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# A degree theory for coupled cell systems with quotient symmetries

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**Abstract**

We introduce a topological degree theory for the study of Hopf bifurcations in coupled cell systems whose quotient systems (obtained by restricting the system to its flow-invariant subspaces) possess various symmetries. To describe the structure of these quotient symmetries, we introduce the concept of a *representation lattice*, which is defined as a lattice of representation spaces of (different) symmetry groups that satisfy a compatibility and a consistence condition. Based on the (twisted) equivariant degree, we define a *lattice-equivariant degree* for maps that are compatible with respect to this representation lattice structure. We apply the lattice-equivariant degree to study a synchrony-breaking Hopf-bifurcation problem in (homogeneous) coupled cell systems and obtain a topological classification of all bifurcating branches of oscillating solutions according to their synchrony types and their symmetric properties.

Mathematics Subject Classification: 34D06, 34C23, 47H11, 34C25

## 1. Introduction

A *coupled cell system* is a finite collection of individual dynamical systems, or *cells*, that are coupled together through interactions, in a sense that the output of a cell affects the time-evolution of other cells. Coupled cell systems provide a large class of dynamical systems which can be used to model collective and synchronized behavior of networks of coupled units in different fields such as physics, biology, chemistry, engineering and social science (see [7, 19, 26] and references therein).

The network architecture of a coupled cell system can be represented by a directed graph, a *coupled cell network*, whose nodes correspond to cells and whose edges indicate interactions between cells. More precisely, a coupled cell network consists of a finite set $\mathcal{C}$ of *cells* and a finite set $\mathcal{E} = \{(c, d) : c, d \in \mathcal{C}\}$ of *edges*, together with two equivalence relations: $\sim_C$ on cells in $\mathcal{C}$ and $\sim_E$ on edges in $\mathcal{E}$ such that if $e_1 \sim_E e_2$, for $e_1 = (c_1, d_1) \in \mathcal{E}$ and $e_2 = (c_2, d_2) \in \mathcal{E}$, then $c_1 \sim_C c_2$ and $d_1 \sim_C d_2$ (see [24]). Two cells $c, d \in \mathcal{C}$ are called *input-equivalent*, if there is an edge-type preserving isomorphism between their input sets. Note that the input-equivalence relation refines the relation of cell-equivalence. Various notions of symmetry on coupled cell networks can be explored using symmetry among input-equivalent cells, such as *symmetry groupoid*, *interior symmetry* and *quotient symmetry*.

A *symmetry groupoid* of a coupled cell network is a collection of input-equivalence relations on cells, which formalizes the notion of 'local symmetries' on coupled cell networks. Between the stringent symmetry and the groupoid symmetry, there is an intermediate notion of symmetry, called *interior symmetry*. An *interior symmetry* (with respect to a subset of cells) of a coupled cell network is a permutation on the subset together with all its input edges, which preserves all its internal dynamics and types of the input edges (see [14]). Parallels exist between coupled cell systems admitting interior symmetry and symmetrically coupled systems. Indeed, analogue of equivariant bifurcation theory including the equivariant branching lemma and the equivariant Hopf theorem has been established in [2, 14] for coupled cell systems with interior symmetry.

A *quotient symmetry* is a shorthand notion for symmetry of a *quotient network*, which in turn is a network obtained by restricting the total coupled cell system to one of its lower dimensional flow-invariant subspaces. Quotient networks exist as a result of the network structure of coupled cell systems. It was shown that the existence of quotient networks has strong implications on synchronized dynamics in coupled cell systems (see [24]). Recent findings in [15, 25] confirm that quotient symmetry is responsible for the existence of periodic solutions with rigid phase-shifts and rigid multirhythms. In a sense, symmetric properties of coupled cell networks characterized by the above mentioned forms of symmetry may give us a key to understanding pattern formations in general coupled cell systems. It should be mentioned that network architecture without any forms of symmetry can also lead to surprising bifurcation behavior on coupled cell networks (see [10]).

A topological degree, in its simplest form, may be thought as a generalization of the *winding number* of a continuous circle map, which counts how many times the image of the map has travelled counterclockwise around the origin. This count remains unchanged, if the map is perturbed slightly. Also the addition of winding numbers corresponds to the conjunction of maps, and the negation of winding numbers can be realized by rewinding the direction of maps. The topological degree is thus usually referred as 'an algebraic count of the zeros of a continuous map'.

