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APPLIED EQUIVARIANT DEGREE. PART II: SYMMETRIC HOPF BIFURCATIONS OF FUNCTIONAL DIFFERENTIAL EQUATIONS

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Abstract. In this paper we apply the equivariant degree method to the Hopf bifurcation problem for a system of symmetric functional differential equations. Local Hopf bifurcation is classified by means of an equivariant topological invariant based on the symmetric properties of the characteristic operator. As examples, symmetric configurations of identical oscillators, with dihedral, tetrahedral, octahedral, and icosahedral symmetries, are analyzed.

1. **Introduction.** This is the second paper in a series devoted to the equivariant degree theory and its applications to non-linear problems admitting a certain (in general, non-abelian) compact Lie group of symmetries (cf. [1]). Our main goal is to study, by means of the equivariant degree theory, the occurrence of Hopf bifurcations in a symmetric system of delayed functional differential equations. Such models appear in many important problems in physics, chemistry, biology, engineering, etc., where the existence of symmetries has an enormous impact on a dynamical process resulting in a formation of various patterns exhibiting certain symmetric properties (e.g. the Turing model of a ring of identical oscillators, cf. [32, 15]). They are also related, for example, to the appearance of turbulence in fluid dynamics (cf. [13]), fluctuations in transmission lines (see [24]), periodic reoccurrence of epidemics, traveling waves in neural networks (cf. [34]), etc. The prediction and classification of the appearing and changing patterns in such systems constitute a complex problem.

The equivariant degree theory (cf. [1, 2, 4, 5, 14, 18, 21, 25, 26]) provides the most effective method for a full analysis of symmetric Hopf bifurcation problems

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(cf. [5, 6, 12, 18, 24, 34, 35]). It allows to directly translate the equivariant spectral properties of the characteristic operator (associated with the system) into an algebraic invariant containing the information related to the topological nature of the occurring Hopf bifurcation, including the symmetric structure of the bifurcating branches of non-constant periodic solutions, and their multiplicities.

More precisely, in the case of a parameterized system of functional differential equations, symmetric with respect to a finite* group Γ, we associate with an isolated center $(\alpha_o, 0)$ and the corresponding purely imaginary characteristic value $i\beta_o$, an element $\omega(\alpha_o, \beta_o)$ of the $\mathbb{Z}$-module $A_1(\Gamma \times S^1)$ generated by the conjugacy classes of the so-called φ-twisted l-folded subgroups $H := K^{\varphi,l} \subset \Gamma \times S^1$ (recall that the group S^1 acts on periodic solutions by shifting the time variable). This element, being the so-called *primary* $\Gamma \times S^1$-equivariant degree of the map $\mathfrak{F}_\varsigma$, associated with the Hopf bifurcation problem (see section 4 for more details), can be written as

$$\omega(\alpha_o, \beta_o) = n_1(H_1) + n_2(H_2) + \cdots + n_r(H_r). \tag{1}$$

The information contained in the element $\omega(\alpha_o, \beta_o)$ describes and classifies the symmetric properties of bifurcating branches of non-constant periodic solutions. For instance, a non-zero coefficient n_k of $\omega(\alpha_o, \beta_o)$ implies the existence of a bifurcating branch of periodic solutions with symmetries at least H_k (as usual, a solution is said to have H-symmetry if it has H as its isotropy group). In the case of the so-called *dominating* orbit types (H_k) (i.e. satisfying a certain maximality condition (cf. Definition 5)), we can predict the existence of bifurcating branches of non-constant periodic solutions with this *exact* type of symmetry, and establish a lower estimate of the number of bifurcating branches.

Let us stress that, in order to compute the primary equivariant degree of $\mathfrak{F}_\varsigma$ and make the above conclusions, we only need the information about the characteristic values of the linearization of a given system around the isolated center, and their symmetries, i.e. the so-called *isotypical decomposition* of the corresponding eigenspaces. The primary degree in question can be computed directly from the tables of the primary degrees of the so-called *basic maps* (i.e. the equivariant maps *canonically assigned* to irreducible G-representations, so that their degrees can be evaluated in advance) combined with the so-called *multiplicativity property* (see Proposition 3). This functorial property of the primary equivariant degree, in turn, appeals to a module $A_1(\Gamma \times S^1)$ over the Burnside ring $A(\Gamma)$ (the addition in the module reflects unions of zeros, while the multiplication corresponds to a Cartesian product of zeros). It should be pointed out that the multiplication structure in $A_1(\Gamma \times S^1)$ can be completely described via appropriate tables (also prepared in advance).

Based on the above arguments, we claim that the (primary) equivariant degree can actually be evaluated without direct connection to its theoretical roots lying in equivariant topology, homotopy theory and bordism theory, and, in fact, all the involved tasks can be completely computerized. The only additional background behind the machinery of computer routines, which is needed for applying the equivariant degree to concrete models, are the representation theory and basic properties of classical groups and their subgroups. However, a proper understanding of any

* Being motivated by the applications presented in this paper, we make this assumption to avoid technical complications related to the so-called "bi-orientability" property (cf. [29, 14, 1]). However, the general results, obtained in [1], allow the applications of the equivariant degree method in the case where Γ is an arbitrary compact Lie group.

applied problem with symmetries requires this knowledge anyway. All this makes us believe that our approach meets the following paradigm: *equivariant topology, via algebra and computer routines, in service of applied mathematics.*

The equivariant degree method that was used for studying symmetric Hopf bifurcation problems in [21, 22, 24, 33, 34, 35] (see also [12, 23, 18]), led to the results based on partial computations only. This was mainly due to technical difficulties related to the absence of a general computational scheme and elaborated algebraic calculations. Using the axiomatic approach to the equivariant degree and the established standards for proper functional settings, which can be equally applied to different kinds of dynamical systems, it is possible to compute (with assistance of the developed Maple© routines) the full bifurcation invariant, based only on the equivariant spectral properties of the linearized system.

After the Introduction the paper is organized as follows. Section 2 contains the equivariant topology and analysis background together with algebraic constructions frequently used throughout the paper. In section 3, we introduce the primary equivariant degree, following an axiomatic approach developed in [1], discuss the multiplicativity property and the Splitting Lemma. In section 4, we present a general parameterized system of symmetric delayed functional differential equations, introduce the notion of the so-called *isotypical crossing number*, and construct the $\Gamma \times S^1$-equivariant mapping $\mathfrak{F}_\varsigma$ associated with the Hopf bifurcation problem. Section 5 is devoted to the computation of the equivariant degree $\omega(\alpha_o, \beta_o) = \Gamma \times S^1$-$\mathrm{Deg}(\mathfrak{F}_\varsigma, \Omega)$ — the equivariant homotopy invariant associated with $\mathfrak{F}_\varsigma$. In section 6, we present, as an example, a system of parameterized symmetric functional differential equations describing a symmetric configuration of identical oscillators. We establish computational formulae for the bifurcation invariant $\omega(\alpha_o, \beta_o)$ in terms of the spectrum of the characteristic operator. In section 7, we present the Hopf bifurcation results, based on the computations of $\omega(\alpha_o, \beta_o)$ for the considered configurations of identical oscillators.

2. **Preliminaries.**

2.1. **Equivariant Jargon and Notations.**

Hereafter, G stands for a compact Lie group. Let H be a subgroup of G (which is always assumed to be closed). In what follows, (H) stands for the conjugacy class of H in G, $N(H)$ for the normalizer of H in G and $W(H) = N(H)/H$ for the Weyl group of H in G. Denote by $\Phi(G)$ the set of all the conjugacy classes (H) in G. For two subgroups H and K of G, we write $(H) \le (K)$ if $H \subset g^{-1}Kg$ for some $g \in G$. The relation $\le$ defines a partial order on the set $\Phi(G)$, which can be extended to a total order. We will assume this total order, also denoted by $\le$, is fixed.

Let V be an orthogonal representation of G and $x \in \mathbb{R}\oplus V$ (with G acting trivially on $\mathbb{R}$). Denote by $G_x = \{g \in G : gx = x\}$ the *isotropy group* of x and we call the conjugacy class (G_x) the *orbit type* of x. Let $X \subset \mathbb{R}\oplus V$ be a G-invariant set. Denote by $\mathcal{J}(X)$ — the set of all orbit types (G_x) for $x \in X$, $X^H := \{x \in X : G_x \supset H\}$ — the set of H-*fixed points* in X and put $X_H = \{x \in X : G_x = H\}$, $X^{(H)} = G(X^H)$ and $X_{(H)} = G(X_H)$.

For the equivariant topology background we refer to [7, 19].

2.2. **Numbers** $n(L, H)$**.**

Given two closed subgroups $L \subset H$ of G, define the set

$$N(L, H) = \Big\{g \in G : gLg^{-1} \subset H\Big\}.$$

It is easy to check that $N(L,H)$ is a compact subset of G, however, in general, it is not a subgroup of G.

Put (cf. [17, 9, 23, 26])

$$n(L,H) = \left|\frac{N(L,H)}{N(H)}\right|, \tag{2}$$

where the symbol $|X|$ stands for the cardinality of the set X.

Proposition 1. (cf. [1]) *Let $L \subset H$ be two closed subgroups of a compact Lie group G such that $\dim W(L) = \dim W(H)$. Then the number $n(L,H)$ is finite and the set $N(L,H)/H$ is a closed submanifold of G/H.*

Remark 1.

1. Let $L \subset H$ be two subgroups of G such that $\dim W(H) = \dim W(L)$. Then the number $n(L,H)$ has a very simple geometric interpretation. It represents the number of different subgroups $\widetilde{H}$ in the conjugacy class (H) such that $L \subset \widetilde{H}$. In particular, if V is an orthogonal G-representation such that (L), $(H) \in \mathcal{J}(V)$, $L \subset H$, then $V^L \cap V_{(H)}$ is a disjoint union of exactly $n(L,H)$ sets V_{H_j}, satisfying $(H_j) = (H)$ (cf. [1]).
2. In the case of two conjugacy classes (L) and (H) such that $L \subset H$, the number $n(L,H)$ is independent of a choice of representatives L and H. Therefore, $n(L,H)$ is well-defined for $(L) \le (H)$ such that $L \subset H$. In the case the orbit types (L) and (H) are not comparable with respect to the partial order relation, put $n(L,H) = 0$.

2.3. **Burnside Ring and Primary Equivariant Degree without Free Parameter.** Assume that Γ is a finite group. The Burnside ring $A(\Gamma)$ is the $\mathbb{Z}$-module generated by $\Phi(\Gamma)$ with the multiplication defined on the generators by the following formula (cf. [3, 21, 31]):

$$(H)\cdot(K) = \sum_{(L)\in\Phi(\Gamma)} n_L \cdot (L), \tag{3}$$

where

$$n_L = \frac{1}{|W(L)|}\left[n(L,H)|W(H)|n(L,K)|W(K)| - \sum_{(\widetilde{L})>(L)} n(L,\widetilde{L})n_{\widetilde{L}}|W(\widetilde{L})|\right]. \tag{4}$$

Given an orthogonal Γ-representation V, an open bounded invariant set $\Omega \subset V$ and a Γ-equivariant map $f : V \to V$ with $f(x) \neq 0$ for all $x \in \partial\Omega$, one can define the (primary) equivariant degree $\Gamma\text{-Deg}\,(f,\Omega) \in A(\Gamma)$ satisfying all the properties expected from any reasonable degree theory: *existence, homotopy, additivity, suspension, multiplicativity* (see, for instance, [23] for details).

2.4. **Isotypical Decompositions.** Consider an orthogonal G-representation V. The representation V can be decomposed into a direct sum

$$V = \tilde{\mathcal{V}}_1 \oplus \tilde{\mathcal{V}}_2 \oplus \cdots \oplus \tilde{\mathcal{V}}_m \tag{5}$$

of irreducible subrepresentations $\tilde{\mathcal{V}}_j$ of V (some of them may be equivalent). This direct decomposition is not "geometrically" unique and is only defined up to isomorphism. Of course, among these irreducible subrepresentations there may be distinct (non-equivalent) subrepresentations which are denoted by $\mathcal{V}_{k_1}$, …, $\mathcal{V}_{k_r}$, including possibly a trivial one-dimensional representation. Let V_{k_j} be the sum of all irreducible subrepresentations $\tilde{\mathcal{V}}_j \subset V$ equivalent to $\mathcal{V}_{k_j}$. Then

$$V = V_{k_1} \oplus \cdots \oplus V_{k_r}, \tag{6}$$

and the direct sum (6) is called the *isotypical decomposition* of V. The isotypical decomposition (6) is unique. The subspaces V_{k_j} are called the *isotypical* components of V (*of type* $\mathcal{V}_{k_j}$, or *modeled* on $\mathcal{V}_{k_j}$).

2.5. **Regular Normal Approximations.** Let V be an orthogonal (or Banach) G-representation, $\Omega \subset \mathbb{R} \oplus V$ a G-invariant bounded open set and $f : \mathbb{R} \oplus V \to V$ a G-equivariant continuous map such that $f(x) \neq 0$ for all $x \in \partial\Omega$. Then, we say that f is Ω-*admissible* and call (f, Ω) an *admissible pair*. Similarly, we define an Ω-*admissible homotopy.*

Definition 1. Let V be an orthogonal G-representation, $\Omega \subset \mathbb{R} \oplus V$ a G-invariant bounded open set and (f, Ω) an admissible pair. Then f is said to be *normal* in Ω, if for every (H) such that $H = G_{x_o}$ for some $x_o \in f^{-1}(0) \cap \Omega$, the following condition is satisfied:
$\forall\, x \in f^{-1}(0) \cap \Omega_H \;\; \exists\, \delta_x > 0 \;\; \forall\, w \in \nu_x(\Omega_{(H)})$

$$\|w\| < \delta_x \implies f(x + w) = f(x) + w = w,$$

where $\nu(\Omega_{(H)})$ denotes the normal bundle to the submanifold $\Omega_{(H)}$ in $\mathbb{R} \oplus V$ and ν_x stands for the normal slice at x. In addition, we say that f is *regular normal* if
(i) f is of class C^1;
(ii) f is normal in Ω;
(iii) for every orbit type (H) in Ω, zero is a regular value of

$$f_H := f_{|\Omega_H} : \Omega_H \to V^H.$$

We have the following (cf. [23, 25, 26, 14])

Theorem 1. (Regular Normal Approximation Theorem) *Let (f, Ω) be an admissible pair. Then, for every $\varepsilon > 0$ there exists a regular normal (in Ω) G-equivariant map $\widetilde{f} : \mathbb{R} \oplus V \to V$ such that $\sup_{x \in \Omega} \|\widetilde{f}(x) - f(x)\| < \varepsilon$. A similar result is true for Ω-admissible homotopies.*

3. Equivariant Degree with One Free Parameter.

3.1. **Twisted Subgroups and Canonical Orientation of Weyl groups.** From now on, Γ stands for a finite group and $G = \Gamma \times S^1$.

Consider the set

$$\Phi_1(G) := \big\{(H) \in \Phi(G) : \dim W(H) = 1\big\}.$$

It is easy to check that the elements of $\Phi_1(G)$ are the conjugacy classes (H) of the so-called φ-*twisted* l-*folded* subgroups of $\Gamma \times S^1$ with $l = 1, 2, \dots$, i.e.

$$H = K^{\varphi,l} := \big\{(\gamma, z) \in K \times S^1 : \varphi(\gamma) = z^l\big\},$$

where K is a subgroup of Γ and $\varphi : K \to S^1$ is a homomorphism. In the case of a φ-twisted 1-folded subgroup $K^{\varphi,1}$, we denote it by K^φ and call it a *twisted* subgroup of $\Gamma \times S^1$. Notice that $N(K^{\varphi,l}) = N_o \times S^1$, where

$$N_o = \{\gamma \in N(K) : \forall_{k \in K} \;\; \varphi(\gamma k \gamma^{-1}) = \varphi(k)\}.$$

For every $(H) \in \Phi_1(G)$, the Weyl group $W(H)$ has a natural invariant (with respect to the right and left translations) orientation. Indeed, we have the natural homomorphism $\varphi_H : S^1 \to W(H)$ defined as the following composition

$$S^1 \hookrightarrow N_o \times S^1 = N(H) \longrightarrow N(H)/H = W(H).$$

Clearly $\ker \varphi_H = \mathbb{Z}_l$, thus φ_H induces the injection

$$\overline{\varphi}_H : S^1 \hookrightarrow W(H).$$

The induced by $\overline{\varphi}_H$ orientation of the connected component of $1 \in W(H)$ can be extended to an invariant orientation of $W(H)$.

Denote by $A_1(G)$ the free $\mathbb{Z}$-module generated by the symbols $(H) \in \Phi_1(G)$. Then, any element $\alpha \in A_1(G)$ can be written as a finite sum

$$\alpha = n_1(H_1) + n_2(H_2) + \cdots + n_r(H_r), \quad n_i \in \mathbb{Z}.$$

3.2. **Positive Orientation in a Slice and Tubular Maps.** Given an orthogonal G-representation V and an invariant subset $X \subset V$, put $\Phi_1(G, X) := \mathcal{J}(X) \cap \Phi_1(G)$. For every $(H) \in \Phi_1(G, V)$ we always assume that:

(i) V^H (and consequently, $\mathbb{R} \oplus V^H$) is equipped with a fixed orientation;
(ii) $W(H)$ is equipped with the canonical orientation described in the previous subsection.

Definition 2. Let V be an orthogonal G-representation and $x_o \in \mathbb{R} \oplus V$ be such that $G_{x_o} = H$ with $(H) \in \Phi_1(G)$. Let S_{x_o} denote the slice to $W(H)x_o$ in $\mathbb{R} \oplus V^H$. We say that S_{x_o} is *positively oriented* if the orientation of the slice followed by the orientation of the orbit $W(H)x_o$ (induced by the fixed orientation of $W(H)$) coincides with the (fixed) orientation of $\mathbb{R} \oplus V^H$ (see, for instance, [1]).

Definition 3. Let V be an orthogonal G-representation and $f : \mathbb{R} \oplus V \to V$ a regular normal map such that $f(x_o) = 0$ with $G_{x_o} = H$ and $(H) \in \Phi_1(G)$. Let $\mathfrak{U}_{G(x_o)}$ be a tube around $G(x_o)$ such that $f^{-1}(0) \cap \mathfrak{U}_{G(x_o)} = G(x_o)$ (notice that, by regular normality of f, $G(x_o)$ is an isolated orbit of zeros of f). Then f is called a *tubular map* around $G(x_o)$.

Consider a tubular map f defined on the tube $\mathfrak{U}_{G(x_o)}$ around $G(x_o)$ with the positively oriented slice S_{x_o}. Denote by f^H the restriction of f on $\mathbb{R} \oplus V^H$. We call $\operatorname{sign} \det Df^H(x_o)|_{S_{x_o}}$ the *local index* of f at x_o in $\mathfrak{U}_{G(x_o)}$.

3.3. **Primary Equivariant Degree with One Free Parameter.** The following theorem (cf. [1]), provides an axiomatic approach to the so-called *primary G-equivariant degree* $G\text{-Deg}\,(f, \Omega) \in A_1(G)$ of an admissible pair (f, Ω).

Theorem 2. *There exists a unique function, denoted by G-Deg, assigning to each admissible pair (f, Ω) an element $G\text{-Deg}\,(f, \Omega) \in \Phi_1(G)$ satisfying the following properties:*

(P1) (Existence) *If $G\text{-Deg}\,(f, \Omega) = \sum_{(H)} n_H(H)$ is such that $n_{H_o} \neq 0$ for some $(H_o) \in \Phi_1(G)$, then there exists $x_o \in \Omega$ with $f(x_o) = 0$ and $G_{x_o} \supset H_o$.*

(P2) (Additivity) *Assume that Ω_1 and Ω_2 are two G-invariant open disjoint subsets of Ω such that $f^{-1}(0) \cap \Omega \subset \Omega_1 \cup \Omega_2$. Then,*

$$G\text{-Deg}\,(f, \Omega) = G\text{-Deg}\,(f, \Omega_1) + G\text{-Deg}\,(f, \Omega_2).$$

(P3) (Homotopy) *Suppose that* $h : [0,1] \times \mathbb{R} \oplus V \to V$ *is an* Ω*-admissible* G*-equivariant homotopy. Then,*

$$G\text{-Deg}\,(h_t, \Omega) = \text{constant}$$

where $h_t := h(t, \cdot)$.

