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G.E. Hutchinson's delay logistic system with symmetries and spatial diffusion

Zalman Balanov[a,b,1], Wieslaw Krawcewicz[a,*,1,2], Haibo Ruan[a,3]

[a] *Department of Mathematical and Statistical Sciences, University of Alberta, Edmonton, Canada T6G 2G1*
[b] *Department of Mathematics and Computer Sciences, Netanya Academic College, Netanya, Israel 42365*

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Abstract

In this paper we apply the equivariant degree method to a Hopf bifurcation problem in a symmetric system of delayed functional parabolic partial differential equations. The equivariant spectral properties of the linearized system are instantaneously translated, with the assistance of a specially developed Maple© package, into a bifurcation invariant providing symmetric classification of the bifurcating branches. This procedure is applied to a symmetric Hutchinson model of an n species ecosystem in a heterogeneous environment. Computational results, indicating the existence, multiplicity and symmetric classification of the solutions, are listed in Tables 1–6.

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1. Introduction

One can easily observe that almost every design (e.g. in architecture, networks, electronical devices) has certain symmetries, which are expressions of our tendencies pursuing elegance and balance. But on the other hand, symmetries in dynamical systems cause a multitude of various types of solutions exhibiting complicated symmetric properties. The problem of "measuring" such impact of symmetries on the complexity of the dynamics constitutes a difficult task.

The main goal of the present paper is to propose the equivariant degree method as a tool providing a full topological picture of local symmetric bifurcation phenomena. To be more specific, we study the existence, multiplicity and symmetric properties of non-trivial periodic solutions, appearing as a result of a local Hopf bifurcation, in a symmetric system of delayed functional parabolic partial differential equations (FPDEs) (see system (14)). Such systems arise

* Corresponding author. Tel.: +1 780 492 0566; fax: +1 780 492 6826.
E-mail addresses: balanov@mail.netanya.ac.il (Z. Balanov), krawcewicz@shaw.ca (W. Krawcewicz), hruan@math.ualberta.ca (H. Ruan).
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naturally in population ecology, for example, they can be derived from the Hutchinson's model (also known as the delayed logistic equation),

$$\dot{u}(t) = \alpha u(t)\left(1 - \frac{u(t-\tau)}{K}\right).$$

Especially, we are interested in investigating an ecosystem composed of n self-inhibiting interacting species (cf. [13]) in a spatially heterogeneous environment. It is also closely related to the differential-difference equation

$$f'(x) = -\alpha f(x-1)\{1 + f(x)\}$$

(see (44)–(45)), which has other interesting applications in number theory and control systems (cf. [16]).

The advantage of the equivariant degree method is based on the fact that it provides a rich topological information about bifurcating branches and their symmetries, which can be easily computed only using the information extracted from the characteristic equation and the symmetric properties of the linearized system. To this end, a computational package of Maple©[4] routines was developed for several types of finite symmetry groups, which takes in the *equivariant data* and outputs the values of bifurcation invariants, in terms of a sequence of integers indexed by subgroups of the whole symmetry group.

Let us briefly explain how to "decode" the information provided by our computational results (see Tables 1–6). Consider a Hutchinson system modelling a population composed of 4 species (see (47)) with the dihedral group D_4 performing as its (spatial) symmetry group (see Section 5.2, Condition (A1) for precise formulations). Recall that the dihedral group D_4 consists of 4 rotations of the square: 1, r, r^2, r^3 and 4 reflections: κ, κr, κr^2, κr^3. Let, further, $\mathbb{F}$ be a "reasonable" space of $\mathbb{R}^4$-valued periodic functions, where periodic solutions to system (47) "live" (cf. Section 4.3). The D_4-action on $\mathbb{R}^4$ combined with the natural S^1-action (induced by the shift of the t-argument) gives rise to the $G = D_4 \times S^1$-representation in $\mathbb{F}$. Given a periodic solution $u \in \mathbb{F}$, we will call a subgroup $H := \{h \in G : hu = u\}$ a *symmetry* of u. It turns out that:

(i) using the isotypical decomposition of the D_4-representation on $\mathbb{R}^4$ and spectral properties of the linearization of (47), one can recognize isolated centers to (47) together with "limit frequencies" of the corresponding periodic solutions bifurcating from the centers;
(ii) given an isolated center $(\alpha_0, 0)$ and the corresponding limit frequency β_0, one can associate with them the (local) bifurcation invariant$\omega(\alpha_0, \beta_0, 0)$ containing a full topological symmetric information on bifurcating solutions (see Section 5, where more "refined" notations are used).

Possible values of $\omega(\alpha_0, \beta_0, 0)$ in the considered case are listed in Table 2. As an example, consider one of these values:

$$\omega(\lambda_0) := (-1)^{\nu+1}\left((\mathbb{Z}_4^t) - (D_2^d) + (\widetilde{D}_2^d) - (\mathbb{Z}_2^-) + (D_1^z) - (\widetilde{D}_1^z) + (D_1) - (\widetilde{D}_1)\right). \tag{1}$$

The subgroups of $D_4 \times S^1$ appearing in (1) are given by

$$\mathbb{Z}_4^t = \{(1,1), (r, \mathrm{i}), (r^2, -1), (r^3, -\mathrm{i})\},$$
$$D_2^d = \{(1,1), (r^2, -1), (\kappa, 1), (\kappa r^2, -1)\},$$
$$\widetilde{D}_2^d = \{(1,1), (r^2, -1), (\kappa r, 1), (\kappa r^3, -1)\},$$
$$D_1^z = \{(1,1), (\kappa, -1)\}, \quad D_1 = \{(1,1), (\kappa, 1)\},$$
$$\widetilde{D}_1^z = \{(1,1), (\kappa r, -1)\}, \quad \widetilde{D}_1 = \{(1,1), (\kappa r, 1)\},$$
$$\mathbb{Z}_2^- = \{(1,1), (r^2, -1)\}.$$

[4] The package is available at http://krawcewicz.net/degree.

These subgroups represent the minimal symmetries of the bifurcating branches. Moreover, for each of the subgroups $\mathbb{Z}_4^t$, D_2^d and $\widetilde{D}_2^d$, we predict the existence of at least 2 branches having the above groups as their *precise* symmetries (cf. Definition 4.4 and Remark 4.5, where an "exclusive" role of the groups in question is explained). Observe that the advanced knowledge of symmetric properties of (periodic) solutions can be very useful in numerical analysis and simulation of the model, since it provides a theoretical basis for prediction and confirmation of the existence and multiplicities of solutions with special properties.

The paper is organized as follows. In Section 2, we summarize several important ingredients of the equivariant degree method which were presented in details in [2–4]. In Section 3, we discuss a general functional setting for an abstract equivariant coincidence problem related to a parameterized family of unbounded Fredholm operators. This setting is specified in Section 4, where the symmetric Hopf bifurcation phenomenon is discussed for a system of functional parabolic PDEs. A system of n coupled Hutchinson's equations, modelling an interactive community ecosystem, is studied in Section 5. In particular, for this system, we carry out a complete analysis of equivariant spectral properties of the linearized system and clarify the important elements of the computational scheme. Moreover (see Section 6), for several geometrically natural group symmetries, we use the Maple© package to establish quantitative results on the associated local bifurcation invariants, minimal number of bifurcating branches of solutions and their symmetries.

2. Preliminaries

2.1. Definitions and notations

Hereafter, $G=\Gamma\times S^1$, where Γ is a *finite* group and S^1 is the unit circle group. For a closed subgroup H of G, denote by (H) the conjugacy class of H in G, $N(H)$—the normalizer of H in G, $W(H)=N(H)/H$—the Weyl group of H in G and $\Phi(G)$—the set of all conjugacy classes in G (which admits a natural partial order: $(K)\leqslant(H)$ if K is conjugate to a subgroup of H).

Let V be an orthogonal (or isometric Banach) G-representation. For $x\in V$, denote by $G_x=\{g\in G : gx=x\}$ the *isotropy group* of x and call the conjugacy class (G_x) the *orbit type* of x in V. For a G-invariant subset $X\subset V$, put $X^H:=\{x\in X : G_x\supset H\}$ and call it the *H-fixed-point subspace*.

Let Ω be an open bounded G-invariant subset of $\mathbb{R}\oplus V$, where we will always assume the trivial G-action on $\mathbb{R}$, and let $f:\mathbb{R}\oplus V\to V$ be a continuous equivariant map in Ω, meaning $f(gx)=gf(x)$ for all $g\in G$ and $x\in\overline{\Omega}$. The f is called *Ω-admissible* if $f(x)\neq 0$ for all $x\in\partial\Omega$, and such a pair (f,Ω) will be called an *admissible pair*. Similarly, a homotopy $h:[0,1]\times\mathbb{R}\oplus V\to V$ is called an *Ω-admissible* G-equivariant homotopy, if $h_t:=h(t,\cdot)$ is an Ω-admissible G-equivariant map for all $t\in[0,1]$.

2.2. Primary equivariant degree with one free parameter

Consider the set

$$\Phi_1(G):=\{(H)\in\Phi(G) : \dim W(H)=1\}.$$

It is easy to check that the elements of $\Phi_1(G)$ are the conjugacy classes (H) of the so-called *φ-twisted l-folded* subgroups of $\Gamma\times S^1$ with $l=1,2,3,\ldots$, i.e.

$$H=K^{\varphi,l}:=\{(\gamma,z)\in K\times S^1 : \varphi(\gamma)=z^l\},$$

where K is a subgroup of Γ and $\varphi:K\to S^1$ is a homomorphism. In the case of a 1-folded φ-twisted subgroup $K^{\varphi,1}$, we will denote it by K^φ and call it simply a *twisted* subgroup of $\Gamma\times S^1$.

Denote by

$$A_1(G):=\mathbb{Z}[\Phi_1(G)]$$

the free $\mathbb{Z}$-module generated by $\Phi_1(G)$, i.e. any element $\alpha\in A_1(G)$ can be written as a finite sum $\alpha=n_{H_1}(H_1)+n_{H_2}(H_2)+\cdots+n_{H_r}(H_r)$, $n_{H_i}\in\mathbb{Z}$.

The $\mathbb{Z}$-module $A_1(G)$ is a range of values of the so-called *primary equivariant degree* $G\text{-Deg}$ defined on admissible pairs (f, Ω) ($\Omega \subset \mathbb{R} \oplus V$ with V being an orthogonal G-representation) and satisfying all the standard properties required from a "reasonable" degree theory. Moreover, the primary equivariant degree theory admits an axiomatic approach (see [2–4] for details). Being limited in size, we list below only those properties which are directly referred to in the present paper.

- Existence: If $G\text{-Deg}\,(f, \Omega) = \sum_{(H)} n_H (H)$ is such that $n_{H_0} \neq 0$ for some $(H_0) \in \Phi_1(G)$, then there exists $x_0 \in \Omega$ with $f(x_0) = 0$ and $G_{x_0} \supset H_0$.
- Homotopy: Suppose that $h : [0, 1] \times \mathbb{R} \oplus V \to V$ is an Ω-admissible G-equivariant homotopy. Then, $G\text{-Deg}\,(h_t, \Omega) = \text{constant}$, where $h_t := h(t, \cdot)$.
- Multiplicativity: Let $A(\Gamma)$ denote the Burnside ring of Γ (see [18,25]). There exists a multiplication $\cdot : A(\Gamma) \times A_1(G) \to A_1(G)$ such that for an orthogonal Γ-representation V_0 and a continuous equivariant map $f_0 : V_0 \to V_0$, one has

$$G\text{-Deg}\,(f \times f_0, \Omega \times \mathscr{B}) = \Gamma\text{-Deg}\,(f_0, \mathscr{B}) \cdot G\text{-Deg}\,(f, \Omega),$$

  where $\mathscr{B} \subset V_0$ is the unit ball, $f_0(x) \neq 0$ for $x \in \partial\mathscr{B}$ and $\Gamma\text{-Deg}$ stands for the equivariant degree without free parameters (see [18] for details).

Remark 2.1. (i) The so-called *basic maps* $\mathfrak{b} : \mathbb{R} \oplus \mathscr{V} \to \mathscr{V}$, associated with orthogonal irreducible G-representations $\mathscr{V}$ (with non-trivial S^1-action), are the simplest homotopically non-trivial equivariant maps for which $G\text{-Deg}$ can be easily evaluated (cf. [1–5]). To be more specific, define $\mathscr{O} := \{(t, v) \in \mathbb{R} \oplus \mathscr{V} : -1 < t < 1, \|v\| < 2\}$ and $\mathfrak{b} : \overline{\mathscr{O}} \to \mathscr{V}$ by

$$\mathfrak{b}(t, v) := (1 - \|v\| + it) \cdot v, \quad (t, v) \in \mathbb{R} \oplus \mathscr{V},$$

and call the primary degree

$$\deg_{\mathscr{V}} := G\text{-Deg}\,(\mathfrak{b}, \mathscr{O})$$

the *basic degree* associated with the irreducible G-representation $\mathscr{V}$. The same notion can be applied to the case without free parameter. Namely, we call $\deg_{\mathscr{V}_o} := \Gamma\text{-Deg}\,(-\mathrm{Id}, \mathscr{B})$ (where $\mathscr{B} \subset \mathscr{V}_o$ is the unit ball) the *basic degree* associated with the irreducible Γ-representation $\mathscr{V}_o$.

(ii) In a standard way, the concept of the primary equivariant degree can be extended to admissible pairs (f, Ω) with $\Omega \subset \mathbb{R} \oplus W$ and $f : \mathbb{R} \oplus W \to W$ being a *completely continuous vector field* on the Banach G-representation $\mathbb{R} \oplus W$ (cf. [15,18]). We will use for it the same symbol.

3. Symmetric bifurcation in parameterized equivariant coincidence problems

The systems of our interest fall into a category of the so-called equivariant *coincidence* problems. Therefore, we start with a brief discussion of an abstract functional setting for the equivariant coincidence problem.

It should be pointed out that the abstract setting and language chosen in this paper go beyond the scope of the applications we are dealing with. This is motivated by the following reasons:

(a) *Setting*: In applied problems there are many kinds of (symmetric functional) parabolic differential equations considered, mainly from the computational point of view, with a little attention given to the proper settings in functional spaces (parameterized systems with fixed boundary, free boundary problems, reaction–diffusion with delay or evolving boundary conditions, to mention a few). To be able to treat potential applications in the same way as it is done in this paper, we need to enclose them in *one setting*, which is discussed in Section 4.3, and to describe a *general* procedure of converting parameterized coincidence problems into fixed-point problems (for which the equivariant degree methods are directly applied). Moreover, although in this paper we only study the local symmetric bifurcation phenomenon, our setting is well-prepared for studying the global symmetric bifurcations as well.

(b) *Language*: As a matter of fact, the linearization of a parameterized parabolic system usually amounts to a family of linear *unbounded* Fredholm operators depending continuously on a point in a parameter space, say, $\mathscr{P}$. In order to better search this continuity (as well as other properties which are important for the applicability of the degree

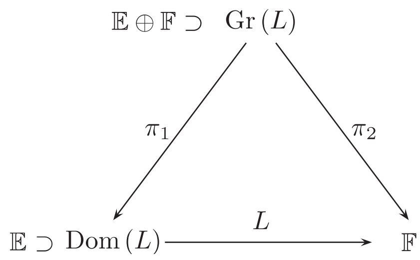

Fig. 1. Commutative diagram related to the graph of L.

methods), it is convenient to consider the family in question as a (locally trivial) vector bundle over $\mathscr{P}$ (with a vector fiber being the graph of the linear operator involved). It should be pointed out that the vector bundle structure is used to study the properties of the linear operators rather than topological properties of the involved spaces. Therefore, the reader should not be confused when we use the "bundle language" even in the case when $\mathscr{P}$ is contractible (meaning that the corresponding fiber bundle is trivial). In fact, the triviality of the related vector bundle can be translated as the equivalence of the corresponding parameterized system of parabolic equations to a system with a *single* elliptic operator (which is independent of the parameter), that provides no practical hints whatsoever for a treatment of a concrete system. On the other hand, (i) in the context relevant to our discussion, vector bundles appear in a very elementary connotation: one can think of them as vector spaces changing continuously; (ii) the use of a vector bundle structure is enough to construct the so-called equivariant *resolvent* for the considered family (a crucial step in converting the coincidence problem to a fixed-point problem), which, in turn, is much more simple than any other attempt to find a concrete trivialization of the bundle.

3.1. Functional setting for equivariant coincidence problems

Let $\mathbb{E}$ and $\mathbb{F}$ be real isometric Banach G-representations, where $G = \Gamma \times S^1$ with Γ being a finite group. Denote by $\mathrm{Op}^G := \mathrm{Op}^G(\mathbb{E} \oplus \mathbb{F})$ the set of all closed G-equivariant linear operators from $\mathbb{E}$ to $\mathbb{F}$. Clearly, for $L \in \mathrm{Op}^G$, the graph $Gr(L)$ is a closed invariant subspace of $\mathbb{E} \oplus \mathbb{F}$, where we assume that G acts diagonally on $\mathbb{E} \oplus \mathbb{F}$. The situation is illustrated in Fig. 1, where $\pi_1 : \mathbb{E} \oplus \mathbb{F} \to \mathbb{E}$ (resp. $\pi_2 : \mathbb{E} \oplus \mathbb{F} \to \mathbb{F}$) are (continuous) equivariant projections on $\mathbb{E}$ (resp. $\mathbb{F}$).

The space $\mathrm{Dom}(L)$ can be equipped with the *graph norm* $\|\cdot\|_L$ defined by

$$\|v\|_L := \|v\|_{\mathbb{E}} + \|Lv\|_{\mathbb{F}}, \tag{2}$$

where $\|\cdot\|_{\mathbb{E}}$ (resp. $\|\cdot\|_{\mathbb{F}}$) denotes the norm in $\mathbb{E}$ (resp. in $\mathbb{F}$), i.e. the norm $\|v\|_L$ is simply the norm of $\pi_1^{-1}(v)$ in $\mathbb{E} \oplus \mathbb{F}$. Notice that the space $\mathrm{Dom}(L)$ equipped with $\|\cdot\|_L$ is a Banach G-representation, and in what follows it will be denoted by $\mathbb{E}_L$. It is clear that $L : \mathbb{E}_L \to \mathbb{F}$ is a continuous equivariant operator.

