to W \oplus U$ be a Σ-interior equivariant homotopy on W for a Σ-equivariant homotopy $h_W : [0,1] \times W \oplus U \to W$ and a continuous homotopy $g : [0,1] \times W \oplus U \to U$. Let $\Omega \subset [0,1] \times W \oplus U$ be open, bounded and Σ-invariant. For every $\varepsilon > 0$, there exists a regular normal Σ-interior equivariant homotopy $\tilde{h}$ such that $\|\tilde{h} - h\| < \varepsilon$ in Ω.*

5.2.3 Definition and Properties

Let Σ be a finite group that acts on W orthogonally and on U trivially. Consider the set of *admissible* pairs (f, Ω) for Σ-interior equivariant maps on W and open, bounded, Σ-invariant domains $\Omega \subset W \oplus U$, i.e.

$$\begin{aligned}\mathcal{P} = \{(f, \Omega) \ : f : W \oplus U \to W \oplus U \text{ is } \Sigma - \text{interior equivariant on } W,\\ \Omega \subset W \oplus U \text{ is open, bounded, } \Sigma\text{-invariant, } f^{-1}(0) \cap \partial\Omega = \emptyset\}.\end{aligned}$$

In the following, we define a map associating to every such pair $(f, \Omega) \in \mathcal{P}$ an algebraic count of zeros of f in Ω according to their *isotropy subgroups*. The count should remain the same against interior equivariant homotopies.

Given $(f, \Omega) \in \mathcal{P}$, define the (Σ, W)*-interior equivariant degree* of f in Ω by assigning to every isotropy type K an integer, i.e.

$$\deg_{(\Sigma, W)}(f, \Omega) = \sum_{K \subset \Sigma} n_K \cdot K, \quad \text{for } n_K \in \mathbb{Z}, \tag{5.18}$$

where $n_K \in \mathbb{Z}$ is to be determined by f.

If f is regular normal in Ω, then zero orbits of different orbit types are isolated and 0 is a regular value of $f|_{\Omega_K}$ for every $K \subset \Sigma$. Thus, we can define

$$n_K = \deg(f|_{\Omega_K}, \Omega_K) \tag{5.19}$$

by the Brouwer degree " deg " of $f|_{\Omega_K}$ in Ω_K.

Otherwise, if f is not regular normal, then apply Proposition 5.2.9 to f, Ω and $\rho := \min\{|f(x)| : x \in \partial\Omega\} > 0$ to obtain a regular normal (Σ, W)-equivariant map $\tilde{f}$ in Ω such that $\|\tilde{f} - f\| < \frac{\rho}{2}$, and define

$$\deg_{(\Sigma,W)}(f, \Omega) = \deg_{(\Sigma,W)}(\tilde{f}, \Omega). \tag{5.20}$$

Remark 5.2.11. It is interesting to note that the interior equivariant degree (5.18) associates an integer to every *isotropy subgroup* in the domain, while the equivariant degree (3.5) associates an integer to every *orbit type* in the domain.

Recall that an orbit type is a set of conjugate isotropy subgroups (cf. (2.7)) and zeros of an equivariant map form group orbits of conjugate isotropies (cf. Remark 3.1.2). Thus, it is sufficient to use one integer n_K for all isotropy subgroups K' that are conjugate to K, i.e. in (3.5), we have

$$n_K = n_{gKg^{-1}}, \quad \forall\, g \in G. \tag{5.21}$$

However, zeros of an interior equivariant map do not form group orbits and (5.21) may not hold (for $G = \Sigma$). Thus, it is important to associate an integer for every isotropy subgroup.

◇

Example 5.2.12. Let $P = W \oplus U = \mathbb{R} \oplus \mathbb{R}$ be a $\mathbb{Z}_2$-representation, where $\mathbb{Z}_2$ acts on W as the antipodal map and on U trivially. Consider the map

$$\begin{aligned} f:\quad & W \oplus U \to W \oplus U \\ & (x, y) \mapsto ((y-1)\sin(x), (x-\pi)^2 + (y-1)) \end{aligned}$$

on the domain $\Omega = (-4, 4) \times (0, 2)$ (cf. Figure 5.7). It can be verified that f

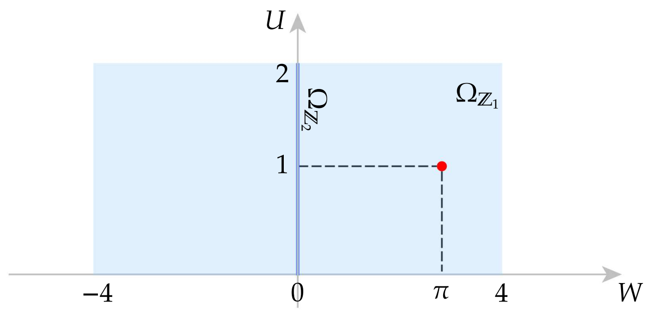


Figure 5.7: The zero (marked by the red dot) of an interior equivariant map in $\Omega = (-4, 4) \times (0, 2)$, which does not form an orbit under $\mathbb{Z}_2$-action.

is a $\mathbb{Z}_2$-interior equivariant map on W and $f^{-1}(0) \cap \Omega = \{(\pi, 1)\}$ which does not include the whole group orbit $\{(\pm\pi, 1)\}$ of $(\pi, 1)$. ◇

Lemma 5.2.13. *The definition of* $\deg_{(\Sigma,W)}(f,\Omega)$ *by (5.18)–(5.20) does not depend on the choice of regular normal approximation maps* $\tilde{f}$.

Proof. Assume that $\tilde{f}_1, \tilde{f}_2$ are two different regular normal approximations of f in Ω such that $\|f - \tilde{f}_i\| < \frac{\rho}{2}$ for $i = 1, 2$, where $\rho := \{|f(x)| : x \in \partial\Omega\} > 0$. Then, $\|\tilde{f}_1 - \tilde{f}_2\| < \rho$ and the straight-line homotopy function $h(t,x) = (1-t)\tilde{f}_1 + t\tilde{f}_2$ gives a Ω-admissible homotopy from $\tilde{f}_1$ to $\tilde{f}_2$, since for $x \in \partial\Omega$,

$$|h(t,x)| = |\tilde{f}_1(x) + t(\tilde{f}_2(x) - \tilde{f}_1(x))| \geq |\tilde{f}_1(x)| - |\tilde{f}_2(x) - \tilde{f}_1(x)| > \rho - \rho = 0.$$

Consequently, for every $K \subset \Sigma$, $\tilde{f}_1|_{\Omega_K}$ is homotopic to $\tilde{f}_2|_{\Omega_K}$. It follows from the homotopy invariance of the Brouwer degree that

$$\deg(\tilde{f}_1|_{\Omega_K}, \Omega_K) = \deg(\tilde{f}_2|_{\Omega_K}, \Omega_K).$$

Thus, they lead to the same definition of n_K. □

We show that the function $\deg_{(\Sigma,W)}$ defined as above satisfies the usual properties of a degree theory.

Theorem 5.2.14. *Let* $\deg_{(\Sigma,W)}$ *be a function defined by (5.18)–(5.20) for* $(f,\Omega) \in \mathcal{P}$. *Then,*

(i) (Existence) *If* $\deg_{(\Sigma,W)}(f,\Omega) = \sum n_K \cdot K$ *with* $n_K \neq 0$, *then*

$$f^{-1}(0) \cap \Omega^K \neq \emptyset.$$

(ii) (Homotopy Invariance) *If* $h : [0,1] \times \overline{\Omega} \to W \oplus U$ *is an* Ω*-admissible* Σ*-equivariant homotopy on* W, *then*

$$\deg_{(\Sigma,W)}(h(t,\cdot),\Omega) = \textit{constant}, \quad \forall t \in [0,1].$$

(iii) (Additivity) *If* $\Omega_1, \Omega_2 \subset \Omega$ *are disjoint open bounded* Σ*-invariant subsets such that* $f^{-1}(0) \cap \Omega \subset \Omega_1 \cup \Omega_2$ *and* f *is* Ω_i*-admissible for* $i = 1, 2$, *then*

$$\deg_{(\Sigma,W)}(f,\Omega) = \deg_{(\Sigma,W)}(f,\Omega_1) + \deg_{(\Sigma,W)}(f,\Omega_2).$$

(iv) (Suspension) *Let* W' *be another orthogonal representation of* Σ *and* $\Omega' \subset W'$ *be an open bounded* Σ*-invariant neighborhood of* 0. *Then,*

$$\deg_{(\Sigma,W)}(f \times \mathrm{Id}, \Omega \times \Omega') = \deg_{(\Sigma,W)}(f,\Omega).$$

Proof. (i) Assume that $\deg_{(\Sigma,W)}(f,\Omega) = \sum n_K \cdot K$ and $n_K \neq 0$ for some $K \subset \Sigma$. Then, $n_K = \deg(\tilde{f}|_{\Omega_K}, \Omega_K) \neq 0$, for a regular normal approximation $\tilde{f}$ of f in Ω. It follows from the existence property of the Brouwer degree that $\tilde{f}^{-1}(0) \cap \Omega_K \neq \emptyset$. Let $x \in \tilde{f}^{-1}(0) \cap \Omega_K$. By the construction of $\tilde{f}$, we have

$x = y + v$ for some $y \in f^{-1}(0) \cap \Omega_{K'}$ and $v \perp y$ with $K' \supset K$ (cf. Proposition 5.2.9). Thus, $\Omega_{K'} \subset \Omega^K$ which implies that $y \in f^{-1}(0) \cap \Omega^K$.

(ii) Let $h : [0,1] \times \overline{\Omega} \to W \oplus U$ be an Ω-admissible Σ-interior equivariant homotopy on W. Without loss of generality, we assume h to be regular normal. Consider an open cover of open intervals for $[0,1]$ constructed as follows: for every $t_o \in [0,1]$ and $\rho = \min\{\|h(t_o, x)\| : x \in \partial\Omega\} > 0$, choose an open interval $i(t_o)$ of t_o in $[0,1]$ such that

$$\|h(t_o, \cdot) - h(t, \cdot)\| < \frac{\rho}{2}, \quad \forall\, t \in i(t_o).$$

In particular, $h(t, \cdot)$ can be regarded as a regular normal approximation of $h(t_o, \cdot)$ and

$$\deg_{(\Sigma,W)}(h(t, \cdot), \Omega) = \deg_{(\Sigma,W)}(h(t_o, \cdot), \Omega), \quad \forall\, t \in i(t_o). \tag{5.22}$$

Define now

$$\mathcal{O} = \{i(t_o) \,:\, t_o \in [0,1]\}$$

which is an open cover of $[0,1]$. Since $[0,1]$ is compact, it contains a finite subcover composed of intervals, say, $i(t_1), \dots i(t_N)$. By (5.22), the value of $\deg_{(\Sigma,W)}(h(t, \cdot), \Omega)$ remains constant, on each of these intervals. Consequently, it remains constant on the whole interval $[0,1]$.