(P4) (Suspension) *Suppose that* W *is another orthogonal* G*-representation and let* U *be an open, bounded* G*-invariant neighborhood of* 0 *in* W*. Then,*

$$G\text{-Deg}\,(f \times \mathrm{Id}\,, \Omega \times U) = G\text{-Deg}\,(f, \Omega).$$

(P5) (Normalization) *Suppose that* $f : \mathbb{R} \oplus V \to V$ *is a tubular map around* $G(x_o)$*,* $H = G_{x_o}$*,* $(H) \in \Phi_1(G)$*, with the local index* n_{x_o} *of* f *at* x_o *in* $\mathfrak{U}_{G(x_o)}$*. Then,*

$$G\text{-Deg}\,(f, \mathfrak{U}_{G(x_o)}) = n_{x_o}(H).$$

(P6) (Elimination) *Suppose that* $f : \mathbb{R} \oplus V \to V$ *is an* Ω*-admissible* G*-equivariant map, which is normal in* Ω *and* $\Omega_H \cap f^{-1}(0) = \emptyset$ *for every* $(H) \in \Phi_1(G)$*. Then,*

$$G\text{-Deg}\,(f, \Omega) = 0.$$

Remark 2. In a standard way (see, for instance, [18, 23]) the primary equivariant degree can be extended to equivariant compact Ω-admissible vector fields defined on infinite dimensional Banach G-representations. We use for this extension the same symbol.

3.4. Burnside Ring Module $A_1(G)$ and Multiplicativity Property.

Proposition 2. *The* $\mathbb{Z}$*-module* $A_1(G)$ *admits a natural structure of an* $A(\Gamma)$*-module, where* $A(\Gamma)$ *denotes the Burnside ring, and the* $A(\Gamma)$*-multiplication on the generators* $(R) \in A(\Gamma)$ *and* $(K^{\varphi,l}) \in A_1(\Gamma \times S^1)$*, is defined by the formula*

$$(R) \cdot (K^{\varphi,l}) = \sum_{(L)} n_L \cdot (L^{\varphi,l}),$$

where the numbers n_L *are computed using the recurrence formula (cf.* [1, 21]*)*

$$n_L = \frac{\left[n(L,R)|W(R)|n(L^{\varphi,l}, K^{\varphi,l})|W(K^{\varphi,l})/S^1| - \displaystyle\sum_{(\widetilde{L})>(L)} n(L^{\varphi,l}, \widetilde{L}^{\varphi,l}) n_{\widetilde{L}} |W(\widetilde{L}^{\varphi,l})/S^1| \right]}{|W(L^{\varphi,l})/S^1|} \tag{7}$$

where $n(L,R)$ *and* $n(L^{\varphi,l}, \widetilde{L}^{\varphi,l})$ *are defined by (2), and* $|Y|$ *stands for the cardinality of* Y*.*

The following *multiplicativity property* of the primary degree plays an important role in practical computations of the primary degree (cf. [5, 21]):

Proposition 3. *Assume that* (f, Ω) *is an admissible pair in* $\mathbb{R} \oplus V$*,* W *is an orthogonal representation of* Γ*,* $\mathfrak{U}$ *an open* Γ*-invariant subset of* W *and* $g : W \to W$ *a* Γ*-equivariant map such that* $g(v) \neq 0$ *for all* $v \in \partial\mathfrak{U}$*. Then,*

(P7) (Multiplicativity) *The product map* $f \times g : \mathbb{R} \oplus V \oplus W \to V \oplus W$ *is* $\Omega \times \mathfrak{U}$*-admissible, and*

$$G\text{-Deg}\,(f \times g, \Omega \times \mathfrak{U}) = \Gamma\text{-Deg}(g, \mathfrak{U}) \cdot G\text{-Deg}\,(f, \Omega),$$

where $\Gamma\text{-Deg}(g, \mathfrak{U}) \in A(\Gamma)$ *denotes the equivariant degree of* g *in* $\mathfrak{U}$ *(without free parameter) and '$\cdot$' stands for the* $A(\Gamma)$*-module multiplication provided by Proposition 2.*

3.5. **Recurrence Formula for Primary Degree.** Let V be an orthogonal G-representation and $f : \mathbb{R} \oplus V \to V$ an Ω-admissible G-equivariant map. Then, the restriction $f^H : \mathbb{R} \oplus V^H \to V^H$ is $W(H)$-equivariant and Ω^H-admissible. Assume, in addition, $(H) \in \Phi_1(G, V)$. According to our choice of the orientation on $W(H)$ (see subsection 3.1), S^1 can be canonically identified with the connected component of $W(H)$. Therefore, f^H is S^1-equivariant and the S^1-degree $S^1\text{-Deg}\,(f^H, \Omega^H) = \sum_k n_k(\mathbb{Z}_k)$ is well-defined (see [1, 18, 23] for more information about the S^1-equivariant degree). Since $W(H)$ acts freely on Ω_H, the orbit type $(\mathbb{Z}_1)$ is the "smallest" one among those occuring in Ω^H. Put $\deg_1(f^H, \Omega^H) := n_1$. The coefficients n_H of the primary equivariant degree

$$G\text{-Deg}\,(f, \Omega) = \sum_{(H)} n_H(H)$$

can be computed using the following *recurrence formula* (cf. [1])

$$n_H = \left[\deg_1(f^H, \Omega^H) - \sum_{(L)>(H)} n_L\, n(L, H)\, |W(L)/S^1|\right] \Big/ \left|\frac{W(H)}{S^1}\right|. \tag{8}$$

3.6. **Splitting Lemma.** The following result established in [1] is used later for the computations of the primary degree:

Lemma 1. (Splitting Lemma) *Let G be a compact Lie group, V_1 and V_2 orthogonal G-representations, $V = V_1 \oplus V_2$. Assume that the isotypical decomposition of V contains only components modeled on irreducible G-representations of complex type. Suppose that $a_j : S^1 \to GL^G(V_j)$, $j = 1, 2$, are two continuous maps and $a : S^1 \to GL^G(V)$ is given by*

$$a(\lambda) = a_1(\lambda) \oplus a_2(\lambda), \quad \lambda \in S^1.$$

Put

$$\mathcal{O}_j := \left\{(\lambda, v_j) \in \mathbb{C} \oplus V_j : \|v_j\| < 2,\ \frac{1}{2} < |\lambda| < 4\right\}, \quad j = 1, 2$$
$$\mathcal{O} := \left\{(\lambda, v) \in \mathbb{C} \oplus V : \|v\| < 2,\ \frac{1}{2} < |\lambda| < 4\right\}.$$

Define the maps $f_{a_j} : \overline{\mathcal{O}}_j \to \mathbb{R} \oplus V_j$, $j = 1, 2$, $f_a : \overline{\mathcal{O}} \to \mathbb{R} \oplus V$ by

$$f_{a_j}(\lambda, v_j) = \left(|\lambda|(\|v_j\| - 1) + \|v_j\| + 1, a_j\left(\frac{\lambda}{|\lambda|}\right) v_j\right),$$
$$f_a(\lambda, v) = \left(|\lambda|(\|v\| - 1) + \|v\| + 1, a\left(\frac{\lambda}{|\lambda|}\right) v\right).$$

Then

$$G\text{-Deg}\,(f_a, \mathcal{O})) = G\text{-Deg}\,(f_{a_1}, \mathcal{O}_1) + G\text{-Deg}\,(f_{a_2}, \mathcal{O}_2). \tag{9}$$

4. **Symmetric Hopf Bifurcation for Functional Differential Equations and Equivariant Degree: General Framework.** Let us discuss a general setting for studying symmetric Hopf bifurcation problems for delayed differential equations with a finite group Γ of symmetries.

Hereafter, V stands for an orthogonal Γ-representation. Given a constant $\tau \geq 0$, denote by $C_{V,\tau}$ the Banach space of continuous functions from $[-\tau, 0]$ into V equipped with the usual supremum norm

$$\|\varphi\| = \sup_{-\tau \leq \theta \leq 0} |\varphi(\theta)|, \qquad \varphi \in C_{V,\tau}.$$

Given a continuous function $x : \mathbb{R} \to V$ and $t \in \mathbb{R}$, define $x_t \in C_{V,\tau}$ by

$$x_t(\theta) = x(t + \theta), \quad \theta \in [-\tau, 0].$$

Clearly, the Γ-action on V induces a natural isometric Banach representation of Γ on the space $C_{V,\tau}$ with the Γ-action given by:

$$(\gamma\varphi)(\theta) := \gamma(\varphi(\theta)), \qquad \gamma \in \Gamma, \;\; \theta \in [-\tau, 0].$$

Consider the following one-parameter family of delayed differential equations

$$\dot{x} = f(\alpha, x_t), \tag{10}$$

where $x : \mathbb{R} \to V$ is a continuous function and $f : \mathbb{R} \oplus C_{V,\tau} \to V$ satisfies the following assumptions:

(A1) f is continuously differentiable;

(A2) f is Γ-equivariant, i.e.

$$f(\alpha, \gamma\varphi) = \gamma f(\alpha, \varphi), \quad \varphi \in C_{V,\tau}, \;\alpha \in \mathbb{R}, \;\gamma \in \Gamma;$$

(A3) $f(\alpha, 0) = 0$ for all $\alpha \in \mathbb{R}$.

In addition, in order to prevent the occurrence of the steady-state bifurcation at a considered point $(\alpha_o, 0) \in \mathbb{R} \oplus V$, assume that

(A4) $\det D_x f(\alpha_o, 0) \neq 0$.

For any $x_o \in V$, we use the same symbol to denote the constant function $x_o(t) \equiv x_o$. Clearly, $(x_o)_t = x_o$ for all $t \in \mathbb{R}$. A point $(\alpha, x_o) \in \mathbb{R} \oplus V$ is said to be a *stationary point* of (10) if $f(\alpha, x_o) = 0$. In particular, by condition (A3), $(\alpha, 0)$ is a stationary point of (10) for all $\alpha \in \mathbb{R}$. Moreover, we say that a stationary point (α, x_o) is *nonsingular* if the restriction of f to the space $\mathbb{R} \oplus V \subset \mathbb{R} \oplus C_{V,\tau}$, still denoted by f, has the derivative $D_x f(\alpha, x_o) : V \to V$ (with respect to $x \in V$), which is an isomorphism. We say that for $\alpha = \alpha_o$ the system (10) has a *Hopf bifurcation* occuring at $(\alpha_o, 0)$ corresponding to the "limit period" $\frac{2\pi}{\beta_o}$, if there exists a family of p_s-periodic non-constant solutions $\{(\alpha_s, x_s(t))\}_{s\in\Lambda}$ (for a proper index set Λ) of (10) satisfying the conditions:

(1) The set $K := \overline{\bigcup_{s\in\Lambda}\{(\alpha_s, x_s(t)) : \; t \in \mathbb{R}\}}$ contains a compact connected set C such that $(\alpha_o, 0) \in C$;

(2) $\forall \varepsilon > 0$, $\exists\, \delta > 0$ such that

$$\forall (\alpha_s, x_s(t)) \in C \;\; \sup_t \|x_s(t)\| < \delta \;\Rightarrow\; \|\alpha_o - \alpha_s\| < \varepsilon \;\text{ and }\; \|p_s - \frac{2\pi}{\beta_o}\| < \varepsilon.$$

4.1. **Characteristic Equation.** Let V^c be a complexification of the vector space V, i.e. $V^c := \mathbb{C} \otimes_{\mathbb{R}} V$. Then, V^c has a natural structure of a complex Γ-representation defined by $\gamma(z \otimes x) = z \otimes \gamma x$ for $z \in \mathbb{C}$ and $x \in V$. Also, a Γ-isotypical decomposition of the real representation V

$$V = V_0 \oplus V_1 \oplus \cdots \oplus V_r, \tag{11}$$

where $V_0 = V^\Gamma$ and V_i is modeled on the real irreducible Γ-representation $\mathcal{V}_i$, gives rise to an isotypical decomposition of the complex Γ-representation V^c

$$V^c = U_0 \oplus U_1 \oplus \cdots \oplus U_s, \tag{12}$$

where $U_0 = (V^c)^\Gamma$ and U_j is modeled on the complex irreducible Γ-representation $\mathcal{U}_j$. Notice that the number s of isotypical components in (12) may be different, in general, from the number r of isotypical components in (11), depending on the type of the irreducible representations $\mathcal{V}_i$ (cf. [8]).

Let (α, x_o) be a stationary point of (10). The linearization of (10) at (α, x_o) leads to the following *characteristic equation* for the stationary point (α, x_o),

$$\det{}_{\mathbb{C}} \triangle_{(\alpha,x_o)} (\lambda) = 0, \tag{13}$$

where

$$\triangle_{(\alpha,x_o)}(\lambda) := \lambda \mathrm{Id} - D_x f(\alpha, x_o)(e^{\lambda \cdot} \cdot)$$

is a complex linear operator from V^c to V^c, with $(e^{\lambda \cdot} \cdot)(\theta, x) = e^{\lambda\theta} x$ (cf. [33]) and $D_x f(\alpha, x_o)(z \otimes x) = z \otimes D_x f(\alpha, x_o)x$ for $z \otimes x \in V^c$. Put $\triangle_\alpha(\lambda) := \triangle_{(\alpha,0)}(\lambda)$.

A solution λ_o to (13) is called a *characteristic root* of (13) at the stationary point (α, x_o). It is clear that (α, x_o) is a nonsingular stationary point if and only if 0 is not a characteristic root of (13) at the stationary point (α, x_o).

In what follows, we say that a nonsingular stationary point (α, x_o) is a *center* if (13) permits a purely imaginary root. We call (α, x_o) an *isolated center* if it is the only center in some neighborhood of (α, x_o) in $\mathbb{R} \oplus V$.

By (A2) and (A3), the operator $\triangle_\alpha(\lambda) : V^c \to V^c$, $\alpha \in \mathbb{R}$, $\lambda \in \mathbb{C}$, is Γ-equivariant. Consequently, for every isotypical component U_j of V^c, $j = 0, 1, \dots, s$ (cf. (12)), $\triangle_\alpha(\lambda)(U_j) \subseteq U_j$. Put

$$\triangle_{\alpha,j}(\lambda) := \triangle_\alpha(\lambda)|_{U_j}.$$

4.2. **Crossing Numbers.** Assume:

(A5) There is an isolated center $(\alpha_o, 0)$ for system (10) such that (13) permits a purely imaginary root $\lambda = i\beta_o$ with $\beta_o > 0$.

Let λ be a complex root of the characteristic equation $\det_{\mathbb{C}} \triangle_{\alpha_o}(\lambda) = 0$. In what follows, we use the following notations:

$$\begin{aligned} E(\lambda) &:= \ker \triangle_{\alpha_o}(\lambda) \subset V^c, \\ E_j(\lambda) &:= E(\lambda) \cap U_j, \\ m_j(\lambda) &:= \dim{}_{\mathbb{C}} E_j(\lambda) / \dim{}_{\mathbb{C}} \mathcal{U}_j. \end{aligned}$$

The integer $m_j(\lambda)$ is called the *$\mathcal{U}_j$-multiplicity* of the characteristic root λ.

Let $B := (0, \delta_1) \times (\beta_o - \delta_2, \beta_o + \delta_2) \subset \mathbb{C}$. Under the assumption (A5), the constants $\delta_1 > 0$, $\delta_2 > 0$ and $\varepsilon > 0$ can be chosen so small that the following condition is satisfied:

(*) For every $\alpha \in [\alpha_o - \varepsilon, \alpha_o + \varepsilon]$, if there is a characteristic root $u + iv \in \partial B$ at $(\alpha, 0)$ then $u + iv = i\beta_o$ and $\alpha = \alpha_o$.

Note that $\triangle_\alpha(\lambda)$ is analytic in $\lambda \in \mathbb{C}$ and continuous in $\alpha \in [\alpha_o - \varepsilon, \alpha_o + \varepsilon]$ (see [16]). It follows that $\det_{\mathbb{C}} \triangle_{\alpha_o \pm \varepsilon}(\lambda) \neq 0$ for all $\lambda \in \partial B$. So the following notation

$$\mathfrak{t}^{\pm}_{j,1}(\alpha_o, \beta_o) := \deg(\det{}_{\mathbb{C}} \triangle_{\alpha_0 \pm \varepsilon, j} (\cdot), B), \tag{14}$$

where deg stands for the usual Brouwer degree, is well-defined for $0 \le j \le s$ (cf. (16) for the use of the lower index "1"). We can now introduce the following important concept (cf. [12, 21, 23, 24], see also [10, 11, 20, 27, 28, 34]):

Definition 4. The *$\mathcal{U}_j$-isotypical crossing number* of $(\alpha_o, 0)$ corresponding to the characteristic root $i\beta_o$ is defined as

$$\mathfrak{t}_{j,1}(\alpha_o, \beta_o) := \mathfrak{t}^{-}_{j,1}(\alpha_o, \beta_o) - \mathfrak{t}^{+}_{j,1}(\alpha_o, \beta_o), \tag{15}$$

where $\mathcal{U}_j$ is the complex Γ-irreducible representation on which is modeled the isotypical component U_j.

The crossing number $\mathfrak{t}_{j,1}$ has a very simple interpretation. In the case $\det_{\mathbb{C}}(\triangle_{\alpha,j}(i\beta_o)) = 0$ (i.e. $i\beta_o$ is a U_j-*characteristic root*), the number $\mathfrak{t}_{j,1}^-$ counts all the U_j-characteristic roots (with multiplicity) in the set B, before α crosses the value α_o, and the number $\mathfrak{t}_{j,1}^+$ counts the U_j-characteristic roots in B, after α crosses α_o. The difference, which is exactly the number $\mathfrak{t}_{j,1}$, represents the net number of the U_j-characteristic roots which 'escaped' (if $\mathfrak{t}_{j,1}$ is positive) or 'entered' (if $\mathfrak{t}_{j,1}$ is negative) the set B when α was crossing α_o. This situation is illustrated in Figure 1.

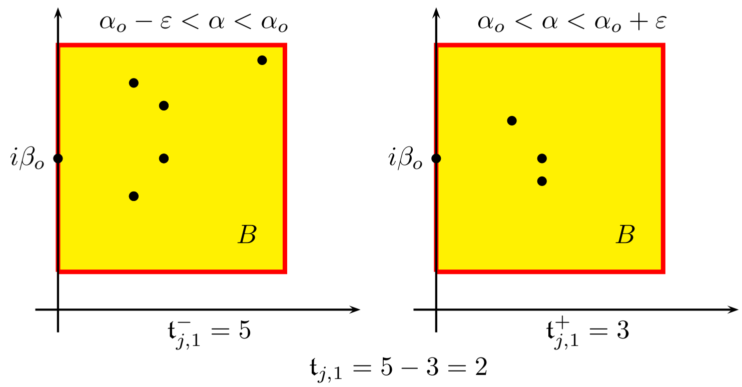


FIGURE 1. The $\mathcal{U}_j$-isotypical crossing number

Remark and Definition 3. For any integer $l > 1$, put

$$\mathfrak{t}_{j,l}(\alpha_o, \beta_o) := \mathfrak{t}_{j,1}(\alpha_o, l\beta_o). \tag{16}$$

Observe that $\mathfrak{t}_{j,l}(\alpha_o, \beta_o) = 0$ if $il\beta_o$ is not a root of (13) (cf. (14) and (15)).

In order to establish the existence of small amplitude periodic solutions bifurcating from the stationary point $(\alpha_o, 0)$, i.e. the occurrence of the Hopf bifurcation at the stationary point $(\alpha_o, 0)$, and to associate with $(\alpha_o, 0)$ a *local bifurcation invariant*, we apply the standard steps for the degree-theoretical approach described in next two subsections.