Equip Op^G with the metric

$$\mathrm{dist}\,(L_1, L_2) = d(Gr(L_1), Gr(L_2)), \quad L_1,\ L_2 \in \mathrm{Op}^G,$$

where $d(\cdot,\cdot)$ is the Hausdorff metric on $\mathrm{Sub}(\mathbb{E} \oplus \mathbb{F})$ (see [18]). Let $\mathfrak{F}_0^G$ be the set of all closed G-equivariant Fredholm operators of index zero from $\mathbb{E}$ to $\mathbb{F}$. It can be verified that $\mathfrak{F}_0^G$ is an open subset of Op^G.

Remark and Notation 3.1. Consider a Fredholm operator $L : \mathrm{Dom}\,(L) \subset \mathbb{E} \to \mathbb{F}$ of index zero. A finite-dimensional map $K : \mathbb{E} \to \mathbb{F}$ is called a (finite-dimensional) *resolvent* of L if the map $L + K : \mathrm{Dom}\,(L) \to \mathbb{F}$ is one-to-one. We will denote by $\mathrm{FR}\,(L)$ the set of all resolvents of L. Since the operator $L : \mathbb{E}_L \to \mathbb{F}$ is continuous and $L + K : \mathbb{E}_L \to \mathbb{F}$ is continuous and one-to-one, by the open mapping theorem, and the fact that a compact linear perturbation of a (bounded) Fredholm operator does not change its index, we obtain that $L + K$ is surjective and $(L + K)^{-1} : \mathbb{F} \to \mathbb{E}_L$ is bounded. Consequently, by applying the natural inclusion $j : \mathbb{E}_L \hookrightarrow \mathbb{E}$, the inverse $(L + K)^{-1} : \mathbb{F} \to \mathbb{E}$ is a bounded operator. Moreover, if the natural inclusion is a compact operator, so is the inverse $(L + K)^{-1}$.

Throughout this section, we assume that $\mathscr{P}$ is a topological space equipped with the trivial G-action, and $\{L_\lambda\}_{\lambda\in\mathscr{P}} \subset \mathrm{Op}^G$ is a *continuous family of equivariant Fredholm operators of index zero, parameterized by* $\mathscr{P}$, i.e. $L_\lambda \in \mathfrak{F}_0^G$ for each $\lambda \in \mathscr{P}$, and the mapping $\eta : \mathscr{P} \to \mathrm{Op}^G$ defined by $\eta(\lambda) = L_\lambda$ for $\lambda \in \mathscr{P}$, is continuous. Define $\pi : \xi \to \mathscr{P}$ as follows:

$$\xi := \{(\lambda, u, y) \in \mathscr{P} \times (\mathbb{E} \oplus \mathbb{F}) : u \in \mathrm{Dom}(L_\lambda),\ \ y = L_\lambda u\},$$
$$\pi(\lambda, u, y) = \lambda \quad \text{for } (\lambda, u, y) \in \xi.$$

It has been shown in [10] that $\pi : \xi \to \mathscr{P}$ determines a locally trivial G-vector bundle.

Let

$$\mathscr{E} := \{(\lambda, u) \in \mathscr{P} \times \mathbb{E} \ : \ u \in \mathbb{E}_{L_\lambda}\} \tag{3}$$

and let $p_1 : \xi \to \mathscr{E}$ be given by $p_1(\lambda, u, y) = (\lambda, u)$, for $(\lambda, u, y) \in \xi$. Since, for every $\lambda \in \mathscr{P}$, the projection $pr_1 : Gr(L_\lambda) \to \mathbb{E}_\lambda := \mathbb{E}_{L_\lambda}$ is an equivariant isometry, the mapping $p_1 : \xi \to \mathscr{E}$ gives us the natural identification of the G-bundles ξ and $\mathscr{E}$ (we use the same symbol for a bundle and its total space).

Now, we can define the vector bundle morphism $L : \mathscr{E} \to \mathbb{F}$, where $\mathbb{F}$ is viewed as a bundle over a one-point space, by

$$L(\lambda, u) = L_\lambda u, \quad (\lambda, u) \in \mathscr{E}. \tag{4}$$

Definition 3.2. Let X be a subset of $\mathscr{P}$ and let L be given by (3) and (4). An *equivariant resolvent* of L over X is a G-vector bundle morphism $K : X \times \mathbb{E} \to \mathbb{F}$ such that

(i) for every $\lambda \in X$, $K_\lambda : \mathbb{E} \to \mathbb{F}$ is a finite-dimensional linear operator;
(ii) for every $\lambda \in X$, $L_\lambda + K_\lambda : \mathbb{E}_\lambda \to \mathbb{F}$ is an isomorphism.

Denote by $\mathscr{R}^G(L, X)$ the set of all equivariant resolvents of L over X.

Remark 3.3. (i) In contrast to the non-equivariant case, given $\lambda_0 \in \mathscr{P}$, one may have $\mathscr{R}^G(L, \{\lambda_0\}) = \emptyset$.
(ii) Also, in general it might happen that $\mathscr{R}^G(L, X) = \emptyset$, while $\mathscr{R}^G(L, \{\lambda_0\}) \neq \emptyset$ for each $\lambda_0 \in X$.

In the light of Remark 3.3, the following result turns out to be useful.

Lemma 3.4 (*cf. [17]*). *Let $X \subset \mathscr{P}$ be a compact contractible set containing a point λ^* such that $\mathscr{R}^G(L, \{\lambda^*\}) \neq \emptyset$. Then, $\mathscr{R}^G(L, X) \neq \emptyset$.*

Assume the following condition to be satisfied:

(H1) *There exists a compact subset $X \subset \mathscr{P}$ such that $\mathscr{R}^G(L, X) \neq \emptyset$.*

Fix $K \in \mathscr{R}^G(L, X)$ and put

$$R_\lambda := (L_\lambda + K_\lambda)^{-1} \tag{5}$$

(cf. Definition 3.2(ii)).

Given a completely continuous G-equivariant map $F : \mathscr{E} \to \mathbb{F}$, consider the associated *parameterized equivariant coincidence problem* (cf. [17]):

$$L_\lambda u = F(\lambda, u), \quad (\lambda, u) \in \mathscr{E}|_{X\times \mathrm{Dom}(L_\lambda)}. \tag{6}$$

Using the resolvent K one can reduce (6) to the following fixed-point problem:

$$y = \mathscr{F}(\lambda, y), \quad (\lambda, y) \in X \times \mathbb{F}, \tag{7}$$

where

$$\mathscr{F}(\lambda, y) = F(\lambda, R_\lambda y) + K_\lambda(R_\lambda y), \quad (\lambda, y) \in X \times \mathbb{F}.$$

By assumption (H1), X is compact, therefore, $\mathscr{F}$ is completely continuous.

3.2. Bifurcation invariant for the equivariant coincidence problem

Throughout this subsection, $\mathbb{E}$, $\mathbb{F}$ stand for Banach G-representations, $\mathscr{P} = \mathbb{R} \times \mathbb{R}_+$ and $\{L_\lambda\}_{\lambda \in \mathscr{P}}$ is a continuous family of G-equivariant Fredholm G-equivariant operators of index zero satisfying condition (H1). Fix $K \in \mathscr{R}^G(L, X)$ with R_λ, $\lambda \in \mathscr{P}$, defined by (5).

Keeping in mind the setting relevant to the functional parabolic parameterized system discussed in the next section, we will specify F from (6), assuming:

(H2) (i) *There exists another real isometric Banach G-representation $\widehat{\mathbb{E}}$ and an injective G-vector bundle morphism $J : \mathscr{E} \to \mathscr{P} \times \widehat{\mathbb{E}}$ such that $J_\lambda := J(\lambda, \cdot)$ is a compact linear operator for every $\lambda \in \mathscr{P}$.*

(ii) *There exists an equivariant C^1-map $\widehat{F} : \mathscr{P} \times \widehat{E} \to \mathbb{F}$.*

Put

$$F := \widehat{F} \circ J. \tag{8}$$

Obviously, F is G-equivariant and completely continuous.

Consider now the coincidence problem (6) with F defined by (8) (see also condition (H2)). Assume, in addition, that there exists a two-dimensional submanifold $M \subset \mathscr{P} \times \mathbb{E}^G$ (thought of as a "bifurcation surface") satisfying the following two conditions:

(H3) *The M is a subset of the solution set to* (6).

(H4) *if $(\lambda_0, u_0) \in M$, then there exist open neighborhoods U_{λ_0} of λ_0 in $\mathscr{P}$ and U_{u_0} of u_0 in $\mathbb{E}^G$, and a C^1-map $\chi : U_{\lambda_0} \to \mathbb{E}^G$ such that*

$$M \cap (U_{\lambda_0} \times U_{u_0}) = Gr(\chi).$$

Bearing in mind condition (H3), call every $(\lambda, u) \in M$ a *trivial solution* to (6). All the other solutions to (6) are called *non-trivial*. A point $(\lambda_0, u_0) \in M$ is called a *bifurcation point* if in each neighborhood of (λ_0, u_0) there exists a non-trivial solution to (6). In what follows we study the existence and multiplicity of branches of non-trivial solutions bifurcating from M, and classify their symmetries.

Remark 3.5. (i) It is clear that (λ, u) is a solution to system (6) if and only if (λ, y) is a solution to system (7), where $y = (L_\lambda + K_\lambda)u$. Moreover, the set of all trivial solutions to (7) can be represented as

$$\widetilde{M} := \{(\lambda, y) \in X \times \mathbb{F} : (\lambda, R_\lambda(y)) \in M\}.$$

Then, condition (H4) translates as:

(H4)′ *If $(\lambda_0, y_0) \in \widetilde{M}$, then there exist open neighborhoods U_{λ_0}* of *λ_0 in $\mathscr{P}$ and U_{y_0} of y_0 in $\mathbb{F}^G$ and a C^1*-map *$\widetilde{\chi} : U_{\lambda_0} \to \mathbb{F}^G$ such that*

$$\widetilde{M} \cap (U_{\lambda_0} \times U_{y_0}) = Gr(\widetilde{\chi}).$$

(ii) *Also*, $(\lambda_0, u_0) \in M$ *is a bifurcation point of* (6) *if and only if* $(\lambda_0, y_0) \in \widetilde{M}$ *is a bifurcation point of* (7), *where* $y_0 = (L_{\lambda_0} + K_{\lambda_0})u_0$.

Now, rewrite system (7) as

$$(\pi - \mathscr{F})(\lambda, y) = 0, \quad (\lambda, y) \in X \times \mathbb{F}, \tag{9}$$

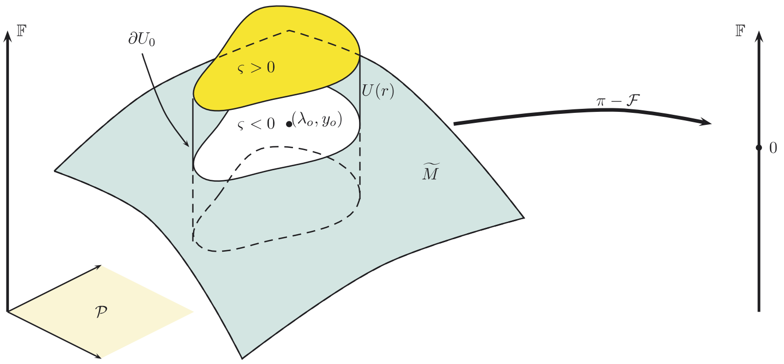

Fig. 2. Functional setting and auxiliary function for the bifurcation problem (9).

where π is the projection map on $\mathbb{F}$, $\pi(\lambda, y) = y$. Clearly, $\pi - \mathscr{F}$ is a G-equivariant completely continuous field of class C^1, and

$$D_y(\pi - \mathscr{F}) = \mathrm{Id} - (D_u F(\lambda, R_\lambda(y))R_\lambda + K_\lambda R_\lambda).$$

Thus, by assumption (H2) (see also (8)), $D_y(\pi - \mathscr{F})$ is a bounded Fredholm operator of index zero. For $(\lambda, y) \in X \times \mathbb{F}^G$, $D_y(\pi - \mathscr{F})(\lambda, y)$ is G-equivariant (so, in particular, it is also G-equivariant for $(\lambda, y) \in \widetilde{M}$). By implicit function theorem, a *necessary* condition for $(\lambda_0, y_0) \in \widetilde{M}$ to be a bifurcation point is that the derivative $D_y(\pi - \mathscr{F})(\lambda_0, y_0)$ is not an isomorphism of $\mathbb{F}$. Such a point $(\lambda_0, y_0) \in \widetilde{M}$ is called *L-singular*. An L-singular point (λ_0, y_0) is said to be *isolated*, if it is the only L-singular point in some neighborhood of (λ_0, y_0) in $\widetilde{M}$.

Finally, assume that:

(H5) *There exists an isolated L-singular point* $(\lambda_0, y_0) \in \widetilde{M}$.

We are now in a position to associate a local bifurcation invariant $\omega(\lambda_0, u_0) \in A_1(G)$ to system (6).

Take a neighborhood $\mathscr{D}_{\lambda_0}$ of (λ_0, y_0) in $\widetilde{M}$ such that (i) (λ_0, y_0) is the only L-singular point in $\mathscr{D}_{\lambda_0}$ and (ii) $\overline{\mathscr{D}_{\lambda_0}} \subset \widetilde{M} \cap (U_{\lambda_0} \times U_{y_0})$ (see (H4)$'$).

Choose a small number $r > 0$, define

$$U(r) := \{(\lambda, y) \in \mathscr{P} \times \mathbb{F} : (\lambda, \widetilde{\chi}(\lambda)) \in \mathscr{D}_{\lambda_0},\ \|y - \widetilde{\chi}(\lambda)\| < r\}, \tag{10}$$

and introduce a G-invariant *auxiliary* function $\varsigma : \overline{U(r)} \to \mathbb{R}$ satisfying the properties

$$\begin{cases} \varsigma(\lambda, y) > 0 & \text{if } \|y - \widetilde{\chi}(\lambda)\| = r, \\ \varsigma(\lambda, y) < 0 & \text{if } (\lambda, y) \in \mathscr{D}_{\lambda_0}. \end{cases} \tag{11}$$

Put

$$\partial U_0 := \{(\lambda, y) \in \overline{U(r)} : (\lambda, \widetilde{\chi}(\lambda)) \in \partial \mathscr{D}_{\lambda_0}\} \subset \partial U(r).$$

Then, by implicit function theorem, the above $r > 0$ can be chosen to be so small that

$$y - \mathscr{F}(\lambda, y) \neq 0 \quad \text{for } (\lambda, y) \in \partial U_0 \backslash \widetilde{M}$$

(see Fig. 2).

Define the map $\mathfrak{F}_\varsigma : \overline{U(r)} \to \mathbb{R} \oplus \mathbb{F}$ by

$$\mathfrak{F}_\varsigma(\lambda, y) := (\varsigma(\lambda, y), (\pi - \mathscr{F})(\lambda, y)), \tag{12}$$

which is clearly a $U(r)$-admissible G-equivariant completely continuous vector field. Therefore, the following *local bifurcation invariant*:

$$\omega(\lambda_0, u_0) := G\text{-Deg}\,(\mathfrak{F}_\varsigma, U(r)) \in A_1(G) \tag{13}$$

is well-defined.

Using the above scheme, one can prove the following G-symmetric (local) bifurcation result for the equivariant coincidence problem (6).

Theorem 3.6 (*Local bifurcation theorem*). *Suppose that assumptions* (H1)–(H5) *are satisfied*, $\omega(\lambda_0, u_0)$ *is given by* (13) (*with* $\mathfrak{F}_\varsigma$ *defined by* (12), $U(r)$ *by* (10) *and* ς *satisfying* (11)). *If*

$$\omega(\lambda_0, u_0) = \sum_{(H)} n_H\,(H) \neq 0,$$

i.e. there is $n_{H_0} \neq 0$ *for some orbit type* (H_0), *then there exists a branch of non-trivial solutions* (λ, u) *to the Eq.* (6) *bifurcating from* (λ_0, u_0) *such that* $G_u \supset H_0$.

Remark 3.7. Theorem 3.6 provides us with a *sufficient* condition for the occurrence of the (symmetric) bifurcation in (6). However, this result does not allow us to obtain reasonable *multiplicity* results. To overcome this obstacle, in what follows we will combine Theorem 3.6 with the concept of *dominating* orbit types (cf. Definition 4.4, Remark 4.5 and Theorem 4.6).

4. Hopf bifurcation for functional parabolic differential equations with symmetries

Let $V := \mathbb{R}^n$ be an orthogonal Γ-representation. Assume that $\Omega \subset \mathbb{R}^m =: V'$ is an open bounded set such that $\partial\Omega$ is C^2-smooth. Clearly, the space $L^2(\mathbb{R} \times \overline{\Omega}; V)$ is an isometric Banach Γ-representation with the Γ-action given by

$$(\gamma u)(t, x) = \gamma(u(t, x)), \quad \gamma \in \Gamma.$$

4.1. Statement of the problem

Consider a system of functional parabolic differential equations on $\mathbb{R} \times \overline{\Omega}$:

$$\begin{cases} \dfrac{\partial}{\partial t} u(t, x) + P(\alpha, x)u = f(\alpha, u_t)(x), & (t, x) \in \mathbb{R} \times \Omega, \\ B(\alpha, x)u(t, x) = 0, & (t, x) \in \mathbb{R} \times \partial\Omega, \end{cases} \tag{14}$$

where $u \in L^2(\mathbb{R} \times \overline{\Omega}; V)$ satisfies appropriate differentiability requirements,[5] $u_t(\theta, x) := u(t + \theta, x)$ for $\theta \in [-\tau, 0]$ ($\tau > 0$ is a fixed constant), $\alpha \in \mathbb{R}$ is a (bifurcation) parameter, $f : \mathbb{R} \times C([-\tau, 0]; L^2(\Omega; V)) \to L^2(\Omega; V)$ is a map of class C^1, which is bounded on bounded sets, $P(\alpha, x) = [P_i(\alpha, x)]_{i=1}^n$ is a vector with components being second-order uniformly elliptic operators, i.e.

$$P_i(\alpha, x) = \nabla^{\mathrm{T}} A_i(\alpha, x)\nabla + a_i(\alpha, x),$$

with $A_i(\alpha, x)$ being a continuously differentiable (with respect to α and x) $n \times n$ symmetric positive definite matrix satisfying the condition

$$\exists c_1, c_2 > 0 \;\; \forall(\alpha, x) \in \mathbb{R} \times \overline{\Omega} \;\; \forall y \in V' \quad c_1\|y\| \leqslant y^{\mathrm{T}} A_i(\alpha, x) y \leqslant c_2\|y\|,$$

where ∇ stands for the gradient operator, and $a_i(\alpha, x)$ is continuous. The boundary operator $B(\alpha, x)$ is defined by either (Dirichlet conditions)

$$B(\alpha, x)u(t, x) = u(t, x)$$

[5] The u is weakly differentiable with respect to t and has weak derivatives of order 2 with respect to $x \in \Omega$. More precisely, we assume here that u is an element of the Sobolev space $H^{1,2}(\mathbb{R} \times \Omega; V)$ of L^2-integrable V-valued functions from $\mathbb{R} \times \Omega$ with weak L^2-integrable derivative in $\mathbb{R}$ and weak L^2-integrable derivatives of order 2 in Ω.

or (mixed Dirichlet/Neumann conditions)

$$B(\alpha, x)u(t, x) = b(\alpha, x)u(t, x) + \frac{\partial}{\partial n}(\alpha, x)u(t, x),$$

where $b \in C^1(\mathbb{R} \times \partial\Omega; \mathbb{R})$, $(\partial/\partial n)(\alpha, x) = [\nu^{\mathrm{T}}(x)A_i(\alpha, x)\nabla]_{i=1}^n$ ($\nu(x)$ is the outward normal vector to $\partial\Omega$ at x).