(iii) Set $\rho_i = \min\{|f(x)| : x \in \partial\Omega_i\}$ for $i = 1, 2$. By applying Proposition 5.2.9 to $f|_{\Omega_i}$ and ρ_i in Ω_i, we obtain a regular normal approximation f_i such that $\|f|_{\Omega_i} - f_i\| < \frac{\rho}{4}$. It follows that

$$\deg_{(\Sigma,W)}(f|_{\Omega_i}, \Omega_i) = \deg_{(\Sigma,W)}(f_i, \Omega_i), \quad i = 1, 2.$$

Next, we construct out of f_1, f_2 a regular normal approximation of f in Ω using Urysohn functions. Without loss of generality, we can assume that $\overline{\Omega}_1 \cap \overline{\Omega}_2 = \emptyset$, since $f^{-1}(0)$ is composed of two disjoint compact sets

$$f^{-1}(0) \cap \Omega_i = f^{-1}(0) \cap \overline{\Omega}_i, \quad \text{for } i = 1, 2.$$

Thus, there exist two disjoint open sets N_1, N_2 such that $N_i \supset \overline{\Omega}_i$, for $i = 1, 2$ (cf. Figure 5.8). Let $\gamma_i : W \oplus U \to [0,1]$ be a Urysohn function such that

$$\gamma_i(x) = \begin{cases} 1, & x \in \Omega_i \\ 0, & x \notin N_i \end{cases}.$$

Then, we can define a continuous extension of f_i to the whole Ω by "gluing" together f_i and f. That is,

$$\tilde{f_i}(x) = \gamma(x) f_i(x) + (1 - \gamma(x)) f(x), \quad \text{for } x \in \Omega,\, i = \{1, 2\}.$$

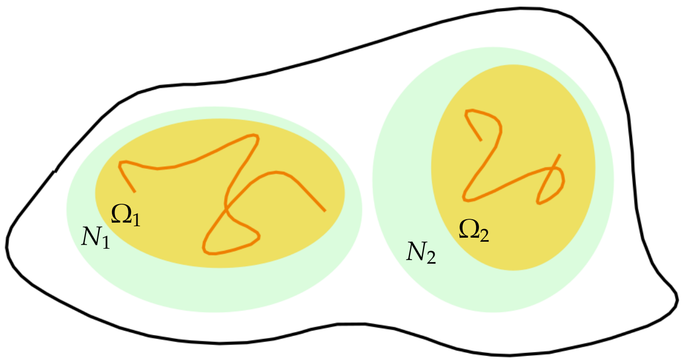


Figure 5.8: Disjoint subsets for zeros of f (cf. Proof of Theorem 5.2.14.

Note that for $x \notin N_i$, we have $\tilde{f}_i(x) = f(x)$. Thus, we can define

$$\tilde{f}(x) = \begin{cases} \tilde{f}_1(x), & x \in N_1 \\ \tilde{f}_2(x), & x \in N_2, \\ f(x), & x \notin N_1 \cup N_2 \end{cases},$$

which is a well-defined continuous map. Moreover, $\tilde{f}$ is regular normal in Ω, since

$$\tilde{f}^{-1}(0) \cap \Omega = (f_1^{-1}(0) \cap \Omega_1) \cup (f_2^{-1}(0) \cap \Omega_2). \tag{5.23}$$

More detailedly, by definition of $\tilde{f}$, we have $\tilde{f}^{-1}(0) \cap \Omega \subset N_1 \cup N_2$. Also, for $x \in N_i \setminus \Omega_i$, we have $\tilde{f}(x) = \tilde{f}_i(x)$ and

$$|\tilde{f}_i(x)| = |f(x) + \gamma_i(x)(f_i(x) - f(x))| \geq |f(x)| - |f_i(x) - f(x)|.$$

If $\delta = \min\{|f(x)| : x \in N_i \setminus \Omega_i\} > 0$, then $|\tilde{f}_i(x)| > \delta - \frac{\rho}{4}$. Now, we can choose ρ such that $\rho < 4\delta$. Thus, (5.23) holds and $\tilde{f}$ is regular normal in Ω. Finally,

$$\begin{aligned} \|f - \tilde{f}\| &\leq \|f - \tilde{f}_1\|_{N_1} + \|f - \tilde{f}_2\|_{N_2} \\ &\leq \|f|_{\Omega_i} - f_1\| + \|f|_{\Omega_i} - f_2\| \\ &< \frac{\rho}{4} + \frac{\rho}{4} = \frac{\rho}{2}. \end{aligned}$$

Therefore,

$$\begin{aligned} \deg_{(\Sigma,W)}(f, \Omega) &= \deg_{(\Sigma,W)}(\tilde{f}, \Omega) = \deg_{(\Sigma,W)}(\tilde{f}, \Omega_1 \cup \Omega_2) \\ &= \deg_{(\Sigma,W)}(f_1, \Omega_1) + \deg_{(\Sigma,W)}(f_2, \Omega_2) \\ &= \deg_{(\Sigma,W)}(f, \Omega_1) + \deg_{(\Sigma,W)}(f, \Omega_2) \end{aligned}$$

(iv) Let W' be another orthogonal representation of Σ and $\Omega' \subset W'$ be an open bounded Σ-invariant neighborhood of 0. Then, $f \times \mathrm{Id}$ is a $(\Sigma, W \oplus W')$-equivariant map, since it can be written as $f \times \mathrm{Id} = (f_W \times \mathrm{Id}, g)$, where $f_W \times \mathrm{Id}$ is Σ-equvariant. Moreover, if $f = (f_W, g)$ is regular normal in Ω, then $f \times \mathrm{Id}$ is regular normal in $\Omega \times \Omega'$. Indeed, if $(w, u, s) \in (f \times \mathrm{Id})^{-1}(0)$, then $s = 0$. Also, since $\mathrm{Iso}\,(0) = \Sigma$, we have

$$(w, u, 0) \in (f \times \mathrm{Id})^{-1}(0) \cap (\Omega \times \Omega')_K \Leftrightarrow (w, u) \in f^{-1}(0) \cap \Omega_K. \tag{5.24}$$

Further, for $v \perp (\Omega \times \Omega')_{(K)}$ at $(w, u, 0)$ in $W \oplus U \oplus W'$, we can write $v = v_1 + v_2$ for $v_1 \in W \oplus U$ and $v_2 \in W'$. Then, $v_1 \perp \Omega_{(K)}$ at (w, u). Since f is normal, for any $(w, u) \in f^{-1}(0) \cap \Omega_K$, there exists a $\delta_{(w,u)} > 0$ such that

$$f((w, u) + v_1) = f((w, u)) + v_1 = v_1$$

is satisfied for all $v_1 \perp \Omega_{(K)}$ at x and $\|v_1\| < \delta_{(w,u)}$. Now, for $(w, u, 0) \in (f \times \mathrm{Id})^{-1}(0) \cap (\Omega \times \Omega')_K$, we can take the same $\delta_{(w,u)}$. Thus, for $v \perp (\Omega \times \Omega')_{(K)}$ at $(w, u, 0)$ with $\|v\| < \delta_{(w,u)}$, we have

$$(f \times \mathrm{Id})((w, u, 0) + v) = f((w, u) + v_1) + v_2 = f((w, u)) + v_1 + v_2 = v_1 + v_2 = v.$$

That is, $f \times \mathrm{Id}$ is normal in $\Omega \times \Omega'$. The regularity of $(f \times \mathrm{Id})|_{(\Omega\times\Omega')_K}$ around its zeros follows from (5.24), the regularity of $f|_{\Omega_K}$ and the fact that $D(f \times \mathrm{Id}) = Df \times \mathrm{Id}$, for the differentiation "D".

Finally, for every $K \subset \Sigma$, we have

$$\begin{aligned}
&\deg\,(f|_{\Omega_K}, \Omega_K) = \sum \{\mathrm{sign}\ \det Df|_{\Omega_K}(x_o) : x_o \in f^{-1}(0) \cap \Omega_K\} \\
&= \sum \{\mathrm{sign}\ \det Df|_{\Omega_K}(x_o) : (x_o, 0) \in (f \times \mathrm{Id})^{-1}(0) \cap (\Omega \times \Omega')_K\}, \\
&= \sum \{\mathrm{sign}\ \det D(f \times \mathrm{Id})|_{(\Omega\times\Omega')_K}(x_o) : (x_o, 0) \in (f \times \mathrm{Id})^{-1}(0) \cap (\Omega \times \Omega')_K\} \\
&= \deg\,((f \times \mathrm{Id})|_{(\Omega\times\Omega')_K}, (\Omega \times \Omega')_K)
\end{aligned}$$

which concludes the desired equality.

□

Like in the case of equivariant degrees, interior equivariant degrees can be calculated by a *recurrence formula*, based on its geometric meaning. Recall that the coefficient n_K in (5.18) stands for an (algebraic) count of zeros of f that have isotropy subgroup K. Since the K-fixed point subspace Ω^K is composed of elements of Ω that have isotropies at least K, the total count of zeros of f in Ω^K is given by the sum of $n_{\tilde{K}}$ for all $\tilde{K} \geq K$. It follows that

$$n_K = \deg\,(f|_{\Omega^K}, \Omega^K) - \sum_{\tilde{K} > K} n_{\tilde{K}}, \tag{5.25}$$

where " deg " stands for the Brouwer degree.

5.2.4 Relation to Equivariant Degree

We show that the interior equivariant degree defined by (5.18)–(5.20) is a natural extension of the equivariant degree without parameters.

Proposition 5.2.15. *The interior equivariant degree (5.18) coincides with equivariant degree (3.5) for equivariant maps. More precisely, let f be a Σ-equivariant admissible map on an open bounded Σ-invariant subset Ω of a finite-dimensional Σ-representation P. Write*

$$\Sigma\text{-Deg}\,(f,\Omega) = \sum_{(K)} n_K \cdot (K)$$

for integer coefficients n_K defined by (3.4). The interior equivariant degree is also well-defined for (f,Ω) by taking $P = W$ and write

$$\deg_{(\Sigma,W)}\,(f,\Omega) = \sum_{K} m_K \cdot K,$$

for integer coefficients m_K's satisfying (5.25). Then,

$$m_K = n_K \cdot |W(K)| \tag{5.26}$$

for every $K \subset \Sigma$, where $W(K)$ is the Weyl group of K.

Proof. Recall the equality (3.2) satisfied by n_K's. Then, we have

$$n_K \cdot |W(K)| = \deg\,(f|_{\Omega^K},\Omega^K) - \sum_{\tilde{K}>K} n_{\tilde{K}} \cdot |W(\tilde{K})|.$$

Thus, $m_K = n_K \cdot |W(K)|$ by induction.

Another way of proving (5.26) is to use the geometric meaning of n_K's and m_K's. Recall that the subset Ω_K of elements of isotropy K is a $W(K)$-free submanifold of Ω (cf. Theorem 2.2.11). Thus, corresponding to n_K number of zero orbits of f having isotropy K, there are $n_K \cdot |W(K)|$ number of zeros of f having isotropy K. Therefore, (5.26) follows.