4.3. **Normalization of the Period.** Normalization of the period is obtained by making the change of variable $u(t) = x(\frac{p}{2\pi}t)$ for $t \in \mathbb{R}$. We obtain the following equation, which is equivalent to (10):

$$\dot{u}(t) = \frac{p}{2\pi} f(\alpha, u_{t,\frac{2\pi}{p}}), \tag{17}$$

where $u_{t,\frac{2\pi}{p}} \in C_{V,\tau}$ is defined by

$$u_{t,\frac{2\pi}{p}}(\theta) = u\big(t + \tfrac{2\pi}{p}\theta\big), \quad \theta \in [-\tau, 0].$$

Evidently, $u(t)$ is a 2π-periodic solution of (17) if and only if $x(t)$ is a p-periodic solution of (10). Introduce $\beta := \frac{2\pi}{p}$ into the equation (17) to obtain

$$\dot{u}(t) = \frac{1}{\beta} f(\alpha, u_{t,\beta}). \tag{18}$$

4.4. $\Gamma \times S^1$**-Equivariant Setting in Functional Spaces.** We use the standard identification $S^1 = \mathbb{R}/2\pi\mathbb{Z}$ (with $t \leftrightarrow e^{it}$) and introduce the operators $L : H^1(S^1; V) \to L^2(S^1; V)$ defined by $Lu(t) = \dot{u}(t)$, $j : H^1(S^1; V) \to C(S^1; V)$ by $j(u(t)) = \widetilde{u}(t)$ and $K : H^1(S^1; V) \to L^2(S^1; V)$ by $Ku(t) = \frac{1}{2\pi}\int_0^{2\pi} u(s)\, ds$, for $u \in H^1(S^1; V)$, $t \in \mathbb{R}$, where $H^1(S^1; V)$ (resp. $C(S^1; V)$) denotes the first Sobolev space of 2π-periodic V-valued functions (resp. the space of continuous 2π-periodic V-valued functions equipped with the usual supremum norm), and $\tilde{u} = u$ a.e. (cf. [30]). Put $\mathbb{R}^2_+ := \mathbb{R} \times \mathbb{R}_+$. It can be easily shown that $(L + K)^{-1} : L^2(S^1; V) \to H^1(S^1; V)$ exists and the map $\mathcal{F} : \mathbb{R}^2_+ \times H^1(S^1; V) \to H^1(S^1; V)$ defined by

$$\mathcal{F}(\alpha, \beta, u) = (L + K)^{-1}\Big[Ku + \tfrac{1}{\beta}\, N_f(\alpha, \beta, j(u))\Big] \tag{19}$$

is completely continuous, where $N_f : \mathbb{R}^2_+ \times C(S^1; V) \to L^2(S^1; V)$ is defined by

$$N_f(\alpha, \beta, v)(t) = f(\alpha, v_{t,\beta}),$$

and $e^{it} \in S^1$, $(\alpha, \beta, v) \in \mathbb{R}^2_+ \times C(S^1; V)$.

Put $W := H^1(S^1; V)$. The space W is an isometric Hilbert representation of the group $G = \Gamma \times S^1$ with the action given by

$$(\gamma, \vartheta)x(t) = \gamma(x(t + \vartheta)), \quad e^{i\vartheta},\, e^{it} \in S^1,\ \gamma \in \Gamma,\ x \in W.$$

The nonlinear operator $\mathcal{F}$ defined by (19) is clearly G-equivariant.

Remark 3. Notice that, $(\alpha, \beta, u) \in \mathbb{R}^2_+ \times W$ is a 2π-periodic solution of (18) if and only if $u = \mathcal{F}(\alpha, \beta, u)$. Consequently, the occurrence of a Hopf bifurcation at $(\alpha_o, 0)$ for the equation (10) is equivalent to a bifurcation of 2π-periodic solutions of (18) from $(\alpha_o, \beta_o, 0)$ for some $\beta_o > 0$. On the other hand, if a bifurcation at $(\alpha_o, \beta_o, 0) \in \mathbb{R}^2_+ \times W$ takes place in (18), then *necessarily* the operator $\mathrm{Id} - D_u\mathcal{F}(\alpha_o, \beta_o, 0) : W \to W$ is not an isomorphism, or equivalently, $il\beta_o$, for some $l \in \mathbb{N}$, is a purely imaginary characteristic root at $(\alpha_o, 0)$, i.e. $\det_{\mathbb{C}} \triangle_{\alpha_o}(il\beta_o) = 0$.

4.5. **Local** $\Gamma \times S^1$**-Invariant for the** Γ**-Symmetric Hopf Bifurcation.** It is convenient to identify $\mathbb{R}^2_+$ with a subset of $\mathbb{C}$, i.e. an element $(\alpha, \beta) \in \mathbb{R}^2_+$ is written as $\lambda = \alpha + i\beta$, and put $\lambda_o = \alpha_o + i\beta_o$. By assumption (A5), $(\alpha_o, 0)$ is an isolated center, thus there exists $\delta > 0$ such that

$$a(\lambda) := \mathrm{Id} - D_u\mathcal{F}(\lambda, 0) : W \to W \tag{20}$$

is an isomorphism for $0 < |\lambda - \lambda_o| \le \delta$. Consequently, by implicit function theorem, there exists ρ, $0 < \rho < \min\{1, \delta\}$, such that for (λ, u) satisfying $|\lambda - \lambda_o| = \delta$ and $0 < \|u\| \le \rho$, we have $u - \mathcal{F}(\lambda, u) \ne 0$. Define the subset $\Omega \subset \mathbb{R}^2_+ \times W$ by

$$\Omega := \Big\{(\lambda, u) \in \mathbb{R}^2_+ \times W : |\lambda - \lambda_o| < \delta,\ \|u\| < \rho\Big\} \tag{21}$$

and put

$$\partial_0 := \overline{\Omega} \cap \big(\mathbb{R}^2_+ \times \{0\}\big) \quad \text{and} \quad \partial_\rho := \{(\lambda, u) \in \overline{\Omega} : \|u\| = \rho\}.$$

Following the standard degree theory treatment of the bifurcation phenomenon (see, for instance, [18]), introduce an *auxiliary* function $\varsigma : \overline{\Omega} \to \mathbb{R}$, which is G-invariant and satisfies the conditions

$$\begin{cases} \varsigma(\lambda, u) > 0 & \text{for } (\lambda, u) \in \partial_\rho, \\ \varsigma(\lambda, u) < 0 & \text{for } (\lambda, u) \in \partial_0. \end{cases}$$

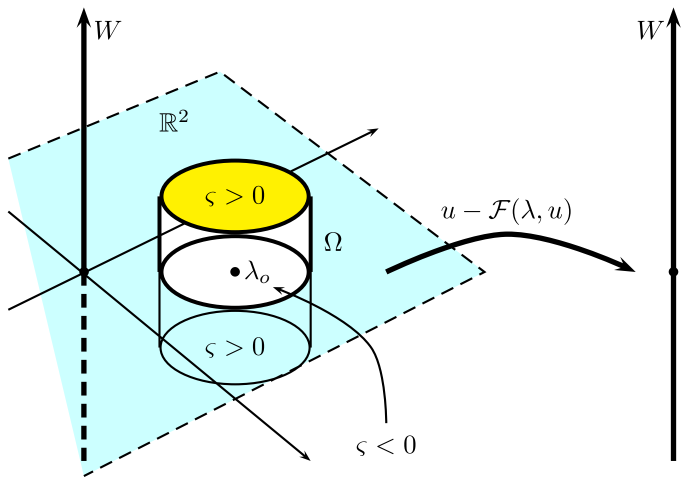


FIGURE 2. Auxiliary Function for Hopf Bifurcation

Such a function ς can be easily constructed, for example,

$$\varsigma(\lambda, u) = |\lambda - \lambda_o|(\|u\| - \rho) + \|u\| - \frac{\rho}{2}; \quad (\lambda, u) \in \overline{\Omega}. \tag{22}$$

Define the map $\mathfrak{F}_\varsigma : \overline{\Omega} \to \mathbb{R} \oplus W$ by

$$\mathfrak{F}_\varsigma(\lambda, u) = \Big(\varsigma(\lambda, u), u - \mathcal{F}(\lambda, u)\Big), \quad (\lambda, u) \in \overline{\Omega} \tag{23}$$

(see formula (19) and Figure 2).

Obviously, $\mathfrak{F}_\varsigma$ is an Ω-admissible G-equivariant compact field, and thus Remark 2 is applied. Put

$$\omega(\lambda_o) := G\text{-Deg}\,(\mathcal{F}_\varsigma, \Omega) \in A_1(G), \tag{24}$$

and we call $\omega(\lambda_o)$ the *local* $\Gamma \times S^1$*-invariant* for the Γ-symmetric Hopf bifurcation at the point $(\lambda_o, 0)$.

4.6. **Dominating Orbit Types.** In order to take advantage of the information provided by the local bifurcation invariant, we need to introduce the following important concept.

Definition 5. An orbit type (H) in W is called *dominating*, if (H) is maximal (with respect to the usual order relation (see subsection 2.1)) in the class of all φ-twisted one-folded orbit types in W (in particular, $H = K^\varphi$).

In what follows, the dominating orbit types are used to estimate the *minimal number* of different periodic solutions (as well as their *symmetries*) to system (10) (see Theorem 4).

Remark 4. (i) Assume there is a solution $u_o \in W$ to (18) (for $\alpha = \alpha_o$ and some $\beta > 0$), for which one has $G_{u_o} \supset H_o$. If (H_o) is a dominating orbit type in W with $H_o = K^\varphi$ for some $K \subset \Gamma$ and $\varphi : K \to S^1$, then, by maximality condition, $(G_{u_o}) = (K^{\varphi,l})$ with $l \geq 1$, and the corresponding orbit $G(u_o)$ is composed of exactly $|G/G_{u_o}|_{S^1}$ different periodic functions (where $|Y|_{S^1}$ denotes the number of S^1-orbits in Y). It is easy to check that the number of

S^1-orbits in G/G_{u_o} is $|\Gamma/K|$ (where $|X|$ stands for the number of elements in X).

On the other hand, let x_o be, say, a p-periodic solution to (10) canonically corresponding to the above u_o. It follows from the definition of l-folding and $\Gamma \times S^1$-action on W that x_o is also a $\frac{p}{l}$-periodic solution to (10). The pair $(x_o, \frac{p}{l})$ canonically determines an element $u_o' \in W$ being a solution to (18) (for $\alpha = \alpha_o$ and some β') satisfying the condition $G_{u_o'} = H_o$. In this way we obtain that (10) has at least $|\Gamma/K|$ different periodic solutions with the orbit type *exactly* (H_o) (considered in W).

(ii) Due to the maximality property of dominating orbit types and the fact that the isotropy groups increase under projections, the dominating orbit types can be easily recognized from the isotropy lattices of the irreducible representations of W.

4.7. **Sufficient Conditions for Symmetric Hopf Bifurcation.** Following the same lines as in the proof of Theorem 3.2 from [12] (see also [21] and [1]), one can easily establish

Theorem 4. *Given system (10), assume conditions (A1)–(A5) to be satisfied. Take $\mathcal{F}$ defined by (19) and construct Ω according to (21). Let $\varsigma : \overline{\Omega} \to \mathbb{R}$ be a G-invariant auxiliary function (see (22)) and let $\mathfrak{F}_\varsigma$ be defined by (23).*

(i) *Assume $\omega_o(\lambda_o) = G\text{-Deg}\,(\mathfrak{F}_\varsigma, \Omega) \neq 0$, i.e.*

$$G\text{-Deg}\,(\mathfrak{F}_\varsigma, \Omega) = \sum_{(H)} n_H (H), \qquad \textit{and} \;\; n_{H_o} \neq 0 \tag{25}$$

for some $(H_o) \in \Phi_1(G)$. Then, there exists a branch of non-trivial solutions to (10) bifurcating from the point $(\alpha_o, 0)$ (with the limit frequency $l\beta_o$ for some $l \in \mathbb{N}$). More precisely, the closure of the set composed of all non-trivial solutions $(\lambda, u) \in \Omega$ to (18), i.e.

$$\overline{\{(\lambda, u) \in \Omega :\; \mathfrak{F}(\lambda, u) = 0,\; u \neq 0\}}$$

contains a compact connected subset C such that

$$(\lambda_o, 0) \in C \quad \textit{and} \quad C \cap \partial_r \neq \emptyset, \quad C \subset \mathbb{R}^2_+ \times W^{H_o},$$

($\lambda_o = \alpha_o + i\beta_o$) which, in particular, implies that for every $(\alpha, \beta, u) \in C$ we have $G_u \supset H_o$.

(ii) *If, in addition, (H_o) is a dominating orbit type in W, then there exist at least $|G/H_o|_{S^1}$ different branches of periodic solutions to the equation (10) bifurcating from $(\alpha_o, 0)$ (with the limit frequency $l\beta_o$ for some $l \in \mathbb{N}$). Moreover, for each (α, β, u) belonging to these branches of (non-trivial) solutions one has $(G_u) = (H_o)$ (considered in the space W).*

Remark 5. (i) It is usually the case that there are more than one dominating orbit types in W contributing to the lower estimate of all bifurcating branches of solutions.

(ii) In addition, if there is also a coefficient $n_K \neq 0$ such that (K) is not a dominating orbit type, but $n_H = 0$ for all dominating orbit types (H) such that $(K) < (H)$, then we can also predict the existence of multiple branches by analyzing all the dominating orbit types (H) larger than (K). However, the exact orbit type of these branches (as well as the corresponding estimate) can not be determined precisely.

5. **Computation of the Local $\Gamma \times S^1$-invariant.** To apply Theorem 4 to classify symmetries of periodic solutions to concrete symmetric FDE's, we use a sequence of reductions that allows us to establish an effective formula for computing/estimating $\omega(\lambda_o)$.

5.1. **Linearization of the Problem.** Assume δ, ρ and Ω are chosen as described in subsection 4.5 (cf. (21)) and define

$$\widetilde{\varsigma}(\lambda, u) = |\lambda - \lambda_o|(\|u\| - \rho) + \|u\| + \frac{\delta}{2}\rho, \quad (\lambda, u) \in \overline{\Omega}.$$

(cf. (22)). It is clear (cf. (23)) that $\mathfrak{F}_\varsigma$ and $\mathfrak{F}_{\widetilde{\varsigma}}$ are homotopic by an Ω-admissible (linear) homotopy, thus

$$G\text{-Deg}\,(\mathfrak{F}_\varsigma, \Omega) = G\text{-Deg}\,(\mathfrak{F}_{\widetilde{\varsigma}}, \Omega). \tag{26}$$

Notice that for $|\lambda - \lambda_o| \leq \frac{\delta}{4}$ and $\|u\| \leq \rho$,

$$\widetilde{\varsigma}(\lambda, u) = \|u\| + \frac{\delta}{2}\rho - |\lambda - \lambda_o|(\rho - \|u\|) \geq \frac{\delta}{2}\rho - \frac{\delta}{4}\rho = \frac{\delta}{4}\rho > 0.$$

Put

$$\Omega_1 := \Big\{(\lambda, u) \in \mathbb{R}^2_+ \times W : \|u\| < \rho,\ \frac{\delta}{4} < |\lambda - \lambda_o| < \delta\Big\}. \tag{27}$$

By excision property of the G-equivariant degree,

$$G\text{-Deg}\,(\mathfrak{F}_{\widetilde{\varsigma}}, \Omega) = G\text{-Deg}\,(\mathfrak{F}_{\widetilde{\varsigma}}, \Omega_1). \tag{28}$$

Define $\widetilde{\mathfrak{F}} : \overline{\Omega}_1 \to \mathbb{R} \oplus W$ by

$$\widetilde{\mathfrak{F}}(\lambda, u) := (\widetilde{\varsigma}(\lambda, u), u - D_u\mathcal{F}(\lambda, 0)u), \quad (\lambda, u) \in \overline{\Omega}_1. \tag{29}$$

By standard linearization argument (cf. (26)),

$$G\text{-Deg}\,(\mathfrak{F}_{\widetilde{\varsigma}}, \Omega_1) = G\text{-Deg}\,(\widetilde{\mathfrak{F}}, \Omega_1) = G\text{-Deg}\,(\widetilde{\mathfrak{F}}, \Omega) = G\text{-Deg}\,(\mathfrak{F}_\varsigma, \Omega). \tag{30}$$

5.2. **Isotypical Decomposition of W.** We start with the following

Remark and Definition 5. For any complex Γ-representation U and $l = 1, 2, \dots,$ define a $\Gamma \times S^1$-action on U by

$$(\gamma, z)w = z^l \cdot (\gamma w), \quad (\gamma, z) \in \Gamma \times S^1, \quad w \in U,$$

where '$\cdot$' is the complex multiplication. This real $\Gamma \times S^1$-representation is denoted by ${}^lU$. Moreover, if U is a complex irreducible Γ-representation, then ${}^lU$ is a real irreducible $\Gamma \times S^1$-representation.

It is easy to observe that the S^1-action on W induces the following S^1-isotypical decomposition of the space W

$$W = V \oplus \overline{\bigoplus_{l=1}^{\infty} W_l}, \tag{31}$$

where $V = W^{S^1}$ is the subspace of the constant functions in W and

$$W_l = \{e^{ilt}(x_n + iy_n) : x_n,\, y_n \in V\}, \quad l = 1, 2, \dots$$

The subspaces V and W_l, $l = 1, 2, \dots,$ are Γ-invariant and W_l, as a complex Γ-representation, is isomorphic to V^c.

Keeping in mind the notations introduced in Remark and Definition 5, observe that the real $\Gamma \times S^1$-representation ${}^l(V^c)$ is equivalent to the $\Gamma \times S^1$-representation

W_l. Thus, the isotypical decomposition (12) of V^c induces, for every $l = 1, 2, \dots,$ the corresponding $\Gamma \times S^1$-isotypical decomposition

$$W_l = V_{0,l} \oplus V_{1,l} \oplus \cdots \oplus V_{s,l}, \tag{32}$$

where $V_{j,l}$ is the isotypical component modeled on the $\Gamma \times S^1$-irreducible representation $\mathcal{V}_{j,l} := {}^l(\mathcal{U}_j)$. On the other hand, (11) gives the $\Gamma \times S^1$-isotypical decomposition of V (here S^1 acts trivially). Consequently, we have the following isotypical decomposition of W:

$$W = V_0 \oplus \cdots \oplus V_r \oplus \overline{\bigoplus_{j,l} V_{j,l}}. \tag{33}$$

For every $j = 0, 1, \dots, s$ and $l = 0, 1, 2, \dots,$ define

$$a_{j,l}(\lambda) := \text{Id} - (L + K)^{-1} \left[K + \tfrac{1}{\beta} D_u N_f(\alpha, \beta, 0) \right] \Big|_{V_{j,l}}, \quad \lambda = \alpha + i\beta, \tag{34}$$

where $|\lambda - \lambda_o| \le \delta$.