Assume that:

(C1) The operators P, B and the map f are Γ-equivariant, i.e. for $\gamma \in \Gamma$,

$$\begin{aligned}
\gamma P(\alpha, x)u &= P(\alpha, x)\gamma u, \quad x \in \Omega, \quad \theta \in [-\tau, 0]\\
\gamma f(\alpha, v(\theta, x)) &= f(\alpha, \gamma v(\theta, x)), \quad x \in \Omega, \quad \theta \in [-\tau, 0]\\
\gamma B(\alpha, x)u &= B(\alpha, x)\gamma u, \quad x \in \partial\Omega.
\end{aligned}$$

In what follows, we use the standard identification $S^1 \simeq \mathbb{R}/2\pi\mathbb{Z}$ and introduce the following notation:

$$\mathscr{H}^{1,2}_{B(\alpha)} = \{\varphi \in H^{1,2}(S^1 \times \Omega; V) : B(\alpha, x)\varphi = 0\}, \tag{15}$$

where $H^{k,\ell}(S^1 \times \Omega; V)$ stands for the Sobolev space of V-valued functions with weak (L^2-integrable) derivatives of order k in S^1 and of order l in Ω. Put

$$\mathbb{E} = \mathbb{F} = L^2(S^1 \times \Omega; V), \quad \mathscr{P} = \mathbb{R} \times \mathbb{R}_+, \quad \widehat{\mathbb{E}} = C(S^1; L^2(\Omega; V)), \tag{16}$$

where $\widehat{\mathbb{E}}$ is equipped with the usual supremum norm.

4.2. Normalization of the period

We are looking for p-periodic solutions to system (14) for an unknown period $p > 0$. For convenience, we translate it into the equivalent 2π-periodic solution problem as follows.

Let $\beta := 2\pi/p$ and

$$v(t, x) = u\left(\frac{1}{\beta}\, t, x\right),$$

then the original periodic problem is reduced to finding non-trivial solutions (α, β, v) for the system

$$\begin{cases}
\dfrac{\partial}{\partial t}\, v(t, x) + \dfrac{1}{\beta}P(\alpha, x)v = \dfrac{1}{\beta}f(\alpha, v_{t,\beta})(x), & (t, x) \in \mathbb{R} \times \Omega,\\
B(\alpha, x)v(t, x) = 0, & (t, x) \in \mathbb{R} \times \partial\Omega,\\
v(t, x) = v(t + 2\pi, x), & (t, x) \in \mathbb{R} \times \Omega,
\end{cases} \tag{17}$$

where

$$v_{t,\beta}(\theta, x) := v(t + \beta\theta, x) \quad \text{for } (\theta, x) \in [-\tau, 0] \times \Omega.$$

4.3. The $\Gamma \times S^1$-setting in functional spaces

Below we reformulate system (17) as an equivariant parameterized coincidence problem based on the general discussion in Section 3.1.

For every $\lambda := (\alpha, \beta) \in \mathscr{P}$, define the subspace

$$\mathrm{Dom}(L_\lambda) := \{u \in \mathbb{E} : u \in \mathscr{H}^{1,2}_{B(\alpha)}\}$$

and the operator

$$L_\lambda : \ \mathrm{Dom}(L_\lambda) \subset \mathbb{E} \to \mathbb{E}$$

(cf. (16)) by

$$L_\lambda v(t,x) := \frac{\partial}{\partial t} v(t,x) + \frac{1}{\beta} P(\alpha, x) v$$

(cf. (15), (16) and (17)).

Notice that $\mathbb{E}$, $H^{1,2}(S^1 \times \Omega; V)$ and $\widehat{\mathbb{E}}$ are isometric Banach G-representations, where S^1 acts in a standard way by shifting the time argument t. It is also clear (see [20]) that each (unbounded) linear operator L_λ, for $\lambda \in \mathscr{P}$, is a closed G-equivariant Fredholm operator of index zero, and the orthogonal projection of the (finite-dimensional) kernel of L_λ is a G-equivariant resolvent K of L_λ. Therefore, $R^G(L, \{\lambda\}) \neq \emptyset$ for any $\lambda \in \mathscr{P}$, and, by Lemma 3.4, condition (H1) is satisfied for every compact subset $X \subset \mathscr{P}$.

On the other hand, since $(1/\beta) f(\alpha, v_{t,\beta}) \in L^2(\Omega; V)$ for $v_{t,\beta} \in C([-\tau, 0]; L^2(\Omega; V))$, we have the continuous map $N_f : \mathscr{P} \times \widehat{\mathbb{E}} \to L^2(\Omega, V)$ with

$$N_f(\alpha, \beta, v)(t) := \frac{1}{\beta} f(\alpha, v_{t,\beta}).$$

Define $\widehat{F} : \mathscr{P} \times \widehat{\mathbb{E}} \to \mathbb{F}$ by

$$\widehat{F}(\lambda, v)(t,x) := i \circ N_f(\alpha, \beta, v)(t)(x) = \frac{1}{\beta}\; f(\alpha, v_{t,\beta})(x), \quad \lambda = (\alpha, \beta),$$

where i denotes the natural embedding $\widehat{\mathbb{E}} \hookrightarrow \mathbb{F}$. The continuous differentiability of f implies that $\widehat{F}$ is continuously differentiable. Since the following composition of the embeddings.

$$H^{1,2}(S^1 \times \Omega; V) \hookrightarrow H^{2/3,0}(S^1 \times \Omega; V) \hookrightarrow C(S^1; L^2(\Omega; V)) = \widehat{\mathbb{E}}$$

is compact (cf. [20]), we have the following embedding:

$$J : \mathscr{E} \longrightarrow \mathscr{P} \times \widehat{\mathbb{E}},$$

where $J_\lambda : \mathbb{E}_\lambda \to \widehat{\mathbb{E}}$ is a compact operator for all $\lambda \in \mathscr{P}$. Thus, $\widehat{F}$ and J satisfy condition (H2) from Section 3. In particular, $F : \mathscr{E} \to \mathbb{F}$ defined by $F = \widehat{F} \circ J$ is a G-equivariant completely continuous map of class C^1.

As a consequence, we obtain that finding a periodic solution $v \in H^{1,2}(S^1 \times \Omega; V)$ for system (17) is equivalent to solving the following parameterized coincidence problem (cf. (6)):

$$L_\lambda v = F(\lambda, v), \quad \lambda \in X, \tag{18}$$

where X is a given compact subset of $\mathscr{P}$.

4.4. The Γ-symmetric steady-state solutions to (14)

In order to describe a manifold of trivial solutions to (14) (cf. conditions (H3) and (H4) from Section 3.2), at which we expect the occurrence of a Hopf bifurcation, observe that the functions $u(t,x)$ from $H^{1,2}(S^1 \times \Omega; V)$, constant with respect to the first variable, can be identified with functions from $H^2(\Omega; V)$, where $H^2(\Omega; V)$ denotes the Sobolev space of V-valued functions with weak (L^2-integrable) derivatives of order 2 in Ω. Clearly, for a function $u(t,x)$, constant with respect to t, we have $u_t(\theta, x) = u(\theta, x)$ for all $t \in \mathbb{R}$.

With these preliminaries on hands, introduce the following

Definition 4.1. Let (α_0, u_0) be a solution to (14) satisfying the following conditions:

(i) $u_0 \in H^2(\Omega, V)$;
(ii) $\gamma u_0 = u_0$ for all $\gamma \in \Gamma$;
(iii) $\begin{cases} P(\alpha_0, x) u_0 = f(\alpha_0, u_0)(x) & \text{in } \Omega, \\ B(\alpha_0, x) u_0 = 0 & \text{on } \partial\Omega. \end{cases}$

Then, (α_0, u_0) is called a Γ-*symmetric steady-state* solution to (14).

Let us introduce the following spaces:

$$\mathfrak{X}_{\alpha_0} := \{\omega \in H^2(\Omega; V) : B(\alpha_0, x)\omega = 0\},$$

$$\mathfrak{X}^{\mathrm{c}}_{\alpha_0} := \{\omega \in H^2(\Omega; V^{\mathrm{c}}) : B(\alpha_0, x)\omega = 0\},$$

$$\mathfrak{C} := C([-\tau, 0]; L^2(\Omega; V)),$$

$$\mathfrak{C}^{\mathrm{c}} := C([-\tau, 0]; L^2(\Omega; V^{\mathrm{c}})),$$

where V^{c} stands for the complexification of the Γ-representation V. Identify $L^2(\Omega; V)$ with the subspace of $\mathfrak{C}$ consisting of constant $L^2(\Omega; V)$-valued functions, and $L^2(\Omega; V^{\mathrm{c}})$ with the subspace of $\mathfrak{C}^{\mathrm{c}}$ consisting of constant $L^2(\Omega; V^{\mathrm{c}})$-valued functions. Denote by $\bar{f}(\alpha, \cdot)$ the restriction of $f(\alpha, \cdot)$ to $L^2(\Omega; V)$ and define

$$\mathscr{L}_{\alpha_0} := P(\alpha_0, x) - \mathfrak{d}f_{\alpha_0} : \mathfrak{X}_{\alpha_0} \subset L^2(\Omega; V) \to L^2(\Omega; V), \tag{19}$$

where $\mathfrak{d}f_{\alpha_0}(\varphi) := D_u\bar{f}(\alpha_0, u_0)\varphi$ for $\varphi \in \mathfrak{X}_{\alpha_0}$.

In what follows, we apply the same symbols to denote the complexified operators $P(\alpha, x)$, $\mathfrak{d}f_\alpha$, and $B(\alpha_0, x)$, i.e. the operators:

$$P(\alpha, x) : \mathfrak{X}^{\mathrm{c}}_\alpha \subseteq L^2(\Omega; V^{\mathrm{c}}) \to L^2(\Omega; V^{\mathrm{c}}),$$

$$\mathfrak{d}f_\alpha : \mathfrak{C}^{\mathrm{c}} \to L^2(\Omega; V^{\mathrm{c}}),$$

$$B(\alpha_0, x) : H^2(\Omega; V^{\mathrm{c}}) \to L^2(\partial\Omega, V^{\mathrm{c}}).$$

Suppose that (α_0, u_0) is a Γ-symmetric steady-state solution of (14) with $\alpha = \alpha_0$. We say that (α_0, u_0) is *non-singular* if $0 \notin \sigma(\mathscr{L}_{\alpha_0})$, where $\sigma(\mathscr{L}(\alpha_0))$ denotes the spectrum of $\mathscr{L}_{\alpha_0}$. Assume that:

(C2) The (α_0, u_0) is a non-singular Γ-symmetric steady-state solution.

Then, by implicit function theorem, there exists a continuously differentiable function $u(\alpha)$ for $\alpha \in (\alpha_0 - \eta, \alpha_0 + \eta)$ (for a sufficiently small $\eta > 0$) such that $(\alpha, u(\alpha))$ is a Γ-symmetric steady-state solution to (14) for each α (the condition $\gamma u(\alpha) = u(\alpha)$ for all $\gamma \in \Gamma$ (cf. Definition 4.1(ii)) is provided by the Γ-equivariance of $\mathscr{L}_\alpha$). In what follows, assume that $\{(\alpha, u(\alpha)) : \alpha \in (\alpha_0 - \eta, \alpha_0 + \eta)\}$ is a fixed family of steady-state Γ-symmetric solutions near (α_0, u_0). Since $(\alpha, \beta, u(\alpha))$, $\alpha \in (\alpha_0 - \eta, \alpha_0 + \eta)$ is clearly a solution to (14) belonging to $\mathscr{P} \times \mathbb{E}^G$, consider it as a *trivial solution*. Moreover, define the map $\chi : (\alpha_0 - \eta, \alpha_0 + \eta) \times \mathbb{R}_+ \to \mathbb{E}^G$ by $\chi(\alpha, \beta) = (\alpha, \beta, u(\alpha))$. The set of (non-singular) Γ-symmetric steady-state solutions to (14) (cf. condition (C2)), gives rise to a manifold $M \subset \mathscr{P} \times \mathbb{E}^G$, $M := \{(\alpha, \beta, u(\alpha)) : \alpha \in (\alpha_0 - \eta, \alpha_0 + \eta),\ \beta \in \mathbb{R}_+\}$ (defined locally satisfying conditions (H3) and (H4).

4.5. Characteristic equation

Let $(\alpha, u(\alpha))$ be a non-singular Γ-symmetric steady-state solution to (14). The linearization of (14) at $(\alpha, u(\alpha))$ leads to the following *characteristic equation* (near $(\alpha_0, u(\alpha_0))$):

$$\triangle_{\alpha;u(\alpha)}(\lambda)w := \lambda w + P(\alpha, x)w - \mathfrak{d}f_\alpha(\mathrm{e}^{\lambda\cdot}w) = 0, \quad \lambda \in \mathbb{C}, \tag{20}$$

where the *characteristic operator* $\triangle_{\alpha;u(\alpha)} : \mathfrak{X}^{\mathrm{c}}_\alpha \to L^2(\Omega; V^{\mathrm{c}})$ is defined (by (20)) using the standard complexifications of $P(\alpha, x)$ and $\mathfrak{d}f_\alpha(\varphi)$.

Observe that $\mathfrak{X}^{\mathrm{c}}_\alpha$ equipped with the H^2-norm is a complex Hilbert space such that the embedding $\mathfrak{X}^{\mathrm{c}}_\alpha \hookrightarrow L^2(\Omega; V^{\mathrm{c}})$ is compact, $P(\alpha, x)$ is an elliptic self-adjoint operator (thus, it is a bounded Fredholm operator of index zero from $\mathfrak{X}^{\mathrm{c}}_\alpha$ to $L^2(\Omega; V^{\mathrm{c}})$) and $\mathfrak{d}f_\alpha(\mathrm{e}^{\lambda\cdot}\cdot)$ is a bounded linear operator for all $\lambda \in \mathbb{C}$, therefore $\triangle_{\alpha;u(\alpha)}(\lambda) : \mathfrak{X}^{\mathrm{c}}_\alpha \to L^2(\Omega; V^{\mathrm{c}})$, where $\mathfrak{X}^{\mathrm{c}}_\alpha$ is equipped with the H^2-norm, is a bounded Fredholm operator of index zero.[6] Consequently, $\triangle_{\alpha;u(\alpha)}(\lambda)$ is a closed (unbounded) Fredholm operator of index zero from $L^2(\Omega; V^{\mathrm{c}})$ to itself.

[6] The $\triangle_{\alpha;u(\alpha)}(\lambda)$ is a sum of the compact operator $\lambda\,\mathrm{Id} - \mathfrak{d}f_\alpha(\mathrm{e}^{\lambda\cdot}\cdot) : \mathfrak{X}^{\mathrm{c}}_\alpha \to L^2(\Omega; V^{\mathrm{c}})$ and the Fredholm operator $P(\alpha, x)$ of index zero.

A number $\lambda \in \mathbb{C}$ is called a *characteristic root* of system (14) at a Γ-symmetric steady-state solution $(\alpha, u(\alpha))$ if $\ker \triangle_{\alpha;u(\alpha)}(\lambda) \neq \{0\}$. It is clear that a Γ-symmetric steady-state solution $(\alpha, u(\alpha))$ is non-singular if and only if 0 is not a characteristic root of (14) at $(\alpha, u(\alpha))$. We say that a non-singular Γ-symmetric steady-state solution (α_0, u_0), $u_0 = u(\alpha_0)$, is a *center* if it has a purely imaginary characteristic root $\mathrm{i}\beta_0$, $\beta_0 > 0$, i.e. $\ker \triangle_{\alpha_0;u_0}(\mathrm{i}\beta_0) \neq \{0\}$. A center (α_0, u_0) is called *isolated* if it is the only center in some neighborhood of (α_0, u_0) in $\mathbb{R} \oplus L^2(\Omega; V)$.

For the purpose of studying the local Hopf bifurcation problem for (14), we assume:

(C3) There exists a Γ-symmetric steady-state solution $(\alpha_0, u_0) \in \mathbb{R} \oplus L^2(\Omega; V)$ which is an isolated center such that $\mathrm{i}\beta_0$ (for $\beta_0 > 0$) is a characteristic root of (14) for $\alpha = \alpha_0$.

Thus, condition (H5) from Section 3.2 is satisfied. Also, notice that (C3) is the necessary condition for the occurrence of the Hopf bifurcation at (α_0, u_0), while (C2) excludes the appearance of the "steady-state" bifurcation.