□

Example 5.2.16. Continuing from Example 3.1.5, consider the negative identity on the unit disk B of the D_3-representation $\mathbb{C}$ as an interior equivariant map. The D_3-equivariant degree is calculated in Example 3.1.5, which is

$$\Sigma\text{-Deg}\,(-\mathrm{Id},B) = (D_3) - 2(D_1) + (\mathbb{Z}_1). \tag{5.27}$$

The isotropy subgroups in B are D_3, D_1, $D_1' := \xi D_1 \xi^{-1}$, $D_1'' := \xi^2 D_1 \xi^{-2}$ and $\mathbb{Z}_1$ (cf. Figure 3.1). Thus, we have

$$\deg_{(\Sigma,W)}\,(-\mathrm{Id},B) = m_{D_3} \cdot D_3 + m_{D_1} \cdot D_1 + m_{D_1'} \cdot D_1' + m_{D_1''} \cdot D_1'' + m_{\mathbb{Z}_1} \cdot \mathbb{Z}_1,$$

for integer coefficients m_*'s. The Weyl groups are (cf. (3.7))

$$W(D_3) = \mathbb{Z}_1, \quad W(D_1) = W(D_1') = W(D_1'') = \mathbb{Z}_1, \quad W(\mathbb{Z}_1) = D_3.$$

Based on the recurrence formula (5.25) and (3.6), we have

$$\begin{aligned} m_{D_3} &= 1, \\ m_{D_1} &= -1 - 1 = -2 = m_{D_1'} = m_{D_1''} \\ m_{\mathbb{Z}_1} &= 1 - 1 - (-2) - (-2) - (-2) = 6. \end{aligned}$$

Thus,

$$\deg_{(\Sigma,W)}(-\mathrm{Id}, B) = D_3 - 2 \cdot D_1 - 2 \cdot D_1' - 2 \cdot D_1'' + 6 \cdot \mathbb{Z}_1.$$

Compared with (5.27), the relation (5.26) is fulfilled. ◇

5.2.5 Interior Equivariant Maps with One Parameter

Similarly, one can define an interior equivariant degree for *interior equivariant maps with one parameter*. In this case, consider $P = W \oplus U$ for a $\Sigma \times S^1$-orthogonal representation W and an S^1-orthogonal representation U. Let $\mathbb{R}$ be a parameter space on which $\Sigma \times S^1$ acts trivially.

Definition 5.2.17. A one-parameter map $F : \mathbb{R} \times P \to P$ is called $\Sigma \times S^1$-*interior equivariant on* W, if $F = (F_W, F_U)$ for a $\Sigma \times S^1$-equivariant map $F_W : \mathbb{R} \times P \to W$ and an S^1-equivariant map $F_U : \mathbb{R} \times P \to U$. That is,

$$\begin{cases} (\sigma, z) F_W(\lambda, w, u) = F_W(\lambda, (\sigma, z)w, u), \\ (\sigma, z) F_U(\lambda, w, u) = F_U(\lambda, zw, u), \end{cases} \quad \forall\, (\sigma, z) \in \Sigma \times S^1$$

◇

Based on the same regular normal approximation scheme, one can define an *interior equivariant degree with one parameter* for all admissible pairs (F, O) such that F is a $\Sigma \times S^1$-interior equivariant map with one parameter and $O \subset \mathbb{R} \times P$ is an open bounded $\Sigma \times S^1$-invariant subset. The necessary regular normal approximations can be similarly proved based on the equivariant regular normal approximations for one-parameter maps. We omit the details of the proof here.

Formally, given such an admissible pair (F, O), define the $(\Sigma \times S^1, W)$-*interior equivariant degree* of F in O by assigning to every twisted isotropy $H \subset \Sigma \times S^1$ an integer, i.e.

$$\deg_{(\Sigma \times S^1, W)}(F, O) = \sum_{H \subset \Sigma \times S^1} n_H \cdot H, \quad \text{for } n_H \in \mathbb{Z}, \tag{5.28}$$

where $H \subset \Sigma \times S^1$ are twisted subgroups of $\Sigma \times S^1$ (cf. Definition 2.3.6).

The coefficients n_H's can be calculated using the following *recurrence formula* (cf. (3.11) for comparison):

$$n_H = \sum_i s_i^H - \sum_{\tilde{H}>H} n_{\tilde{H}}, \tag{5.29}$$

where s_i^H is the $(\mathbb{Z}_i)$-coefficient in $S^1\text{-Deg}\,(F|_{O^H}, O^H)$ (cf. (3.9)).

The interior equivariant degree (5.28)–(5.29) is a natural extension of the equivariant degree with one parameter.

Proposition 5.2.18. *The interior equivariant degree (5.28) coincides with equivariant degree (3.12) for equivariant maps. More precisely, let F be a $\Sigma \times S^1$-equivariant admissible map on an open bounded $\Sigma \times S^1$-invariant subset O of a finite-dimensional $\Sigma \times S^1$-representation $\mathbb{R} \times P$. Write*

$$\Sigma \times S^1\text{-Deg}\,(F, O) = \sum_{(H)} n_H \cdot (H)$$

for integer coefficients n_H defined by (3.11). The interior equivariant degree is also well-defined for (F, O) by taking $P = W$ and write

$$\deg_{(\Sigma\times S^1, W)}(F, O) = \sum_H m_H \cdot H,$$

for integer coefficients m_H's satisfying (5.29). Then,

$$m_H = n_H \cdot |W(H)/S^1| \tag{5.30}$$

for every twisted subgroup $H \subset \Sigma \times S^1$, where $W(H)$ is the Weyl group of H.

Proof. It can be proved in the same way as Proposition 5.2.15. □

Remark 5.2.19. The interior equivariant degree with one parameter defined by (5.28) can be extended in a standard way to infinite-dimensional $\Sigma \times S^1$-representations for compact $\Sigma \times S^1$-interior equivariant vector fields, based on the equivariant Schauder projection (cf. Lemma 4.2.7). ◇

5.3 Bifurcations in Coupled Cell Systems

We outline the major steps how to apply the interior equivariant degree theory to interior equivariant bifurcations in coupled cell networks. To compare, we refer to Section 3.2 for equivariant bifurcations and Section 4.3 for synchrony-breaking bifurcations with or without quotient symmetries.

Consider a coupled cell system given by

$$\dot{x} = f(\lambda, x), \tag{5.31}$$

where $\lambda \in \mathbb{R}$ is a bifurcation parameter, $x \in P$ is the total state variable of all cells and $f : \mathbb{R} \times P \to P$ is continuously differentiable.

Assume that $x = 0$ is an equilibrium of (5.31). We are interested in studying the Hopf bifurcations of 0 as λ varies. The basic assumptions are (E1), (B1), (B2) from Section 3.2. Moreover, we assume

(I) There is a phase space decomposition

$$P = W \oplus U \tag{5.32}$$

of P such that f is Σ-interior equivariant on W. More precisely, f can be written as $f = (f_W, g)$ for a Σ-equivariant map $f_W : \mathbb{R} \times P \to W$ and a C^1 map $g : \mathbb{R} \times P \to U$

Let $(\lambda_o, 0)$ be the isolated bifurcation center given by (B1). By interior symmetry, the linearization of f at $(\lambda_o, 0)$ is of form (cf. (5.9))

$$D_x f(\lambda_o, 0) = \begin{pmatrix} A & 0 \\ C & B \end{pmatrix}, \tag{5.33}$$

where $A = D_w f_W(\lambda_o, 0)$, $B = D_u g(\lambda_o, 0)$ and $C = D_w g(\lambda_o, 0)$.

5.3.1 Degree Approach

Following the same lines of Subsection 3.2.1, we define a bifurcation invariant for $(\lambda_o, 0)$ using the interior equivariant degree with one parameter. The phase space decomposition (5.32) carries over to the functional spaces.

Let $p > 0$ be the unknown period of the bifurcating solution x of (5.31). Let $\beta = \frac{2\pi}{p}$ and $y(t) = x(\frac{1}{\beta}t)$. Then,

$$\begin{cases} \dot{y} = \frac{1}{\beta} f(\lambda, y) \\ y(0) = y(2\pi) \end{cases}.$$

Using the phase decomposition (5.32) and the usual operators L, j and N_f (cf. (3.16)–(3.18)), the above can be reformulated as

$$\dot{y} = \frac{1}{\beta} f(\lambda, y) \quad \Leftrightarrow \quad \begin{cases} \dot{w} = \frac{1}{\beta} f_W(\lambda, y) \\ \dot{u} = \frac{1}{\beta} g(\lambda, y) \end{cases} \quad \Leftrightarrow \quad \begin{cases} Lw = \frac{1}{\beta} N_{f_W}(\lambda, j(y)) \\ Lu = \frac{1}{\beta} N_g(\lambda, j(y)) \end{cases},$$

where $L : H^1(S^1, P) \to L^2(S^1, P)$ is the differentiation operator given by (3.16), $j : H^1(S^1, P) \to C(S^1, P)$ is the compact embedding given by (3.17) and $N_f : \mathbb{R} \times C(S^1, X) \to L^2(S^1, X)$ is the Nemyskii operator defined by (3.18) for $X \in \{W, U\}$. Let $K : H^1(S^1, P) \to L^2(S^1, P)$ be the integration operator. Then,

$$\begin{cases} w = (L + K)^{-1}(\frac{1}{\beta} N_{f_W}(\lambda, j(y)) + Kw) := F_W(\lambda, \beta, y) \\ u = (L + K)^{-1}(\frac{1}{\beta} N_g(\lambda, j(y)) + Ku) := F_U(\lambda, \beta, y) \end{cases},$$

where F_W is $\Sigma \times S^1$-equivariant and F_U is S^1-equivariant. That is, we have reformulated the bifurcation problem as a fixed point problem of a compact map in an interior equivariant setting.

By (B1), $(\lambda_o, 0)$ is an isolated bifurcation center. Assume that $i\beta_o \in \sigma(D_x f(\lambda_o, 0))$ for some $\beta_o > 0$. Let O be defined by

$$\begin{aligned} O :=& \{(\lambda, \beta, y) : \sqrt{(\lambda - \lambda_o)^2 + (\beta - \beta_o)^2} < \varepsilon, \ \|y\| < r\} \\ & \subset \mathbb{R}^2 \times H^1(S^1, P), \end{aligned} \tag{5.34}$$

for appropriate $\varepsilon, r > 0$ and $\zeta : \overline{O} \to \mathbb{R}$ be an auxiliary function given by

$$\zeta(\lambda, \beta, y) = \sqrt{(\lambda - \lambda_o)^2 + (\beta - \beta_o)^2}(\|y\| - r) + \|y\| - \frac{r}{2}, \tag{5.35}$$

(cf. Figure 3.2). Define

$$\begin{aligned} F_\zeta : \qquad & \overline{O} \to \mathbb{R} \times H^1(S^1, W) \times H^1(S^1, U) \\ & (\lambda, \beta, y) \mapsto (\zeta(\lambda, \beta, y), w - F_W(\lambda, \beta, y), u - F_U(\lambda, \beta, y)), \end{aligned} \tag{5.36}$$

which is a $\Sigma \times S^1$-interior equivariant in $H^1(S^1, W)$). Using the interior equivariant degree (5.28) with one parameter, we define

$$\omega(\lambda_o, 0) := \deg_{(\tilde{\Sigma}, \tilde{W})}(F_\zeta, O),$$

for $\tilde{W} := H^1(S^1, W)$ and call it the *bifurcation invariant* around $(\lambda_o, 0)$.