Equivariant degree theory is a topological degree theory that is concerned with *equivariant maps*, that is, maps that commute with the actions of a group on their space of domain and image. A main objective of the equivariant degree theory is to attain the topological structure of the zeros of an equivariant map and their algebraic properties induced by the equivariance. The past two decades have witnessed continuous progress in the development of equivariant degree, both in theory and in practice (see [6, 8, 11, 12, 17, 21] and references therein). Among others, a *twisted equivariant degree*, which is a truncated part of the full equivariant degree turns out to be the most 'computable' part of the equivariant degree, and serves as an effective topological tool in the study of equivariant systems, including the symmetric Hopf-bifurcation problems in equivariant dynamical systems.

The main advantage of using equivariant degree theory lies in both the topological and algebraic properties of the equivariant degree. On the one hand, equivariant degree is a topological invariant, which remains unchanged against all (admissible) equivariant homotopies. This allows, in practice, flexibility and freedom in computations of the equivariant degree. A concrete example is that equivariant degree theory for studying equivariant bifurcations can be also applied, in the case when critical eigenvalues of the Jacobian carry higher multiplicity. In other words, as a topological invariant, equivariant degree sees no additional complication in treating multiple eigenvalues. On the other hand, equivariant degree is algebraic, in a sense that it is compatible with respect to homomorphisms between groups (of symmetry). In application, this results in simplicity in treating change of equivariance in systems, which is also what we will use in this paper. Further applications of degree theory in coupled cell networks may be found in predicting periodic solutions in both variational and non-variational systems (see [16, 20] for equivariant systems), as well as for studying global bifurcations (see [3] for equivariant systems).

The main goal of this paper is to introduce a degree theory that is suitable for studying general coupled cell systems with quotient symmetries, where different quotient symmetries are brought together and fit in the integral picture of the total influence of quotient symmetries on the network dynamics. It is known that the set of all balanced equivalence relations (thus their induced synchrony subspaces) on a network forms a lattice (see [23]). Therefore, we are interested in defining a degree theory for maps which keep a given lattice of linear subspaces (that are also representations of individual symmetry groups) invariant and which are equivariant with respect to these group actions on the subspaces.

More precisely, a finite collection $\mathcal{L}$ of closed linear subspaces of a Banach space $X$ is called a *lattice*, if $X \in \mathcal{L}$ and $\mathcal{L}$ is closed under set intersections. A lattice $\mathcal{L}$ is called a *representation lattice*, if every $U \in \mathcal{L}$ is an isometric Banach representation of a compact Lie group $G_U$ such that for every $U_1 \subsetneq U_2$, there exists a group homomorphism $\mathsf{h}_{U_1,U_2} : G_{U_2} \to G_{U_1}$ satisfying

$$g.\, x = \mathsf{h}_{U_1,U_2}(g)_\circ\, x, \qquad \forall x \in U_1, \quad g \in G_{U_2}, \tag{1}$$

where '$.$' denotes the $G_{U_2}$-action and '$_\circ$' denotes the $G_{U_1}$-action. Moreover, $\mathsf{h}_{U_1,U_2} \circ \mathsf{h}_{U_2,U_3} = \mathsf{h}_{U_1,U_3}$ for every $U_1 \subseteq U_2 \subseteq U_3$. The *compatibility* condition (1) is essential for our consideration, since it allows us to 'lift' a $G_{U_1}$-orbit in $U_1$ to $G_{U_2}$-orbits in $U_2$. As we will see later, this enables an inductive definition of degree.