Apply the "eigenspace" reduction to describe the characteristic roots of system (14) at the Γ-symmetric steady-state solution $(\alpha, u(\alpha))$. Denote by $\sigma_\alpha \subset \mathbb{R}$ the spectrum of the self-adjoint operator $P(\alpha, x) : \mathfrak{X}^{\mathrm{c}}_\alpha \subseteq L^2(\Omega; V^{\mathrm{c}}) \to L^2(\Omega; V^{\mathrm{c}})$. Since $P(\alpha, x)$ is a uniformly elliptic differential operator, the spectrum σ_α is discrete and all the eigenvalues $\mu^\alpha_{\mathfrak{k}}$ are real of finite multiplicity and such that

$$\mu^\alpha_0 < \mu^\alpha_1 < \cdots < \mu^\alpha_{\mathfrak{k}} < \cdots .$$

Using the fact that for every $r > 0$, the number $\mathrm{i}r$ is not in the spectrum σ_α of $P(\alpha, x)$, the (auxiliary) operator $S : L^2(\Omega; V^{\mathrm{c}}) \to L^2(\Omega; V^{\mathrm{c}})$ defined by

$$Sw = \mathrm{i}rw, \quad w \in L^2(\Omega; V^{\mathrm{c}})$$

is a Γ-equivariant resolvent (cf. Remark and Notation 3.1 and Definition 3.2) of $P(\alpha, x)$, i.e. the (bounded) inverse $\widetilde{R}_{\alpha,r} := [P(\alpha, x) + S]^{-1} : L^2(\Omega; V^{\mathrm{c}}) \to L^2(\Omega; V^{\mathrm{c}})$ exists for all $\alpha \in \mathbb{R}$ and is Γ-equivariant. On the other hand, since $P(\alpha, x) + S$ as a (bounded) operator from the space $\mathfrak{X}^{\mathrm{c}}_\alpha$ (equipped with the Sobolev H^2-norm) is also invertible and the embedding $\mathfrak{X}^{\mathrm{c}}_\alpha \hookrightarrow L^2(\Omega; V^{\mathrm{c}})$ is compact, we obtain that the inverse $\widetilde{R}_{\alpha,r} : L^2(\Omega; V^{\mathrm{c}}) \to L^2(\Omega; V^{\mathrm{c}})$ (i.e. $\widetilde{R}_{\alpha,r}$ is considered here as the inverse of the *unbounded* operator $P(\alpha, x) + S$ from $L^2(\Omega; V^{\mathrm{c}})$ into itself) is a compact Γ-equivariant operator. Eq. (20) can be rewritten as

$$\widetilde{\triangle}^r_{\alpha;u(\alpha)}(\lambda)w := w - \mathfrak{d}f_\alpha(\mathrm{e}^{\lambda\cdot}\widetilde{R}_{\alpha,r}(w)) + (\lambda - \mathrm{i}r)\widetilde{R}_{\alpha,r}(w) = 0. \tag{21}$$

It is clear that $\lambda \in \mathbb{C}$ is a characteristic root of system (14) at the steady-state solution $(\alpha, u(\alpha))$ if and only if $\ker \widetilde{\triangle}^r_{\alpha;u(\alpha)}(\lambda) \neq \{0\}$. Also, $\widetilde{\triangle}^r_{\alpha;u(\alpha)}(\lambda)$ is an analytic function in λ (cf. [26]), hence all the characteristic roots λ are isolated. Moreover, $\widetilde{\triangle}^r_{\alpha;u(\alpha)}(\lambda)$ is a Γ-equivariant completely continuous field, thus it is a bounded Γ-equivariant Fredholm operator of index zero.

Denote by $E^\alpha_{\mathfrak{k}} \subset L^2(\Omega; V^{\mathrm{c}})$ the eigenspace of $P(\alpha, x)$ corresponding to $\mu^\alpha_{\mathfrak{k}} \in \sigma_\alpha$, and let $\mathfrak{p}^\alpha_{\mathfrak{k}} : L^2(\Omega; V^{\mathrm{c}}) \to E^\alpha_{\mathfrak{k}}$ be the orthogonal projection. Consequently, for every $w \in L^2(\Omega; V^{\mathrm{c}})$, we have $w = \sum_{\mathfrak{k}=0}^\infty \mathfrak{p}^\alpha_{\mathfrak{k}}(w)$. By substituting $w = \sum_{\mathfrak{k}=0}^\infty \mathfrak{p}^\alpha_{\mathfrak{k}}(w)$ into (21), we obtain

$$\sum_{\mathfrak{k}=0}^{\infty}\left[\mathfrak{p}^\alpha_{\mathfrak{k}}(w) - \frac{1}{\mu^\alpha_{\mathfrak{k}} + \mathrm{i}r}\mathfrak{d}f_\alpha(\mathrm{e}^{\lambda\cdot}\mathfrak{p}^\alpha_{\mathfrak{k}}(w)) + \frac{\lambda - \mathrm{i}r}{\mu^\alpha_{\mathfrak{k}} + \mathrm{i}r}\mathfrak{p}^\alpha_{\mathfrak{k}}(w)\right] = 0. \tag{22}$$

Denote by $F^\alpha_{\mathfrak{k}}$ the subspace of $\mathfrak{C}$ spanned by functions of the type $t \to \varphi(t)w$, where $\varphi \in C([-\tau, 0]; \mathbb{C})$ and $w \in E^\alpha_{\mathfrak{k}}$. We need the following hypothesis (cf. [17,23]):

(C4) The $\mathfrak{d}f_\alpha(F^\alpha_{\mathfrak{k}}) \subset E^\alpha_{\mathfrak{k}}$ for all steady-state solutions $(\alpha, u(\alpha))$ and $\mathfrak{k} = 0, 1, 2, \ldots$.

Remark 4.2. Condition (C4) is required mainly to simplify the computation of the characteristic roots through a reduction to isotypical components of $L^2(\Omega; V^{\mathrm{c}})$ (see also [21,22]). One can check that the reaction–diffusion systems with delay of the type considered in [7–9] satisfy (C4). In the case of a parabolic system of Γ-symmetric PDEs (without delay) or the reaction–diffusion logistic equation with delay (cf. [14]), (C4) is automatically satisfied.

Under assumption (C4), Eq. (22) can be reduced to the following sequence of equations:

$$\mathfrak{p}_{\mathfrak{k}}^{\alpha}(w) - \frac{1}{\mu_{\mathfrak{k}}^{\alpha} + \mathrm{i}r}\mathfrak{d} f_{\alpha}(\mathrm{e}^{\lambda \cdot}\mathfrak{p}_{\mathfrak{k}}^{\alpha}(w)) + \frac{\lambda - \mathrm{i}r}{\mu_{\mathfrak{k}}^{\alpha} + \mathrm{i}r}\mathfrak{p}_{\mathfrak{k}}^{\alpha}(w) = 0, \quad \mathfrak{k} = 0, 1, \ldots \ . \tag{23}$$

Eq. (23) can be written in the equivalent form as

$$(\mu_{\mathfrak{k}}^{\alpha} + \lambda)\mathfrak{p}_{\mathfrak{k}}^{\alpha}(w) + \mathfrak{d} f_{\alpha}(\mathrm{e}^{\lambda \cdot}\rho_{\mathfrak{k}}^{\alpha}(w)) = 0, \quad \mathfrak{k} = 0, 1, \ldots \ . \tag{24}$$

4.6. Local bifurcation $\Gamma \times S^1$-invariant and its computation

Under assumptions (C1)–(C4), for any compact subset $X \subset \mathscr{P}$, system (14) leads to a parameterized equivariant coincidence problem of type (6) satisfying conditions (H1)–(H5) (cf. (18)). Hence, given an isolated center (α_0, u_0) with the corresponding characteristic root $\mathrm{i}\beta_0$ (cf. Condition (C3)), and following the scheme outlined in Section 3 (cf. (10)–(13)), one can associate to (α_0, β_0, u_0) the local bifurcation invariant $\omega(\lambda_0, u_0) = \omega(\alpha_0, \beta_0, u_0) \in A_1(\Gamma \times S^1)$.

In order to obtain an effective computational formula for $\omega(\alpha_0, \beta_0, u_0)$, we need to discuss the so-called *negative spectrum* and *crossing numbers*.

4.6.1. Negative spectrum

Consider the real (resp. complex) Γ-isotypical decomposition of V (resp. V^{c}):

$$V = V_0 \oplus V_1 \oplus \cdots \oplus V_r, \quad V^{\mathrm{c}} = U_0 \oplus U_1 \oplus \cdots \oplus U_s, \tag{25}$$

where the isotypical components V_j (resp. U_j) are modelled on the real (resp. complex) Γ-irreducible representations $\mathscr{V}_j$ (resp. $\mathscr{U}_j$). For technical reasons, we assume that these decompositions correspond to a complete list of the irreducible Γ-representations:

$$\text{real irreducible } \Gamma\text{-representations: } \mathscr{V}_0, \mathscr{V}_1, \ldots, \mathscr{V}_r, \tag{26}$$

$$\text{complex irreducible } \Gamma\text{-representations: } \mathscr{V}_0, \mathscr{V}_1, \ldots, \mathscr{V}_s, \tag{27}$$

where $\mathscr{V}_0$ and $\mathscr{U}_0$ denote the trivial Γ-representations. We do not exclude that some of these Γ-isotypical components are trivial. Notice that it is possible that there is a different number of isotypical components in the isotypical decompositions of V and V^{c} (cf. [6]). In the case of specific groups Γ, the lists (26) and (27), which are given in [1], will be used later by the Maple© routines to create the tables of results (cf. Section 6, Tables 1–6).

Clearly, (25) induces the Γ-isotypical decompositions

$$L^2(\Omega; V) = \bigoplus_{i=0}^{r}\mathfrak{V}_i, \quad L^2(\Omega; V^{\mathrm{c}}) = \bigoplus_{j=0}^{s}\mathfrak{U}_j, \tag{28}$$

where $\mathfrak{V}_i := L^2(\Omega; \mathscr{V}_i)$ and $\mathfrak{U}_j := L^2(\Omega; \mathscr{U}_j)$.

Table 1
Equivariant classification of the Hopf bifurcation with D_3 symmetries

${}^j\xi_0$	$\varepsilon_0, \varepsilon_1$	$\omega(\alpha_{v,m,k}, \beta_{v,m,k}, 0)_1$	# Branches
${}^0\xi_0$	00	$(-1)^v((D_3))$	1
${}^0\xi_0$	01	$(-1)^v((D_3) - (\mathbb{Z}_3))$	1
${}^0\xi_0$	10	$(-1)^{v+1}((D_3))$	1
${}^0\xi_0$	11	$(-1)^{v+1}((D_3) - (\mathbb{Z}_3))$	1
${}^1\xi_1$	00	$(-1)^v((\mathbb{Z}_3^z) + (D_1^z) + (D_1) - (\mathbb{Z}_1))$	6
${}^1\xi_1$	01	$(-1)^v((\mathbb{Z}_3^z) - (D_1^z) - (D_1) + (\mathbb{Z}_1))$	6
${}^1\xi_1$	10	$(-1)^{v+1}((\mathbb{Z}_3^z) + (D_1^z) + (D_1) - (\mathbb{Z}_1))$	6
${}^1\xi_1$	11	$(-1)^{v+1}((\mathbb{Z}_3^z) - (D_1^z) - (D_1) + (\mathbb{Z}_1))$	6

Table 2
Equivariant classification of the Hopf bifurcation with D_4 symmetries

${}^{j}\xi_0$	$\varepsilon_0, \varepsilon_1, \varepsilon_2$	$\omega(\alpha_{v,m,k}, \beta_{v,m,k}, 0)_1$	# Branches
${}^{0}\xi_0$	000	$(-1)^v((D_4))$	1
${}^{0}\xi_0$	001	$(-1)^v((D_4)-(D_2))$	1
${}^{0}\xi_0$	010	$(-1)^v((D_4)-(D_1)-(\widetilde{D}_1)+(\mathbb{Z}_1))$	1
${}^{0}\xi_0$	011	$(-1)^v((D_4)-(D_2)+(D_1)-(\widetilde{D}_1))$	1
${}^{0}\xi_0$	100	$(-1)^{v+1}((D_4))$	1
${}^{0}\xi_0$	101	$(-1)^{v+1}((D_4)-(D_2))$	1
${}^{0}\xi_0$	110	$(-1)^{v+1}((D_4)-(D_1)-(\widetilde{D}_1)+(\mathbb{Z}_1))$	1
${}^{0}\xi_0$	111	$(-1)^{v+1}((D_4)-(D_2)+(D_1)-(\widetilde{D}_1))$	1
${}^{1}\xi_1$	000	$(-1)^v((\mathbb{Z}_4^t)+(D_2^d)+(\widetilde{D}_2^d)-(\mathbb{Z}_2^-))$	6
${}^{1}\xi_1$	001	$(-1)^v((\mathbb{Z}_4^t)-(D_2^d)+(\widetilde{D}_2^d)-(\mathbb{Z}_2^-))$	6
${}^{1}\xi_1$	010	$(-1)^v((\mathbb{Z}_4^t)+(D_2^d)+(\widetilde{D}_2^d)-(\mathbb{Z}_2^-)-(D_1^z)-(\widetilde{D}_1^z)-(D_1)-(\widetilde{D}_1)+2(\mathbb{Z}_1))$	6
${}^{1}\xi_1$	011	$(-1)^v((\mathbb{Z}_4^t)-(D_2^d)-(\widetilde{D}_2^d)-(\mathbb{Z}_2^-)+(D_1^z)-(\widetilde{D}_1^z)+(D_1)-(\widetilde{D}_1))$	6
${}^{1}\xi_1$	100	$(-1)^{v+1}((\mathbb{Z}_4^t)+(D_2^d)+(\widetilde{D}_2^d)-(\mathbb{Z}_2^-))$	6
${}^{1}\xi_1$	101	$(-1)^{v+1}((\mathbb{Z}_4^t)-(D_2^d)+(\widetilde{D}_2^d)-(\mathbb{Z}_2^-))$	6
${}^{1}\xi_1$	110	$(-1)^{v+1}((\mathbb{Z}_4^t)+(D_2^d)+(\widetilde{D}_2^d)-(\mathbb{Z}_2^-)-(D_1^z)-(\widetilde{D}_1^z)-(D_1)-(\widetilde{D}_1)+2(\mathbb{Z}_1))$	6
${}^{1}\xi_1$	111	$(-1)^{v+1}((\mathbb{Z}_4^t)-(D_2^d)+(\widetilde{D}_2^d)-(\mathbb{Z}_2^-)+(D_1^z)-(\widetilde{D}_1^z)+(D_1)-(\widetilde{D}_1))$	6
${}^{3}\xi_2$	000	$(-1)^v((D_4^d))$	1
${}^{3}\xi_2$	001	$(-1)^v((D_4^d)-(D_2))$	2
${}^{3}\xi_2$	010	$(-1)^v((D_4^d)-(D_1^z)-(D_1)+(\mathbb{Z}_1))$	2
${}^{3}\xi_2$	011	$(-1)^v((D_4^d)-(D_2)-(\widetilde{D}_1^z)-(D_1))$	2
${}^{3}\xi_2$	100	$(-1)^{v+1}((D_4^d))$	2
${}^{3}\xi_2$	101	$(-1)^{v+1}((D_4^d)-(D_2))$	2
${}^{3}\xi_2$	110	$(-1)^{v+1}((D_4^d)-(\widetilde{D}_1^z)-(D_1)+(\mathbb{Z}_1))$	2
${}^{3}\xi_2$	111	$(-1)^{v+1}((D_4^d)-(D_2)-(D_1^z)+(D_1))$	2

Consider the operator $P(\alpha_0, x) : \mathfrak{X}_{\alpha_0} \subseteq L^2(\Omega; V) \to L^2(\Omega; V)$ and let K be the orthogonal projection on its kernel. Then, K is a Γ-equivariant resolvent of $P(\alpha_0, x)$. Put $\widetilde{R}_{\alpha_0} := [P(\alpha_0, x) + K]^{-1}$ and define

$$\mathscr{A} := \mathrm{Id} - \frac{1}{\beta_0}\widetilde{R}_{\alpha_0} \circ D_u \bar{f}(\alpha_0, u_0) - \widetilde{R}_{\alpha_0} K : L^2(\Omega; V) \to L^2(\Omega; V), \tag{29}$$

which is clearly a completely continuous field. Put $\mathscr{A}^i := \mathscr{A}_{|\mathfrak{V}_i} : \mathfrak{V}_i \to \mathfrak{V}_i$ and let $\sigma_-(\mathscr{A})$ denote the set of all negative eigenvalues of the operator $\mathscr{A}$. Since $\mathscr{A}$ is a completely continuous field, thus it is a Fredholm operator of index zero, the set $\sigma_-(\mathscr{A})$ is finite and all the eigenvalues in $\sigma_-(\mathscr{A})$ are of finite multiplicity. Thus, for $\mu \in \sigma_-(\mathscr{A})$, define

$$E(\mu) := \bigcup_{k=1}^{\infty} \ker[\mathscr{A} - \mu\,\mathrm{Id}]^k,$$

$$E_i(\mu) := \bigcup_{k=1}^{\infty} \ker[\mathscr{A}^i - \mu\,\mathrm{Id}_{|\mathfrak{V}_i}]^k,$$

$$m_i(\mu) := \dim\, E_i(\mu)/\dim\, \mathscr{V}_i, \tag{30}$$

where the subspace $E(\mu)$ (resp. $E_i(\mu)$) is referred to as a *generalized* (resp. $\mathscr{V}_i$*-isotypical*) *eigenspace* of the operator $\mathscr{A}$ and the integer $m_i(\mu)$ will be called the $\mathscr{V}_i$*-multiplicity* of μ.

In all the examples considered in Section 5 the following condition is satisfied.