5.3.2 Homotopy to An Equivariant Bifurcation Problem

We show that F_ζ defined by (5.36) is $\Sigma \times S^1$-interior equivariant homotopic to an equivariant map that is associated to an equivariant bifurcation problem. This results in accessible computations of $\omega(\lambda_o, 0)$ using the existing EDML command `showdegree` explained in Subsection 3.2.3.

Consider another bifurcation problem in $P = W \oplus U$

$$\begin{cases} \dot{w} = f_W(\lambda, w, 0) \\ \dot{u} = g(\lambda, 0, u) \end{cases} \tag{5.37}$$

near the same equilibrium $0 \in P$.

The bifurcation center $(\lambda_o, 0)$ of (5.31) is also an (isolated) bifurcation center for (5.37), which fulfills (B2) as well, since the linearization of (5.37) has the same spectrum as that of (5.31) at $(\lambda_o, 0)$.

Moreover, due to the special form of the vector field of (5.37) and the interior equivariance assumption (I) on (5.31), (5.37) is in fact, an equivariant bifurcation problem. Thus, (5.37) also satisfies (I) in particular.

The relation between the bifurcation problems (5.31) and (5.37) around $(\lambda_o, 0)$ can be described as follows.

Theorem 5.3.1. *The bifurcating branches of (5.31) under the assumptions (E), (B1), (B2), (I) are $\Sigma \times S^1$-interior equivariant homotopic to those of (5.37) near the equilibrium $(\lambda_o, 0)$.*

Proof. For convenience, let $a_W(\lambda, w) = f_W(\lambda, w, 0)$ and $b(\lambda, u) = g(\lambda, 0, u)$. Then, following the same procedure in Subsection 5.3.1, the bifurcation problem of (5.37) can be reformulated as finding zeros of

$$\begin{aligned} G_\zeta : \qquad & \overline{O} \to \mathbb{R} \times H^1(S^1, W) \times H^1(S^1, U) \\ & (\lambda, \beta, y) \mapsto (\zeta(\lambda, \beta, y), w - G_W(\lambda, \beta, y), u - G_U(\lambda, \beta, y)), \end{aligned} \tag{5.38}$$

where O, ζ are given by (5.34)–(5.35) and

$$\begin{cases} G_w(\lambda, \beta, y) = (L + K)^{-1}(\frac{1}{\beta} N_{a_W}(\lambda, j(w)) + Kw) \\ G_u(\lambda, \beta, y) = (L + K)^{-1}(\frac{1}{\beta} N_b(\lambda, j(u)) + Ku) \end{cases}.$$

We show that F_ζ and G_ζ are $\Sigma \times S^1$-interior equivariant homotopic on O.

Step 1. (A modified auxiliary function) Let

$$\tilde{\zeta}(\lambda, \beta, y) = \sqrt{(\lambda - \lambda_o)^2 + (\beta - \beta_o)^2}(\|y\| - r) + \|y\| + \varepsilon \frac{r}{2},$$

where $\varepsilon, r > 0$ are measurement of O. Compared with ζ, the function $\tilde{\zeta}$ is still always positive for $\|y\| = r$, but not always negative for $\|y\| = 0$. Define $F_{\tilde{\zeta}}$ and $G_{\tilde{\zeta}}$ just like how F_ζ and G_ζ are defined, by replacing ζ with $\tilde{\zeta}$. Since F_ζ and $F_{\tilde{\zeta}}$ do not point to the opposite directions on the boundary of O, one can use a straight-line homotopy to connect them. Thus, F_ζ and $F_{\tilde{\zeta}}$ are homotopic on O. The homotopy is also $\Sigma \times S^1$-interior equivariant, since it only involves auxiliary functions which are always $\Sigma \times S^1$-equivariant. In the same way, G_ζ and $G_{\tilde{\zeta}}$ are $\Sigma \times S^1$-interior equivariant homotopic on O. That is,

$$(F_\zeta, O) \sim (F_{\tilde{\zeta}}, O), \quad (G_\zeta, O) \sim (G_{\tilde{\zeta}}, O)$$

Step 2. (A modified domain) If (λ, β) is close enough to (λ_o, β_o), then $\tilde{\zeta}$ is always positive. Indeed, for $\sqrt{(\lambda - \lambda_o)^2 + (\beta - \beta_o)^2} \leq \frac{\varepsilon}{4}$ and $\|y\| \leq r$, we have

$$\tilde{\zeta}(\lambda, \beta, y) \geq \sqrt{(\lambda - \lambda_o)^2 + (\beta - \beta_o)^2} \cdot (-r) + \varepsilon \frac{r}{2} \geq \frac{\varepsilon}{4} \cdot (-r) + \varepsilon \frac{r}{2} > 0. \tag{5.39}$$

Let $O_1 \subset \mathbb{R}^2 \times H^1(S^1, P)$ be defined by

$$O_1 = \{(\lambda, \beta, y) : \frac{\varepsilon}{4} < \sqrt{(\lambda - \lambda_o)^2 + (\beta - \beta_o)^2} < \varepsilon, \ \|y\| < r\}$$

Then, $F_{\tilde{\zeta}}$ and $G_{\tilde{\zeta}}$ are homotopic on O if and only if they are homotopic on O_1, since $F_{\tilde{\zeta}}$ and $G_{\tilde{\zeta}}$ are always homotopic on $O \setminus O_1$ due to (5.39). That is,

$$(F_{\tilde{\zeta}}, O) \sim (G_{\tilde{\zeta}}, O) \quad \Leftrightarrow \quad (F_{\tilde{\zeta}}, O_1) \sim (G_{\tilde{\zeta}}, O_1)$$

Step 3. (Linearization) Denote by $F := (F_W, F_U)$ and $G := (G_W, G_U)$. Then,

$$\begin{aligned} D_yF(\lambda,\beta,0) &= \begin{pmatrix} D_wF_W(\lambda,\beta,0) & D_uF_W(\lambda,\beta,0) \\ D_wF_U(\lambda,\beta,0) & D_uF_U(\lambda,\beta,0) \end{pmatrix} \\ &= \begin{pmatrix} (L+K)^{-1}(\frac{1}{\beta}D_wf_W(\lambda,0)+K) & 0 \\ (L+K)^{-1}(\frac{1}{\beta}D_wg(\lambda,0)) & (L+K)^{-1}(\frac{1}{\beta}D_ug(\lambda,0)+K) \end{pmatrix} \end{aligned}$$

Since $a_W(\lambda,w) = f_W(\lambda,w,0)$, we have $D_wf_W(\lambda,0) = D_wa_W(\lambda,0)$. Similarly, $D_ug(\lambda,0) = D_ub(\lambda,0)$. Thus,

$$D_yF(\lambda,\beta,0) = \begin{pmatrix} (L+K)^{-1}(\frac{1}{\beta}D_wa_W(\lambda,0)+K) & 0 \\ (L+K)^{-1}(\frac{1}{\beta}D_wg(\lambda,0)) & (L+K)^{-1}(\frac{1}{\beta}D_ub(\lambda,0)+K) \end{pmatrix}$$

On the other hand, we have

$$D_yG(\lambda,\beta,0) = \begin{pmatrix} (L+K)^{-1}(\frac{1}{\beta}D_wa_W(\lambda,0)+K) & 0 \\ 0 & (L+K)^{-1}(\frac{1}{\beta}D_ub(\lambda,0)+K) \end{pmatrix}$$

The above two linear operators have the same spectrum, thus are homotopic on O_1. The homotopy is also $\Sigma \times S^1$-interior equivariant, since it involves only a map with image in $H^1(S^1, U)$. Define

$$\begin{aligned} A_{\tilde{\zeta}} : \qquad & \overline{O} \to \mathbb{R} \times H^1(S^1, P) \\ & (\lambda,\beta,y) \mapsto (\tilde{\zeta}(\lambda,\beta,y), y - D_yF(\lambda,\beta,0)y), \end{aligned}$$

and

$$\begin{aligned} B_{\tilde{\zeta}} : \qquad & \overline{O} \to \mathbb{R} \times H^1(S^1, P) \\ & (\lambda,\beta,y) \mapsto (\tilde{\zeta}(\lambda,\beta,y), y - D_yG(\lambda,\beta,0)y). \end{aligned}$$

Thus, by linearization, we have

$$(F_{\tilde{\zeta}}, O_1) \sim (A_{\tilde{\zeta}}, O_1) \sim (B_{\tilde{\zeta}}, O_1) \sim (G_{\tilde{\zeta}}, O_1).$$

Consequently, F_ζ and G_ζ are $\Sigma \times S^1$-interior equivariant homotopic maps on O, thus have zeros which are $\Sigma \times S^1$-interior equivariant homotopic to each other. The statement follows.

□

Corollary 5.3.2. *Let ω, ω_e be the bifurcation invariants associated to (5.31) and (5.37) respectively. That is,*

$$\omega = \deg_{(\tilde{\Sigma}, \tilde{W})}(F_\zeta, O),$$

and

$$\omega_e = \Sigma \times S^1\text{-Deg}\,(G_\zeta, O),$$

where O, F_ζ, G_ζ are defined by (5.34), (5.36), (5.38) and $\Sigma \times S^1\text{-Deg}$ stands for equivaraint degree (cf. Subsection 3.2.1). Then,

$$\omega = \omega_e$$

using the identification (5.30) in Proposition 5.2.18.

Proof. It follows from Theorem 5.3.1 and the homotopy invariance of the interior equivariant degree. □

5.3.3 An Example

We use the example from Subsection 4.3.3 to show how to obtain bifurcating branches based on Corollary 5.3.2.

Consider the coupled cell system (4.26) admissible to the network of Figure 4.4. As indicated by Example 2.1.12, the regular network has an interior symmetry of D_3 on the subset $\mathcal{S} = \{1, 2, 4\}$. The adjacency matrix A has eigenvalues $2, 1, -1$, where the multiplicity of -1 is partly related to this interior symmetry.

We are interested in finding out whether there are *additional* bifurcating branches related to the interior symmetry than those predicted in Subsection 4.3.3 using quotient symmetries, around the bifurcation center

$$(\lambda_o, \beta_o, 0) = (0, 1, 0).$$

Interior Symmetry By Theorem 5.3.1, bifurcating branches of (4.26) are $D_3 \times S^1$-interior equivariant homotopic to those of the following system

$$\begin{cases} \dot{x}_1 &= f_o(\lambda, x_1, x_2, x_4) \\ \dot{x}_2 &= f_o(\lambda, x_2, x_1, x_4) \\ \dot{x}_4 &= f_o(\lambda, x_4, x_1, x_2) \\ \dot{x}_3 &= f_o(\lambda, x_3, 0, x_5) \\ \dot{x}_5 &= f_o(\lambda, x_5, 0, x_3), \end{cases} \tag{5.40}$$

which is D_3-equivariant and admissible to the network in Figure 5.9.

Using the D_3-equivariance of (5.40), one can define a bifurcation invariant ω_e for every Hopf bifurcation center, following the procedure outlined in Subsection 3.2.1. The value of ω_e can be obtained by

$$\texttt{showdegree}[D3](n_0, n_1, n_2, m_0, m_1, m_2), \quad \text{for } n_i, m_j \in \mathbb{Z},$$

(cf. Subsection 3.2.3). We explain in detail.