Using the same argument as in [22], we obtain that

$$a_{j,l}(\lambda) = \tfrac{1}{il\beta} \triangle_{\alpha,j} (il\beta), \quad j = 0, 1, 2, \dots, s \tag{35}$$

and

$$a_{j,0}(\lambda) = -\frac{1}{\beta} D_x f(\alpha, 0)|_{V_j}, \quad j = 0, 1, \dots, r. \tag{36}$$

5.3. **Computation of G-Deg $(\widetilde{\mathfrak{F}}, \Omega_1)$: Reduction to the Product Formula.** Put

$$W_o := \overline{\bigoplus_{l=1}^{\infty} W_l}$$

(cf. (31)). By applying the standard finite-dimensional reduction, we can assume without loss of generality that W_o is finite-dimensional. Also, without loss of generality one can assume that $\Omega_1 = \mathcal{B} \times \Omega_o$, where $\mathcal{B}$ stands for the unit ball in V and

$$\Omega_o = \Omega_1 \cap (\mathbb{R}^2 \oplus W_o). \tag{37}$$

Consider two operators $\overline{\mathfrak{F}} := -\frac{1}{\beta_o} D_x f(\alpha_o, 0) : V \to V$ (cf. (36)), which is clearly Γ-equivariant, and $\mathfrak{F}_o : \overline{\Omega}_o \to \mathbb{R} \oplus W_o$, defined by

$$\mathfrak{F}_o(\lambda, u_o) = (\widetilde{\varsigma}(\lambda, u_o), u_o - D_u \mathcal{F}(\lambda, 0) u_o), \quad (\lambda, u_o) \in \overline{\Omega}_o.$$

It is easy to verify that the product map $\overline{\mathfrak{F}} \times \mathfrak{F}_o$ is homotopic to $(\widetilde{\mathfrak{F}}, \Omega_1)$ by a $\Gamma \times S^1$-equivariant Ω_1-admissible homotopy, therefore

$$G\text{-Deg}\,(\mathfrak{F}_\varsigma, \Omega) = G\text{-Deg}\,(\widetilde{\mathfrak{F}}, \Omega_1) = G\text{-Deg}\,(\overline{\mathfrak{F}} \times \mathfrak{F}_o, \mathcal{B} \times \Omega_o)$$

(cf. (30)). By applying multiplicativity property of the equivariant degree (cf. Proposition 3), we obtain that

$$G\text{-Deg}\,(\overline{\mathfrak{F}} \times \mathfrak{F}_o, \mathcal{B} \times \Omega_o) = \Gamma\text{-Deg}\,(\overline{\mathfrak{F}}, \mathcal{B}) \cdot G\text{-Deg}\,(\mathfrak{F}_o, \Omega_o). \tag{38}$$

5.4. **Computation of Γ-Deg $(\overline{\mathfrak{F}}, \mathcal{B})$.** For every negative eigenvalue μ of the linear Γ-equivariant operator $\overline{\mathfrak{F}}$, which is clearly a positive eigenvalue of the operator $D_x f(\alpha_o, 0)$, consider the Γ-isotypical decomposition of the eigenspace $E(\mu)$

$$E(\mu) = E_0(\mu) \oplus E_1(\mu) \oplus \cdots \oplus E_r(\mu),$$

where the component $E_i(\mu)$ is modeled on the irreducible Γ-representation $\mathcal{V}_i$. Put

$$m_i(\mu) := \dim_{\mathbb{R}} E_i(\mu) / \dim_{\mathbb{R}} \mathcal{V}_i. \tag{39}$$

Let σ_+ denote the set of all negative eigenvalues of $\overline{\mathfrak{F}}$. Take $\mu \in \sigma_+$ and suppose $E(\mu)$ contains an isotypical component $E_i(\mu)$ modeled on $\mathcal{V}_i$. Denote by $\deg_{\mathcal{V}_i}$ the Γ-equivariant degree of the operator $-\mathrm{Id} : \mathcal{V}_i \to \mathcal{V}_i$ on the unit ball in $\mathcal{V}_i$. Then, by multiplicativity property of the Γ-equivariant degree (in the case without parameter (see [23])), we obtain (cf. (39)):

$$\Gamma\text{-Deg}\,(\overline{\mathfrak{F}}, \mathcal{B}) = \prod_{\mu \in \sigma_+} \prod_{i=0}^{r} \Big(\deg_{\mathcal{V}_i} \Big)^{m_i(\mu)}, \tag{40}$$

where the multiplication is taken in the Burnside ring $A(\Gamma)$ (cf. formulae (3) and (4)).

5.5. **Computation of** G-Deg $(\mathfrak{F}_o, \Omega_o)$**.** Define $\widetilde{\mathfrak{F}}_o : \overline{\Omega}_o \to \mathbb{R} \oplus W_o$ (cf. (37) and (27)) by

$$\widetilde{\mathfrak{F}}_o(\lambda, u_o) := (|\lambda|(\|u_o\| - \rho) + \|u_o\| + 1, \mathfrak{a}(\lambda) u_o),$$

where $\mathfrak{a}(\lambda) := a\left(\lambda_o + \frac{(\lambda - \lambda_o)\delta}{2|\lambda - \lambda_o|}\right)$ and $a(\lambda) : W_o \to W_o$ is given by (20) (recall, W_o is assumed to be finite-dimensional, see subsection 5.3). By excision and homotopy properties of the equivariant degree,

$$G\text{-Deg}\,(\mathfrak{F}_o, \Omega_o) = G\text{-Deg}\,(\widetilde{\mathfrak{F}}_o, \Omega_o).$$

To compute the latter degree, consider the $\Gamma \times S^1$-isotypical decomposition

$$W_o = \bigoplus_{j,l} V_{j,l}.$$

Then, we have the following decomposition of the map $\mathfrak{a}$ with respect to this representation

$$\mathfrak{a}(\lambda) = \bigoplus_{j,l} \mathfrak{a}_{j,l}(\lambda),$$

where $\mathfrak{a}_{j,l}(\lambda) := \mathfrak{a}(\lambda)|_{V_{j,l}}$ (see (35)).

Define

$$\Omega_{j,l} := \Big\{ (\lambda, v) \in \mathbb{R}^2_+ \times V_{j,l} : \|v\| < \rho,\ \frac{\delta}{4} < |\lambda - \lambda_o| < \delta \Big\},$$

and $\widetilde{\mathfrak{F}}_{j,l} : \overline{\Omega}_{j,l} \to \mathbb{R} \oplus V_{j,l}$ by

$$\widetilde{\mathfrak{F}}_{j,l}(\lambda, v) := (|\lambda|(\|v\| - \rho) + \|v\| + 1, \mathfrak{a}_{j,l}(\lambda) v), \quad (\lambda, v) \in \overline{\Omega}_{j,l}.$$

By applying the Splitting Lemma, we obtain that

$$G\text{-Deg}\,(\widetilde{\mathfrak{F}}_o, \Omega_o) = \sum_{j,l} G\text{-Deg}\,(\widetilde{\mathfrak{F}}_{j,l}, \Omega_{j,l}). \tag{41}$$

On the other hand, define

$$\mathcal{O}_{j,l} = \{(t, v) \in \mathbb{R} \oplus \mathcal{V}_{j,l} : \frac{1}{2} < \|v\| < 2,\ \ -1 < t < 1\},$$

and the so-called *basic map* $\mathfrak{b} : \overline{\mathcal{O}}_{j,l} \to \mathcal{V}_{j,l}$ on $\mathcal{V}_{j,l}$, given by

$$\mathfrak{b}(t, v) = (1 - \|v\| + it) \cdot v, \quad (t, v) \in \overline{\mathcal{O}}_{j,l}. \tag{42}$$

Then, using (35) and the standard argument, we obtain (cf. [23] for the computation of crossing numbers, [21]-[22] for the reduction to the basic maps)

$$G\text{-Deg}\,(\widetilde{\mathfrak{F}}_{j,l}, \Omega_{j,l}) = \mathfrak{t}_{j,l}(\alpha_o, \beta_o)\ \deg_{\mathcal{V}_{j,l}}, \tag{43}$$

where $\deg_{\mathcal{V}_{j,l}} := G\text{-Deg}\,(\mathfrak{b}, \mathcal{O}_{j,l})$ and $\mathfrak{t}_{j,l}(\alpha_o, \beta_o)$ is the $\mathcal{V}_{j,l}$-isotypical crossing number of $(\alpha_o, 0)$ corresponding to $i\beta_o$ (see Definition 4 and (16)).

Remark 6. Notice that (41) may contain a large number of (j, l)-summands depending on a Lerray-Schauder approximation. However, not all of them are essential. By Remark 3 and formula (43), $G\text{-Deg}\,(\widetilde{\mathfrak{F}}_{j,l}, \Omega_{j,l}) = 0$ for all l such that $il\beta_o$ is not a characteristic root of (13) at the stationary point $(\alpha_o, 0)$. Therefore, we know in advance exactly which $\deg_{\mathcal{V}_{j,l}}$ may contribute to the non-triviality of (41). Consequently, we obtain

$$G\text{-Deg}\,(\mathfrak{F}_o, \Omega_o) = \sum_{j,l} \mathfrak{t}_{j,l}(\alpha_o, \beta_o)\ \deg_{\mathcal{V}_{j,l}}, \tag{44}$$

where l runs over "reasonable" positive integer values.

Combining (44) with (40), we obtain the following result:

$$G\text{-Deg}\,(\widetilde{\mathfrak{F}}, \Omega) = \prod_{\mu \in \sigma_+} \prod_{i=0}^{r} \Big(\deg_{\mathcal{V}_i} \Big)^{m_i(\mu)} \cdot \sum_{j,l} \mathfrak{t}_{j,l}(\alpha_o, \beta_o)\ \deg_{\mathcal{V}_{j,l}}, \tag{45}$$

where $j = 0, 1, \dots, s$ and l runs over positive integers such that $il\beta_o$ is a characteristic root of (13) at the stationary point $(\alpha_o, 0)$.

6. Symmetric Configurations of Identical Oscillators as Γ-symmetric FDEs.

6.1. **The Model.** Consider a network of n identical cells (for example, being chemical oscillators) coupled symmetrically by diffusion between certain selected cells. Denote by $x^j(t)$, the concentration of the chemical substance in the j-th cell. Assume that the coupling is taking place between adjacent cells connected by the edges of a regular polygon or polyhedron, describing the geometrical configuration of this network. More precisely, the coupling strength between cells is represented by a function $h : \mathbb{R} \to \mathbb{R}$, in general, nonlinear, which is continuously differentiable and $h(x) \neq 0$ for all $x \in \mathbb{R}$. Since the coupling strength between the adjacent j-th and i-th cells may be nonlinear as well, and depending on the concentrations x^j and x^i, we assume that it is of the form

$$h(x^j(t))\Big(g(\alpha, x_t^j) - g(\alpha, x_t^i)\Big).$$

This term is supported by the ordinary law of diffusion, which simply means that the chemical substance moves from a region of greater concentration to a region of less concentration, at the rate proportional to the gradient of the concentration. Suppose also that the kinetic law obeyed by the concentration x^j in every cell is described by a certain functional f.

In summary, assume that

(H1) $g : \mathbb{R} \times C([-\tau, 0]; \mathbb{R}) \to \mathbb{R}$, $\tau > 0$, is a continuously differentiable map such that $g(\alpha, 0) = 0$ for all $\alpha \in \mathbb{R}$;

(H2) $f : \mathbb{R} \times C([-\tau, 0]; \mathbb{R}) \to \mathbb{R}$ is continuously differentiable;
(H3) $f(\alpha, 0) = 0$ for all $\alpha \in \mathbb{R}$.

A dynamical system describing such a configuration, is of the type

$$\frac{d}{dt}x(t) = F(\alpha, x_t) + H(x(t)) \cdot C(G(\alpha, x_t)), \tag{46}$$

where

$$x = \begin{bmatrix} x^1 \\ x^2 \\ \vdots \\ x^n \end{bmatrix}, \; x_t = \begin{bmatrix} x_t^1 \\ x_t^2 \\ \vdots \\ x_t^n \end{bmatrix}, \; F(\alpha, x_t) = \begin{bmatrix} f(\alpha, x_t^1) \\ f(\alpha, x_t^2) \\ \vdots \\ f(\alpha, x_t^n) \end{bmatrix}, \; G(\alpha, x_t) = \begin{bmatrix} g(\alpha, x_t^1) \\ g(\alpha, x_t^2) \\ \vdots \\ g(\alpha, x_t^n) \end{bmatrix},$$

$$H(x) = \begin{bmatrix} h(x^1) \\ h(x^2) \\ \vdots \\ h(x^n) \end{bmatrix}, \quad x \cdot y = \begin{bmatrix} x^1 \\ x^2 \\ \vdots \\ x^n \end{bmatrix} \cdot \begin{bmatrix} y^1 \\ y^2 \\ \vdots \\ y^n \end{bmatrix} = \begin{bmatrix} x^1 y^1 \\ x^2 y^2 \\ \vdots \\ x^n y^n \end{bmatrix},$$

and C is a symmetric $n \times n$-matrix.

Clearly, by (H3), $(\alpha, 0)$ is a stationary point of (46).

Suppose in addition, that

(H4) The geometrical configuration of these cells has a symmetry group Γ. The group Γ permutes the vertices of the related polygon or polyhedron, which means that it acts on $\mathbb{R}^n$ by permuting the coordinates of the vectors $x \in \mathbb{R}^n$, and C commutes with this action.

Clearly, the system (46) is symmetric with respect to the Γ-action on $V := \mathbb{R}^n$. In this way, we are dealing here with a Γ-symmetric system of FDEs.

6.2. **Examples.** In the subsequent examples, we discuss concrete configurations of such identical cells coupled symmetrically by diffusion between adjacent cells, modeled on the regular n-gon, tetrahedron, octahedron, and dodecahedron. In each case, the symmetry group Γ of the system is composed of the orthogonal symmetries corresponding to the given n-gon or polyhedron. To simplify the presentation, in the case of symmetry groups modeled on the above polyhedrons, we consider only those orthogonal symmetries T for which $\det T = 1$. This assumption is not essential, and in the general case, similar results can be easily derived based on the computations here. We refer the reader to our paper [3] for applied aspects related to these configurations.

Dihedral Configurations of Identical Oscillators. Consider a ring of n identical oscillators, where the interaction takes place only between the neighboring oscillators. In this case, the matrix C is of the type

$$C = \begin{bmatrix} c & d & 0 & \dots & 0 & d \\ d & c & d & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ d & 0 & 0 & \dots & d & c \end{bmatrix}. \tag{47}$$

It is easy to check that the system (46) is symmetric with respect to the action of the dihedral group D_n.

Tetrahedral Configuration of Identical Oscillators. Consider four identical inter-connected oscillators having exactly the same linear interaction between all the other oscillators. In this case, the matrix C is of the type

$$C = \begin{bmatrix} c & d & d & d \\ d & c & d & d \\ d & d & c & d \\ d & d & d & c \end{bmatrix} \tag{48}$$

It is also clear that this system of differential equations (46) is symmetric with respect to the tetrahedral group $\mathbb{T} = A_4$.

Octahedral Configuration of Identical Oscillators. Suppose that the identical cells (oscillators) are arranged in a configuration corresponding to the vertices of a cube. Assume that the interaction takes place between those oscillators that are connected by an edge of the cube. This configuration leads to the system of eight equations with the matrix C of the type

$$C = \begin{bmatrix} c & d & 0 & d & 0 & d & 0 & 0 \\ d & c & d & 0 & 0 & 0 & d & 0 \\ 0 & d & c & d & 0 & 0 & 0 & d \\ d & 0 & d & c & d & 0 & 0 & 0 \\ 0 & 0 & 0 & d & c & d & 0 & d \\ d & 0 & 0 & 0 & d & c & d & 0 \\ 0 & d & 0 & 0 & 0 & d & c & d \\ 0 & 0 & d & 0 & d & 0 & d & c \end{bmatrix}. \tag{49}$$

It is clear that in this case the system (46) is symmetric with respect to the octahedral symmetry group $\mathbb{O}$ which is isomorphic to the symmetric group S_4.

Icosahedral Configuration of Identical Oscillators. Consider an arrangement of identical oscillators based on the inter-connections given by the edges of a dodecahedron. It is clear that the group of symmetries of the dodecahedron, which is the icosahedral group $\mathbb{I}$, is the symmetry group of the system (46). Let us point out that the icosahedral group $\mathbb{I}$ is isomorphic to the alternating group A_5. In this case, we have the system (46) composed of 20 equations, where the matrix C is of the type

$$C = \begin{bmatrix}
c & d & 0 & 0 & d & 0 & 0 & d & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
d & c & d & 0 & 0 & 0 & 0 & 0 & 0 & d & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & d & c & d & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & d & c & d & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d & 0 & 0 & 0 & 0 & 0 & 0 \\
d & 0 & 0 & d & c & d & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & d & c & d & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & d & c & d & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d & 0 & 0 & 0 \\
d & 0 & 0 & 0 & 0 & 0 & d & c & d & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & d & c & d & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d & 0 & 0 \\
0 & d & 0 & 0 & 0 & 0 & 0 & 0 & d & c & d & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d & c & d & 0 & 0 & 0 & 0 & 0 & 0 & d & 0 \\
0 & 0 & d & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d & c & d & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d & c & d & 0 & 0 & 0 & 0 & 0 & d \\
0 & 0 & 0 & d & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d & c & d & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & d & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d & c & d & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d & c & d & 0 & 0 & d \\
0 & 0 & 0 & 0 & 0 & 0 & d & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d & c & d & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d & c & d & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d & 0 & 0 & 0 & 0 & 0 & 0 & d & c & d \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d & 0 & 0 & d & 0 & 0 & d & c
\end{bmatrix}. \tag{50}$$

Of course, other configurations of identical oscillators could also be considered, for example, those based on octahedron, icosahedron or other higher dimensional polyhedra.

6.3. **Characteristic Equation for a Symmetric Configuration of Identical Oscillators.** Consider the linearization of the system (46) at $(\alpha, 0)$, i.e.

$$\frac{d}{dt}x(t) = D_xF(\alpha, 0)x_t + h(0)C(D_xG(\alpha, 0)x_t).$$

Since $D_xG(\alpha, 0)$ is diagonal and C has constant coefficients, $CD_xG(\alpha, 0) = D_xG(\alpha, 0)C$. Put $K(\alpha) := h(0)D_xG(\alpha, 0)$, i.e. the linearized system (46) can be written as

$$\frac{d}{dt}x(t) = D_xF(\alpha, 0)x_t + K(\alpha)Cx_t.$$

A number $\lambda \in \mathbb{C}$ is a characteristic root of (13) at the stationary solution $(\alpha, 0) \in \mathbb{R} \oplus V$ if there exists a nonzero vector $z \in V^c$ such that

$$\triangle_\alpha(\lambda)z := \lambda z - D_xF(\alpha, 0)(e^{\lambda \cdot} z) - K(\alpha)C\big(e^{\lambda \cdot} z\big) = 0. \tag{51}$$

Since the matrix C is symmetric, it is completely diagonalizable by using a basis composed of its eigenvectors. Thus, suppose that $\sigma(C) = \{\xi_1, \xi_2, \dots, \xi_k\}$ is the set of all eigenvalues of C. Then (cf. (13) and (51)), we have the following

Proposition 4. *Given system (46), a number $\lambda \in \mathbb{C}$ is a characteristic root of (13) at the stationary solution $(\alpha, 0)$ for the system (46) if and only if*

$$\det{}_{\mathbb{C}} \triangle_\alpha(\lambda) = \prod_{i=1}^{k} \Big[\lambda - D_xf(\alpha, 0)e^{\lambda \cdot} - \xi_i K(\alpha)e^{\lambda \cdot}\Big] = 0,$$

where $\xi_1, \xi_2, \dots, \xi_k$ are the eigenvalues of the matrix C.

Since the characteristic operator $\triangle_\alpha(\lambda) : V^c \to V^c$ is Γ-equivariant, its eigenspaces are Γ-invariant.

Remark 7. To be compatible with the general setting described in section 4, assume that system (46) satisfies the corresponding analogs of conditions (A4) and (A5), which is referred to as (H5) and (H6).

6.4. **Applications of the Equivariant Degree.** By following the steps, which were explained in section 4, associate with the point (α_o, β_o) a local bifurcation invariant $\omega(\alpha_o, \beta_o, 0) := G\text{-Deg}\,(\mathfrak{F}_\varsigma, \Omega)$, where $G = \Gamma \times S^1$, $\Omega \subset \mathbb{R}^2_+ \times W$ is an open neighborhood of $(\alpha_o, \beta_o, 0)$ defined by (21), $W := H^1(S^1; V)$, and $\mathfrak{F}_\varsigma : \mathbb{R}^2_+ \times W \to \mathbb{R} \oplus W$ is the mapping associated with the bifurcation problem (46) (cf. (23)). This bifurcation invariant can be evaluated by applying the standard steps, which were explained in subsections 5.3–5.5.

Consider, for instance, the argument given in subsection 5.4. Put $A(\alpha_o) := D_xF(\alpha_o, 0) + K(\alpha_o)C : V \to V$. It is easy to check that a number μ belongs to the spectrum $\sigma(A(\alpha_o))$ if and only if for some eigenvalue ξ of the matrix C we have

$$\mu = D_xf(\alpha_o, 0)(1) + k(\alpha_o)\xi,$$

where $k(\alpha_o) = h(0)D_xg(\alpha, 0)(1)$ and $D_xf(\alpha_o, 0)(1)$ are constants (here, $1 \in C([-\tau, 0]; \mathbb{R})$ is the constant function). Consequently, we obtain

$$\sigma(A(\alpha_o)) = \Big\{\mu_i : \mu_i := D_xf(\alpha_o, 0)(1) + k(\alpha_o)\xi_i,\ i = 1, 2, \dots, k\Big\}.$$

Let $\sigma_+(A(\alpha_o))$ be the set of all positive eigenvalues of $A(\alpha_o)$. Then (cf. (40)), formula (45) is applied with σ_+ replaced by $\sigma_+(A(\alpha_o))$.

Under the assumptions (H1)-(H6), Theorem 4 can be applied.