Table 3
Equivariant classification of the Hopf bifurcation with D_5 symmetries

${}^j\xi_0$	$\varepsilon_0, \varepsilon_1, \varepsilon_2$	$\omega(\alpha_{v,m,k}, \beta_{v,m,k}, 0)_1$	# Branches
${}^0\xi_0$	010	$(-1)^v((D_5) - 2(D_1) + (\mathbb{Z}_1))$	1
${}^0\xi_0$	011	$(-1)^v((D_5))$	1
${}^0\xi_0$	101	$(-1)^{v+1}((D_5) - 2(D_1) + (\mathbb{Z}_1))$	1
${}^0\xi_0$	110	$(-1)^{v+1}((D_5) - 2(D_1) + (\mathbb{Z}_1))$	1
${}^0\xi_0$	111	$(-1)^{v+1}((D_5))$	1
${}^1\xi_1$	010	$(-1)^v((\mathbb{Z}_5^{t_1}) - (D_1^z) - (D_1) + (\mathbb{Z}_1))$	8
${}^1\xi_1$	011	$(-1)^v((\mathbb{Z}_5^{t_1}) + (D_1^z) + (D_1) - (\mathbb{Z}_1))$	8
${}^1\xi_1$	100	$(-1)^{v+1}((\mathbb{Z}_5^{t_1}) + (D_1^z) + (D_1) - (\mathbb{Z}_1))$	8
${}^1\xi_1$	101	$(-1)^{v+1}((\mathbb{Z}_5^{t_1}) - (D_1^z) - (D_1) + (\mathbb{Z}_1))$	8
${}^1\xi_1$	111	$(-1)^{v+1}((\mathbb{Z}_5^{t_1}) + (D_1^z) + (D_1) - (\mathbb{Z}_1))$	8
${}^2\xi_2$	010	$(-1)^v((\mathbb{Z}_5^{t_2}) - (D_1^z) - (D_1) + (\mathbb{Z}_1))$	8
${}^2\xi_2$	011	$(-1)^v((\mathbb{Z}_5^{t_2}) + (D_1^z) + (D_1) - (\mathbb{Z}_1))$	8
${}^2\xi_2$	100	$(-1)^{v+1}((\mathbb{Z}_5^{t_2}) + (D_1^z) + (D_1) - (\mathbb{Z}_1))$	8
${}^2\xi_2$	101	$(-1)^{v+1}((\mathbb{Z}_5^{t_2}) - (D_1^z) - (D_1) + (\mathbb{Z}_1))$	8
${}^2\xi_2$	110	$(-1)^{v+1}((\mathbb{Z}_5^{t_2}) - (D_1^z) - (D_1) + (\mathbb{Z}_1))$	8
${}^2\xi_2$	111	$(-1)^{v+1}((\mathbb{Z}_5^{t_2}) + (D_1^z) + (D_1) - (\mathbb{Z}_1))$	8

Table 4
Equivariant classification of the Hopf bifurcation with A_4 symmetries

${}^j\xi_0$	$\varepsilon_0, \varepsilon_3$	$\omega(\alpha_{v,m,k}, \beta_{v,m,k}, 0)_1$	# Branches
${}^0\xi_0$	00	$(-1)^v((A_4))$	1
${}^0\xi_0$	01	$(-1)^v((A_4) - 2(\mathbb{Z}_3) - (\mathbb{Z}_2) + (\mathbb{Z}_1))$	1
${}^0\xi_0$	10	$(-1)^{v+1}((A_4))$	1
${}^0\xi_0$	11	$(-1)^{v+1}((A_4) - 2(\mathbb{Z}_3) - (\mathbb{Z}_2) + (\mathbb{Z}_1))$	1
${}^3\xi_1$	00	$(-1)^v((V_4^-) + (\mathbb{Z}_3^{t_1}) + (\mathbb{Z}_3^{t_2}) + (\mathbb{Z}_3) - (\mathbb{Z}_1))$	12
${}^3\xi_1$	01	$(-1)^v((V_4^-) - (\mathbb{Z}_3^{t_1}) - (\mathbb{Z}_3^{t_2}) - (\mathbb{Z}_3) - (\mathbb{Z}_2^-) - (\mathbb{Z}_2) + 2(\mathbb{Z}_1))$	12
${}^3\xi_1$	10	$(-1)^{v+1}((V_4^-) + (\mathbb{Z}_3^{t_1}) + (\mathbb{Z}_3^{t_2}) + (\mathbb{Z}_3) - (\mathbb{Z}_1))$	12
${}^3\xi_1$	11	$(-1)^{v+1}((V_4^-) - (\mathbb{Z}_3^{t_1}) - (\mathbb{Z}_3^{t_2}) - (\mathbb{Z}_3) - (\mathbb{Z}_2^-) - (\mathbb{Z}_2) + 2(\mathbb{Z}_1))$	12

Condition (R): (i) Decomposition (25) contains isotypical components modelled only on irreducible representations of *real* type (in particular, $r = s$).

(ii) For each $\mu \in \sigma_-(\mathscr{A})$, there exists a *single* isotypical component $\mathfrak{V}_i := \mathfrak{V}_{i_\mu}$ in (28) which (completely) contains the generalized eigenspace $E(\mu)$.

Therefore, formula (30) of the $\mathscr{V}_i$-multiplicity $m_i(\mu)$ reduces to

$$m_i(\mu) = \begin{cases} \dim\ E(\mu)/\dim\ \mathscr{V}_i, & i = i_\mu, \\ 0, & i \neq i_\mu. \end{cases} \tag{31}$$

4.6.2. Crossing numbers

Put $\widetilde{\triangle}^r_{\alpha;u(\alpha),j}(\lambda) := \widetilde{\triangle}^r_{\alpha;u(\alpha)}(\lambda)|_{\mathfrak{U}_j}$ (cf. (21) and (25)). For a characteristic root λ of system (14) at the Γ-symmetric steady-state solution (α_0, u_0), we use the following notations:

$$E_j(\lambda) := \bigcup_{k=1}^{\infty} \ker[\widetilde{\triangle}^r_{\alpha;u(\alpha),j}(\lambda)]^k,$$

$$m_j(\lambda) := \dim\ E_j(\lambda)/\dim\ \mathscr{U}_j, \tag{32}$$

Table 5
Equivariant classification of the Hopf bifurcation with S_4 symmetries

${}^{j}\xi_0$	$\varepsilon_0,\varepsilon_1,\varepsilon_3,\varepsilon_4$	$\omega(\alpha_{v,m,k},\beta_{v,m,k},0)_1$	# Branches
${}^{0}\xi_0$	0011	$(-1)^v((S_4)-2(D_3)-(D_2)-(\mathbb{Z}_4)+(\mathbb{Z}_3)+2(D_1)+(\mathbb{Z}_2)-(\mathbb{Z}_1))$	1
${}^{0}\xi_0$	0100	$(-1)^v((S_4)-2(A_4))$	1
${}^{0}\xi_0$	0101	$(-1)^v((S_4)-(A_4)-(\mathbb{Z}_4)+(\mathbb{Z}_3)-(D_1)+(\mathbb{Z}_1))$	1
${}^{0}\xi_0$	0110	$(-1)^v((S_4)-(A_4)-2(D_3)-(D_2)+2(\mathbb{Z}_3)+3(D_1)+(\mathbb{Z}_2)-2(\mathbb{Z}_1))$	1
${}^{0}\xi_0$	0111	$(-1)^v((S_4)-(A_4)-2(D_3)-(D_2)-(\mathbb{Z}_4)+(\mathbb{Z}_3)+2(D_1)+(\mathbb{Z}_2)-(\mathbb{Z}_1))$	1
${}^{0}\xi_0$	1011	$(-1)^{v+1}((S_4)-2(D_3)-(D_2)-(\mathbb{Z}_4)+(\mathbb{Z}_3)+2(D_1)+(\mathbb{Z}_2)-(\mathbb{Z}_1))$	1
${}^{0}\xi_0$	1100	$(-1)^{v+1}((S_4)-(A_4))$	1
${}^{0}\xi_0$	1101	$(-1)^{v+1}((S_4)-(A_4)-(\mathbb{Z}_4)+(\mathbb{Z}_3)-(D_1)+(\mathbb{Z}_2))$	1
${}^{0}\xi_0$	1110	$(-1)^{v+1}((S_4)-(A_4)-2(D_3)-(D_2)+2(\mathbb{Z}_3)+3(D_1)+(\mathbb{Z}_2)-2(\mathbb{Z}_1))$	1
${}^{0}\xi_0$	1111	$(-1)^{v+1}((S_4)-(A_4)-2(D_3)-(D_2)-(\mathbb{Z}_4)+(\mathbb{Z}_3)+2(D_1)+(\mathbb{Z}_2)-(\mathbb{Z}_1))$	1
${}^{1}\xi_1$	0100	$(-1)^v((S_4^-)-(A_4))$	1
${}^{1}\xi_1$	0101	$(-1)^v((S_4^-)-(A_4)-(\mathbb{Z}_4^-)+(\mathbb{Z}_3)-(D_1^z)+(\mathbb{Z}_2))$	1
${}^{1}\xi_1$	0110	$(-1)^v((S_4^-)-(A_4)-2(D_3^z)-(D_2^z)+2(\mathbb{Z}_3)+3(D_1^z)+(\mathbb{Z}_2)-2(\mathbb{Z}_1))$	1
${}^{1}\xi_1$	0111	$(-1)^v((S_4^-)-(A_4)-2(D_3^z)-(D_2^z)-(\mathbb{Z}_4^-)+(\mathbb{Z}_3)+2(D_1^z)+(\mathbb{Z}_2)-(\mathbb{Z}_1))$	1
${}^{1}\xi_1$	1010	$(-1)^{v+1}((S_4^-)-2(D_3^z)-(D_2^z)+3(D_1^z)-(\mathbb{Z}_1))$	1
${}^{1}\xi_1$	1011	$(-1)^{v+1}((S_4^-)-2(D_3^z)-(D_2^z)-(\mathbb{Z}_4^-)+(\mathbb{Z}_3)+2(D_1^z)+(\mathbb{Z}_2)-(\mathbb{Z}_1))$	1
${}^{1}\xi_1$	1100	$(-1)^{v+1}((S_4^-)-(A_4))$	1
${}^{1}\xi_1$	1101	$(-1)^{v+1}((S_4^-)-(A_4)-(\mathbb{Z}_4^-)+(\mathbb{Z}_3)-(D_1^z)+(\mathbb{Z}_2))$	1
${}^{1}\xi_1$	1110	$(-1)^{v+1}((S_4^-)-(A_4)-2(D_3^z)-(D_2^z)+2(\mathbb{Z}_3)+3(D_1^z)+(\mathbb{Z}_2)-2(\mathbb{Z}_1))$	1
${}^{1}\xi_1$	1111	$(-1)^{v+1}((S_4^-)-(A_4)-2(D_3^z)-(D_2^z)-(\mathbb{Z}_4^-)+(\mathbb{Z}_3)+2(D_1^z)+(\mathbb{Z}_2)-(\mathbb{Z}_1))$	1
${}^{3}\xi_2$	0100	$(-1)^v((D_4^d)+(D_3)+(D_2^d)+(\mathbb{Z}_4^c)-(V_4^-)-(\mathbb{Z}_3^t)-(\mathbb{Z}_3)-(D_1)-(\mathbb{Z}_2^-)+(\mathbb{Z}_1))$	24
${}^{3}\xi_2$	0101	$(-1)^v((D_4^d)+(D_3)+(D_2^d)-(\mathbb{Z}_4^c)-(\mathbb{Z}_4^-)-(V_4^-)+(\mathbb{Z}_3^t)-(D_1^z)-3(D_1)+(\mathbb{Z}_2^-)+(\mathbb{Z}_2)+(\mathbb{Z}_1))$	24
${}^{3}\xi_2$	0110	$(-1)^v((D_4^d)-(D_3)-(D_2^d)-(D_2)+(\mathbb{Z}_4^c)-(V_4^-)+(\mathbb{Z}_3^t)+(\mathbb{Z}_3)+(D_1^z)+3(D_1)+(\mathbb{Z}_2^-)+(\mathbb{Z}_2)-3(\mathbb{Z}_1))$	24
${}^{3}\xi_2$	0111	$(-1)^v((D_4^d)-(D_3)-(D_2^d)-(D_2)-(\mathbb{Z}_4^c)-(\mathbb{Z}_4^-)-(V_4^-)-(\mathbb{Z}_3^t)+(D_1)+(\mathbb{Z}_2^-)+(\mathbb{Z}_2))$	24
${}^{3}\xi_2$	1011	$(-1)^v(-(D_4^d)+(D_3)+(D_2^d)+(D_2)+(\mathbb{Z}_4^c)+(\mathbb{Z}_4^-)-(\mathbb{Z}_3^t)-(\mathbb{Z}_3)-(D_1)-(\mathbb{Z}_2^-)-(\mathbb{Z}_2)+(\mathbb{Z}_1))$	24
${}^{3}\xi_2$	1100	$(-1)^v(-(D_4^d)-(D_3)-(D_2^d)-(\mathbb{Z}_4^c)+(\mathbb{Z}_4^-)+(\mathbb{Z}_3^t)+(\mathbb{Z}_3)+(D_1)+(\mathbb{Z}_2^-)-(\mathbb{Z}_1))$	24
${}^{3}\xi_2$	1101	$(-1)^v(-(D_4^d)-(D_3)-(D_2^d)+(\mathbb{Z}_4^c)+(\mathbb{Z}_4^-)+(V_4^-)-(\mathbb{Z}_3^t)+(D_1^z)+3(D_1)-(\mathbb{Z}_2^-)-(\mathbb{Z}_2)-(\mathbb{Z}_1))$	24
${}^{3}\xi_2$	1110	$(-1)^v(-(D_4^d)+(D_3)+(D_2^d)+(D_2)-(\mathbb{Z}_4^c)+(V_4^-)-(\mathbb{Z}_3^t)-(\mathbb{Z}_3)-(D_1^z)-3(D_1)-(\mathbb{Z}_2^-)-(\mathbb{Z}_2)+3(\mathbb{Z}_1))$	24
${}^{3}\xi_2$	1111	$(-1)^v(-(D_4^d)+(D_3)+(D_2^d)+(\mathbb{Z}_4^c)+(\mathbb{Z}_4^-)+(V_4^-)+(\mathbb{Z}_3^t)-(D_1)-(\mathbb{Z}_2^-)-(\mathbb{Z}_2))$	24
${}^{4}\xi_3$	0011	$(-1)^v((D_4^z)-(D_3^z)-(D_2^d)-(D_2^z)-(\mathbb{Z}_4^c)-(\mathbb{Z}_4)+(\mathbb{Z}_3^t)+(\mathbb{Z}_3)+(D_1^z)+(\mathbb{Z}_2^-)+(\mathbb{Z}_2)-(\mathbb{Z}_1))$	24
${}^{4}\xi_3$	0100	$(-1)^v((D_4^z)+(D_3^z)+(D_2^d)+(\mathbb{Z}_4^c)-(V_4^-)-(\mathbb{Z}_3^t)-(\mathbb{Z}_3)-(D_1^z)-(\mathbb{Z}_2^-)+(\mathbb{Z}_1))$	24
${}^{4}\xi_3$	0110	$(-1)^v((D_4^z)-(D_3^z)-(D_2^d)-(D_2^z)+(\mathbb{Z}_4^c)-(V_4^-)+(\mathbb{Z}_3^t)+(\mathbb{Z}_3)+3(D_1^z)+(D_1)+(\mathbb{Z}_2^-)+(\mathbb{Z}_2)-3(\mathbb{Z}_1))$	24
${}^{4}\xi_3$	0111	$(-1)^v((D_4^z)-(D_3^z)-(D_2^d)-(D_2^z)-(\mathbb{Z}_4^c)-(\mathbb{Z}_4)-(V_4^-)-(\mathbb{Z}_3^t)+(D_1^z)+(\mathbb{Z}_2^-)+(\mathbb{Z}_2))$	24
${}^{4}\xi_3$	1010	$(-1)^v(-(D_4^z)+(D_3^z)+(D_2^d)+(D_2^z)-(\mathbb{Z}_4^c)+(\mathbb{Z}_3^t)-3(D_1^z)-(D_1)+(\mathbb{Z}_2^-)+(\mathbb{Z}_1))$	24
${}^{4}\xi_3$	1011	$(-1)^v(-(D_4^z)+(D_3^z)+(D_2^d)+(D_2^z)+(\mathbb{Z}_4^c)+(\mathbb{Z}_4)-(\mathbb{Z}_3^t)-(\mathbb{Z}_3)-(D_1^z)-(\mathbb{Z}_2^-)-(\mathbb{Z}_2)+(\mathbb{Z}_1))$	24
${}^{4}\xi_3$	1100	$(-1)^v(-(D_4^z)-(D_3^z)-(D_2^d)-(\mathbb{Z}_4^c)+(V_4^-)+(\mathbb{Z}_3^t)+(\mathbb{Z}_3)+(D_1^z)+(\mathbb{Z}_2^-)-(\mathbb{Z}_1))$	24
${}^{4}\xi_3$	1110	$(-1)^v(-(D_4^z)+(D_3^z)+(D_2^d)+(D_2^z)-(\mathbb{Z}_4^c)+(V_4^-)-(\mathbb{Z}_3^t)-(\mathbb{Z}_3)-3(D_1^z)-(D_1)-(\mathbb{Z}_2^-)-(\mathbb{Z}_2)+3(\mathbb{Z}_1))$	24
${}^{4}\xi_3$	1111	$(-1)^v(-(D_4^z)+(D_3^z)+(D_2^d)+(D_2^z)+(\mathbb{Z}_4^c)+(\mathbb{Z}_4)+(V_4^-)+(\mathbb{Z}_3^t)-(D_1^z)-(\mathbb{Z}_2^-)-(\mathbb{Z}_2))$	24

where the subspace $E_j(\lambda)$ is referred to as a *generalized kernel* of the operator $\widetilde{\triangle}^r_{\alpha;u(\alpha),j}(\lambda)$ and the integer $m_j(\lambda)$ will be called the $\mathscr{U}_j$-*multiplicity* of the characteristic root λ. Notice that, since $\widetilde{\triangle}^r_{\alpha;u(\alpha),j}(\lambda)$ is a Fredholm operator of index zero, $m_j(\lambda)<\infty$ for all characteristic roots λ.