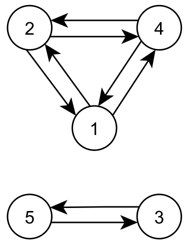


Figure 5.9: The network structure of the system (5.40).

For $(\lambda_o, \beta_o, 0) = (0, 1, 0)$, the entries n_i's and m_j's can be determined based on the spectrum of the Jacobian. The adjacency matrix in this case is

$$A_e = \left(\begin{array}{ccc|cc} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right) = \left(\begin{array}{c|c} A_1 & 0 \\ \hline 0 & A_2 \end{array}\right) \tag{5.41}$$

which has the same spectrum as that of A in (4.23):

$$\sigma(A_e) = \sigma(A_1) \cup \sigma(A_2) = \{2, -1, -1\} \cup \{1, -1\}.$$

Consequently, the spectrum of the Jacobian of (5.40) at $(\lambda_o, 0)$ given by

$$J_e(\lambda_o) = \alpha(\lambda_o) \otimes I_5 + \beta \otimes A_e,$$

agrees with that of the Jacobian of (4.26), as listed in (4.27).

The Integers n_0, n_1, n_2 Based on the symbols of irreducible representations of D_3 listed in Example 2.2.15, we have

$$(\mathbb{R}^2)^5 = \mathcal{V}_0 \times \mathcal{V}_0 \times \mathcal{V}_0 \times \mathcal{V}_0 \times \mathcal{V}_0 \times \mathcal{V}_0 \times \mathcal{V}_2 \times \mathcal{V}_2,$$

where $\mathcal{V}_0$ is 1-dimensional and $\mathcal{V}_2$ is 2-dimensional.

The eigenvalues of $J_e(\lambda_o)$ with positive real part are

$$\sigma_{1,2} = 3 \pm 4i, \quad \sigma_{3,4} = 2 \pm 3i.$$

The pair $\sigma_{1,2}$ is related to the valency 2 of the upper bi-direction ring in Figure 5.9, and the pair $\sigma_{3,4}$ is related to the valency 1 of the lower bi-direction ring in Figure 5.9. Thus, their eigenspaces are composed of trivial representations $\mathcal{V}_0$ only. It follows that

$$n_0 = 4, \quad n_1 = n_2 = 0.$$

The Integers m_0, m_1, m_2 Using the irreducible complex representations of D_3 (cf. Example 2.2.15), we have

$$(\mathbb{C}^2)^5 = \mathcal{U}_0 \times \mathcal{U}_0 \times \mathcal{U}_0 \times \mathcal{U}_0 \times \mathcal{U}_0 \times \mathcal{U}_0 \times \mathcal{U}_2 \times \mathcal{U}_2,$$

where $\mathcal{U}_0$ is 1-dimensional and $\mathcal{U}_2$ is 2-dimensional.

The eigenspace $E^c(i\beta_o)$ is related to the eigenvalue -1 of A_e. Thus, we have

$$E^c(i\beta_o) = \mathcal{U}_0 \times \mathcal{U}_0 \times \mathcal{U}_2 \times \mathcal{U}_2,$$

where $\mathcal{U}_0$'s are related to the eigenvalue -1 of A_2 and $\mathcal{U}_2$'s are related to the eigenvalue -1 of A_1 which is relevant to the symmetry D_3. It follows that

$$m_0 = 1, \quad , m_1 = 0, \quad m_2 = 2.$$

Therefore, we have

$$\begin{aligned} \omega_e &= \pm\texttt{showdegree}[D3](4,0,0,1,0,2) \\ &= \pm\Big(2(\mathbb{Z}_3^t) + (D_3) + 2(D_1^z) + 2(D_1) - 2(\mathbb{Z}_1)\Big) \end{aligned} \tag{5.42}$$

The dominating orbit types are (D_3), $(\mathbb{Z}_3^t)$, (D_1^z) and we list the bifurcating branches of solutions with their symmetry and form in Table 5.1.

Orbit Type	**Symmetry**	**Form of Periodic Solutions** (for some period T)
(D_3)	D_3	$\big(x(t), x(t), y(t), x(t), z(t)\big)$
$(\mathbb{Z}_3^t)$	$\mathbb{Z}_3^t$	$\big(x(t), x(t+\frac{T}{3}), y(t), x(t+\frac{2T}{3}), z(t)\big)$
	$\kappa\mathbb{Z}_3^t\kappa^{-1}$	$\big(x(t), x(t+\frac{2T}{3}), y(t), x(t+\frac{T}{3}), z(t)\big)$
(D_1^z)	D_1^z	$\big(x(t), y(t), z_(t), x(t+\frac{T}{2}), w(t)\big)$, for $y(t) = y(t+\frac{T}{2})$
	$\xi D_1^z\xi^{-1}$	$\big(x(t), x(t+\frac{T}{2}), z_(t), y(t), w(t)\big)$, for $y(t) = y(t+\frac{T}{2})$
	$\xi^2 D_1^z\xi^{-2}$	$\big(x(t), y(t), z(t), y(t+\frac{T}{2}), w(t)\big)$, for $x(t) = x(t+\frac{T}{2})$

Table 5.1: The summary of symmetric properties of bifurcating branches of solutions from $x_o = 0$ of system (4.26) (Part II), using the interior symmetry on $\{1, 2, 4\}$.

Unfortunately, none of the solutions listed in Table 5.1 are necessarily *new* branches compared with those listed in Table 4.1. For example, the first branch in Table 5.1 of form

$$\big(x(t), x(t), y(t), x(t), z(t)\big)$$

may coincide with the branch of synchrony type Δ_2 in Table 4.1, if $z(t) = y(t + \frac{T}{2})$. The same holds for the rest of the branches in Table 5.1, since they may coincide with the branches of synchrony type Δ_1 (with the same symmetry) in Table 4.1.

Quotient Interior Symmetry The network has a quotient interior symmetry D_3, which is related to the quotient network induced by $\{\{1\}, \{2\}, \{3, 5\}, \{4\}\}$ (cf. Example 2.1.18). The quotient system is of form

$$\begin{cases} \dot{x}_1 &= f_o(\lambda, x_1, x_2, x_4) \\ \dot{x}_2 &= f_o(\lambda, x_2, x_1, x_4) \\ \dot{x}_3 &= f_o(\lambda, x_3, x_1, x_3) \\ \dot{x}_4 &= f_o(\lambda, x_4, x_1, x_2) \end{cases} \tag{5.43}$$

obtained by restricting the flow onto $\Delta_{01} = \{x_3 = x_5\}$ in (4.26) (cf. Figure 2.6).

By Theorem 5.3.1, bifurcating branches of (5.43) are $D_3 \times S^1$-interior equivariant homotopic to those of the system

$$\begin{cases} \dot{x}_1 &= f_o(\lambda, x_1, x_2, x_4) \\ \dot{x}_2 &= f_o(\lambda, x_2, x_1, x_4) \\ \dot{x}_4 &= f_o(\lambda, x_4, x_1, x_2) \\ \dot{x}_3 &= f_o(\lambda, x_3, 0, x_3) \end{cases} \tag{5.44}$$

The adjacency matrix of (5.44) is

$$A_q = \left(\begin{array}{ccc|c} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array} \right) = \left(\begin{array}{c|c} Q_1 & 0 \\ \hline 0 & Q_2 \end{array} \right) \tag{5.45}$$

where $\sigma(Q_1) = \{2, -1, -1\}$ and $\sigma(Q_2) = \{1\}$. Compare with (5.41), the linearization has effectively lost an eigenvalue -1 through the quotient. The eigenvalues -1 that are related to the D_3-symmetry remain.

Following the same procedure as the previous subsection, we have

$$\begin{aligned} \omega_q &= \pm\texttt{showdegree}[D3](4, 0, 0, 0, 0, 2) \\ &= \pm\Big(2(\mathbb{Z}_3^t) + 2(D_1^z) + 2(D_1) - 2(\mathbb{Z}_1)\Big) \end{aligned} \tag{5.46}$$

where ω_q stands for the bifurcation invariant associated to the quotient system (5.43).

Using the dominating orbit types (D_3), $(\mathbb{Z}_3^t)$ and (D_1^z), we list the bifurcating branches of solutions with their synchrony and symmetry in Table 5.2.

The last branch in Table 5.2 coincides with the branch predicted for Δ_{01} in Table 4.1, and the branch in Δ_3 may coincide with the branch predicted for Δ_{13} in Table 4.1. All other branches are *new* branches.

Synchrony	**Symmetry**	**Form of Periodic Solutions** (for some period T)
Δ_{01} (a,b,c,d,c)	$\mathbb{Z}_3^t$	$\big(x(t), x(t+\frac{T}{3}), y(t), x(t+\frac{2T}{3}), y(t)\big)$
	$\kappa\mathbb{Z}_3^t\kappa^{-1}$	$\big(x(t), x(t+\frac{2T}{3}), y(t), x(t+\frac{T}{3}), y(t)\big)$
	D_1	$\big(x(t), y(t), z(t), x(t), z(t)\big) \in \Delta_5$
	$\xi D_1 \xi^{-1}$	$\big(x(t), x(t), z(t), y(t), z(t)\big) \in \Delta_6$
	$\xi^2 D_1 \xi^{-2}$	$\big(x(t), y(t), z(t), y(t), z(t)\big) \in \Delta_3$
	D_1^z	$\big(x(t), y(t), z(t), x(t+\frac{T}{2}), z(t)\big)$, for $y(t) = y(t+\frac{T}{2})$
	$\xi D_1^z \xi^{-1}$	$\big(x(t), x(t+\frac{T}{2}), z(t), y(t), z(t)\big)$, for $y(t) = y(t+\frac{T}{2})$
	$\xi^2 D_1^z \xi^{-2}$	$\big(x(t), y(t), z(t), y(t+\frac{T}{2}), z(t)\big)$, for $x(t) = x(t+\frac{T}{2})$

Table 5.2: The summary of symmetric properties of bifurcating branches of solutions from $x_o = 0$ of system (4.26) (Part III), using the quotient interior symmetry on $\Delta_{01} = \{x_3 = x_5\}$.

Summary Using quotient symmetry and lattice equivariant degree, we obtained in Subsection 4.3.3 **18** distinct bifurcating branches for the system (4.26), as listed in Table 4.1. Using quotient interior symmetry and interior equivariant degree, we obtained additional **6** bifurcating branches (cf. Table 5.2). We summarize these branches by following through the synchrony subspaces listed in Figure 2.6, from top to bottom and left to right. See Table 5.3.