7. **Hopf Bifurcation Results for Concrete Configurations of Identical Oscillators.** In the previous sections, we outlined a general approach to studying the symmetric Hopf bifurcation occuring in (symmetric) FDEs. This approach is applied to a particular case of system (46) in the presence of specific finite symmetry groups. Our final results are presented in tables containing, in particular, information about a minimal number of bifurcating solutions and their symmetries.

7.1. **Model and Equivariant Degree Data.**

7.1.1. *Model.* We consider the following system of delayed differential equations

$$\frac{d}{dt}x(t) = -\alpha x(t) - \alpha H(x(t)) \cdot C(G(x(t-1))), \tag{52}$$

where

$$x = \begin{bmatrix} x^1 \\ x^2 \\ \vdots \\ x^n \end{bmatrix}, \quad H(x) = \begin{bmatrix} h(x^1) \\ h(x^2) \\ \vdots \\ h(x^n) \end{bmatrix}, \quad G(x) = \begin{bmatrix} g(x^1) \\ g(x^2) \\ \vdots \\ g(x^n) \end{bmatrix},$$

and the product '$\cdot$' is defined on the vectors by component-wise multiplication, where

(G1) The functions $h, g : \mathbb{R} \to \mathbb{R}$ are continuously differentiable, $h(t) \neq 0$ for all $t \in \mathbb{R}$, $g(0) = 0$, $g'(0) > 0$ and C is a symmetric $n \times n$-matrix, which commutes with an orthogonal Γ-representation (Γ is finite).

Therefore, conditions (H1)-(H4) are satisfied for system (52).

7.1.2. *Characteristic Values.* To specify conditions (H5)-(H6), we consider the linearization of the system (52) at $(\alpha, 0)$:

$$\frac{d}{dt}x(t) = -\alpha x(t) - \alpha h(0)g'(0)C(x(t-1)), \tag{53}$$

and put

$$\eta := h(0)g'(0). \tag{54}$$

Thus, the condition (H5), which is an analog to (A4) for the system (46), amounts to

$$\prod_{i=1}^{k} \Big[-\alpha - \alpha\eta\xi_i \Big] \neq 0, \tag{55}$$

where $\xi_1, \xi_2, \ldots, \xi_k$ are the eigenvalues of the matrix C. Moreover,

$$\triangle_\alpha(\lambda) = (\lambda + \alpha)\mathrm{Id} + \alpha\eta e^{-\lambda}C$$

(cf. (51)). Therefore, by Proposition 4, a number $\lambda \in \mathbb{C}$ is a characteristic root of (13) at the stationary point $(\alpha_o, 0)$ if and only if

$$\det\nolimits_{\mathbb{C}} \triangle_\alpha(\lambda) = \prod_{i=1}^{k} \Big[\lambda + \alpha + \alpha\eta\xi_i e^{-\lambda}\Big] = 0. \tag{56}$$

To find a root $\lambda \in \mathbb{C}$ of the system (56), consider the following equation

$$\lambda + \alpha + \alpha\eta\xi_o e^{-\lambda} = 0, \tag{57}$$

where ξ_o is an eigenvalue of C. Obviously, $\xi_o \neq 0$ (otherwise $\lambda = -\alpha \in \mathbb{R}$ can not be purely imaginary). A similar reason forces

$$\alpha \neq 0. \tag{58}$$

The equation (57) can be written as the system

$$\begin{cases} u + \alpha + \alpha\eta\xi_o e^{-u}\cos v = 0 \\ v - \alpha\eta\xi_o e^{-u}\sin v = 0, \end{cases} \tag{59}$$

where $\lambda = u + iv$.

Since we are interested in purely imaginary characteristic roots $\lambda = i\beta_o$, by substituting $u = 0$ and $v = \beta$ into the system (59), we obtain

$$\begin{cases} \alpha + \alpha\eta\xi_o\cos\beta = 0 \\ \beta - \alpha\eta\xi_o\sin\beta = 0, \end{cases}$$

which can be easily transformed to

$$\begin{cases} \cos\beta = -\frac{1}{\eta\xi_o} \\ \sin\beta = \frac{1}{\alpha\eta\xi_o}\beta. \end{cases} \tag{60}$$

If $\left|\frac{1}{\xi_o\eta}\right| < 1$, then there exists $\beta_o \in (0,\pi]$ such that $\cos\beta_o = -\frac{1}{\eta\xi_o}$, and, in addition, it is possible to find a unique $\alpha_o = -\beta_o\cot\beta_o$. Therefore, we obtain a pair of solutions (α_o, β_o) to (60) so that (H6) is satisfied.

In what follows we always assume that

(G2) $|\frac{1}{\xi\eta}| < 1$ for all non-zero $\xi \in \sigma(C)$.

Notice that under the assumption (G2), the condition (H5) is also satisfied for $\alpha_o \neq 0$ (cf. Remark 7 and condition (58)).

7.1.3. *Crossing Numbers.* In order to determine the value of the crossing number associated with a purely imaginary characteristic root $\lambda_o = i\beta_o$, we compute (by implicit differentiation) $\frac{d}{d\alpha}u(\alpha)$. By differentiating the system (59) with respect to α,

$$\begin{cases} u'(1 - \alpha\eta\xi_o e^{-u}\cos v) - v'(\alpha\eta\xi_o e^{-u}\sin v) = -\eta\xi_o e^{-u}\cos v - 1 \\ u'(\alpha\eta\xi_o e^{-u}\sin v) + v'(1 - \alpha\eta\xi_o e^{-u}\cos v) = \eta\xi_o e^{-u}\sin v, \end{cases}$$

which, by (59), leads to

$$\begin{cases} u'(1 + u + \alpha) - v'v = \frac{u+\alpha}{\alpha} - 1 \\ u'v + v'(1 + u + \alpha) = \frac{v}{\alpha}. \end{cases} \tag{61}$$

By substituting $\alpha = \alpha_o$, $u = 0$ and $v = \beta_o$ into the system (61), we obtain

$$\begin{cases} u'(1 + \alpha_o) - v'\beta_o = 0 \\ u'\beta_o + v'(1 + \alpha_o) = \frac{\beta_o}{\alpha_o}. \end{cases} \tag{62}$$

The system (62) yields

$$\frac{d}{d\alpha}u|_{\alpha=\alpha_o} = \frac{\beta_o^2}{\alpha_o((\alpha_o + 1)^2 + \beta_o^2)}, \tag{63}$$

thus

$$\text{sign } \frac{d}{d\alpha}u|_{\alpha=\alpha_o} = \text{sign } \alpha_o. \tag{64}$$

Consider a non-zero eigenvalue $\xi_d \in \sigma(C)$. Then, by condition (G2), there always exists a purely imaginary characteristic root $i\beta_d$, $\beta_d > 0$, of the characteristic equation (56) for $\alpha = \alpha_d$, where

$$\cos\beta_d = -\frac{1}{\eta\xi_d}, \quad \alpha_d = \frac{\beta_d}{\eta\xi_d\sin\beta_d}.$$

In what follows, we assume that
(G3) $h(0) > 0$.

Consequently (see formula (54) and condition (G1)), $\eta > 0$. Thus, $\operatorname{sign}\alpha_d = \operatorname{sign}(\eta\xi_d)$. Therefore, using Definition 4 and the fact that $\det_{\mathbb{C}} \triangle_\alpha(\lambda)$ is an analytic function in λ (cf. [16]), we have:

$$\text{if} \quad \alpha_d > 0 \quad \text{then} \quad \mathfrak{t}_{j,1}(\alpha_d, \beta_d) = -m_j(i\beta_d), \tag{65}$$

$$\text{if} \quad \alpha_d < 0 \quad \text{then} \quad \mathfrak{t}_{j,1}(\alpha_d, \beta_d) = m_j(i\beta_d). \tag{66}$$

7.1.4. *Positive Eigenvalues.* We use the same notations as in section 6. We have $A(\alpha) = -\alpha\mathrm{Id} - \alpha h(0)g'(0)C = -\alpha\mathrm{Id} - \alpha\eta C$, so

$$\sigma(A(\alpha)) = \Big\{\mu : \mu = -\alpha - \alpha\eta\xi,\ \xi \in \sigma(C)\Big\}.$$

In order to determine all the positive eigenvalues of the operator $A(\alpha)$, we divide the spectrum $\sigma(C)$ into two parts $\sigma_a(C)$ and $\sigma_b(C)$:

$$\sigma_a(C) = \{\xi \in \sigma(C) : -1 < \eta\xi\}, \tag{67}$$

$$\sigma_b(C) = \{\xi \in \sigma(C) : \eta\xi < -1\}. \tag{68}$$

By condition (G2), $1+\eta\xi \neq 0$ for all eigenvalues ξ of C, thus $\sigma(C) = \sigma_a(C) \cup \sigma_b(C)$. Put

$$\Sigma(C) := \begin{cases} \sigma_a(C) & \text{if } \alpha < 0, \\ \sigma_b(C) & \text{if } \alpha > 0. \end{cases} \tag{69}$$

Then, by (67)-(68), (58) and condition (G3), the set $\sigma_+(A(\alpha))$ of all positive eigenvalues of $A(\alpha)$ can be identified as

$$\sigma_+(A(\alpha)) = \{\mu : \mu = -\alpha(1+\eta\xi),\ \xi \in \Sigma(C)\}. \tag{70}$$

7.2. **Equivariant Degree: First Coefficients.** The local bifurcation invariant $\omega(\lambda_o)$ defined by (24) provides a complete description of the symmetric Hopf bifurcation at $(\alpha_o, 0)$, i.e. (see Theorem 4(i)) every non-zero coefficient n_{H_o} in (25) indicates a "topological obstruction" resulting in the existence of a branch of non-trivial periodic solutions to (52) of the orbit type at least (H_o). Although, the entire value of the degree $\Gamma\times S^1\text{-Deg}(\mathfrak{F}_\varsigma, \Omega)$ should be considered as the *equivariant invariant* classifying the symmetric Hopf bifurcation, in order to simplify the exposition (by reducing the number of additional cases) we will restrict our computations to the coefficients $n_{H_o} = n_{L^{\varphi,1}}$, which are called the *first coefficients*, and we denote the corresponding part of the equivariant degree (24) by $\Gamma \times S^1\text{-Deg}(\mathfrak{F}_\varsigma, \Omega)_1$.

It follows immediately from formula (45) that under the assumptions (G1)-(G3) and (58) (which we always assume), one has

$$\omega(\lambda_o)_1 = \Gamma\times S^1\text{-Deg}(\mathfrak{F}_\varsigma, \Omega)_1 = \prod_{\mu\in\sigma_+(A(\alpha_o))} \prod_{i=0}^{r} \Big(\deg_{\mathcal{V}_i}\Big)^{m_i(\mu)} \cdot \sum_{j=0}^{s} \mathfrak{t}_{j,1}(\alpha_o, \beta_o)\ \deg_{\mathcal{V}_{j,1}}, \tag{71}$$

and Theorem 4 can be applied.

For convenience, we describe below the scheme one has to follow in order to compute $\omega(\lambda_o)_1$:

(a) Take a non-zero $\xi_o \in \sigma(C)$. By condition (G2), one can find a solution (α_o, β_o) to system (60) so that $(\alpha_o, 0)$ is an isolated center of (52) and $\det_{\mathbb{C}} \triangle_{\alpha_o}(i\beta_o) = 0$.

(b) Given a real orthogonal Γ-representation V, take its complexification V^c and consider the isotypical decomposition (12).
(c) Using formula (32), define on V^c a $\Gamma \times S^1$-action with $l = 1$ and denote the obtained (real) representation by W_1. Then, formula (12) induces a (real) $\Gamma \times S^1$-isotypical decomposition:

$$W_1 = V_{0,1} \oplus V_{1,1} \oplus \cdots \oplus V_{s,1}, \tag{72}$$

where $V_{j,1}$ is modeled on $\mathcal{V}_{j,1}$.
(d) Put $E(i\beta_o) := \ker \triangle_{\alpha_o}(i\beta_o)$ and $E_{j,1}(i\beta_o) := E(i\beta_o) \cap V_{j,1}$.
(e) Evaluate $m_j(i\beta_o) := \dim E_{j,1}(i\beta_o)/\dim \mathcal{V}_{j,1}$.
(f) Evaluate $\mathcal{V}_{j,1}$-crossing numbers of $(\alpha_o, 0)$ according to (65)-(66):

$$\mathfrak{t}_{j,1}(\alpha_o, \beta_o) = -\text{sign}\,(\alpha_o) \cdot m_j(i\beta_o).$$

(g) For all $j = 0, 1, \dots, s$ such that $m_j(i\beta_o) \neq 0$, take $\deg_{\mathcal{V}_{j,1}}$ from the list of basic degrees provided by [3].
(h) Identify $\sigma_+(A(\alpha_o))$ (see (67)-(70) with α replaced by α_o).
(i) For each $\mu \in \sigma_+(A(\alpha_o))$, choose the corresponding $\xi \in \sigma(C)$ (cf. (70)) and let $\tilde{E}(\xi)$ be the eigenspace corresponding to ξ. Then the isotypical decomposition (11) yields

$$\tilde{E}(\xi) = \tilde{E}_0(\xi) \oplus \tilde{E}_1(\xi) \oplus \cdots \oplus \tilde{E}_r(\xi).$$

(j) Compute $m_i(\mu) := \dim \tilde{E}_i(\xi)/\dim \mathcal{V}_i$.
(k) For each $i = 0, 1, \dots, r$ such that $m_i(\mu) \neq 0$, take $\deg_{\mathcal{V}_i}$ (see subsection 5.4) from the list of the degrees provided by [3].
(l) Use the multiplication tables for $\cdot : A(\Gamma) \times A_1(\Gamma \times S^1) \to A_1(\Gamma \times S^1)$ given in [3] to compute (71).

We are now in a position to discuss concrete examples of the system (52), admitting dihedral, tetrahedral, octahedral and icosahedral group symmetries.

7.3. **Usage of Maple© Routines.** By putting together all the elements discussed in the previous subsection, we can compute for the system (52) the value of the invariant $\Gamma \times S^1$-Deg$\,(\mathfrak{F}_\varsigma, \Omega)_1$ according to the computational formula (71). Then, by applying Theorem 4 and Remark 5, we can classify the bifurcating branches according to their symmetries. For each of the considered group Γ, we assume that all the real (resp. complex) irreducible Γ-representations $\mathcal{V}_j$ (resp. $\mathcal{U}_j$) are listed in a specific order (see [3]). This order, as well as the notations for twisted subgroups, are compatible with the data format that should be used in the Maple© package for the computation of the equivariant degree.

For a non-zero $\xi_o \in \sigma(C)$, consider the corresponding isolated center $(\alpha_o, 0)$ with a purely imaginary characteristic value $i\beta_o$. Then, the (real) eigenspace $E_o := E(\xi_o)$ is also the eigenspace of the operator $\bar{\tilde{\mathfrak{F}}}$ (defined for (α_o, β_o)) associated to the eigenvalue $\mu_o = -\alpha_o - \alpha_o\eta\xi_o$.

In all the examples considered in the sequel, the following condition is satisfied.
Condition (R)

(i) Decomposition (11) contains isotypical components modeled only on irreducible representations of *real* type (in particular (cf. (11) and (12)), $r = s$).
(ii) For each $\xi_o \in \sigma(C)$ there exists a *single* isotypical component V_j in (11) which (completely) contains the eigenspace $E(\xi_o)$.

Remark and Notation 6. By technical reasons related to the usage of Maple© routines, we present the computational results in such a way that the Γ-isotypical decomposition (11) corresponds to a *complete* list of the irreducible Γ-representations $\mathcal{V}_0$, $\mathcal{V}_1$, …, $\mathcal{V}_r$ given in [3], i.e. we do not exclude the possibility that $V_j = \{0\}$ for some j. Also, given $\xi_o \in \sigma(C)$ and assuming Condition (R) to be satisfied, in what follows we use the notation ξ_o^j to indicate a component V_j in (11) containing $E(\xi_o)$.

In order to compute Γ-Deg$(\bar{\mathfrak{F}}, \mathcal{B})$, we need to organize the equivariant spectral data. As is well-known, for any $\deg_{\mathcal{V}} \in A(\Gamma)$ one has $(\deg_{\mathcal{V}})^2 = (\Gamma)$. Therefore, we associate with $\sigma_+(A(\alpha_o))$ the sequence $(\varepsilon_0, \varepsilon_1, \dots, \varepsilon_r)$ defined by

$$\varepsilon_i = \sum_{\mu \in \sigma_+(A(\alpha_o))} m_i(\mu) \qquad (\text{mod } 2).$$

Then, the formula (40) can be reduced to

$$\Gamma\text{-Deg}(\bar{\mathfrak{F}}, \mathcal{B}) = \prod_{i=0}^{r} \Big(\deg_{\mathcal{V}_i} \Big)^{\varepsilon_i}.$$

On the other hand, assuming the condition (R) to be satisfied, take $\xi_o^j \in \sigma(C)$ and the pair (α_o, β_o) associated with it. Then, $\mathfrak{t}_{j,1}(\alpha_o, \beta_o) = -\text{sign}\,(\alpha_o) \cdot m_j(i\beta_o)$, while $\mathfrak{t}_{j',1}(\alpha_o, \beta_o) = 0$ for $j \neq j'$. Thus, (71) takes the following form:

$$\omega(\alpha_o, \beta_o)_1 = \Gamma \times S^1\text{-Deg}\,(\mathfrak{F}_\varsigma, \Omega)_1 = \prod_{i=0}^{r} \Big(\deg_{\mathcal{V}_i} \Big)^{\varepsilon_i} \cdot \big(- \text{sign}\,(\alpha_o) \big)\, m_j(i\beta_o) \deg_{\mathcal{V}_{j,1}}. \tag{73}$$

In this way, the Maple© input data for the computation of the invariant $\omega(\alpha_o, \beta_o)_1$ consists of the two sequences:

$$\{\varepsilon_0, \varepsilon_1, \dots, \varepsilon_r\}, \quad \{\mathfrak{t}_0, \mathfrak{t}_1, \dots, \mathfrak{t}_r\},$$

where $\mathfrak{t}_j = \mathfrak{t}_{j,1}(\alpha_o, \beta_o)$, $j = 0, 1, \dots, r$, and we can use the Maple© package as follows:

$$\omega(\alpha_o, \beta_o)_1 = \texttt{showdegree}\,[\Gamma](\varepsilon_0, \varepsilon_1, \dots, \varepsilon_r, \mathfrak{t}_0, \mathfrak{t}_1, \dots, \mathfrak{t}_r).$$

For simplicity, we assume that $\text{sign}\,\alpha_o < 0$ (for $\text{sign}\,\alpha_o > 0$, the value of $\omega(\alpha_o, \beta_o)_1$ can be obtained simply by reversing its sign (cf. (65)-(66) and (71))).

Remark 8. In general, it might happen that there are several eigenvalues ξ_o^j corresponding to the same isotypical component V_j. However, by formula (73), they contribute equivalently in the value of $\omega(\alpha_o, \beta_o)_1$, meaning that in this case, $\omega(\alpha_o, \beta_o)_1$ only depends on the isotypical component V_j associated with ξ_o^j and the sequence $(\varepsilon_0, \varepsilon_1, \dots, \varepsilon_r)$. Therefore, we present the results in a form of a matrix

ξ_o^j	$\varepsilon_{i_1}, \varepsilon_{i_2}, \dots, \varepsilon_{i_m}$	$\omega(\alpha_o, \beta_o)_1$	No. of branches

where in the sequence $\{\varepsilon_{i_1}, \varepsilon_{i_2}, \dots, \varepsilon_{i_m}\} \subset \{\varepsilon_0, \varepsilon_1, \dots, \varepsilon_r\}$ we only list those ε_j which can realize the value 1.

7.4. **Hopf Bifurcation in a System with Dihedral Symmetries.** Consider here the system (52) with the matrix C of the type (47). This system is symmetric with respect to the dihedral group $\Gamma = D_n$ acting on $V = \mathbb{R}^n$. Denote by $\rho := e^{\frac{2\pi}{n} i}$ the generator of $\mathbb{Z}_n$ and $\kappa := \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ the operator of complex conjugation. Notice that ρ acts on a vector $x = (x^0, x^1, \dots, x^{n-1})$ by sending the k-th coordinate of x to the $k+1$ (mod n) coordinate, and κ acts by reversing the order of the components of x.