Let $\mathcal{L}$ be a representation lattice in $\mathbb{R}^n$ and $\mathbb{R}$ be a parameter space (on which all groups act trivially). For our purpose[1], we assume $G_U = \Gamma_U \times S^1$ for a finite group $\Gamma_U$, for every $U \in \mathcal{L}$. Let $\Omega \subset \mathbb{R} \times \mathbb{R}^n$ be an $\mathcal{L}$-*invariant* open bounded set, that is, $\Omega \cap (\mathbb{R} \times U)$ is $G_U$-invariant, for all $U \in \mathcal{L}$. Consider a continuous map $f : \overline{\Omega} \to \mathbb{R}^n$ such that $f^{-1}(0) \cap \partial\Omega = \emptyset$ (in which case, we say the pair $(f, \Omega)$ is *admissible*) and $f$ is $\mathcal{L}$-*equivariant*, meaning that $f(\overline{\Omega} \cap (\mathbb{R} \times U)) \subset U$ and

$$f_U := f|_{\overline{\Omega} \cap (\mathbb{R} \times U)} : \overline{\Omega} \cap (\mathbb{R} \times U) \to U$$

is $G_U$-equivariant, for all $U \in \mathcal{L}$. Then, the twisted equivariant degree of $f_U$ on $\Omega \cap (\mathbb{R} \times U)$ is well-defined, for every $U \in \mathcal{L}$.

A *lattice-equivariant degree*, denoted by $\mathcal{L}$-$\mathrm{Deg}^t$, is an assignment of a formal sum $\sum_{U \in \mathcal{L}} (U, a_U)$ to every admissible pair $(f, \Omega)$, where $a_U$ is inductively defined by

$$a_U := G_U\text{-}\mathrm{Deg}^t\,(f_U, \Omega \cap (\mathbb{R} \times U)) - \sum_{U' \subsetneq U} \mathsf{H}_{U',U}(a_{U'}),$$

where '$G_U$-$\mathrm{Deg}^t$' stands for the twisted equivariant degree and $\mathsf{H}_{U',U}$ is certain ring homomorphism induced by $\mathsf{h}_{U',U}$, which 'lift' $G_{U'}$-orbits in $U'$ to $G_U$-orbits in $U$. The geometric meaning of $a_U$ is that it gives an algebraic count of those $G_U$-orbits of zeros of $f$ that lie in $\Omega \cap (\mathbb{R} \times U)$ but not in $\Omega \cap (\mathbb{R} \times U')$ for any $U' \subsetneq U$.

[1] In the context of coupled cell systems with quotient symmetries, $\Gamma_U$ describes the coupling symmetry of the quotient network obtained by restricting the system to $U$, and the unit circle group $S^1$ describes the temporal symmetry of possible periodic states of the quotient network.

We show that the lattice-equivariant degree satisfies usual properties of a degree theory such as the *existence, homotopy invariance, additivity* and the *suspension*. Further, we show that it also has algebraic properties related to the *inclusion* and the *product* of lattices. Following the standard procedure, we also extend this degree to infinite-dimensional Banach spaces for compact vector fields.

Using the lattice-equivariant degree, we study the synchrony-breaking bifurcations in homogeneous coupled cell systems. More precisely, a coupled cell system is called *homogeneous*, if it consists of only one type of cell (in particular, every cell receives the same types of input arrows); that is, it can be described by

$$\begin{aligned} \dot{x}_1 &= f_o(x_1; x_{i_1}, \dots, x_{i_s}), \\ \dot{x}_2 &= f_o(x_2; x_{j_1}, \dots, x_{j_s}), \\ &\cdots \\ \dot{x}_n &= f_o(x_n; x_{k_1}, \dots, x_{k_s}), \end{aligned} \tag{2}$$

where $x_i \in \mathbb{R}^k$, $f_o : \mathbb{R}^k \times (\mathbb{R}^k)^s \to \mathbb{R}^k$ is of class $C^1$ and $k \in \mathbb{N}$ is the dimension of internal dynamics (the first argument in $f_o$ indicates the internal cell dynamics and the remaining variables indicate external couplings). Then,

$$\Delta_0 = \{x \in (\mathbb{R}^k)^n \ : \ x_1 = x_2 = \cdots = x_n\}$$

is a flow-invariant subspace of (2), independent of specific forms of $f_o$. Elements of $\Delta_0$ are called *fully synchronous*. In general, depending on the network structure, (2) may have a number of flow-invariant subspaces of *partial synchrony*, that is, they are given by equalities of the internal state of some of the cells. A *synchrony-breaking bifurcation* refers to a type of bifurcation, where a fully synchronous equilibrium loses its stability and bifurcates to states of less synchrony. If the bifurcating states are all oscillating, then it is called a *synchrony-breaking Hopf bifurcation*. We show how the lattice-equivariant degree can be used to give a topological treatment of synchrony-breaking Hopf bifurcations (see theorem 5.3).