Synchrony	**Form of Periodic Solutions** (for some period T)	**Symmetry**
Δ_{43} (a,b,a,a,b)	$\big(x(t), y(t), x(t), x(t), y(t)\big)$	
Δ_{41} (a,a,b,b,a)	$\big(x(t), x(t), y(t), y(t), x(t)\big)$	
Δ_{13} (a,b,b,b,b)	$\big(x(t), y(t), y(t), y(t), y(t)\big)$	
Δ_{12} (a,b,b,a,a)	$\big(x(t), y(t), y(t), x(t), x(t)\big)$	
Δ_{11} (a,a,a,b,b)	$\big(x(t), x(t), x(t), y(t), y(t)\big)$	
Δ_4 (a,b,c,c,b)	$\big(x(t), x(t+\frac{T}{3}), x(t+\frac{2T}{3}), x(t+\frac{2T}{3}), x(t+\frac{T}{3})\big)$	$\mathbb{Z}_3^t$
	$\big(x(t), x(t+\frac{2T}{3}), x(t+\frac{T}{3}), x(t+\frac{T}{3}), x(t+\frac{2T}{3})\big)$	$\kappa\mathbb{Z}_3^t\kappa^{-1}$
	$\big(x(t), y(t), x(t+\frac{T}{2}), x(t+\frac{T}{2}), y(t)\big)$, for $y(t) = y(t+\frac{T}{2})$	D_1^z
	$\big(x(t), x(t+\frac{T}{2}), y(t), y(t), x(t+\frac{T}{2})\big)$, for $y(t) = y(t+\frac{T}{2})$	$\xi D_1^z\xi^{-1}$
	$\big(x(t), y(t), y(t+\frac{T}{2}), y(t+\frac{T}{2}), y(t)\big)$, for $x(t) = x(t+\frac{T}{2})$	$\xi^2 D_1^z\xi^{-2}$
Δ_2 (a,a,b,a,c)	$\big(x(t), x(t), y(t), x(t), y(t+\frac{T}{2})\big)$	$\mathbb{Z}_2^-$
Δ_6 (a,a,b,c,b)	$\big(x(t), x(t), y(t), z(t), y(t)\big)$	
Δ_5 (a,b,c,a,c)	$\big(x(t), y(t), z(t), x(t), z(t)\big)$	
Δ_3 (a,b,c,b,c)	$\big(x(t), y(t), z(t), y(t), z(t)\big)$	$\mathbb{Z}_1$
Δ_1 (a,b,b,c,c)	$\big(x(t), x(t+\frac{T}{3}), x(t+\frac{T}{3}), x(t+\frac{2T}{3}), x(t+\frac{2T}{3})\big)$	$\mathbb{Z}_3^t$
	$\big(x(t), x(t+\frac{2T}{3}), x(t+\frac{2T}{3}), x(t+\frac{T}{3}), x(t+\frac{T}{3})\big)$	$\kappa\mathbb{Z}_3^t\kappa^{-1}$
	$\big(x(t), y(t), y(t), x(t+\frac{T}{2}), x(t+\frac{T}{2})\big)$, for $y(t) = y(t+\frac{T}{2})$	D_1^z
	$\big(x(t), x(t+\frac{T}{2}), x(t+\frac{T}{2}), y(t), y(t)\big)$, for $y(t) = y(t+\frac{T}{2})$	$\xi D_1^z\xi^{-1}$
	$\big(x(t), y(t), y(t), y(t+\frac{T}{2}), y(t+\frac{T}{2})\big)$, for $x(t) = x(t+\frac{T}{2})$	$\xi^2 D_1^z\xi^{-2}$
Δ_{01} (a,b,c,d,c)	$\big(x(t), x(t+\frac{T}{3}), y(t), x(t+\frac{2T}{3}), y(t)\big)$	$\mathbb{Z}_3^t$
	$\big(x(t), x(t+\frac{2T}{3}), y(t), x(t+\frac{T}{3}), y(t)\big)$	$\kappa\mathbb{Z}_3^t\kappa^{-1}$
	$\big(x(t), y(t), z(t), x(t+\frac{T}{2}), z(t)\big)$, for $y(t) = y(t+\frac{T}{2})$	D_1^z
	$\big(x(t), x(t+\frac{T}{2}), z(t), y(t), z(t)\big)$, for $y(t) = y(t+\frac{T}{2})$	$\xi D_1^z\xi^{-1}$
	$\big(x(t), y(t), z(t), y(t+\frac{T}{2}), z(t)\big)$, for $x(t) = x(t+\frac{T}{2})$	$\xi^2 D_1^z\xi^{-2}$

Table 5.3: The summary of distinct bifurcating branches of solutions from $x_o = 0$ of system (4.26), based on Table 4.1, Table 5.1 and Table 5.2.

Appendix: Computation of the Bifurcation Invariant

We give the details of the computation of the bifurcation invariant

$$\omega(\lambda_o, \beta_o, x_o) = \mathcal{T}\text{-Deg}^t(F_\zeta, O),$$

associated with the bifurcation center $(\lambda_o, \beta_o, x_o) = (0, 1, 0)$ of the system (4.26).

Recall that by definition,

$$\mathcal{T}\text{-Deg}^t(F_\zeta, O) = \sum_{U \in \mathcal{L}} (\breve{U}, a_{\breve{U}}),$$

where

$$a_{\breve{U}} = \Gamma_U \times S^1\text{-Deg}^t\,(F_\zeta|_{\mathbb{R}^2 \times \breve{U}}, O \cap (\mathbb{R}^2 \times \breve{U})) - \sum_{U' < U} \mathsf{H}_{U',U}(a_{\breve{U}'}).$$

Let $\Gamma = \Gamma_U$, $F = F_\zeta|_{\mathbb{R}^2 \times \breve{U}}$ and $\Omega = O \cap (\mathbb{R}^2 \times \breve{U})$ for some $U \in \mathcal{L}$. The twisted degree $\Gamma \times S^1\text{-Deg}^t\,(F, \Omega)$ can be computed from the following formula (cf. [16])

$$\Gamma \times S^1\text{-Deg}^t\,(F, \Omega) = \prod_{\mu \in \sigma_+(J(\lambda_o))} \prod_i (\deg_{\mathcal{V}_i})^{m_i(\mu)} \cdot \sum_{j,l} t_{j,l}(\lambda_o, \beta_o) \deg_{\mathcal{V}_{j,l}},$$

where $\sigma_+(J(\lambda_o))$ is the positive spectrum of $J(\lambda_o)$, $\deg_{\mathcal{V}_i}$ is the basic degree of the i-th irreducible representation of Γ over reals, $m_i(\mu) = \dim(E(\mu) \cap V_i)/\dim \mathcal{V}_i$ is the algebraic multiplicity of μ when restricted to the i-th isotypical component of the eigenspace $E(\mu)$, $\deg_{\mathcal{V}_{j,l}}$ is the basic degree of the (j,l)-th irreducible representation of $\Gamma \times S^1$ over complex numbers, and $t_{j,l}(\lambda_o, \beta_o)$ is the (j,l)-th isotypical crossing number of (λ_o, β_o).

In our example, since $\sigma_+(J(\lambda_o)) = \emptyset$ and $il\beta_o$ is only a critical eigenvalue for $l = 1$, we have

$$\Gamma \times S^1\text{-Deg}^t\,(F, \Omega) = \sum_j t_{j,1}(\lambda_o, \beta_o) \deg_{\mathcal{V}_{j,1}}.$$

Computation of $a_{\check{\Delta}}$ In this case, $\Gamma = \Gamma_\Delta = \mathbb{Z}_1$. Consider $\Delta^c = \mathbb{R}^2 \otimes \mathbb{C} \simeq \mathbb{C}^2$ as a complex $\mathbb{Z}_1$-representation. Then, the $\mathbb{Z}_1$-isotypical decomposition of Δ^c is

$$\Delta^c = U_0 \oplus U_0,$$

where U_0 is the trivial (complex) $\mathbb{Z}_1$-representation. Thus,

$$\Gamma_\Delta \times S^1\text{-Deg}^t\,(F_\zeta|_{\mathbb{R}^2\times\check{\Delta}}, \mathcal{O} \cap (\mathbb{R}^2 \times \check{\Delta})) = t_{0,1}(\lambda_o, \beta_o)\deg_{\mathcal{V}_{0,1}}.$$

Consider $J(\lambda)$ as a complex linear map in Δ^c. Then,

$$J(\lambda) = \alpha(\lambda) + 2\beta, \quad \text{and} \quad \sigma(J(\lambda)) = \{\sigma_{1,2}\}.$$

Since $\sigma(J(\lambda_o)) \cap i\mathbb{R} = \emptyset$, there are no eigenvalues crossing the purely imaginary axis, as λ crosses λ_o. Thus, $t_{0,1}(\lambda_o, \beta_o) = 0$ and consequently,

$$a_{\check{\Delta}} = \Gamma_\Delta \times S^1\text{-Deg}^t\,(F_\zeta|_{\mathbb{R}^2\times\check{\Delta}}, \mathcal{O} \cap (\mathbb{R}^2 \times \check{\Delta})) = 0.$$

Computation of $a_{\check{\Delta}_{21}}$ In this case, $\Gamma = \Gamma_{\Delta_{21}} = \mathbb{Z}_1$. Similarly, we have

$$\Delta_{21}^c = U_0 \oplus U_0 \oplus U_0 \oplus U_0,$$

where U_0 is the trivial (complex) $\mathbb{Z}_1$-representation. Thus,

$$\Gamma_{\Delta_{21}} \times S^1\text{-Deg}^t\,(F_\zeta|_{\mathbb{R}^2\times\check{\Delta}_{21}}, \mathcal{O} \cap (\mathbb{R}^2 \times \check{\Delta}_{21})) = t_{0,1}(\lambda_o, \beta_o)\deg_{\mathcal{V}_{0,1}}.$$

Consider $J(\lambda)$ as a complex linear map in Δ_{21}^c. Then,

$$J(\lambda) = \alpha \otimes \mathbb{R}^2 + \beta \otimes \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix}, \quad \text{and} \quad \sigma(J(\lambda)) = \{\sigma_{1,2}, \sigma_{3,4}\}.$$

Since $\sigma(J(\lambda_o)) \cap i\mathbb{R} = \emptyset$, we have $t_{0,1}(\lambda_o, \beta_o) = 0$. Thus,

$$a_{\check{\Delta}_{21}} = \Gamma_{\Delta_{21}} \times S^1\text{-Deg}^t\,(F_\zeta|_{\mathbb{R}^2\times\check{\Delta}_{21}}, \mathcal{O} \cap (\mathbb{R}^2 \times \check{\Delta}_{21})) - 0 = 0.$$

Computation of $a_{\check{\Delta}_2}$ In this case, $\Gamma = \Gamma_{\Delta_2} = \mathbb{Z}_2$. Consider $\Delta_2^c = \mathbb{R}^6 \otimes \mathbb{C} \simeq \mathbb{C}^6$ as a complex $\mathbb{Z}_2$-representation. Then, the $\mathbb{Z}_2$-isotypical decomposition of Δ_2^c is

$$\Delta_2^c = U_0 \oplus U_0 \oplus U_0 \oplus U_0 \oplus U_1 \oplus U_1,$$

where U_0 is the trivial (complex) $\mathbb{Z}_2$-representation and U_1 is the $\mathbb{Z}_2$-representation given by antipodal action. Thus,

$$\Gamma_{\Delta_2}\times S^1\text{-Deg}^t\,(F_\zeta|_{\mathbb{R}^2\times\check{\Delta}_2}, \mathcal{O}\cap(\mathbb{R}^2\times\check{\Delta}_2)) = t_{0,1}(\lambda_o, \beta_o)\deg_{\mathcal{V}_{0,1}} + t_{1,1}(\lambda_o, \beta_o)\deg_{\mathcal{V}_{1,1}}.$$