In the case n is odd, the D_n-isotypical components of V are: V_0, V_i, $1 \le i < \frac{n}{2}$, where V_0 is equivalent to the trivial one-dimensional representation $\mathcal{V}_0$, V_i is equivalent to the irreducible two-dimensional representation $\mathcal{V}_i$ defined on $\mathbb{R}^2 = \mathbb{C}$ by $\gamma z := \gamma^i \cdot z$, $\gamma \in \mathbb{Z}_n$, $z \in \mathbb{C}$, $1 \le i < \frac{n}{2}$, $\kappa z := \overline{z}$ (here "$\cdot$" stands for the complex multiplication).

In the case n is even, we have *additionally* $V_{\frac{n}{2}+1}$, which is equivalent to the one-dimensional representation $\mathcal{V}_{\frac{n}{2}+1}$ (cf. [3]) given by the homomorphism $d : D_n \to \mathbb{Z}_2 \subset O(1)$ with $\ker d = D_{\frac{n}{2}}$. Obviously, Condition (R) is satisfied and each V_i is an eigenspace of the matrix C. Moreover, for $0 \le i < \frac{n}{2}$, the corresponding eigenvalue is

$$\xi_i := c + 2d \cos \frac{i2\pi}{n}. \tag{74}$$

In addition, for n even, the eigenspace $V_{\frac{n}{2}+1}$ corresponds to the eigenvalue

$$\xi_{\frac{n}{2}+1} = c - 2d. \tag{75}$$

It seems to be a difficult task to completely evaluate $D_n \times S^1$-Deg $(\mathfrak{F}_\varsigma, \Omega)_1$ for an arbitrary n. However, in the case of the Hopf bifurcation with the symmetry group D_n, it is possible to determine the coefficients n_{H_o} of the invariant $D_n \times S^1$-Deg $(\mathfrak{F}_\varsigma, \Omega)_1 = \sum_{(H)} n_H(H)$ corresponding to the dominating orbit types (H_o) (cf. Definition 5). For this purpose, we need the following

Lemma 2. *Let V and $\mathcal{V}_i$ be the D_n-representations described above and let (H_o) be a dominating orbit type in W_1 (cf. Definition 5 and (32)). Then, the coefficient of (H_o) in $\deg_{\mathcal{V}_i} \cdot (H_o)$ is different from zero for all i (here "$\cdot$" stands for the multiplication described in Proposition 2).*

Proof. By Remark 4 (ii), the dominating orbit types in W_1 can be easily recognized from lattices of orbit types in the irreducible $D_n \times S^1$-representations appearing in the isotypical decomposition of W_1:

$$W_1 = \bigoplus V_{j,1}$$

where $V_{j,1}$ is the isotypical component modeled on the $D_n \times S^1$-irreducible representation $\mathcal{V}_{j,1}$. For each $0 < j < \frac{n}{2}$, put

$$h' = gcd(j,n), \quad m' = \frac{n}{h'}. \tag{76}$$

From the lattices of orbit types provided by [3], all possible choices of (H_o) are

(c1) $(\mathbb{Z}_n^{t_j})$, for $0 < j < \frac{n}{2}$;
(c2) $(D_{h'})$, $(D_{h'}^z)$ if m' is odd;
(c3) $(D_{2h'}^d)$ if m' is even;
(c4) $(D_{2h'}^{\hat{d}})$ if m' is even but $4h' \nmid n$;
(c5) $(\tilde{D}_{2h'}^d)$ if m' is even and $4h' \mid n$;
(c6) (D_n^d) , for $j = \frac{n}{2} + 1$ (if n is even).

On the other hand, according to the list of basic degrees given in [3], $\deg_{\mathcal{V}_i} = \delta(D_n) + \sum \alpha(K)$, where $\alpha \in \mathbb{Z}$, $(K) \ne (D_n)$ and $\delta \in \{-1, 1\}$ depending on the irreducible representation $\mathcal{V}_i$. Since $(D_n) \cdot (H_o) = (H_o)$ in any case, the only possibility for the coefficient of (H_o) in $\deg_{\mathcal{V}_i} \cdot (H_o)$ to be zero is some of the $(K) \cdot (H_o)$ contains a non-trivial term $\alpha'(H_o)$ and a cancelation of the coefficients of (H_o) occurs.

Suppose that $H_o = K_o^\varphi$, then the product $(K) \cdot (K_o^\varphi)$ contains $\alpha'(H_o)$ with $\alpha' \neq 0$ only if $(K) > (K_o)$. For convenience, given a dominating orbit type (H_o) in W_1, we refer to (K) satisfying the conditions:

(i) (K) appears with a non-trivial coefficient in a $\deg_{\mathcal{V}_i}$, for some i;
(ii) $(K) \neq (D_n)$;
(iii) $(K) > (K_o)$,

as *satisfying condition* (†) *for* (H_o).

We look into the following cases.

The case $i = 0$. We have the one-dimensional trivial representation $\mathcal{V}_0$ and $\deg_{\mathcal{V}_0} = -(D_n)$. Consequently, $\deg_{\mathcal{V}_0} \cdot (H_o) = -(H_o)$ for any dominating orbit type (H_o).

The case $1 \leq i < \frac{n}{2}$. Put

$$h = gcd(i, n), \quad m = \frac{n}{h}. \tag{77}$$

The subcase m is odd. We have

$$\deg_{\mathcal{V}_i} = (D_n) - 2(D_h) + (\mathbb{Z}_h). \tag{78}$$

By (77), it is clear that $h < n$, so $(\mathbb{Z}_h) \not> (\mathbb{Z}_n)$, thus $(\mathbb{Z}_h)$ does not satisfy condition (†) for the case (c1) of (H_o). Also, since $(\mathbb{Z}_h) \not> (D_k)$ for any k, $(\mathbb{Z}_h)$ does not satisfy condition (†) for the cases (c2)-(c6) as well. Thus $(\mathbb{Z}_h) \cdot (H_o)$ does not contain a non-trivial (H_o)-term.

Also, by the fact that $h < n$, clearly $(D_h) \not> (\mathbb{Z}_n)$ and $(D_h) \not> (D_n)$, so (D_h) does not satisfy condition (†) for the cases (c1) and (c6) of (H_o). For the remaining possibilities of (H_o), using the fact that $(D_h) > (D_k)$ if $k \mid h \mid n$ and $(D_h) > (\tilde{D}_k)$ if $2k \mid h \mid n$ and $2h \nmid n$, (c2)-(c5) yield the following results for (D_h): (D_h) satisfies condition (†) for

(r1) $(D_{h'})$, $(D_{h'}^z)$ if $h' \mid h \mid n$ and m' is odd;
(r2) $(D_{2h'}^d)$ if $2h' \mid h \mid n$ (m' is even);
(r3) $(D_{2h'}^{\hat{d}})$ if $2h' \mid h \mid n$ but $4h' \nmid n$ (m' is even);
(r4) $(\tilde{D}_{2h'}^d)$ if $4h' \mid h \mid n$ and $2h \nmid n$ (m' is even).

Notice that all the possible cases for $(H_o) = (K_o^\varphi)$ listed in (r1)-(r4) force $(K_o) = (D_k)$ for an appropriate k. Thus, the (H_o)-coefficient n_o in the product $(D_h) \cdot (H_o)$ can be computed via the formula (7) as follows

$$\begin{aligned} n_o &= \frac{n(D_k, D_h)|W(D_h)|n(H_o, H_o)|W(H_o)/S^1|}{|W(H_o)/S^1|} \\ &= n(D_k, D_h)|W(D_h)|n(H_o, H_o) \\ &= 1 \cdot |W(D_h)| \cdot 1 \\ &= |W(D_h)|. \end{aligned} \tag{79}$$

Combining (79) with the fact that $W(D_h) \cong \mathbb{Z}_1$ in D_n if m is odd, we arrive at $n_o = 1$. Therefore (see (78)),

$$\deg_{\mathcal{V}_i} \cdot (H_o) = -(H_o) + \text{ other terms,}$$

where (H_o) runs over (r1)-(r4).

The subcase m is even. We have

$$\deg_{\mathcal{V}_i} = (D_n) - (D_h) - (\tilde{D}_h) + (\mathbb{Z}_h).$$

By the same reason as discussed in the previous subcase, $(\mathbb{Z}_h)\cdot(H_o)$ does not provide a non-trivial (H_o)-term.

According to (76) and (77), if $h' \mid h$ we immediately have $m \mid m'$. Thus, in the case m is even, m' has also to be even, which implies that the case (r1) is excluded for (D_h). However, the remaining cases (r2)-(r4) are still valid for (D_h). So, combining (79) with the fact that $W(D_h) \cong \mathbb{Z}_2$ in D_n if m is even, we have $(D_h)\cdot(H_o) = 2(H_o) +$ other terms, where (H_o) is from (r2)-(r4).

Again, by the fact that $h < n$, $(\tilde{D}_h) \not> (\mathbb{Z}_n)$ and $(\tilde{D}_h) \not> (D_n)$, so $(\tilde{D}_h)$ does not satisfy condition (†) for the cases (c1) and (c6) of (H_o). Also, since $(\tilde{D}_h) \not> (D_k)$ for any k, $(\tilde{D}_h)$ does not satisfy condition (†) for the cases (c2)-(c4) of (H_o). Finally, in the case (c5), $(\tilde{D}_h)$ satisfies condition (†) for $(\tilde{D}^d_{2h'})$ if and only if $4h' \mid 2h \mid n$, and we will refer to such a case as (c5)'. Namely, $(\tilde{D}_h)$ satisfies condition (†) for

(c5)' $(\tilde{D}^d_{2h'})$ if $4h' \mid 2h \mid n$, m' is even and $4h' \mid n$.

Moreover, under the conditions mentioned in (c5)', we have the following formula (cf. (79)) for the $(\tilde{D}^d_{2h'})$-coefficient $n_o{}'$ in the product $(\tilde{D}_h)\cdot(\tilde{D}^d_{2h'})$:

$$\begin{aligned}
n_o{}' &= \frac{n(\tilde{D}_{2h'},\tilde{D}_h)|W(\tilde{D}_h)|n(\tilde{D}^d_{2h'},\tilde{D}^d_{2h'})|W(\tilde{D}^d_{2h'})/S^1|}{|W(\tilde{D}^d_{2h'})/S^1|}\\
&= n(\tilde{D}_{2h'},\tilde{D}_h)|W(\tilde{D}_h)|n(\tilde{D}^d_{2h'},\tilde{D}^d_{2h'})\\
&= 1\cdot|W(\tilde{D}_h)|\cdot 1\\
&= |W(\tilde{D}_h)|
\end{aligned}$$

Since $W(\tilde{D}_h) \cong \mathbb{Z}_2$ in D_n, we have under (c5)' the following: $(\tilde{D}_h)\cdot(\tilde{D}^d_{2h'}) = 2(\tilde{D}^d_{2h'}) +$ other terms. Notice that (r4) and (c5)' can not be satisfied simultaneously, therefore

$$\deg_{\mathcal{V}_i}\cdot(H_o) = -(H_o) + \text{ other terms},$$

where (H_o) runs over (r2)-(r4) and (c5)'.

The case $i = \frac{n}{2}+1$ (if n is even). We have

$$\deg_{\mathcal{V}_{\frac{n}{2}+1}} = (D_n) - (D_{\frac{n}{2}}).$$

By replacing h with $\frac{n}{2}$ and analyzing the possibility of each condition listed in (r1)-(r4), we have that $(D_{\frac{n}{2}})$ satisfies condition (†) only for (r2). By (79) and the fact that $W(D_{\frac{n}{2}}) \cong \mathbb{Z}_2$ in D_n, we have $(D_{\frac{n}{2}})\cdot(D^d_{2h'}) = 2(D^d_{2h'}) +$ other terms. Therefore,

$$\deg_{\mathcal{V}_i}\cdot(H_o) = -(H_o) + \text{ other terms},$$

where (H_o) is from (r2).

□

As an immediate consequence of Lemma 2, we have

Proposition 5. *Assume that the system (52) admits the symmetry group* $\Gamma = D_n$, *i.e. the matrix* C *is of type (47), and let the invariant* $D_n \times S^1\text{-Deg}\,(\mathfrak{F}_\varsigma,\Omega)_1$ *be defined by (73). Let* $(H_o) \in \mathcal{V}_{j,1}$ *be a dominating orbit type appearing in* $\deg_{\mathcal{V}_{j,1}}$ *with a non-zero coefficient. Then,* (H_o) *will also appear with a non-zero coefficient in* $D_n \times S^1\text{-Deg}\,(\mathfrak{F}_\varsigma,\Omega)_1$.

Remark 9. In general, an $A(\Gamma)$-multiplication in $A_1(\Gamma\times S^1)$ can lead to a cancelation of a maximal (in $\Phi_1(\Gamma\times S^1)$) conjugacy class. For example, in $A_1(S_4\times S^1)$, we have $(2(S_4)-(A_4))\cdot(A_4^t) = 0$. Therefore, it is possible that the conclusion of

Proposition 5 may not be true for $\Gamma \neq D_n$. Thus, in general, it is necessary to use the complete value of $\Gamma \times S^1$-Deg$(\mathfrak{F}_\varsigma, \Omega)_1$ to detect branches of solutions and classify their symmetries.

We are now in a position to present the following general result (cf. [22]).

Theorem 7. *Assume that the system (52) admits the symmetry group $\Gamma = D_n$, i.e. the matrix C is of type (47). Assume also that the system (52) satisfies (G1)-(G3) and (58) as described in subsection 7.1. Take $\xi_i \in \sigma(C)$ (cf. (74) and (75)), and let $(\alpha_i, \beta_i) \in \mathbb{R}^2_+$ satisfy the relations*

$$\cos\,\beta_i = -\frac{1}{\eta \xi_i}, \quad \alpha_i = \frac{\beta_i}{\eta \xi_i\,\sin \beta_i},$$

(cf. subsubsection 7.1.2).

Then there exists a branch of non-constant $\frac{2\pi}{\beta}$-periodic solutions to (52) bifurcating from $(\alpha_i, 0)$ with the "limit period" $\frac{2\pi}{\beta_i}$ (cf. section 4). In addition,*

(i) *If $i = 0$, then there exists at least 1 branch of non-constant periodic solutions with symmetries at least (D_n).*

(ii) *If $1 \leq i < \frac{n}{2}$, let h and m be defined as in (77).*

  (a) *If m is odd, then there exist at least 2 branches of non-constant periodic solutions with symmetries at least $(\mathbb{Z}_n^{t_i})$, $\frac{n}{h}$ branches of non-constant periodic solutions with symmetries at least (D_h), and $\frac{n}{h}$ branches of non-constant periodic solutions with symmetries at least (D_h^z);*

  (b) *If $m \equiv 2$ (mod 4), then there exist at least 2 branches of non-constant periodic solutions with symmetries at least $(\mathbb{Z}_n^{t_i})$, $\frac{n}{2h}$ branches of non-constant periodic solutions with symmetries at least (D_{2h}^d), and $\frac{n}{2h}$ branches of non-constant periodic solutions with symmetries at least $(D_{2h}^{\widehat{d}})$;*

  (c) *If $m \equiv 0$ (mod 4), then there exist at least 2 branches of non-constant periodic solutions with symmetries at least $(\mathbb{Z}_n^{t_i})$, $\frac{n}{2h}$ branches of non-constant periodic solutions with symmetries at least (D_{2h}^d), and $\frac{n}{2h}$ branches of non-constant periodic solutions with symmetries at least $(\widetilde{D}_{2h}^d)$.*

(iii) *If $i = \frac{n}{2} + 1$ (for n even), then there exists at least 1 branch of non-constant periodic solutions with symmetries at least (D_n^d).*

(We assume, of course, that each orbit type appearing in the statement is considered in the setting when it is dominating (cf. Remark 5).)

Proof. Theorem 7 is a direct consequence of Theorem 4, formula (73), the dominating orbit type list (c1)-(c6) in the proof of Lemma 2, Proposition 5 and the following list of basic degrees provided by [3]:

$$\deg_{\mathcal{V}_{0,1}} = (D_n),$$

$$\deg_{\mathcal{V}_{i,1}} = \begin{cases} (\mathbb{Z}_n^{t_i}) + (D_h) + (D_h^z) - (\mathbb{Z}_h) & (\text{ if } \ m \text{ is odd}), \\ (\mathbb{Z}_n^{t_i}) + (D_{2h}^d) + (D_{2h}^{\widehat{d}}) - (\mathbb{Z}_{2h}^d) & (\text{ if } \ m \equiv 2 \text{ (mod 4)}) \\ (\mathbb{Z}_n^{t_i}) + (D_{2h}^d) + (\widetilde{D}_{2h}^d) - (\mathbb{Z}_{2h}^d) & (\text{ if } \ m \equiv 0 \text{ (mod 4)}), \end{cases} \quad \text{for } 1 \leq i < \frac{n}{2},$$

$$\deg_{\mathcal{V}_{\frac{n}{2}+1,1}} = (D_n^d) \ \ (\text{ if } \ n \text{ is even}).$$

□

* We do not assume the period to be minimal.

Remark 10. The advantage of Theorem 7 rests on the fact that it can be applied to study the D_n-symmetric Hopf bifurcation phenomena for an *arbitrary* n. However, Theorem 7 does not take into account the possible contribution of *non-dominating* orbit types. To be more specific, according to formula (73), the complete value of $D_n \times S^1\text{-Deg}(\mathfrak{F}_\varsigma, \Omega)_1$ may contain a non-dominating orbit type (with a non-trivial coefficient), which is not smaller than any dominating orbit types appearing in $\deg_{\mathcal{V}_{j,1}}$. The appearance of a such a non-dominating orbit type contributes to at least 1 more branch of non-constant periodic solutions.

Below, we discuss 3 examples of dihedral groups D_n ($n = 3, 4, 5$), for which we obtain a classification of the symmetric Hopf bifurcation, in terms of the invariant $\omega(\alpha_j, \beta_j)_1 = D_n \times S^1\text{-Deg}(\mathfrak{F}_\varsigma, \Omega)_1$, with non-dominating orbit types being taken into account (for notations see Remark and Notation 6 and [3]).

Hopf Bifurcation with D_3 Symmetries. In this case,

$$V = V_0 \oplus V_1, \quad V_0 = \mathcal{V}_0,\ V_1 = \mathcal{V}_1,$$

$\sigma(C) = \{\xi_0^0 = c + 2d,\ \xi_1^1 = c - d\}$, with each of the eigenvalues* ξ_j^j ($j = 0, 1$) corresponding to a pair (α_j, β_j) such that $i\beta_j$ is a purely imaginary characteristic root of (56) at the point $(\alpha_j, 0)$. The dominating orbit types in W (cf. (31)) are (D_3), $(\mathbb{Z}_3^t) := (\mathbb{Z}_3^{t_1})$ and (D_1^z). Apply the invariant $D_3 \times S^1\text{-Deg}(\mathfrak{F}_\varsigma, \Omega)_1$ to classify the Hopf bifurcation at the point $(\alpha_j, 0)$ according to the symmetries of the corresponding branches. We summarize in Table 1 the topological invariants $\omega(\alpha_j, \beta_j)_1 = D_3 \times S^1\text{-Deg}(\mathfrak{F}_\varsigma, \Omega)_1$ corresponding to possible values of (α_j, β_j) and variations of $\sigma_+(A(\alpha_j))$ (cf. (67)-(70)), represented by $(\varepsilon_0, \varepsilon_1)$.

Hopf Bifurcation with D_4 Symmetries. Here,

$$V = V_0 \oplus V_1 \oplus V_3, \quad V_0 = \mathcal{V}_0,\ V_1 = \mathcal{V}_1,\ V_3 = \mathcal{V}_3,$$

$\sigma(C) = \left\{\xi_0^0 = c + 2d, \xi_1^1 = c, \xi_2^3 = c - 2d\right\}$ and the dominating orbit types in W are (D_4), $(\mathbb{Z}_4^t) := (\mathbb{Z}_4^{t_1})$, (D_2^d), $(\widetilde{D}_2^d)$ and (D_4^d). We summarize in Table 2 the corresponding results for D_4-symmetric Hopf bifurcation.