Let $(\alpha_0,u_0)\in\mathbb{R}\oplus L^2(\Omega;V)$ be an isolated center with $\mathrm{i}\beta_0$ ($\beta_0>0$) being the corresponding characteristic root as described in condition (C3) from Section 4.5. Define the set

$$\mathscr{S}=\{\tau+\mathrm{i}\beta:0<\tau<\delta,\ |\beta-\beta_0|<\varepsilon\}\subset\mathbb{C},$$

Table 6
Equivariant classification of the Hopf bifurcation with A_5 symmetries

${}^j\xi_0$	$\varepsilon_0, \varepsilon_1, \varepsilon_2, \varepsilon_3$	$\omega(\alpha_{v,m,k}, \beta_{v,m,k}, 0)_1$	# Branches
${}^0\xi_0$	0000	$(-1)^v(A_5)$	1
${}^0\xi_0$	0001	$(-1)^v((A_5)-(\mathbb{Z}_5)-(\mathbb{Z}_3)-(\mathbb{Z}_2)+(\mathbb{Z}_1))$	1
${}^0\xi_0$	0010	$(-1)^v((A_5)-2(D_5)-2(D_3)+3(\mathbb{Z}_2)-(\mathbb{Z}_1))$	1
${}^0\xi_0$	0011	$(-1)^v((A_5)-2(D_5)-2(D_3)+(\mathbb{Z}_5)+(\mathbb{Z}_3)+4(\mathbb{Z}_2)-2(\mathbb{Z}_1))$	1
${}^0\xi_0$	0101	$(-1)^v((A_5)-2(A_4)-2(D_3)-(\mathbb{Z}_5)+2(\mathbb{Z}_3)+2(\mathbb{Z}_2)-(\mathbb{Z}_1))$	1
${}^0\xi_0$	1000	$(-1)^{v+1}(A_5)$	1
${}^0\xi_0$	1001	$(-1)^v(-(A_5)+(\mathbb{Z}_5)+(\mathbb{Z}_3)+(\mathbb{Z}_2)-(\mathbb{Z}_1))$	1
${}^0\xi_0$	1010	$(-1)^v(-(A_5)+2(D_5)+2(D_3)-3(\mathbb{Z}_2)+(\mathbb{Z}_1))$	1
${}^0\xi_0$	1011	$(-1)^v(-(A_5)+2(D_5)+2(D_3)-(\mathbb{Z}_5)-(\mathbb{Z}_3)-4(\mathbb{Z}_2)+2(\mathbb{Z}_1))$	1
${}^0\xi_0$	1111	$(-1)^v(-(A_5)+2(A_4)+2(D_5)-(\mathbb{Z}_5)-2(\mathbb{Z}_3)-3(\mathbb{Z}_2)+2(\mathbb{Z}_1))$	1
${}^0\xi_0$	0000	$(-1)^v((A_4)+(D_3^z)+(D_3)+(\mathbb{Z}_5^{t_1})+(\mathbb{Z}_5^{t_2})+(V_4^-)+(\mathbb{Z}_3^t)-(\mathbb{Z}_3)-(\mathbb{Z}_2^-)-(\mathbb{Z}_2))$	55
${}^1\xi_0$	0001	$(-1)^v((A_4)+(D_3^z)+(D_3)-(\mathbb{Z}_5^{t_1})-(\mathbb{Z}_5^{t_2})+(V_4^-)-(\mathbb{Z}_3^t)-3(\mathbb{Z}_3)$ $-3(\mathbb{Z}_2^-)-3(\mathbb{Z}_2)+4(\mathbb{Z}_1))$	55
${}^1\xi_0$	0010	$(-1)^v((A_4)-(D_3^z)-(D_3)-(\mathbb{Z}_5^{t_1})-(\mathbb{Z}_5^{t_2})+(V_4^-)-(\mathbb{Z}_3^t)-(\mathbb{Z}_3)-(\mathbb{Z}_2^-)-(\mathbb{Z}_2)+2(\mathbb{Z}_1))$	55
${}^1\xi_0$	0011	$(-1)^v((A_4)-(D_3^z)-(D_3)+(\mathbb{Z}_5^{t_1})+(\mathbb{Z}_5^{t_2})+(V_4^-)+(\mathbb{Z}_3^t)+(\mathbb{Z}_3)$ $+(\mathbb{Z}_2^-)+(\mathbb{Z}_2)-2(\mathbb{Z}_1))$	55
${}^1\xi_0$	0101	$(-1)^v(-(A_4)-(D_3^z)-(D_3)-(\mathbb{Z}_5^{t_1})-(\mathbb{Z}_5^{t_2})-(V_4^-)-(\mathbb{Z}_3^t)+(\mathbb{Z}_3)+(\mathbb{Z}_2^-)+(\mathbb{Z}_2))$	55
${}^1\xi_0$	1000	$(-1)^v(-(A_4)-(D_3^z)-(D_3)-(\mathbb{Z}_5^{t_1})-(\mathbb{Z}_5^{t_2})-(V_4^-)-(\mathbb{Z}_3^t)+(\mathbb{Z}_3)$ $+(\mathbb{Z}_2^-)+(\mathbb{Z}_2))$	55
${}^1\xi_0$	1001	$(-1)^v(-(A_4)-(D_3^z)-(D_3)+(\mathbb{Z}_5^{t_1})+(\mathbb{Z}_5^{t_2})-(V_4^-)+(\mathbb{Z}_3^t)+3(\mathbb{Z}_3)+3(\mathbb{Z}_2^-)+3(\mathbb{Z}_2)-4(\mathbb{Z}_1))$	55
${}^1\xi_0$	1010	$(-1)^v(-(A_4)+(D_3^z)+(D_3)+(\mathbb{Z}_5^{t_1})+(\mathbb{Z}_5^{t_2})-(V_4^-)+(\mathbb{Z}_3^t)+(\mathbb{Z}_3)$ $+(\mathbb{Z}_2^-)+(\mathbb{Z}_2)-2(\mathbb{Z}_1))$	55
${}^1\xi_0$	1011	$(-1)^v(-(A_4)+(D_3^z)+(D_3)-(\mathbb{Z}_5^{t_1})-(\mathbb{Z}_5^{t_2})-(V_4^-)-(\mathbb{Z}_3^t)-(\mathbb{Z}_3)$ $-(\mathbb{Z}_2^-)-(\mathbb{Z}_2)+2(\mathbb{Z}_1))$	55
${}^1\xi_0$	1111	$(-1)^v((A_4)-(D_3^z)-(D_3)-(\mathbb{Z}_5^{t_1})-(\mathbb{Z}_5^{t_2})+(V_4^-)-(\mathbb{Z}_3^t)-(\mathbb{Z}_3)$ $-(\mathbb{Z}_2^-)-(\mathbb{Z}_2)+2(\mathbb{Z}_1))$	55
${}^2\xi_0$	0000	$(-1)^v((A_4^{t_1})+(A_4^{t_2})+(D_5)+(D_3)+(\mathbb{Z}_5^{t_1})+(\mathbb{Z}_5^{t_2})+(V_4^-)-2(\mathbb{Z}_2))$	50
${}^2\xi_0$	0001	$(-1)^v((A_4^{t_1})+(A_4^{t_2})+(D_5)+(D_3)-(\mathbb{Z}_5^{t_1})-(\mathbb{Z}_5^{t_2})-(\mathbb{Z}_5)+(V_4^-)$ $-4(\mathbb{Z}_3^t)-(\mathbb{Z}_3)-2(\mathbb{Z}_2^-)-5(\mathbb{Z}_2)+5(\mathbb{Z}_1))$	50
${}^2\xi_0$	0010	$(-1)^v((A_4^{t_1})+(A_4^{t_2})-(D_5)-(D_3)-(\mathbb{Z}_5^{t_1})-(\mathbb{Z}_5^{t_2})+(V_4^-)-4(\mathbb{Z}_3^t)-2(\mathbb{Z}_2^-)-(\mathbb{Z}_2)+3(\mathbb{Z}_1))$	50
${}^2\xi_0$	0011	$(-1)^v((A_4^{t_1})+(A_4^{t_2})-(D_5)-(D_3)+(\mathbb{Z}_5^{t_1})+(\mathbb{Z}_5^{t_2})+(\mathbb{Z}_5)+(V_4^-)$ $+(\mathbb{Z}_3)+2(\mathbb{Z}_2)-2(\mathbb{Z}_1))$	50
${}^2\xi_0$	0101	$(-1)^v(-(A_4^{t_1})-(A_4^{t_2})+(D_5)-(D_3)-(\mathbb{Z}_5^{t_1})-(\mathbb{Z}_5^{t_2})-(\mathbb{Z}_5)-(V_4^-)+(\mathbb{Z}_1))$	50
${}^2\xi_0$	1000	$(-1)^v(-(A_4^{t_1})-(A_4^{t_2})-(D_5)-(D_3)-(\mathbb{Z}_5^{t_1})-(\mathbb{Z}_5^{t_2})-(V_4^-)+2(\mathbb{Z}_2))$	50
${}^2\xi_0$	1001	$(-1)^v(-(A_4^{t_1})-(A_4^{t_2})-(D_5)-(D_3)+(\mathbb{Z}_5^{t_1})+(\mathbb{Z}_5^{t_2})+(\mathbb{Z}_5)-(V_4^-)$ $+4(\mathbb{Z}_3^t)+(\mathbb{Z}_3)+2(\mathbb{Z}_2^-)+5(\mathbb{Z}_2)-5(\mathbb{Z}_1))$	50
${}^2\xi_0$	1010	$(-1)^v(-(A_4^{t_1})-(A_4^{t_2})+(D_5)+(D_3)+(\mathbb{Z}_5^{t_1})+(\mathbb{Z}_5^{t_2})-(V_4^-)+4(\mathbb{Z}_3^t)$ $+2(\mathbb{Z}_2^-)+(\mathbb{Z}_2)-3(\mathbb{Z}_1))$	50
${}^2\xi_0$	1011	$(-1)^v(-(A_4^{t_1})-(A_4^{t_2})+(D_5)+(D_3)-(\mathbb{Z}_5^{t_1})-(\mathbb{Z}_5^{t_2})-(\mathbb{Z}_5)-(V_4^-)-(\mathbb{Z}_3)$ $-2(\mathbb{Z}_2)+2(\mathbb{Z}_1))$	50
${}^2\xi_0$	1111	$(-1)^v((A_4^{t_1})+(A_4^{t_2})+(D_5)-(D_3)-(\mathbb{Z}_5^{t_1})-(\mathbb{Z}_5^{t_2})-(\mathbb{Z}_5)+(V_4^-)-4(\mathbb{Z}_3^t)$ $-2(\mathbb{Z}_2^-)-3(\mathbb{Z}_2)+4(\mathbb{Z}_1))$	50
${}^3\xi_0$	0000	$(-1)^v((D_5^z)+(D_3^z)+(\mathbb{Z}_5^{t_1})+(V_4^-)+(\mathbb{Z}_3^t)-2(\mathbb{Z}_2^-))$	48
${}^3\xi_0$	0001	$(-1)^v((D_5^z)+(D_3^z)-(\mathbb{Z}_5^{t_1})-(\mathbb{Z}_5)+(V_4^-)-(\mathbb{Z}_3^t)-(\mathbb{Z}_3)-4(\mathbb{Z}_2^-)-(\mathbb{Z}_2)+3(\mathbb{Z}_1))$	48
${}^3\xi_0$	0010	$(-1)^v(-(D_5^z)-(D_3^z)-(\mathbb{Z}_5^{t_1})+(V_4^-)-(\mathbb{Z}_3^t)-(\mathbb{Z}_2)+(\mathbb{Z}_1))$	48
${}^3\xi_0$	0011	$(-1)^v(-(D_5^z)-(D_3^z)+(\mathbb{Z}_5^{t_1})+(\mathbb{Z}_5)+(V_4^-)+(\mathbb{Z}_3^t)+(\mathbb{Z}_3)+2(\mathbb{Z}_2^-)-2(\mathbb{Z}_2))$	48
${}^3\xi_0$	0101	$(-1)^v((D_5^z)-(D_3^z)-(\mathbb{Z}_5^{t_1})-(V_4^-)-(\mathbb{Z}_3^t)+(\mathbb{Z}_1))$	48
${}^3\xi_0$	1000	$(-1)^v(-(D_5^z)-(D_3^z)-(\mathbb{Z}_5^{t_1})-(V_4^-)-(\mathbb{Z}_3^t)+2(\mathbb{Z}_2^-))$	48
${}^3\xi_0$	1001	$(-1)^v(-(D_5^z)-(D_3^z)+(\mathbb{Z}_5^{t_1})+(\mathbb{Z}_5)-(V_4^-)+(\mathbb{Z}_3^t)+(\mathbb{Z}_3)+4(\mathbb{Z}_2^-)+(\mathbb{Z}_2)-3(\mathbb{Z}_1))$	48
${}^3\xi_0$	1010	$(-1)^v((D_5^z)+(D_3^z)+(\mathbb{Z}_5^{t_1})-(V_4^-)+(\mathbb{Z}_3^t)+(\mathbb{Z}_2)-(\mathbb{Z}_1))$	48
${}^3\xi_0$	1011	$(-1)^v((D_5^z)+(D_3^z)-(\mathbb{Z}_5^{t_1})-(\mathbb{Z}_5)-(V_4^-)-(\mathbb{Z}_3^t)-(\mathbb{Z}_3)-2(\mathbb{Z}_2^-)+2(\mathbb{Z}_1))$	48
${}^3\xi_0$	1111	$(-1)^v((D_5^z)-(D_3^z)-(\mathbb{Z}_5^{t_1})-(\mathbb{Z}_5)+(V_4^-)-(\mathbb{Z}_3^t)-2(\mathbb{Z}_2^-)-(\mathbb{Z}_2)+2(\mathbb{Z}_1))$	48

where $\delta > 0$ and $\varepsilon > 0$ are so small numbers that for all $\tau + \mathrm{i}\beta \in \partial\mathscr{S}$ and $\alpha \in [\alpha_0 - \varepsilon, \alpha_0 + \varepsilon]$, $\ker \triangle_{\alpha;u(\alpha)}(\tau + \mathrm{i}\beta) \neq \{0\}$ implies $\alpha = \alpha_0$ and $\tau + \mathrm{i}\beta = \mathrm{i}\beta_0$. Put $\alpha_\pm := \alpha_0 \pm \varepsilon$ and denote by $\mathfrak{s}_\pm$ the set of all characteristic roots $\lambda \in \mathscr{S}$ for $\alpha = \alpha_\pm$, i.e.

$$\mathfrak{s}_\pm := \{\lambda \in \mathscr{S} : \ker \triangle_{\alpha_\pm;u(\alpha_\pm)}(\lambda) \neq \{0\}\}.$$

Since $\ker \triangle_{\alpha_\pm;u(\alpha_\pm)}(\lambda) = \ker \widetilde{\triangle}^r_{\alpha_\pm;u(\alpha_\pm)}(\lambda)$ and $\widetilde{\triangle}^r_{\alpha_\pm;u(\alpha_\pm)}(\lambda)$ is an analytic function in λ, the sets $\mathfrak{s}_\pm$ are finite. Then, for $j = 0, 1, 2, \ldots, s$ (corresponding to the complex Γ-irreducible representations $\mathscr{U}_j$), put

$$\mathrm{t}_j^\pm(\alpha_0, \beta_0, u_0) := \sum_{\lambda \in \mathfrak{s}_\pm} m_j(\lambda), \tag{33}$$

(cf. (32)).

Definition 4.3. The *$\mathscr{U}_j$-isotypical crossing number* of (α_0, β_0, u_0) is defined as

$$\mathrm{t}_{j,1}(\alpha_0, \beta_0, u_0) := \mathrm{t}_j^-(\alpha_0, \beta_0, u_0) - \mathrm{t}_j^+(\alpha_0, \beta_0, u_0), \tag{34}$$

where $\mathrm{t}_j^\pm(\alpha_0, \beta_0, u_0)$ are given by (33). In the case $l\beta_0$ is also a characteristic root of (14) at (α_0, u_0) for some integer $l > 1$, put (cf. [2])

$$\mathrm{t}_{j,l}(\alpha_0, \beta_0, u_0) := \mathrm{t}_{j,1}(\alpha_0, l\beta_0, u_0).$$

By applying the standard finite-dimensional reduction and using the arguments similar to those in [2], one can establish

$$\mathrm{t}_{j,l}(\alpha_0, \beta_0, u_0) = -\mathrm{sign}\, \frac{\mathrm{d}}{\mathrm{d}\alpha} w(\alpha)|_{\alpha=\alpha_0} m_j(\mathrm{i}l\beta_0), \tag{35}$$

where $w(\alpha)$ stands for the real part of the characteristic root of (14) at $(\alpha, u(\alpha))$.

Under condition (R), each $E(\mathrm{i}\beta_0)$ is completely contained in a *single* isotypical component $\mathfrak{U}_j$ for some $j = j_{\beta_0}$ in (28). Thus,

$$m_j(\mathrm{i}\beta_0) = \begin{cases} \dim_{\mathbb{C}} E(\mathrm{i}\beta_0)/\dim_{\mathbb{C}} \mathscr{U}_j, & j = j_{\beta_0}, \\ 0, & j \neq j_{\beta_0}. \end{cases}$$

Therefore, by (35), for $l = 1$, we have

$$\mathrm{t}_{j,1}(\alpha_0, \beta_0) = \begin{cases} -\mathrm{sign}\, \dfrac{\mathrm{d}}{\mathrm{d}\alpha} w(\alpha)|_{\alpha=\alpha_0} \dim_{\mathbb{C}} E(\mathrm{i}\beta_0)/\dim_{\mathbb{C}} \mathscr{U}_j, & j = j_{\beta_0}, \\ 0, & j \neq j_{\beta_0}. \end{cases} \tag{36}$$

Based on the homotopy and multiplicativity properties of the equivariant degree (cf. Section 2.2), one can establish the following computational formula (for details and justification of the derivation, we refer to [2]):

$$\omega(\alpha_0, \beta_0, u_0) := \left(\prod_{\mu \in \sigma_-(\mathscr{A})} \prod_{i=0}^{r} ((\deg_{\mathscr{V}_i})^{m_i(\mu)}) \right) \cdot \sum_{j,l} \mathrm{t}_{j,l}(\alpha_0, \beta_0, u_0) \deg_{\mathscr{V}_{j,l}}. \tag{37}$$

4.7. Dominating orbit types and first coefficients

In order to take advantage of the information provided by the local bifurcation invariant $\omega(\alpha_0, \beta_0, u_0)$, we need the following important concept (cf. [2]).

Definition 4.4. An orbit type (H) in $\mathbb{F}$ is called *dominating*, if (H) is maximal (with respect to the usual order relation (see Section 2.1)) in the class of all φ-twisted 1-folded orbit types in $\mathbb{F}$ (in particular, $H = K^\varphi$).

In what follows, the dominating orbit types will be used to estimate the *minimal number* of different periodic solutions to system (14) as well as their *symmetries*.

Remark 4.5. (i) Due to the maximality property of dominating orbit types and the fact that the isotropy groups increase under projections, the dominating orbit types can be easily recognized from the isotropy lattices of the irreducible subrepresentations of $\mathbb{F}$.

(ii) Assume there is a solution $u_0 \in \mathbb{F}$ to (17) (for $\alpha = \alpha_0$ and some $\beta > 0$), for which one has $G_{u_0} \supset H_0$. If (H_0) is a dominating orbit type in $\mathbb{F}$ with $H_0 = K^{\varphi}$ for some $K \subset \Gamma$ and $\varphi : K \to S^1$, then, by the maximality condition, $(G_{u_0}) = (K^{\varphi,l})$ with $l \geqslant 1$, and the corresponding orbit $G(u_0)$ is composed of exactly $|G/G_{u_0}|_{S^1}$ different periodic functions (where $|Y|_{S^1}$ denotes the number of S^1-orbits in Y). It is easy to check that the number of S^1-orbits in G/G_{u_0} is $|\Gamma/K|$ (where $|X|$ stands for the number of elements in X).