Consider $J(\lambda)$ as a complex linear map in Δ_2^c. Then,

$$J(\lambda) = \alpha \otimes \mathbb{R}^2 + \beta \otimes \begin{pmatrix} 2 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}, \quad \text{and} \quad \sigma(J(\lambda)) = \{\sigma_{1,2}, \sigma_{3,4}, \sigma_{5,6}\}.$$

Thus, $t_{0,1}(\lambda_o, \beta_o) = 0$ and $t_{1,1}(\lambda_o, \beta_o) = -1$. Therefore,

$$a_{\check{\Delta}_2} = \Gamma_{\Delta_2} \times S^1\text{-Deg}^t\,(F_\zeta|_{\mathbb{R}^2\times\check{\Delta}_2}, O \cap (\mathbb{R}^2 \times \check{\Delta}_2)) - 0 - 0 = -\deg_{\mathcal{V}_{1,1}}.$$

Here $\mathcal{V}_{1,1} \simeq U_1$ is a $\mathbb{Z}_2 \times S^1$-representation given by "complexifying" the $\mathbb{Z}_2$-action on U_1, that is $(\xi, z)w := z\xi w$, for $\xi \in \mathbb{Z}_2$, $z \in S^1$, $w \in \mathbb{C}$. The orbit type of $w \neq 0$ is $\mathbb{Z}_2^- := \{(1,1), (-1,-1)\}$. Thus, $\deg_{\mathcal{V}_{1,1}} = (\mathbb{Z}_2^-)$ and so

$$a_{\check{\Delta}_2} = -\deg_{\mathcal{V}_{1,1}} = -(\mathbb{Z}_2^-).$$

Computation of $a_{\check{\Delta}_4}$ In this case, $\Gamma = \Gamma_{\Delta_4} = D_3$. Consider $\Delta_4^c = \mathbb{R}^6 \otimes \mathbb{C} \simeq \mathbb{C}^6$ as a complex D_3-representation. Then, the D_3-isotypical decomposition of Δ_4^c is

$$\Delta_4^c = U_0 \oplus U_0 \oplus U_1 \oplus U_1,$$

where U_0 is the trivial D_3-representation, $U_1 \simeq \mathbb{C} \oplus \mathbb{C}$ is the complex D_3-representation given by $\xi(z_1, z_2) = (\xi z_1, \xi^{-1} z_2)$, $\kappa(z_1, z_2) = (z_2, z_1)$, for $z_1, z_2 \in \mathbb{C}$. Thus,

$$\Gamma_{\Delta_4}\times S^1\text{-Deg}^t\,(F_\zeta|_{\mathbb{R}^2\times\check{\Delta}_4}, O\cap(\mathbb{R}^2\times\check{\Delta}_4)) = t_{0,1}(\lambda_o, \beta_o)\deg_{\mathcal{V}_{0,1}} + t_{1,1}(\lambda_o, \beta_o)\deg_{\mathcal{V}_{1,1}}.$$

Consider $J(\lambda)$ as a complex linear map in Δ_4^c. Then,

$$J(\lambda) = \alpha \otimes \mathbb{R}^2 + \beta \otimes \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}, \quad \text{and} \quad \sigma(J(\lambda)) = \{\sigma_{1,2}, \sigma_{5,6,7,8}\}.$$

Thus, $t_{0,1}(\lambda_o, \beta_o) = 0$ and $t_{1,1}(\lambda_o, \beta_o) = -2$. Therefore,

$$a_{\check{\Delta}_4} = \Gamma_{\Delta_4} \times S^1\text{-Deg}^t\,(F_\zeta|_{\mathbb{R}^2\times\check{\Delta}_4}, O \cap (\mathbb{R}^2 \times \check{\Delta}_4)) - 0 = -2\deg_{\mathcal{V}_{1,1}}.$$

It was shown in [16] that (cf. Example 2.3.7 for the definition of $\mathbb{Z}_3^t$ and D_1^z)

$$\deg_{\mathcal{V}_{1,1}} = (\mathbb{Z}_3^t) + (D_1) + (D_1^z) - (\mathbb{Z}_1).$$

Thus,

$$a_{\check{\Delta}_4} = -2(\mathbb{Z}_3^t) - 2(D_1) - 2(D_1^z) + 2(\mathbb{Z}_1).$$

Computation of $a_{\check{\Delta}_1}$ In this case, $\Gamma = \Gamma_{\Delta_1} = D_3$. Consider $\Delta_1^c = \mathbb{R}^6 \otimes \mathbb{C} \simeq \mathbb{C}^6$ as a complex D_3-representation. Similar as the case for Δ_4, the D_3-isotypical decomposition of Δ_1^c is

$$\Delta_1^c = U_0 \oplus U_0 \oplus U_1 \oplus U_1.$$

Thus,

$$\Gamma_{\Delta_1}\times S^1\text{-Deg}^t\,(F_\zeta|_{\mathbb{R}^2\times\check{\Delta}_1}, O\cap(\mathbb{R}^2\times\check{\Delta}_1)) = t_{0,1}(\lambda_o, \beta_o)\deg_{\mathcal{V}_{0,1}} + t_{1,1}(\lambda_o, \beta_o)\deg_{\mathcal{V}_{1,1}}.$$

Consider $J(\lambda)$ as a complex linear map in Δ_1^c. Then,

$$J(\lambda) = \alpha \otimes \mathbb{R}^2 + \beta \otimes \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}, \quad \text{and} \quad \sigma(J(\lambda)) = \{\sigma_{1,2}, \sigma_{5,6,7,8}\}.$$

Thus, $t_{0,1}(\lambda_o, \beta_o) = 0$ and $t_{1,1}(\lambda_o, \beta_o) = -2$. Therefore,

$$\begin{aligned} a_{\check{\Delta}_1} &= \Gamma_{\Delta_1} \times S^1\text{-Deg}^t\,(F_\zeta|_{\mathbb{R}^2\times\check{\Delta}_1}, O \cap (\mathbb{R}^2 \times \check{\Delta}_1)) - 0 = -2\deg_{\mathcal{V}_{1,1}} \\ &= -2(\mathbb{Z}_3^t) - 2(D_1) - 2(D_1^z) + 2(\mathbb{Z}_1). \end{aligned}$$

Computation of $a_{\check{\Delta}_3}$ In this case, $\Gamma = \Gamma_{\Delta_3} = \mathbb{Z}_1$. Consider $\Delta_3^c = \mathbb{R}^6 \otimes \mathbb{C} \simeq \mathbb{C}^6$ as a complex $\mathbb{Z}_1$-representation. Then, , the $\mathbb{Z}_1$-isotypical decomposition of Δ_3^c is

$$\Delta_1^c = U_0 \oplus U_0 \oplus U_0 \oplus U_0 \oplus U_0 \oplus U_0.$$

Thus,

$$\Gamma_{\Delta_3} \times S^1\text{-Deg}^t\,(F_\zeta|_{\mathbb{R}^2\times\check{\Delta}_3}, O \cap (\mathbb{R}^2 \times \check{\Delta}_3)) = t_{0,1}(\lambda_o, \beta_o)\deg_{\mathcal{V}_{0,1}}.$$

Consider $J(\lambda)$ as a complex linear map in Δ_3^c. Then,

$$J(\lambda) = \alpha \otimes \mathbb{R}^2 + \beta \otimes \begin{pmatrix} 0 & 2 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \quad \text{and} \quad \sigma(J(\lambda)) = \{\sigma_{1,2}, \sigma_{3,4}, \sigma_{5,6}\}.$$

Thus, $t_{0,1}(\lambda_o, \beta_o) = -1$. Therefore,

$$\begin{aligned} a_{\check{\Delta}_3} &= \Gamma_{\Delta_3} \times S^1\text{-Deg}^t\,(F_\zeta|_{\mathbb{R}^2\times\check{\Delta}_3}, O \cap (\mathbb{R}^2 \times \check{\Delta}_3)) - 0 = -\deg_{\mathcal{V}_{0,1}} \\ &= -(\mathbb{Z}_1). \end{aligned}$$

Computation of $a_{\check{\Delta}_{00}}$ In this case, $\Gamma = \Gamma_{\Delta_{00}} = \mathbb{Z}_2$. Consider $\Delta_{00}^c = \mathbb{R}^8 \otimes \mathbb{C} \simeq \mathbb{C}^8$ as a complex $\mathbb{Z}_2$-representation. Then, the $\mathbb{Z}_2$-isotypical decomposition of Δ_{00}^c is

$$\Delta_{00}^c = U_0 \oplus U_0 \oplus U_0 \oplus U_0 \oplus U_0 \oplus U_0 \oplus U_1 \oplus U_1,$$

where U_0 is the trivial (complex) $\mathbb{Z}_2$-representation and U_1 is the $\mathbb{Z}_2$-representation given by antipodal action. Thus,

$$\Gamma_{\Delta_{00}} \times S^1\text{-Deg}^t\,(F_\zeta|_{\mathbb{R}^2\times\check{\Delta}_{00}}, O \cap (\mathbb{R}^2 \times \check{\Delta}_{00})) = t_{0,1}(\lambda_o, \beta_o)\deg_{\mathcal{V}_{0,1}} + t_{1,1}(\lambda_o, \beta_o)\deg_{\mathcal{V}_{1,1}}.$$

Consider $J(\lambda)$ as a complex linear map in Δ_{00}^c. Then,

$$J(\lambda) = \alpha \otimes \mathbb{R}^2 + \beta \otimes \begin{pmatrix} 0 & 2 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}, \quad \text{and} \quad \sigma(J(\lambda)) = \{\sigma_{1,2}, \sigma_{3,4}, \sigma_{5,6,7,8}\}.$$

Thus, $t_{0,1}(\lambda_o, \beta_o) = 0$ and $t_{1,1}(\lambda_o, \beta_o) = -2$. Therefore,

$$\Gamma_{\Delta_{00}} \times S^1\text{-Deg}^t\,(F_\zeta|_{\mathbb{R}^2\times\check{\Delta}_{00}}, O \cap (\mathbb{R}^2 \times \check{\Delta}_{00})) = -2\deg_{\mathcal{V}_{1,1}} = -2(\mathbb{Z}_2^-).$$

On the other hand, $\mathsf{H}_{\Delta_2,\Delta_{00}} = \text{Id}$, since $\mathsf{h}_{\Delta_2,\Delta_{00}} = \text{Id}$. Consequently,

$$a_{\check{\Delta}_{00}} = -2(\mathbb{Z}_2^-) - \mathsf{H}_{\Delta_2,\Delta_{00}}\Big(-(\mathbb{Z}_2^-)\Big) = -2(\mathbb{Z}_2^-) + (\mathbb{Z}_2^-) = -(\mathbb{Z}_2^-).$$