Hopf Bifurcation with D_5 Symmetries. We have

$$V = V_0 \oplus V_1 \oplus V_2, \quad V_0 = \mathcal{V}_0,\ V_1 = \mathcal{V}_1,\ V_2 = \mathcal{V}_2,$$

$\sigma(C) = \left\{\xi_0^0 = c + 2d, \xi_1^1 = c + 2d\frac{\sqrt{5}-1}{4}, \xi_2^2 = c - 2d\frac{\sqrt{5}+1}{4}\right\}$ and the dominating orbit types in W are (D_5), $(\mathbb{Z}_5^{t_1})$, $(\mathbb{Z}_5^{t_2})$ and (D_1^z). We summarize in Table 3 the corresponding results for D_5-symmetric Hopf bifurcation.

7.5. **Hopf Bifurcation in a System with Tetrahedral Symmetries.** We consider here the system (52) with the matrix C of the type (48), which commutes with the tetrahedral group $\Gamma = A_4$-action on the space $V = \mathbb{R}^4$. The group A_4 acts on $\mathbb{R}^4$ by permuting the coordinates of the vectors in the same way as the symmetries of a tetrahedron in $\mathbb{R}^3$ permute its four vertices. We have (see [3] for notations and details)

$$V = V_0 \oplus V_3, \quad V_0 = \mathcal{V}_0,\ V_3 = \mathcal{V}_3,$$

where $\mathcal{V}_0$ stands for the trivial one-dimensional representation and $\mathcal{V}_3$ denotes the standard three-dimensional representation of the tetrahedral group. Also, $\sigma(C) = \{\xi_0^0 = c + 3d, \xi_1^3 = c - d\}$. The dominating orbit types in W are (A_4), $(\mathbb{Z}_3^{t_1})$, $(\mathbb{Z}_3^{t_2})$

* We use the lower index j to enumerate the eigenvalues, which, in general, may not be related to the order of the isotypical components

ξ_o^j	$\varepsilon_0, \varepsilon_1$	$\omega(\alpha_j, \beta_j)_1$	# Branches
ξ_o^0	00	(D_3)	1
ξ_o^0	10	$-(D_3)$	1
ξ_o^0	01	$(D_3) - 2(D_1) + (\mathbb{Z}_1)$	1
ξ_o^0	11	$-(D_3) + 2(D_1) - (\mathbb{Z}_1)$	1
ξ_o^1	00	$(\mathbb{Z}_3^t) + (D_1^z) + (D_1) - (\mathbb{Z}_1)$	6
ξ_o^1	10	$-(\mathbb{Z}_3^t) - (D_1^z) - (D_1) + (\mathbb{Z}_1)$	6
ξ_o^1	01	$(\mathbb{Z}_3^t) - (D_1) - (D_1^z) + (\mathbb{Z}_1)$	6
ξ_o^1	11	$-(\mathbb{Z}_3^t) + (D_1) + (D_1^z) - (\mathbb{Z}_1)$	6

TABLE 1. Equivariant classification of the Hopf bifurcation with D_3 symmetries

and (V_4^-) . We summarize in Table 4 the corresponding results for A_4-symmetric Hopf bifurcation.

7.6. **Hopf Bifurcation in a System with Octahedral Symmetries.** Consider the system (52) with the matrix C of the type (49), which is symmetric with respect to the octahedral group $\Gamma = S_4$, where S_4 acts on the space $V := \mathbb{R}^8$ by permuting the coordinates of the vectors in the same way as the symmetries of a cube in $\mathbb{R}^3$ permute its eight vertices. The representation V has the following S_4-isotypical decomposition (see [3] for details)

$$V = V_0 \oplus V_1 \oplus V_3 \oplus V_4,\ V_0 = \mathcal{V}_0,\ V_1 = \mathcal{V}_1,\ V_3 = \mathcal{V}_3,\ V_4 = \mathcal{V}_4,$$

(here $\mathcal{V}_0$ stands for the trivial one-dimensional S_4-representation; $\mathcal{V}_1$ is the one-dimensional representation corresponding to the homomorphism $S^4 \to \mathbb{Z}_2 \subset O(1)$; $\mathcal{V}_3$ denotes the standard three-dimensional S_4-representation (where S_4 acts as a subgroup of $SO(3)$); $\mathcal{V}_4 = \mathcal{V}_1 \otimes V_3$). Also, all the above irreducible representations are of real type. In addition, the spectrum of C is given by

$$\sigma(C) = \{\xi_0^0 = c + 3d, \xi_1^1 = c - 3d, \xi_2^3 = c + d, \xi_3^4 = c - d\}.$$

The dominating orbit types in W are (S_4), (S_4^-), (D_4^d), (D_2^d), $(\mathbb{Z}_4^c)$, $(\mathbb{Z}_3^t)$ and (D_4^z). We summarize in Tables 5 the corresponding results for S_4-symmetric Hopf bifurcation.

7.7. **Hopf Bifurcation in a System with Icosahedral Symmetries.** Finally, consider the system (52) with the matrix C of the type (50), which commutes with the $\Gamma = A_5$-action on the space $V := \mathbb{R}^{20}$ permuting the coordinates of the vectors in the same way as the symmetries of a dodecahedron in $\mathbb{R}^3$ permute its 20 vertices. The A_5-isotypical decomposition of V is given by (cf. [3])

$$V = V_0 \oplus V_1 \oplus V_2 \oplus V_3, \quad V_0 = \mathcal{V}_0, ; V_1 = \mathcal{V}_1 \oplus \mathcal{V}_1,\ V_2 = \mathcal{V}_2,\ V_3 = \mathcal{V}_3,\ V_4 = \mathcal{V}_4,$$

and

$$\sigma(C) := \left\{\xi_0^0 = c + 3d, \xi_1^1 = c, \xi_2^1 = c - 2d, \xi_3^2 = c + d, \xi_4^3 = c + \sqrt{5}d, \xi_5^4 = c - \sqrt{5}d\right\}.$$

For a pair (α_j, β_j) such that $i\beta_j$ is a purely imaginary root of (56) at $(\alpha_j, 0)$, there are the following possible combinations of the eigenvalues μ_k, $k = 0, 1, 2, 3, 4, 5$, in $\sigma_+(A(\alpha_j))$, with the corresponding to them sequences $\{\varepsilon_0, \varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4\}$:

ξ_o^j	$\varepsilon_0, \varepsilon_1, \varepsilon_3$	$\omega(\alpha_j, \beta_j)_1$	# Branches
ξ_o^0	000	(D_4)	1
ξ_o^0	100	$-(D_4)$	1
ξ_o^0	001	(D_4)	1
ξ_o^0	110	$-(D_4)+(D_1)+(\widetilde{D}_1)-(\mathbb{Z}_1)$	1
ξ_o^0	011	$(D_4)-(D_1)-(\widetilde{D}_1)+(\mathbb{Z}_1)$	1
ξ_o^0	111	$-(D_4)+(D_1)+(\widetilde{D}_1)-(\mathbb{Z}_1)$	1
ξ_o^1	000	$(\mathbb{Z}_4^t)+(D_2^d)+(\widetilde{D}_2^d)-(\mathbb{Z}_2^-)$	6
ξ_o^1	100	$-(\mathbb{Z}_4^t)-(D_2^d)-(\widetilde{D}_2^d)+(\mathbb{Z}_2^-)$	6
ξ_o^1	001	$(\mathbb{Z}_4^t)+(D_2^d)+(\widetilde{D}_2^d)-(\mathbb{Z}_2^-)$	6
ξ_o^1	110	$-(\mathbb{Z}_4^t)-(D_2^d)-(\widetilde{D}_2^d)+(\mathbb{Z}_2^-)+(D_1^z)+(\widetilde{D}_1^z)+(D_1)+(\widetilde{D}_1)-2(\mathbb{Z}_1)$	6
ξ_o^1	011	$(\mathbb{Z}_4^t)+(D_2^d)+(\widetilde{D}_2^d)-(\mathbb{Z}_2^-)-(D_1^z)-(\widetilde{D}_1^z)-(D_1)-(\widetilde{D}_1)+2(\mathbb{Z}_1)$	6
ξ_o^1	111	$-(\mathbb{Z}_4^t)-(D_2^d)-(\widetilde{D}_2^d)+(\mathbb{Z}_2^-)+(D_1^z)+(\widetilde{D}_1^z)+(D_1)+(\widetilde{D}_1)-2(\mathbb{Z}_1)$	6
ξ_o^3	000	(D_4^d)	1
ξ_o^3	100	$-(D_4^d)$	1
ξ_o^3	001	(D_4^d)	1
ξ_o^3	110	$-(D_4^d)+(\widetilde{D}_1^z)+(D_1)-(Z_1)$	1
ξ_o^3	011	$(D_4^d)-(\widetilde{D}_1^z)-(D_1)+(Z_1)$	1
ξ_o^3	111	$-(D_4^d)+(\widetilde{D}_1^z)+(D_1)-(Z_1)$	1

TABLE 2. Equivariant classification of the Hopf bifurcation with D_4 symmetries

ξ_o^j	$\varepsilon_0, \varepsilon_1, \varepsilon_2$	$\omega(\alpha_j, \beta_j)_1$	# Branches
ξ_o^0	000	(D_5)	1
ξ_o^0	100	$-(D_5)$	1
ξ_o^0	001	$(D_5)-2(D_1)+(\mathbb{Z}_1)$	1
ξ_o^0	110	$-(D_5)+2(D_1)-(\mathbb{Z}_1)$	1
ξ_o^0	011	(D_5)	1
ξ_o^0	111	$-(D_5)$	1
ξ_o^1	000	$(\mathbb{Z}_5^{t_1})+(D_1^z)+(D_1)-(\mathbb{Z}_1)$	8
ξ_o^1	100	$-(\mathbb{Z}_5^{t_1})-(D_1^z)-(D_1)+(\mathbb{Z}_1)$	8
ξ_o^1	001	$(\mathbb{Z}_5^{t_1})-(D_1^z)-(D_1)+(\mathbb{Z}_1)$	8
ξ_o^1	110	$-(\mathbb{Z}_5^{t_1})+(D_1^z)+(D_1)-(\mathbb{Z}_1)$	8
ξ_o^1	011	$(\mathbb{Z}_5^{t_1})-(D_1^z)-(D_1)+(\mathbb{Z}_1)$	8
ξ_o^1	111	$-(\mathbb{Z}_5^{t_1})-(D_1^z)-(D_1)+(\mathbb{Z}_1)$	8
ξ_o^2	000	$(\mathbb{Z}_5^{t_2})+(D_1^z)+(D_1)-(\mathbb{Z}_1)$	8
ξ_o^2	100	$-(\mathbb{Z}_5^{t_2})-(D_1^z)-(D_1)+(\mathbb{Z}_1)$	8
ξ_o^2	001	$(\mathbb{Z}_5^{t_2})-(D_1^z)-(D_1)+(\mathbb{Z}_1)$	8
ξ_o^2	110	$-(\mathbb{Z}_5^{t_2})+(D_1^z)+(D_1)-(\mathbb{Z}_1)$	8
ξ_o^2	011	$(\mathbb{Z}_5^{t_2})+(D_1^z)+(D_1)-(\mathbb{Z}_1)$	8
ξ_o^2	111	$-(\mathbb{Z}_5^{t_2})-(D_1^z)-(D_1)+(\mathbb{Z}_1)$	8

TABLE 3. Equivariant classification of the Hopf bifurcation with D_5 symmetries

$\sigma_+(A(\alpha_j))$	$\{\varepsilon_0, \varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4\}$	$\sigma_+(A(\alpha_j))$	$\{\varepsilon_0, \varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4\}$
$\{\mu_5\}$	00001	$\{\mu_0\}$	10000
$\{\mu_2, \mu_5\}$	01001	$\{\mu_0, \mu_4\}$	10010
$\{\mu_1, \mu_2, \mu_5\}$	00001	$\{\mu_0, \mu_3, \mu_4\}$	10110
$\{\mu_1, \mu_2, \mu_3, \mu_5\}$	00101	$\{\mu_0, \mu_1, \mu_3, \mu_4\}$	11110
$\{\mu_1, \mu_2, \mu_3, \mu_4, \mu_5\}$	00111	$\{\mu_0, \mu_1, \mu_2, \mu_3, \mu_4\}$	10110
$\{\mu_0, \mu_1, \mu_2, \mu_3, \mu_4, \mu_5\}$	10111	$\emptyset$	00000

ξ_o^j	$\varepsilon_0,\varepsilon_3$	$\omega(\alpha_j,\beta_j)_1$	# Branches
ξ_o^0	00	(A_4)	1
ξ_o^0	10	$-(A_4)$	1
ξ_o^0	01	$(A_4)-2(\mathbb{Z}_3)-(\mathbb{Z}_2)+(\mathbb{Z}_1)$	1
ξ_o^0	11	$-(A_4)+2(\mathbb{Z}_3)+(\mathbb{Z}_2)-(\mathbb{Z}_1)$	1
ξ_o^3	00	$(V_4^-)+(\mathbb{Z}_3^{t_1})+(\mathbb{Z}_3^{t_2})+(\mathbb{Z}_3)-(\mathbb{Z}_1)$	12
ξ_o^3	10	$-(V_4^-)-(\mathbb{Z}_3^{t_1})-(\mathbb{Z}_3^{t_2})-(\mathbb{Z}_3)+(\mathbb{Z}_1)$	12
ξ_o^3	01	$(V_4^-)-(\mathbb{Z}_3^{t_1})-(\mathbb{Z}_3^{t_2})-(\mathbb{Z}_3)-2(\mathbb{Z}_2^z)-(\mathbb{Z}_3)+(\mathbb{Z}_1)$	12
ξ_o^3	11	$-(V_4^-)+(\mathbb{Z}_3^{t_1})+(\mathbb{Z}_3^{t_2})+(\mathbb{Z}_3)+2(\mathbb{Z}_2^z)+(\mathbb{Z}_3)-(\mathbb{Z}_1)$	12

TABLE 4. Equivariant classification of the Hopf bifurcation with A_4 symmetries

ξ_o^j	$\varepsilon_0,\varepsilon_1,\varepsilon_3,\varepsilon_4$	$\omega(\alpha_j,\beta_j)_1$	# Branches
ξ_0^0	0000	(S_4)	1
ξ_0^0	1000	$-(S_4)$	1
ξ_0^0	0100	$(S_4)-(A_4)$	1
ξ_0^0	1010	$-(S_4)+2(D_3)+(D_2)-3(D_1)+(\mathbb{Z}_1)$	1
ξ_0^0	0101	$(S_4)-(A_4)-(\mathbb{Z}_4)+(\mathbb{Z}_3)-(D_1)+(\mathbb{Z}_1)$	1
ξ_0^0	1011	$-(S_4)+2(D_3)+(D_2)+(\mathbb{Z}_4)-(\mathbb{Z}_3)-2(D_1)-(\mathbb{Z}_2)+(\mathbb{Z}_1)$	1
ξ_0^0	0111	$-(S_4)-(A_4)-2(D_3)-(D_2)-(\mathbb{Z}_4)+(\mathbb{Z}_3)+2(D_1)+(\mathbb{Z}_2)-(\mathbb{Z}_1)$	1
ξ_0^0	1111	$(S_4)+(A_4)+2(D_3)+(D_2)+(\mathbb{Z}_4)-(\mathbb{Z}_3)-2(D_1)-(\mathbb{Z}_2)+(\mathbb{Z}_1)$	1
ξ_1^1	0000	(S_4^-)	1
ξ_1^1	1000	$-(S_4^-)$	1
ξ_1^1	0100	$(S_4^-)-(A_4)$	1
ξ_1^1	1010	$-(S_4^-)+2(D_3^z)+(D_2^z)-3(D_1^z)+(\mathbb{Z}_1)$	1
ξ_1^1	0101	$(S_4^-)-(A_4)-(\mathbb{Z}_4^-)+(\mathbb{Z}_3)-(D_1^z)+(\mathbb{Z}_2)$	1
ξ_1^1	1011	$-(S_4^-)+2(D_3^z)+(D_2^z)+(\mathbb{Z}_4^-)-(\mathbb{Z}_3)-2(D_1^z)-(\mathbb{Z}_2)+(\mathbb{Z}_1)$	1
ξ_1^1	0111	$(S_4^-)-(A_4)-2(D_3^z)-(D_2^z)-(\mathbb{Z}_4^-)+(\mathbb{Z}_3)+2(D_1^z)+(\mathbb{Z}_2)-(\mathbb{Z}_1)$	1
ξ_1^1	1111	$-(S_4^-)+(A_4)+2(D_3^z)+(D_2^z)+(\mathbb{Z}_4^-)-(\mathbb{Z}_3)-2(D_1^z)-(\mathbb{Z}_2)+(\mathbb{Z}_1)$	1
ξ_2^3	0000	$(D_4^d)+(D_3)+(D_2^d)+(\mathbb{Z}_4^c)+(\mathbb{Z}_3^t)-(D_1)-(\mathbb{Z}_2^-)$	24
ξ_2^3	1000	$-(D_4^d)-(D_3)-(D_2^d)-(\mathbb{Z}_4^c)-(\mathbb{Z}_3^t)+(D_1)+(\mathbb{Z}_2^-)$	24
ξ_2^3	0100	$(D_4^d)+(D_3)+(D_2^d)+(\mathbb{Z}_4^c)-(V_4^-)-(\mathbb{Z}_3^t)-(\mathbb{Z}_3)-(D_1)-(\mathbb{Z}_2^-)+(\mathbb{Z}_1)$	24
ξ_2^3	1010	$-(D_4^d)+(D_3)+(D_2^d)-(\mathbb{Z}_4^c)+(\mathbb{Z}_3^t)-(D_1^z)-3(D_1)+(\mathbb{Z}_2^-)+(\mathbb{Z}_1)$	24
ξ_2^3	0101	$(D_4^d)+(D_3)+(D_2^d)+(D_2)-(\mathbb{Z}_4^c)-(\mathbb{Z}_4^-)-(V_4^-)+(\mathbb{Z}_3^t)-(D_1^z)-3(D_1)+(\mathbb{Z}_2^-)+(\mathbb{Z}_2)+(\mathbb{Z}_1)$	24
ξ_2^3	1011	$-(D_4^d)+(D_3)+(D_2^d)+(D_2)+(\mathbb{Z}_4^c)+(\mathbb{Z}_4^-)-(\mathbb{Z}_3^t)-(\mathbb{Z}_3)-(D_1)-(\mathbb{Z}_2^-)-(\mathbb{Z}_2)+(\mathbb{Z}_1)$	24
ξ_2^3	0111	$(D_4^d)-(D_3)-(D_2^d)-(D_2)-(\mathbb{Z}_4^c)-(\mathbb{Z}_4^-)-(V_4^-)-(\mathbb{Z}_3^t)+(D_1)+(\mathbb{Z}_2^-)+(\mathbb{Z}_2)$	24
ξ_2^3	1111	$-(D_4^d)+(D_3)+(D_2^d)+(D_2)+(\mathbb{Z}_4^c)+(\mathbb{Z}_4^-)+(V_4^-)+(\mathbb{Z}_3^t)-(D_1)-(\mathbb{Z}_2^-)-(\mathbb{Z}_2)$	24
ξ_3^4	0000	$(D_4^z)+(D_3^z)+(D_2^d)+(\mathbb{Z}_4^c)+(\mathbb{Z}_3^t)-(D_1^z)-(\mathbb{Z}_2^-)$	24
ξ_3^4	1000	$-(D_4^z)-(D_3^z)-(D_2^d)-(\mathbb{Z}_4^c)-(\mathbb{Z}_3^t)+(D_1^z)+(\mathbb{Z}_2^-)$	24
ξ_3^4	0100	$(D_4^z)+(D_3^z)+(D_2^d)+(\mathbb{Z}_4^c)-(V_4^-)-(\mathbb{Z}_3^t)-(\mathbb{Z}_3)-(D_1^z)-(\mathbb{Z}_2^-)+(\mathbb{Z}_1)$	24
ξ_3^4	1010	$-(D_4^z)+(D_3^z)+(D_2^d)+(D_2^z)-(\mathbb{Z}_4^c)+(\mathbb{Z}_3^t)-3(D_1^z)-(D_1)+(\mathbb{Z}_2^-)+(\mathbb{Z}_1)$	24
ξ_3^4	0101	$(D_4^z)+(D_3^z)+(D_2^d)-(\mathbb{Z}_4^c)-(\mathbb{Z}_4)-(V_4^-)+(\mathbb{Z}_3^t)-3(D_1^z)-(D_1)+(\mathbb{Z}_2^-)+(\mathbb{Z}_2)+(\mathbb{Z}_1)$	24
ξ_3^4	1011	$-(D_4^z)+(D_3^z)+(D_2^d)+(D_2^z)+(\mathbb{Z}_4^c)+(\mathbb{Z}_4)-(\mathbb{Z}_3^t)-(\mathbb{Z}_3)-(D_1^z)-(\mathbb{Z}_2^-)-(\mathbb{Z}_2)+(\mathbb{Z}_1)$	24
ξ_3^4	0111	$(D_4^z)-(D_3^z)-(D_2^d)-(D_2^z)-(\mathbb{Z}_4^c)-(\mathbb{Z}_4)-(V_4^-)-(\mathbb{Z}_3^t)+(D_1^z)+(\mathbb{Z}_2^-)+(\mathbb{Z}_2)$	24
ξ_3^4	1111	$-(D_4^z)+(D_3^z)+(D_2^d)+(D_2^z)+(\mathbb{Z}_4^c)+(\mathbb{Z}_4)+(V_4^-)+(\mathbb{Z}_3^t)-(D_1^z)-(\mathbb{Z}_2^-)-(\mathbb{Z}_2)$	24

TABLE 5. Equivariant classification of the Hopf bifurcation with S_4 symmetries

The dominating orbit types are (A_5), (D_3^z), (V_4^-), $(\mathbb{Z}_5^{t_1})$, $(\mathbb{Z}_5^{t_2})$, $(A_4^{t_1})$, $(A_4^{t_2})$ and (D_5^z). We summarize in Tables 6 and 7 the corresponding results for A_5-symmetric Hopf bifurcation.