On the other hand, let x_0 be a, say, p-periodic solution to (14) canonically corresponding to the above u_0. It follows from the definition of l-folding and $\Gamma \times S^1$-action on $\mathbb{F}$ that x_0 is also a p/l-periodic solution to (14). The pair $(x_0, p/l)$ canonically determines an element $u_0' \in \mathbb{F}$ being a solution to (17) (for $\alpha = \alpha_0$ and some β') satisfying the condition $G_{u_0'} = H_0$. In this way we obtain that (14) has at least $|\Gamma/K|$ different periodic solutions with the orbit type *exactly* (H_0) (considered in $\mathbb{F}$).

Combining the concept of the dominating orbit types with Theorem 3.6 (see also Remark 4.5(ii)) and using the same argument as in [2], one can easily establish:

Theorem 4.6. *Suppose that system* (14) *satisfies assumptions* (C1) *and* (C4), *and suppose that* (α_0, u_0) *is a* Γ*-symmetric steady-state solution to* (14) (*cf. Definition* 4.1) *satisfying* (C2)–(C3), $\omega(\alpha_0, \beta_0, u_0)$ *is given by* (13) (*with* $\lambda_0 = (\alpha_0, \beta_0)$, $\mathfrak{F}_\varsigma$ *defined by* (12), $U(r)$ *by* (10) *and* ς *satisfying* (11)). *Assume* (*cf.* (37)) $\omega(\alpha_0, \beta_0, u_0) \neq 0$, *i.e.*

$$\omega(\alpha_0, \beta_0, u_0) = \sum_{(H)} n_H(H) \quad \textit{and} \quad n_{H_0} \neq 0 \tag{38}$$

for some $(H_0) \in \Phi_1(G)$.

(i) *Then*, *there exists a branch of non-trivial solutions to* (14) *with symmetries at least* H_0 (*considered in the space* $\mathbb{F}$ (*cf.* (16))) *bifurcating from the point* (α_0, u_0) (*with the limit frequency* $l\beta_0$ *for some* $l \in \mathbb{N}$).
(ii) *If*, *in addition*, (H_0) *is a dominating orbit type in* $\mathbb{F}$, *then there exist at least* $|G/H_0|_{S^1}$ *different branches of periodic solutions to the Eq.* (14) *bifurcating from* (α_0, u_0). *Moreover*, *for each* (α, β, u) *belonging to these branches of (non-trivial) solutions one has* $(G_u) = (H_0)$ (*considered in the space* $\mathbb{F}$).

Although the entire value of the invariant $\omega(\alpha_0, \beta_0, u_0)$ should be considered to fully classify the symmetric Hopf bifurcation branches for system (14), in order to simplify our exposition (by reducing the number of additional cases), we restrict our computations to the coefficients $n_{H_0} = n_{K_0^{\varphi,1}}$, which are called *first coefficients*, and we will denote the corresponding part of the invariant $\omega(\alpha_0, \beta_0, u_0)$ by $\omega(\alpha_0, \beta_0, u_0)_1$. Thus, by (37),

$$\omega(\alpha_0, \beta_0, u_0)_1 := \left(\prod_{\mu \in \sigma_-(\mathscr{A}^i)} \prod_{i=0}^{r} (\deg_{\mathscr{V}_i})^{m_i(\mu)} \right) \cdot \sum_j t_{j,1}(\alpha_0, \beta_0, u_0) \deg_{\mathscr{V}_{j,1}}. \tag{39}$$

The first coefficients turn out to be sufficient to detect the solutions corresponding to the dominating orbit types.

5. Symmetric system of Hutchinson model in population dynamics

5.1. A Hutchinson model of an n species ecosystem

We start with the standard model for the dynamics of a simple (single) population[7] in terms of its density—the Verhulst equation (cf. [13,12]):

$$\dot{v} = \alpha v \left(1 - \frac{v}{K}\right),$$

[7] For population ecology background, we refer to [13,24,11].

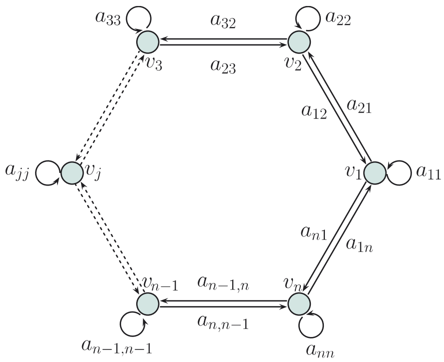

Fig. 3. System with dihedral symmetries.

which is based on the idea that the population grows exponentially at low densities and saturates towards the carrying capacity K (of resources) at high densities. By taking into account a delayed response to the remaining resources, the Hutchinson's model (of a single species) is obtained:

$$\dot{v}(t) = \alpha v(t)\left(1 - \frac{v(t-\tau)}{K}\right), \tag{40}$$

where $\tau > 0$ is a presumed delay constant and α refers to the intrinsic growth rate.

Now, consider an ecosystem composed of n species interacting with each other (according to a certain symmetry) by competing (or cooperating) over shared resources such as food and habitats, while maintaining a self-inhibiting nature (meaning self-limiting in response to rare resources and self-reproducing to abundant resources). A mathematical treatment for such a community model was developed by Levins in [19], where one attaches a loop diagram in order to carry out a loop analysis for this community-type situation (Fig. 3).

Here, a_{jj} describes the self-inhibiting nature of the jth species, and $a_{ij} < 0$ (resp. $a_{ij} > 0$) is the competing (resp. cooperating) coefficient between species i and j. Also, observe that $a_{ij} = a_{ji}$.

Introduce

$$C = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \tag{41}$$

and call it the *community matrix*. This community ecosystem can be described by the following equations:

$$\dot{v}(t) = \alpha C v(t) \cdot \left(1 - \frac{v(t-\tau)}{K}\right), \tag{42}$$

where

$$v(t) = \begin{bmatrix} v_1(t) \\ v_2(t) \\ \vdots \\ v_n(t) \end{bmatrix}, \quad v(t)\cdot u(t) = \begin{bmatrix} v_1(t)u_1(t) \\ v_2(t)u_2(t) \\ \vdots \\ v_n(t)u_n(t) \end{bmatrix}. \tag{43}$$

By applying the standard transformation

$$v(t) = K(1 + u(t)), \tag{44}$$

to system (42), one obtains the equivalent system

$$\dot{u}(t) = -\alpha C u(t-\tau) \cdot [1 + u(t)], \tag{45}$$

where $u(t) = v(t)/K - 1$ is, in fact, a population saturation index with respect to the available resources.

Finally, to study system (45) in a heterogeneous environment, we add to (45) a spatial diffusion term, which leads to the following reaction–diffusion equations:

$$\frac{\partial}{\partial t}u(x,t) = \mathrm{d}\frac{\partial^2}{\partial x^2}u(x,t) - \alpha C u(x,t-1)[1+u(x,t)], \tag{46}$$

where $d > 0$ is a spatial diffusion coefficient.

5.2. A symmetric system of the Hutchinson model

Consider a symmetric system of n species Hutchinson model of form (46) (for $t > 0$ and $x \in (0,\pi)$):

$$\begin{cases} \frac{\partial}{\partial t}u(x,t) = \mathrm{d}\frac{\partial^2}{\partial x^2}u(x,t) - \alpha C u(x,t-1)\cdot[1+u(x,t)], \\ \frac{\partial}{\partial x}u(x,t) = 0, \quad x = 0, \pi, \end{cases} \tag{47}$$

where $u : [0,\pi]\times\mathbb{R} \to \mathbb{R}^n$ is a population saturation index (cf. (45)), '$\cdot$' is defined by (43), $d > 0$ is a spatial diffusion coefficient, $\alpha \neq 0$ is the intrinsic growth rate (cf. (40)), which will be considered as a bifurcation parameter, and C is a (symmetric) community matrix describing the interaction among the species.

We will assume that:

(A1) The geometrical configuration described by system (47) has a (finite) symmetry group Γ. The group Γ permutes the vertices of the related polygon or polyhedron, which means it acts on $\mathbb{R}^n$ by permuting the coordinates of the vectors $x \in \mathbb{R}^n$. The (symmetric) matrix C commutes with this Γ-action and $0 \notin \sigma(C)$.

For concrete examples of configurations such as regular n-gon, tetrahedron, octahedron, dodecahedron and related matrices C, we refer to [1,2].

Under assumption (A1), the space $V := \mathbb{R}^n$ becomes an orthogonal Γ-representation and condition (C1) from Section 4.1 is satisfied by system (47).

5.3. Characteristic equation and isolated centers

At the Γ-symmetric steady-state solution $(\alpha, 0)$ system (47) has the linearization

$$\begin{cases} \frac{\partial}{\partial t}u(x,t) = \mathrm{d}\frac{\partial^2}{\partial x^2}u(x,t) - \alpha C\, u(x,t-1), \\ \frac{\partial}{\partial x}u(x,t) = 0, \quad x = 0, \pi. \end{cases} \tag{48}$$

Since the matrix C is symmetric, it is completely diagonalizable with respect to a basis composed of its eigenvectors. Consider the spectrum $\sigma(C) = \{\xi_1, \xi_2, \dots, \xi_q\}$ of the matrix C and denote by $\mathfrak{E}(\xi_k) \subset V$ the eigenspace of ξ_k. Then,

$$L^2([0,\pi]; V) = \bigoplus_{k=1}^{q} L^2([0,\pi]; \mathfrak{E}(\xi_k)), \tag{49}$$

and $w \in L^2([0,\pi]; V)$ can be represented as $w(x) = \sum_k w_k(x)$, where $w_k \in L^2([0,\pi]; \mathfrak{E}(\xi_k))$. In a similar way, one also has

$$L^2([0,\pi]; V^{\mathrm{c}}) = \bigoplus_{k=1}^{q} L^2([0,\pi]; \mathfrak{E}^{\mathrm{c}}(\xi_k)), \tag{50}$$

where $\mathfrak{E}^{\mathrm{c}}(\xi_k)$ denotes the complexification of the eigenspace $\mathfrak{E}(\xi_k)$.

Notice that $(\alpha, 0)$ is a Γ-symmetric steady-state solution to (47) for all (non-zero) α. Thus, we can take the set $(\alpha, \beta, 0)$, $\alpha \neq 0$, for the manifold $M \subset \mathscr{P} \times \mathbb{E}^G$ described in Section 3.2 (see also Section 4.4). Moreover, $(\alpha_0, 0)$

is non-singular if $0 \notin \sigma(\mathscr{L}_{\alpha_0})$, where $\mathscr{L}_{\alpha_0} := \mathrm{d}\partial^2/\partial x^2 - \alpha_0 C : H_0^2([0,\pi]; V) \to L^2([0,\pi); V]$ with $H_0^2([0,\pi]; V)$ being the subspace of $H^2([0,\pi]; V)$ consisting of functions u satisfying $u(0) = u(\pi) = 0$. One can easily verify that if

$$-\frac{\alpha_0 \xi_k}{d} \neq m^2 \quad \text{for all } k = 1, 2, \ldots, q \text{ and } m = 0, 1, 2, \ldots,$$

then $(\alpha_0, 0)$ is a non-singular Γ-symmetric steady-state solution, i.e. $(\alpha_0, 0)$ satisfies condition (C2) from Section 4.4.

A number $\lambda \in \mathbb{C}$ is a characteristic root of system (47) at a Γ-symmetric steady-state solution $(\alpha, 0)$ if there exists a non-zero function $v \in L^2([0,\pi]; V^c)$ such that

$$\triangle_\alpha(\lambda) v(x) := \lambda v(x) - \mathrm{d}\frac{\partial^2}{\partial x^2} v(x) + \alpha\, \mathrm{e}^{-\lambda} C\, v(x) = 0, \tag{51}$$

where we put $\triangle_\alpha := \triangle_{\alpha;0}$ (cf. (20)).

By using decomposition (50), v can be written as $v(x) = \sum_{k=1}^q v_k(x)$, for $v_k(x) \in E(\xi_k)$.

Consequently, (51) yields

$$\triangle_\alpha(\lambda) v(x) = \sum_k \left(\lambda v_k(x) - \mathrm{d}\frac{\partial^2}{\partial x^2} v_k(x) + \alpha\, \mathrm{e}^{-\lambda} \xi_k\, v_k(x) \right) = 0. \tag{52}$$

Next, by using the point spectrum $\{\zeta_m := dm^2\}_{m=0}^\infty$ of the (scalar-valued) Laplace operator $L := -\mathrm{d}\,\partial^2/\partial x^2$ and the corresponding eigenspaces $E(\zeta_m)$, we can write $v_k(x) = \sum_m v_{k,m}(x)$, for $v_{k,m} \in E(\zeta_m)$, thus

$$\triangle_\alpha(\lambda) v(x) = \sum_{k,m} (\lambda v_{k,m}(x) + dm^2 v_{k,m}(x) + \alpha\, \mathrm{e}^{-\lambda} \xi_k\, v_{k,m}(x)) = 0. \tag{53}$$

Therefore, one obtains that $\lambda \in \mathbb{C}$ is a characteristic root of (48) at the Γ-symmetric steady-state solution $(\alpha, 0)$, if

$$\lambda + dm^2 + \alpha\, \xi_k e^{-\lambda} = 0 \quad \text{for } k = 1, \ldots, q \text{ and } m = 0, 1, \ldots\ . \tag{54}$$

5.4. Computations for the local bifurcation $\Gamma \times S^1$-invariant

In order to find the values α_0 for which the condition (C3) from Section 4.5 holds, we need to find purely imaginary roots $\lambda = \mathrm{i}\beta$ $(\beta > 0)$ of (54). Assume that $(\alpha, 0)$ is a non-singular steady-state solution to (47) (in particular, $\alpha \neq 0$).

• *Computation for purely imaginary roots $\lambda = \mathrm{i}\beta$ $(\beta > 0)$:*

By substituting $\lambda = \mathrm{i}\beta$ into (54),

$$\begin{cases} dm^2 + \alpha\xi_k \cos\beta = 0 \\ \beta - \alpha\xi_k \sin\beta = 0 \end{cases} \quad \text{for } k = 1, \ldots, q. \tag{55}$$

In the case $m = 0$,

$$\begin{cases} \beta := \beta_{v,0,k} = \dfrac{\pi}{2} + v\pi, \\ \alpha := \alpha_{v,0,k} = (-1)^v \dfrac{\beta}{\xi_k}, \end{cases}$$

for $k = 1, \ldots, q$ and $v = 0, 1, \ldots$. Consequently,

$$\text{sign } \alpha_{v,0,k} = (-1)^v \text{ sign } \xi_k. \tag{56}$$

In the case $m \neq 0$ (thus $\cos\beta \neq 0$ by the first equation in (55)),

$$\tan\beta = -\frac{\beta}{dm^2}, \tag{57}$$

$$\alpha = -\frac{dm^2}{\xi_k \cos\beta}, \tag{58}$$

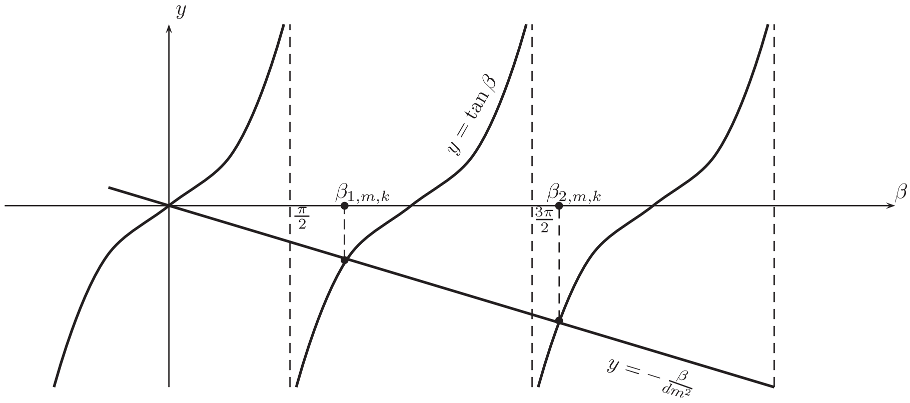

Fig. 4. Purely imaginary roots of the characteristic equation.

Eq. (57) has infinitely many positive solutions, which are denoted by $\{\beta_{v,m,k}\}_{v=1}^{\infty}$ (see Fig. 4). The corresponding solution α of (58) is denoted by $\alpha_{v,m,k}$.

Also, notice that sign $\cos\beta_{v,m,k} = (-1)^v$, thus by (58),

$$\operatorname{sign}\alpha_{v,m,k} = (-1)^{v+1}\operatorname{sign}\xi_k. \tag{59}$$

• *Computation for* sign $(\mathrm{d}/\mathrm{d}\alpha)w(\alpha)|_{\alpha=\alpha_{v,m,k}}$: Put $\alpha_0 := \alpha_{v,m,k}$ and $\beta_0 := \beta_{v,m,k}$. In order to determine the value of the crossing number $t_{j,1}(\alpha_0, \beta_0, 0)$, compute $(\mathrm{d}/\mathrm{d}\alpha)w(\alpha)|_{\alpha=\alpha_0}$ by implicit differentiation (cf. (36)).