Computation of $a_{\check{\Delta}_{02}}$ This is a similar case as for Δ_{00}. We have $\Gamma = \Gamma_{\Delta_{02}} = \mathbb{Z}_2$ and the $\mathbb{Z}_2$-isotypical decomposition of Δ_{02}^c is

$$\Delta_{02}^c = U_0 \oplus U_0 \oplus U_0 \oplus U_0 \oplus U_0 \oplus U_0 \oplus U_1 \oplus U_1,$$

where U_0 is the trivial (complex) $\mathbb{Z}_2$-representation and U_1 is the $\mathbb{Z}_2$-representation given by antipodal action. Thus,

$$\Gamma_{\Delta_{02}}\times S^1\text{-Deg}^t\,(F_\zeta|_{\mathbb{R}^2\times\check{\Delta}_{02}}, O\cap(\mathbb{R}^2\times\check{\Delta}_{02})) = t_{0,1}(\lambda_o,\beta_o)\deg_{\mathcal{V}_{0,1}} + t_{1,1}(\lambda_o,\beta_o)\deg_{\mathcal{V}_{1,1}}.$$

Consider $J(\lambda)$ as a complex linear map in Δ_{02}^c. Then,

$$J(\lambda) = \alpha \otimes \mathbb{R}^2 + \beta \otimes \begin{pmatrix} 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 2 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix}, \quad \text{and} \quad \sigma(J(\lambda)) = \{\sigma_{1,2}, \sigma_{3,4}, \sigma_{5,6,7,8}\}.$$

Thus, $t_{0,1}(\lambda_o, \beta_o) = 0$ and $t_{1,1}(\lambda_o, \beta_o) = -2$. Therefore,

$$\Gamma_{\Delta_{02}} \times S^1\text{-Deg}^t\,(F_\zeta|_{\mathbb{R}^2\times\check{\Delta}_{02}}, O \cap (\mathbb{R}^2 \times \check{\Delta}_{02})) = -2\deg_{\mathcal{V}_{1,1}} = -2(\mathbb{Z}_2^-).$$

On the other hand, $\mathsf{H}_{\Delta_2,\Delta_{02}} = \text{Id}$, since $\mathsf{h}_{\Delta_2,\Delta_{02}} = \text{Id}$. Consequently,

$$a_{\check{\Delta}_{02}} = -2(\mathbb{Z}_2^-) - \mathsf{H}_{\Delta_2,\Delta_{02}}\Big(-(\mathbb{Z}_2^-)\Big) = -2(\mathbb{Z}_2^-) + (\mathbb{Z}_2^-) = -(\mathbb{Z}_2^-).$$

Computation of $a_{\check{\Delta}_{03}}$ Similar to Δ_{00} and Δ_{02}, we have

$$\Gamma_{\Delta_{03}}\times S^1\text{-Deg}^t\,(F_\zeta|_{\mathbb{R}^2\times\check{\Delta}_{03}}, O\cap(\mathbb{R}^2\times\check{\Delta}_{03})) = t_{0,1}(\lambda_o,\beta_o)\deg_{\mathcal{V}_{0,1}} + t_{1,1}(\lambda_o,\beta_o)\deg_{\mathcal{V}_{1,1}}.$$

Consider $J(\lambda)$ as a complex linear map in Δ_{03}^c. Then,

$$J(\lambda) = \alpha \otimes \mathbb{R}^2 + \beta \otimes \begin{pmatrix} 1 & 1 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}, \quad \text{and} \quad \sigma(J(\lambda)) = \{\sigma_{1,2}, \sigma_{3,4}, \sigma_{5,6,7,8}\}.$$

Thus, $t_{0,1}(\lambda_o, \beta_o) = 0$ and $t_{1,1}(\lambda_o, \beta_o) = -2$. Therefore,

$$\Gamma_{\Delta_{03}} \times S^1\text{-Deg}^t\,(F_\zeta|_{\mathbb{R}^2\times\check{\Delta}_{03}}, O \cap (\mathbb{R}^2 \times \check{\Delta}_{03})) = -2\deg_{\mathcal{V}_{1,1}} = -2(\mathbb{Z}_2^-),$$

and

$$a_{\check{\Delta}_{03}} = -2(\mathbb{Z}_2^-) - \mathsf{H}_{\Delta_2,\Delta_{03}}\Big(- (\mathbb{Z}_2^-)\Big) = -2(\mathbb{Z}_2^-) + (\mathbb{Z}_2^-) = -(\mathbb{Z}_2^-).$$

Computation of $a_{\check{\Delta}_{01}}$ In this case, $\Gamma = \Gamma_{\Delta_{01}} = \mathbb{Z}_2$. Similar to the case for $\Delta_{00}, \Delta_{02}, \Delta_{03}$, the $\mathbb{Z}_2$-isotypical decomposition of Δ_{01}^c is

$$\Delta_{01}^c = U_0 \oplus U_0 \oplus U_0 \oplus U_0 \oplus U_0 \oplus U_0 \oplus U_1 \oplus U_1,$$

where U_0 is the trivial (complex) $\mathbb{Z}_2$-representation and U_1 is the $\mathbb{Z}_2$-representation given by antipodal action. Thus,

$$\Gamma_{\Delta_{01}} \times S^1\text{-Deg}^t\,(F_\zeta|_{\mathbb{R}^2\times\check{\Delta}_{01}}, O\cap(\mathbb{R}^2\times\check{\Delta}_{01})) = t_{0,1}(\lambda_o,\beta_o)\deg_{\mathcal{V}_{0,1}} + t_{1,1}(\lambda_o,\beta_o)\deg_{\mathcal{V}_{1,1}}.$$

Consider $J(\lambda)$ as a complex linear map in Δ_{01}^c. Then,

$$J(\lambda) = \alpha \otimes \mathbb{R}^2 + \beta \otimes \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix}, \quad \text{and} \quad \sigma(J(\lambda)) = \{\sigma_{1,2}, \sigma_{3,4}, \sigma_{5,6,7,8}\}.$$

Thus, $t_{0,1}(\lambda_o,\beta_o) = 0$ and $t_{1,1}(\lambda_o,\beta_o) = -2$. Therefore,

$$\begin{aligned}\Gamma_{\Delta_{01}} \times S^1\text{-Deg}^t\,(F_\zeta|_{\mathbb{R}^2\times\check{\Delta}_{01}}, O\cap(\mathbb{R}^2\times\check{\Delta}_{01})) &= -2\deg_{\mathcal{V}_{1,1}} \\ &= -2(\mathbb{Z}_2^-).\end{aligned}$$

Consequently,

$$a_{\check{\Delta}_{01}} = -2(\mathbb{Z}_2^-) - \mathsf{H}_{\Delta_3,\Delta_{01}}(a_{\check{\Delta}_3}) = -2(\mathbb{Z}_2^-) + (\mathbb{Z}_2).$$

Computation of a_W In this case, $\Gamma = \Gamma_V = \mathbb{Z}_1$. Consider $V^c = \mathbb{R}^{10}\otimes\mathbb{C} \simeq \mathbb{C}^{10}$ as a complex $\mathbb{Z}_1$-representation. Then, the $\mathbb{Z}_1$-isotypical decomposition of V^c is

$$V^c = U_0 \oplus U_0 \oplus U_0 \oplus U_0 \oplus U_0 \oplus U_0 \oplus U_0 \oplus U_0 \oplus U_0 \oplus U_0,$$

where $U_0 \simeq \mathbb{C}$ is the trivial $\mathbb{Z}_1$-representation. Thus,

$$\Gamma_V \times S^1\text{-Deg}^t\,(F_\zeta|_{\mathbb{R}^2\times W}, O\cap(\mathbb{R}^2\times W)) = t_{0,1}(\lambda_o,\beta_o)\deg_{\mathcal{V}_{0,1}}.$$

Consider $J(\lambda)$ as a complex linear map in V^c. Then,

$$J(\lambda) = \alpha \otimes \mathbb{R}^2 + \beta \otimes A, \quad \text{and} \quad \sigma(J(\lambda)) = \{\sigma_{1,2}, \sigma_{3,4}, \sigma_{5,6,7,8,9,10}\}.$$

Thus, $t_{0,1}(\lambda_o,\beta_o) = -3$. Therefore,

$$\Gamma_V \times S^1\text{-Deg}^t\,(F_\zeta|_{\mathbb{R}^2\times W}, O\cap(\mathbb{R}^2\times W)) = -3\deg_{\mathcal{V}_{0,1}} = -3(\mathbb{Z}_1).$$

Consequently,

$$\begin{aligned}
a_W &= -3(\mathbb{Z}_1) - \mathsf{H}_{\Delta_1,V}(a_{\check{\Delta}_1}) - \mathsf{H}_{\Delta_2,V}(a_{\check{\Delta}_2}) - \mathsf{H}_{\Delta_3,V}(a_{\check{\Delta}_3}) - \mathsf{H}_{\Delta_4,V}(a_{\check{\Delta}_4})\\
&\quad - \mathsf{H}_{\Delta_{00},V}(a_{\check{\Delta}_{00}}) - \mathsf{H}_{\Delta_{01},V}(a_{\check{\Delta}_{01}}) - \mathsf{H}_{\Delta_{02},V}(a_{\check{\Delta}_{02}}) - \mathsf{H}_{\Delta_{03},V}(a_{\check{\Delta}_{03}})\\
&= -3(\mathbb{Z}_1) - 2\mathsf{H}_{\Delta_1,V}(a_{\check{\Delta}_1}) - \mathsf{H}_{\Delta_3,V}(a_{\check{\Delta}_3}) - 4\mathsf{H}_{\Delta_2,V}(a_{\check{\Delta}_2}) - \mathsf{H}_{\Delta_{01},V}(a_{\check{\Delta}_{01}})\\
&= (-3 - 2\cdot(-4) - (-1) - 4\cdot(-1) - (-2+1))(\mathbb{Z}_1) = 11(\mathbb{Z}_1),
\end{aligned}$$

where the last equality used the fact that if $\mathsf{h} : \mathbb{Z}_1 \to G$ is the inclusion homomorphism, then by definition of H, $\mathsf{H}\big((K)\big) = \chi_c(G/K)(\mathbb{Z}_1)$, for $(K) \in \Phi(G)$.

In summary, we have

$$\begin{aligned}
\mathcal{T}\text{-Deg}^t(F_\zeta, O) &= \Big(\boldsymbol{\check{\Delta}_1}, -2(\mathbf{Z}_3^{\mathbf{t}}) - 2(\mathbf{D}_1) - 2(\mathbf{D}_1^z) + 2(\mathbb{Z}_1)\Big) + \Big(\boldsymbol{\check{\Delta}_2}, -(\mathbb{Z}_2^-)\Big) + \Big(\boldsymbol{\check{\Delta}_3}, -(\mathbb{Z}_1)\Big)\\
&+ \Big(\boldsymbol{\check{\Delta}_4}, -2(\mathbf{Z}_3^{\mathbf{t}}) - 2(\mathbf{D}_1) - 2(\mathbf{D}_1^z) + 2(\mathbb{Z}_1)\Big) + \Big(\check{\Delta}_{00}, -(\mathbb{Z}_2^-)\Big) + \Big(\boldsymbol{\check{\Delta}_{01}}, -2(\mathbb{Z}_2^-) + (\mathbb{Z}_2)\Big)\\
&+ \Big(\check{\Delta}_{02}, -(\mathbb{Z}_2^-)\Big) + \Big(\check{\Delta}_{03}, -(\mathbb{Z}_2^-)\Big) + (W, 11(\mathbb{Z}_1)),
\end{aligned}$$

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