Remark 11. Let us explain how to decode the information provided by the invariant $\omega(\alpha_j,\beta_j)_1 = \Gamma\times S^1\text{-Deg}(\mathfrak{F}\varsigma,\Omega)_1$ from the corresponding table. For example, in the case $\Gamma = A_4$, $\xi_o^3 = \xi_1^3 = c-d$, $(\varepsilon_0,\varepsilon_3) = (1,0)$, we have (as listed in Table 4)

$$\omega(\alpha_j,\beta_j)_1 = -(V_4^-)-(\mathbb{Z}_3^{t_1})-(\mathbb{Z}_3^{t_2})-(\mathbb{Z}_3)+(\mathbb{Z}_1).$$

ξ_o^j	$\varepsilon_0, \varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4$	$\omega(\alpha_j, \beta_j)_1$	# Branches
ξ_0^0	00000	(A_5)	1
ξ_0^0	00001	$(A_5) - (\mathbb{Z}_5) - (\mathbb{Z}_3) - (\mathbb{Z}_2) + (\mathbb{Z}_1)$	1
ξ_0^0	00111	$(A_5) - 2(D_5) - 2(D_3) + 3(\mathbb{Z}_2) - (\mathbb{Z}_1)$	1
ξ_0^0	00101	$(A_5) - 2(D_5) - 2(D_3) + (\mathbb{Z}_5) + (\mathbb{Z}_3) + 4(\mathbb{Z}_2) - 2(\mathbb{Z}_1)$	1
ξ_0^0	01001	$(A_5) - 2(A_4) - 2(D_3) - (\mathbb{Z}_5) + 2(\mathbb{Z}_3) + 2(\mathbb{Z}_2) - (\mathbb{Z}_1)$	1
ξ_0^0	10000	$-(A_5)$	1
ξ_0^0	10010	$-(A_5) + (\mathbb{Z}_5) + (\mathbb{Z}_3) + (\mathbb{Z}_2) - (\mathbb{Z}_1)$	1
ξ_0^0	10111	$-(A_5) + 2(D_5) + 2(D_3) - 3(\mathbb{Z}_2) + (\mathbb{Z}_1)$	1
ξ_0^0	10110	$-(A_5) + 2(D_5) + 2(D_3) - (\mathbb{Z}_5) - (\mathbb{Z}_3) - 4(\mathbb{Z}_2) + 2(\mathbb{Z}_1)$	1
ξ_0^0	11110	$-(A_5) + 2(A_4) + 2(D_5) - (\mathbb{Z}_5) - 2(\mathbb{Z}_3) - 3(\mathbb{Z}_2) + 2(\mathbb{Z}_1)$	1
ξ_0^1	00000	$(A_4) + (D_3^z) + (D_3) + (\mathbb{Z}_5^{t_1}) + (\mathbb{Z}_5^{t_2}) + (V_4^-) + (\mathbb{Z}_3^t) - (\mathbb{Z}_3) - (\mathbb{Z}_2^-) - (\mathbb{Z}_2)$	55
ξ_o^1	00001	$(A_4) + (D_3^z) + (D_3) - (\mathbb{Z}_5^{t_1}) - (\mathbb{Z}_5^{t_2}) + (V_4^-) - (\mathbb{Z}_3^t) - 3(\mathbb{Z}_3)$ $-3(\mathbb{Z}_2^-) - 3(\mathbb{Z}_2) + 4(\mathbb{Z}_1)$	55
ξ_o^1	00111	$(A_4) - (D_3^z) - (D_3) - (\mathbb{Z}_5^{t_1}) - (\mathbb{Z}_5^{t_2}) + (V_4^-) - (\mathbb{Z}_3^t) - (\mathbb{Z}_3)$ $-(\mathbb{Z}_2^-) - (\mathbb{Z}_2) + 2(\mathbb{Z}_1)$	55
ξ_o^1	00101	$(A_4) - (D_3^z) - (D_3) + (\mathbb{Z}_5^{t_1}) + (\mathbb{Z}_5^{t_2}) + (V_4^-) + (\mathbb{Z}_3^t) + (\mathbb{Z}_3)$ $+(\mathbb{Z}_2^-) + (\mathbb{Z}_2) - 2(\mathbb{Z}_1)$	55
ξ_o^1	01001	$-(A_4) - (D_3^z) - (D_3) - (\mathbb{Z}_5^{t_1}) - (\mathbb{Z}_5^{t_2}) - (V_4^-) - (\mathbb{Z}_3^t) + (\mathbb{Z}_3)$ $+(\mathbb{Z}_2^-) + (\mathbb{Z}_2)$	55
ξ_o^1	10000	$-(A_4) - (D_3^z) - (D_3) - (\mathbb{Z}_5^{t_1}) - (\mathbb{Z}_5^{t_2}) - (V_4^-) - (\mathbb{Z}_3^t) + (\mathbb{Z}_3)$ $+(\mathbb{Z}_2^-) + (\mathbb{Z}_2)$	55
ξ_o^1	10010	$-(A_4) - (D_3^z) - (D_3) + (\mathbb{Z}_5^{t_1}) + (\mathbb{Z}_5^{t_2}) - (V_4^-) + (\mathbb{Z}_3^t) + 3(\mathbb{Z}_3)$ $+3(\mathbb{Z}_2^-) + 3(\mathbb{Z}_2) - 4(\mathbb{Z}_1)$	55
ξ_o^1	10111	$-(A_4) + (D_3^z) + (D_3) + (\mathbb{Z}_5^{t_1}) + (\mathbb{Z}_5^{t_2}) - (V_4^-) + (\mathbb{Z}_3^t) + (\mathbb{Z}_3)$ $+(\mathbb{Z}_2^-) + (\mathbb{Z}_2) - 2(\mathbb{Z}_1)$	55
ξ_o^1	10110	$-(A_4) + (D_3^z) + (D_3) - (\mathbb{Z}_5^{t_1}) - (\mathbb{Z}_5^{t_2}) - (V_4^-) - (\mathbb{Z}_3^t) - (\mathbb{Z}_3)$ $-(\mathbb{Z}_2^-) - (\mathbb{Z}_2) + 2(\mathbb{Z}_1)$	55
ξ_o^1	11110	$(A_4) - (D_3^z) - (D_3) - (\mathbb{Z}_5^{t_1}) - (\mathbb{Z}_5^{t_2}) + (V_4^-) - (\mathbb{Z}_3^t) - (\mathbb{Z}_3)$ $-(\mathbb{Z}_2^-) - (\mathbb{Z}_2) + 2(\mathbb{Z}_1)$	55
ξ_3^2	00000	$(A_4^{t_1}) + (A_4^{t_2}) + (D_5) + (D_3) + (\mathbb{Z}_5^{t_1}) + (\mathbb{Z}_5^{t_2}) + (V_4^-) - 2(\mathbb{Z}_2)$	50
ξ_3^2	00001	$(A_4^{t_1}) + (A_4^{t_2}) + (D_5) + (D_3) - (\mathbb{Z}_5^{t_1}) - (\mathbb{Z}_5^{t_2}) - (\mathbb{Z}_5) + (V_4^-)$ $-4(\mathbb{Z}_3^t) - (\mathbb{Z}_3) - 2(\mathbb{Z}_2^-) - 5(\mathbb{Z}_2) + 5(\mathbb{Z}_1)$	50
ξ_3^2	00111	$(A_4^{t_1}) + (A_4^{t_2}) - (D_5) - (D_3) - (\mathbb{Z}_5^{t_1}) - (\mathbb{Z}_5^{t_2}) + (V_4^-) - 4(\mathbb{Z}_3^t)$ $-2(\mathbb{Z}_2^-) - (\mathbb{Z}_2) + 3(\mathbb{Z}_1)$	50
ξ_3^2	00101	$(A_4^{t_1}) + (A_4^{t_2}) - (D_5) - (D_3) + (\mathbb{Z}_5^{t_1}) + (\mathbb{Z}_5^{t_2}) + (\mathbb{Z}_5) + (V_4^-)$ $+(\mathbb{Z}_3) + 2(\mathbb{Z}_2) - 2(\mathbb{Z}_1)$	50
ξ_3^2	01001	$-(A_4^{t_1}) - (A_4^{t_2}) + (D_5) - (D_3) - (\mathbb{Z}_5^{t_1}) - (\mathbb{Z}_5^{t_2}) - (\mathbb{Z}_5) - (V_4^-) + (\mathbb{Z}_1)$	50
ξ_3^2	10000	$-(A_4^{t_1}) - (A_4^{t_2}) - (D_5) - (D_3) - (\mathbb{Z}_5^{t_1}) - (\mathbb{Z}_5^{t_2}) - (V_4^-) + 2(\mathbb{Z}_2)$	50
ξ_3^2	10010	$-(A_4^{t_1}) - (A_4^{t_2}) - (D_5) - (D_3) + (\mathbb{Z}_5^{t_1}) + (\mathbb{Z}_5^{t_2}) + (\mathbb{Z}_5) - (V_4^-)$ $+4(\mathbb{Z}_3^t) + (\mathbb{Z}_3) + 2(\mathbb{Z}_2^-) + 5(\mathbb{Z}_2) - 5(\mathbb{Z}_1)$	50
ξ_3^2	10111	$-(A_4^{t_1}) - (A_4^{t_2}) + (D_5) + (D_3) + (\mathbb{Z}_5^{t_1}) + (\mathbb{Z}_5^{t_2}) - (V_4^-) + 4(\mathbb{Z}_3^t)$ $+2(\mathbb{Z}_2^-) + (\mathbb{Z}_2) - 3(\mathbb{Z}_1)$	50
ξ_3^2	10110	$-(A_4^{t_1}) - (A_4^{t_2}) + (D_5) + (D_3) - (\mathbb{Z}_5^{t_1}) - (\mathbb{Z}_5^{t_2}) - (\mathbb{Z}_5) - (V_4^-) - (\mathbb{Z}_3)$ $-2(\mathbb{Z}_2) + 2(\mathbb{Z}_1)$	50
ξ_3^2	11110	$(A_4^{t_1}) + (A_4^{t_2}) + (D_5) - (D_3) - (\mathbb{Z}_5^{t_1}) - (\mathbb{Z}_5^{t_2}) - (\mathbb{Z}_5) + (V_4^-) - 4(\mathbb{Z}_3^t)$ $-2(\mathbb{Z}_2^-) - 3(\mathbb{Z}_2) + 4(\mathbb{Z}_1)$	50
ξ_4^3	00000	$(D_5^z) + (D_3^z) + (\mathbb{Z}_5^{t_1}) + (V_4^-) + (\mathbb{Z}_3^t) - 2(\mathbb{Z}_2^-)$	48
ξ_4^3	00001	$(D_5^z) + (D_3^z) - (\mathbb{Z}_5^{t_1}) - (\mathbb{Z}_5) + (V_4^-) - (\mathbb{Z}_3^t) - (\mathbb{Z}_3) - 4(\mathbb{Z}_2^-) - (\mathbb{Z}_2) + 3(\mathbb{Z}_1)$	48
ξ_4^3	00111	$-(D_5^z) - (D_3^z) - (\mathbb{Z}_5^{t_1}) + (V_4^-) - (\mathbb{Z}_3^t) - (\mathbb{Z}_2) + (\mathbb{Z}_1)$	48
ξ_4^3	00101	$-(D_5^z) - (D_3^z) + (\mathbb{Z}_5^{t_1}) + (\mathbb{Z}_5) + (V_4^-) + (\mathbb{Z}_3^t) + (\mathbb{Z}_3) + 2(\mathbb{Z}_2^-) - 2(\mathbb{Z}_2)$	48
ξ_4^3	01001	$(D_5^z) - (D_3^z) - (\mathbb{Z}_5^{t_1}) - (V_4^-) - (\mathbb{Z}_3^t) + (\mathbb{Z}_1)$	48
ξ_4^3	10000	$-(D_5^z) - (D_3^z) - (\mathbb{Z}_5^{t_1}) - (V_4^-) - (\mathbb{Z}_3^t) + 2(\mathbb{Z}_2^-)$	48
ξ_4^3	10010	$-(D_5^z) - (D_3^z) + (\mathbb{Z}_5^{t_1}) + (\mathbb{Z}_5) - (V_4^-) + (\mathbb{Z}_3^t) + (\mathbb{Z}_3) + 4(\mathbb{Z}_2^-) + (\mathbb{Z}_2) - 3(\mathbb{Z}_1)$	48
ξ_4^3	10111	$(D_5^z) + (D_3^z) + (\mathbb{Z}_5^{t_1}) - (V_4^-) + (\mathbb{Z}_3^t) + (\mathbb{Z}_2) - (\mathbb{Z}_1)$	48
ξ_4^3	10110	$(D_5^z) + (D_3^z) - (\mathbb{Z}_5^{t_1}) - (\mathbb{Z}_5) - (V_4^-) - (\mathbb{Z}_3^t) - (\mathbb{Z}_3) - 2(\mathbb{Z}_2^-) + 2(\mathbb{Z}_1)$	48
ξ_4^3	11110	$(D_5^z) - (D_3^z) - (\mathbb{Z}_5^{t_1}) - (\mathbb{Z}_5) + (V_4^-) - (\mathbb{Z}_3^t) - 2(\mathbb{Z}_2^-) - (\mathbb{Z}_2) + 2(\mathbb{Z}_1)$	48

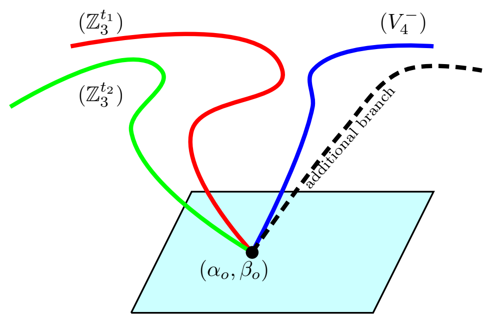
TABLE 6. Equivariant classification of the Hopf bifurcation with A_5 symmetries – Part I.

The dominating orbit types in $\omega(\alpha_j, \beta_j)_1$ with non-trivial coefficients are $(\mathbb{Z}_3^{t_1})$, $(\mathbb{Z}_3^{t_2})$ and (V_4^-). Therefore, by Theorem 4, there is a Hopf bifurcation occuring at $(\alpha_0, 0)$. More specifically, one can expect the occurrence of at least 3 branches of non-constant periodic solutions with the symmetries at least (V_4^-), 4 branches of non-constant periodic solutions with the symmetries at least $(\mathbb{Z}_3^{t_1})$ and 4 branches of non-constant periodic solutions with the symmetries at least $(\mathbb{Z}_3^{t_2})$. Moreover, the

ξ_o^j	$\varepsilon_0,\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4$	$\omega(\alpha_j,\beta_j)_1$	# Branches
ξ_5^4	00000	$(D_5^z)+(D_3^z)+(\mathbb{Z}_5^{t_2})+(V_4^-)+(\mathbb{Z}_3^t)-2(\mathbb{Z}_2^-)$	48
ξ_5^4	00001	$(D_5^z)+(D_3^z)-(\mathbb{Z}_5^{t_2})-(\mathbb{Z}_5)+(V_4^-)-(\mathbb{Z}_3^t)-(\mathbb{Z}_3)-4(\mathbb{Z}_2^-)-(\mathbb{Z}_2)+3(\mathbb{Z}_1)$	48
ξ_5^4	00111	$-(D_5^z)-(D_3^z)-(\mathbb{Z}_5^{t_2})+(V_4^-)-(\mathbb{Z}_3^t)-(\mathbb{Z}_2)+(\mathbb{Z}_1)$	48
ξ_5^4	00101	$-(D_5^z)-(D_3^z)+(\mathbb{Z}_5^{t_2})+(\mathbb{Z}_5)+(V_4^-)+(\mathbb{Z}_3^t)+(\mathbb{Z}_3)+2(\mathbb{Z}_2^-)-2(\mathbb{Z}_2)$	48
ξ_5^4	01001	$(D_5^z)-(D_3^z)-(\mathbb{Z}_5^{t_2})-(V_4^-)-(\mathbb{Z}_3^t)+(\mathbb{Z}_1)$	48
ξ_5^4	10000	$-(D_5^z)-(D_3^z)-(\mathbb{Z}_5^{t_2})-(V_4^-)-(\mathbb{Z}_3^t)+2(\mathbb{Z}_2^-)$	48
ξ_5^4	10010	$-(D_5^z)-(D_3^z)+(\mathbb{Z}_5^{t_2})+(\mathbb{Z}_5)-(V_4^-)+(\mathbb{Z}_3^t)+(\mathbb{Z}_3)+4(\mathbb{Z}_2^-)+(\mathbb{Z}_2)-3(\mathbb{Z}_1)$	48
ξ_5^4	10111	$(D_5^z)+(D_3^z)+(\mathbb{Z}_5^{t_2})-(V_4^-)+(\mathbb{Z}_3^t)+(\mathbb{Z}_2)-(\mathbb{Z}_1)$	48
ξ_5^4	10110	$(D_5^z)+(D_3^z)-(\mathbb{Z}_5^{t_2})-(\mathbb{Z}_5)-(V_4^-)-(\mathbb{Z}_3^t)-(\mathbb{Z}_3)-2(\mathbb{Z}_2^-)+2(\mathbb{Z}_1)$	48
ξ_5^4	11110	$(D_5^z)-(D_3^z)-(\mathbb{Z}_5^{t_2})-(\mathbb{Z}_5)+(V_4^-)-(\mathbb{Z}_3^t)-2(\mathbb{Z}_2^-)-(\mathbb{Z}_2)+2(\mathbb{Z}_1)$	48

TABLE 7. Equivariant classification of the Hopf bifurcation with A_5 symmetries – Part II.

non-trivial $(\mathbb{Z}_3)$-term also contributes to at least one more branch of non-constant periodic solutions with the symmetry at least $(\mathbb{Z}_3)$ (cf. Remark 10). In this way, we predict the existence of at least 12 branches of non-constant periodic solutions. We illustrate this situation on a diagram below.



Remark 12. Computations of the equivariant degrees, which were applied to estimate of the number of non-constant periodic solutions of the systems (52) with different symmetries, were done with the assistance of the Maple© package "EquivariantDegree". This package, which contains the complete multiplication tables and the equivariant degrees of the basic maps for the groups $D_3\times S^1$, $D_4\times S^1$, $D_5\times S^1$, $A_4\times S^1$, $S_4\times S^1$ and $A_5\times S^1$, was created by Adrian Biglands, and is available at the web site:

`http://www.math.ualberta.ca/~wkrawcew/degree` or
`http://krawcewicz.net/degree`

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