By substituting $\lambda = w + \mathrm{i}v$ into (54),

$$\begin{cases} w + dm^2 + \alpha\xi_k \mathrm{e}^{-w}\cos v = 0, \\ v - \alpha\xi_k\mathrm{e}^{-w}\sin v = 0, \end{cases} \tag{60}$$

then, differentiating (60) with respect to α,

$$\begin{cases} \dfrac{\mathrm{d}w}{\mathrm{d}\alpha} - \alpha\xi_k\mathrm{e}^{-w}\left(\dfrac{\mathrm{d}w}{\mathrm{d}\alpha}\cos v + \dfrac{\mathrm{d}v}{\mathrm{d}\alpha}\sin v\right) = -\xi_k\mathrm{e}^{-w}\cos v, \\ \dfrac{\mathrm{d}v}{\mathrm{d}\alpha} + \alpha\xi_k\mathrm{e}^{-w}\left(\dfrac{\mathrm{d}w}{\mathrm{d}\alpha}\sin v - \dfrac{\mathrm{d}v}{\mathrm{d}\alpha}\cos v\right) = \xi_k\mathrm{e}^{-w}\sin v, \end{cases} \tag{61}$$

which is equivalent to

$$\begin{cases} \dfrac{\mathrm{d}w}{\mathrm{d}\alpha}(\alpha\xi_k\mathrm{e}^{-w} - \cos v) + \dfrac{\mathrm{d}v}{\mathrm{d}\alpha}\sin v = \xi_k\mathrm{e}^{-w}, \\ \dfrac{\mathrm{d}w}{\mathrm{d}\alpha}\sin v + \dfrac{\mathrm{d}v}{\mathrm{d}\alpha}(\cos v - \alpha\xi_k\mathrm{e}^{-w}) = 0. \end{cases} \tag{62}$$

Thus,

$$\frac{\mathrm{d}w}{\mathrm{d}\alpha} = -\frac{\xi_k\mathrm{e}^{-w}(\cos v - \alpha\xi_k\mathrm{e}^{-w})}{\alpha^2\xi_k^2\mathrm{e}^{-2w} - 2\alpha\xi_k\mathrm{e}^{-w}\cos v + 1}. \tag{63}$$

By substituting $\alpha = \alpha_0$, $w = 0$ and $v = \beta_0$,

$$\begin{aligned} \left.\frac{\mathrm{d}w}{\mathrm{d}\alpha}\right|_{\alpha=\alpha_0} &= -\frac{\xi_k(\cos\beta_0 - \alpha_0\xi_k)}{\alpha_0^2\xi_k^2 - 2\alpha_0\xi_k\cos\beta_0 + 1} \\ &= -\frac{\xi_k\cos\beta_0 - \alpha_0\xi_k^2}{\alpha_0^2\xi_k^2 - 2\alpha_0\xi_k\cos\beta_0 + 1}. \end{aligned}$$

Replacing $\xi_k \cos\beta_0$ with $-dm^2/\alpha_0$ in the last equality (cf. (58)),

$$\left.\frac{\mathrm{d}w}{\mathrm{d}\alpha}\right|_{\alpha=\alpha_0} = \frac{1}{\alpha_0}\cdot\frac{dm^2+\alpha_0^2\xi_k^2}{\alpha_0^2\xi_k^2+2dm^2+1}.$$

Consequently,

$$\operatorname{sign}\left.\frac{\mathrm{d}w}{\mathrm{d}\alpha}\right|_{\alpha=\alpha_0} = \operatorname{sign}\alpha_0.$$

Hence, by (56) and (59),

$$\operatorname{sign}\left.\frac{\mathrm{d}w}{\mathrm{d}\alpha}\right|_{\alpha=\alpha_0=\alpha_{v,m,k}} = \begin{cases}(-1)^v \operatorname{sign}\xi_k & \text{if } m=0,\\ (-1)^{v+1}\operatorname{sign}\xi_k & \text{if } m=1,2,\ldots\ .\end{cases} \tag{64}$$

Therefore, combining (64) with (36), we have for $m \neq 0$[8]

$$\mathrm{t}_{j,1}(\alpha_0,\beta_0) = \begin{cases}(-1)^v \operatorname{sign}\xi_k \dim_{\mathbb{C}} E(i\beta_0)/\dim_{\mathbb{C}}\mathscr{U}_j, & j = j_{\beta_0}\\ 0, & j \neq j_{\beta_0}.\end{cases} \tag{65}$$

6. Usage of Maple© package and concrete results for selected symmetry groups

In this section, assuming conditions (C1)–(C4) to be satisfied by system (47), we will present quantitative results for some specific symmetry group Γ, where Γ takes values from the dihedral groups D_3, D_4, D_5, the tetrahedral group A_4, the octahedral group S_4 and the icosahedral group A_5.

Below we will briefly summarize our discussions presented in Sections 5.3 and 5.4, and describe the *input data* to the Maple© package used to compute $\omega(\alpha_0,\beta_0,0)_1$.

Recall that, by (39),

$$\omega(\alpha_0,\beta_0,0)_1 = \omega_\Gamma\cdot\omega_G,$$

where $\omega_\Gamma = \prod_{\mu\in\sigma_-(\mathscr{A})}\prod_i(\deg_{\mathscr{V}_i})^{m_i(\mu)}$, $\omega_G = \sum_j \mathrm{t}_{j,1}(\alpha_0,\beta_0,0)\deg_{\mathscr{V}_{j,1}}$, and $\mathscr{A}$ is defined for $(\alpha_0,\beta_0,0) = (\alpha_{v,m,k},\beta_{v,m,k},0)$ (by formula (29)).

By formula (31), we have

$$\omega_\Gamma = \prod_{i=0}^{r}(\deg_{\mathscr{V}_i})^{\sum_{\mu\in\sigma_-(\mathscr{A})} m_i(\mu)}. \tag{66}$$

Since $(\deg_{\mathscr{V}_i})^2 = (\Gamma)$ for $i=0,1,\ldots,r$ (cf. [2]), we can associate with $\sigma_-(\mathscr{A})$ the sequence $(\varepsilon_0,\varepsilon_1,\ldots,\varepsilon_r)$ defined by

$$\varepsilon_i := \sum_{\mu\in\sigma_-(\mathscr{A})} m_i(\mu)\ (\mathrm{mod}\ 2),\quad i=0,1,\ldots,r.$$

Then, formula (66) can be reduced to

$$\omega_\Gamma = \prod_{i=0}^{r}(\deg_{\mathscr{V}_i})^{\varepsilon_i}.$$

Clearly, the sequence $\{\varepsilon_0,\varepsilon_1,\ldots,\varepsilon_r\}$ permits only possibly finitely many different values.

---

[8] Throughout the rest of this section, we carry out the computation of the local $\Gamma\times S^1$-invariant $\omega(\alpha_0,\beta_0,0)_1 = \omega(\alpha_{v,m,k},\beta_{v,m,k},0)_1$ for $m\neq 0$. In the case $m=0$, one only needs to change the formula for $\operatorname{sign}(\mathrm{d}w/\mathrm{d}\alpha)|_{\alpha=\alpha_0}$ according to (64).

By formula (65),

$$\omega_G = (-1)^{\nu} \dim_{\mathbb{C}} E(\mathrm{i}\beta_{\nu,m,k}) / \dim_{\mathbb{C}} \mathscr{U}_{j_{\beta_{\nu,m,k}}} \deg_{\mathscr{V}_{j_{\beta_{\nu,m,k}},1}}.$$

We will use the notation $\mathfrak{m}_{j_{\beta_{\nu,m,k}}} := \dim_{\mathbb{C}} E^{c}(\mathrm{i}\beta_{\nu,m,k}) / \dim_{\mathbb{C}} \mathscr{U}_{j_{\beta_{\nu,m,k}}}$, which stands for the $\mathscr{U}_j$-*multiplicity* of $i\beta_{\nu,m,k}$. Thus, $\mathfrak{m}_{j_{\beta_{\nu,m,k}}}$ also permits only possibly finitely many different values.

Therefore, we have the following formula for the first coefficients of the local bifurcation invariant:

$$\omega(\alpha_{\nu,m,k}, \beta_{\nu,m,k}, 0)_1 = (-1)^{\nu} \prod_{i=0}^{r} (\deg_{\mathscr{V}_i})^{\varepsilon_i} \cdot \mathfrak{m}_{j_{\beta_{\nu,m,k}}} \deg_{\mathscr{V}_{j_{\beta_{\nu,m,k}},1}}. \tag{67}$$

The input data for the computation of the local invariant consists of two finite sequences:

$$\{\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_r\}, \quad \{\mathfrak{m}_0, \mathfrak{m}_1, \ldots, \mathfrak{m}_r\},$$

which are forwarded to the following command from the Maple© package:

$$\omega(\alpha_{\nu,m,k}, \beta_{\nu,m,k}, 0)_1 := (-1)^{\nu}\, \texttt{showdegree}[\Gamma](\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_r, \mathfrak{m}_0, \mathfrak{m}_1, \ldots, \mathfrak{m}_r).$$

Remark and Notation 6.1. Given $\xi_0 \in \sigma(C)$ and assuming condition (R) to be satisfied, in what follows we will use the notation ξ_0^i to indicate that $\mathfrak{E}(\xi_0) \subset V_i$ and ${}^{j}\xi_0$, when $\mathfrak{E}^c(\xi_0) \subset U_j$ (here we consider the matrix C acting on V^c). In such a case we also write ${}^{j}\xi_0^i$. Observe that if condition (R) is not satisfied, then it is possible that $i \neq j$ (cf. (25)). Since the value of $\mathfrak{m}_{j_{\beta_{\nu,m,k}}}$, by condition (R), is equal to the $\mathscr{U}_{j_{\beta_{\nu,m,k}}}$-multiplicity $\dim_{\mathbb{C}} (\mathfrak{E}^c(\xi_k) \cap U_{j_{\beta_{\nu,m,k}}}) / \dim_{\mathbb{C}} \mathscr{U}_{j_{\beta_{\nu,m,k}}}$ of the eigenvalue ${}^{j_{\beta_{\nu,m,k}}}\xi_k$ of the complexified matrix C, and $E(i\beta_{\nu,m,k}) \subset \mathfrak{U}_{j_{\beta_{\nu,m,k}}}$, it is convenient to present our quantitative results in the form of a matrix:

${}^{j}\xi_0$	$\varepsilon_{i_1}, \varepsilon_{i_2}, \ldots, \varepsilon_{i_l}$	$\omega(\lambda_0)_1$	# Branches

where we only list $\{\varepsilon_{i_1}, \varepsilon_{i_2}, \ldots, \varepsilon_{i_l}\} \subset \{\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_r\}$ for those ε_{i_ι}, which can realize the value 1.

Remark 6.2. Although we are dealing with *infinitely* many isolated centers

$$(\alpha_0, \beta_0, 0) \in \{(\alpha_{\nu,m,k}, \beta_{\nu,m,k}, 0)\}_{\nu,m,k},$$

only *finitely* many different values of $\omega(\alpha_0, \beta_0, 0)_1$ may occur, which is related to the fact that the value of $\omega(\alpha_0, \beta_0, 0)_1$ is determined by only possibly *finitely* many different choices of the values of the two sequences $\{\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_r\}$ and $\{\mathfrak{m}_0, \mathfrak{m}_1, \ldots, \mathfrak{m}_r\}$.

The case $\Gamma = D_3$: We assume here that the matrix C is of type

$$C = \begin{bmatrix} c & d & d \\ d & c & d \\ d & d & c \end{bmatrix}$$

with $c = -3$ and $d = -1$. In this case, $\sigma(C) = \{{}^{0}\xi_0^0 = -5, {}^{1}\xi_1^1 = -2\}$, $m(\xi_0) = m(\xi_1) = 1$. The bifurcation invariants $\omega(\alpha_{\nu,m,k}, \beta_{\nu,m,k}, 0)_1$ in this case are listed in Table 1, which was established by using the Maple© routines for the group $\Gamma = D_3$, in the following way:

$$\omega(\alpha_{\nu,m,k}, \beta_{\nu,m,k}, 0)_1 = (-1)^{\nu}\, \texttt{showdegree[D3]}(\varepsilon_0, \varepsilon_1, 0, m_0(\zeta_k), m_1(\zeta_k), 0).$$

The case $\Gamma = D_4$: We assume here that the matrix C is of type

$$C = \begin{bmatrix} c & d & 0 & d \\ d & c & d & 0 \\ 0 & d & c & d \\ d & 0 & d & c \end{bmatrix}$$

with $c=-3$ and $d=-1$. In this case, $\sigma(C)=\{{}^0\xi_0^0=-5, {}^1\xi_1^1=-3, {}^3\xi_3^3=-1\}$, $m(\xi_0)=m(\xi_1)=m(\xi_3)=1$. Some selected bifurcation invariants $\omega(\alpha_{v,m,k},\beta_{v,m,k},0)_1$ in this case are listed in Table 2. The remaining results can be easily established by using the Maple© routines for the group $\Gamma=D_4$:

$$\omega(\alpha_{v,m,k},\beta_{v,m,k},0)_1 \\ =(-1)^v\,\texttt{showdegree[D4]}(\varepsilon_0,\varepsilon_1,0,\varepsilon_2,0,m_0(\zeta_k),m_1(\zeta_k),0,m_2(\zeta_k),0).$$

The case $\Gamma=D_5$: We assume here that the matrix C is of type

$$C=\begin{bmatrix} c & d & 0 & 0 & d\\ d & c & d & 0 & 0\\ 0 & d & c & d & 0\\ 0 & 0 & d & c & d\\ d & 0 & 0 & d & c\end{bmatrix}$$

with $c=-3$ and $d=-1$. In this case, the eigenvalues of C are: $\sigma(C)=\{{}^0\xi_0^0=-5, {}^1\xi_1^1=-3+2(\sqrt{5}-1)/4, {}^2\xi_2^2=-3+2(\sqrt{5}+1)/4\}$, $m(\xi_0^0)=m(\xi_1)=m(\xi_2)=1$. Some selected bifurcation invariants $\omega(\alpha_{v,m,k},\beta_{v,m,k},0)_1$ in this case are listed in Table 3, which were computed by using the Maple© routines for the group $\Gamma=D_5$:

$$\omega(\alpha_{v,m,k},\beta_{v,m,k},0)_1 \\ =(-1)^v\,\texttt{showdegree[D5]}(\varepsilon_0,\varepsilon_1,\varepsilon_2,0,m_0(\zeta_k),m_1(\zeta_k),m_3(\zeta_k),0).$$

The case $\Gamma=A_4$: We assume here that the matrix C is of type

$$C=\begin{bmatrix} c & d & d & d\\ d & c & d & d\\ d & d & c & d\\ d & d & d & c\end{bmatrix}$$

with $c=-4$ and $d=1$. Clearly, C is A_4-equivariant. In this case, the eigenvalues of C are: $\sigma(C)=\{-1,-5\}$. We classify the eigenvalues of C as ${}^0\xi_0^0=-1, {}^3\xi_1^3=-5$, and their multiplicities: $m(\xi_0)=m(\xi_1)=1$. Sample invariants $\omega(\alpha_{v,m,k},\beta_{v,m,k},0)_1$ in this case are listed in Table 4. To obtain the other invariants, use the Maple© routines for the group $\Gamma=A_4$:

$$\omega(\alpha_{v,m,k},\beta_{v,m,k},0)_1=(-1)^v\,\texttt{showdegree[A4]}(\varepsilon_0,0,\varepsilon_3,m_0(\zeta_k),0,0,m_3(\zeta_k)).$$

The case $\Gamma=S_4$: We assume that the matrix C is of the type

$$C=\begin{bmatrix} c & d & 0 & d & 0 & d & 0 & 0\\ d & c & d & 0 & 0 & 0 & d & 0\\ 0 & d & c & d & 0 & 0 & 0 & d\\ d & 0 & d & c & d & 0 & 0 & 0\\ 0 & 0 & 0 & d & c & d & 0 & d\\ d & 0 & 0 & 0 & d & c & d & 0\\ 0 & d & 0 & 0 & 0 & d & c & d\\ 0 & 0 & d & 0 & d & 0 & d & c\end{bmatrix}$$

with $c=-6$, and $d=1$. In this case we have $\sigma(C)=\{{}^0\xi_0^0=-2, {}^1\xi_1^1=-8, {}^3\xi_2^3=-4, {}^4\xi_3^4=-6\}$, $m(\xi_0)=m(\xi_1)=m(\xi_2)=m(\xi_3)=1$. Some selected bifurcation invariants $\omega(\alpha_{v,m,k},\beta_{v,m,k},0)_1$ in this case are listed in Table 5. Use the Maple© routines for the group $\Gamma=S_4$ to obtain the rest of invariants:

$$\omega(\alpha_{v,m,k},\beta_{v,m,k},0)_1 \\ =(-1)^v\,\texttt{showdegree[S4]}(\varepsilon_0,\varepsilon_1,0,\varepsilon_3,\varepsilon_4,m_0(\zeta_k),m_1(\zeta_k),0,m_3(\zeta_k),m_4(\zeta_k)).$$

The case $\Gamma = A_5$: We assume that the matrix C is of the type

$$C = \begin{bmatrix}
c & d & 0 & 0 & d & 0 & 0 & d & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
d & c & d & 0 & 0 & 0 & 0 & 0 & 0 & d & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & d & c & d & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & d & c & d & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d & 0 & 0 & 0 & 0 & 0 & 0 \\
d & 0 & 0 & d & c & d & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & d & c & d & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & d & c & d & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d & 0 & 0 & 0 \\
d & 0 & 0 & 0 & 0 & 0 & d & c & d & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & d & c & d & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d & 0 & 0 \\
0 & d & 0 & 0 & 0 & 0 & 0 & 0 & d & c & d & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d & c & d & 0 & 0 & 0 & 0 & 0 & 0 & d & 0 \\
0 & 0 & d & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d & c & d & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d & c & d & 0 & 0 & 0 & 0 & 0 & d \\
0 & 0 & 0 & d & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d & c & d & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & d & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d & c & d & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d & c & d & 0 & 0 & d \\
0 & 0 & 0 & 0 & 0 & 0 & d & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d & c & d & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d & c & d & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d & 0 & 0 & 0 & 0 & 0 & 0 & d & c & d \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d & 0 & 0 & d & 0 & 0 & d & c
\end{bmatrix}$$

with $c = -6$, and $d = 1$. In this case,

$$\sigma(C) := \{{}^0\xi_0^0 = -3, {}^1\xi_1^1 = -6, {}^1\xi_2^1 = -4, {}^2\xi_3^2 = -5, {}^3\xi_4^3 = -6 - \sqrt{5}, {}^4\xi_5^4 = -6 + \sqrt{5}\},$$

and $m(\xi_k) = 1$, $k = 0, 1, 2, 3, 4, 5$. Some selected bifurcation invariants $\omega(\alpha_{\nu,m,k}, \beta_{\nu,m,k}, 0)_1$ in this case are listed in Table 6, which was established by using the Maple© routines for the group $\Gamma = A_5$:

$$\omega(\alpha_{\nu,m,k}, \beta_{\nu,m,k}, 0)_1 \\ = (-1)^{\nu}\,\texttt{showdegree[A5]}(\varepsilon_0, \varepsilon_1, \varepsilon_2, \varepsilon_3, 0, m_0(\zeta_k), m_1(\zeta_k), m_2(\zeta_k), m_3(\zeta_k), 0).$$

The remaining invariants can be easily evaluated by using the Maple© package.

Remark 6.3. We refer to our previous paper [2] (and also [1]) for the interpretation of the presented tables including the explanation of the symbols representing the subgroups in $\Gamma \times S^1$.

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