As an example, we consider a 5-cell homogeneous coupled cell system which admits a large variety of symmetric quotient networks and show that these quotient symmetries lead to 25 bifurcating branches of oscillating states, characterized by their distinct synchrony types and symmetric properties.

## 2. Preliminaries

In this section, we give a brief review on compact Lie groups and their representations. We elaborate on the Euler ring associated with compact Lie groups, which is related to the range of lattice-equivariant degrees.

### *2.1. Compact Lie Groups and the Euler Ring*

**Definition 2.1 (see [9]).** A *topological group* $G$ is a group together with a topology on $G$ such that the binary operation and the inverse operation of $G$ are continuous with respect to this topology. Let $G$ be a topological group and $X$ be a topological space. A (left) *action* of $G$ on $X$ is a continuous map $\varphi : G \times X \to X$ such that

(i) $\varphi(g, \varphi(h, x)) = \varphi(gh, x)$ for all $g, h \in G$ and $x \in X$;
(ii) $\varphi(e, x) = x$ for all $x \in X$, where $e$ is the identity element of $G$.

A (left) *G-space* is a pair $(X, \varphi)$ consisting of a space $X$ together with a (left) action $\varphi$ of $G$ on $X$. We usually denote $(X, \varphi)$ just by the underlying topological space $X$. It is also convenient to denote $\varphi(g, x)$ by $gx$.

Let $G$ be a topological group and $X$ be a $G$-space. For a closed subgroup $L \subset G$, denote by $(L)$ the conjugacy class of $L$, $X/G$ the orbit space, $G/L$ the left coset of $L$ and $N(L)$ the normalizer of $L$ in $G$. Note that $(L) \simeq G/L$ are canonically isomorphic. Write $(L_1) \geqslant (L_2)$, if $L_1 \supseteq gL_2g^{-1}$ for some $g \in G$. Set

$$X_L := \{x\ :\ gx = x \iff g \in L\} \tag{3}$$

$$X^L := \{x\ :\ gx = x,\ \text{if } g \in L\} \tag{4}$$

$$X_{(L)} := \{gx\ :\ x \in X_L,\ g \in G\}. \tag{5}$$

Let $x \in X$. If $x \in X_L$, then $L$ is called the *isotropy type* of $x$. If $x \in X^L$, then $x$ is called an *$L$-fixed point*. By *orbit* of $x$, we mean $\{gx\ :\ g \in G\}$. If $x \in X_{(L)}$, then $(L)$ is called the *orbit type* of $x$. Note that $X_{(L)}$ is a $G$-invariant subspace of $X$.

**Example 2.2.** Let $G$ be a topological group and $H \subset G$ be a closed subgroup. Then, $(H) \simeq G/H$ is a $G$-space with respect to the action given by

$$G \times G/H \to G/H, \qquad (g', gH) \mapsto (g'g)H. \tag{6}$$

The isotropy type of the element $gH \in G/H$ is given by $gHg^{-1}$. Similarly, let $K \subset G$ be another closed subgroup. Consider the action defined on the product $G$-space $G/H \times G/K$ by

$$G \times \big(G/H \times G/K\big) \to G/H \times G/K, \qquad \big(g', (g_1H, g_2K)\big) \mapsto \big((g'g_1)H, (g'g_2K)\big). \tag{7}$$

The isotropy type of the element $(g_1H, g_2K) \in G/H \times G/K$ is then given by

$$g_1Hg_1^{-1} \cap g_2Kg_2^{-1}. \qquad \diamondsuit$$

**Definition 2.3.** A *Lie group* $G$ is a group which is also a finite-dimensional smooth manifold, in which the binary operation and the inverse operation of $G$ are smooth maps. A Lie group $G$ which is also compact with respect to this smooth structure is called a *compact Lie group*.

**Definition 2.4 (see [9]).** Let $G$ be a compact Lie group and $\Phi(G)$ be the set of conjugacy classes of closed subgroups of $G$. Let $A(G) := \mathbb{Z}[\Phi(G)]$ be the free $\mathbb{Z}$-module generated by $\Phi(G)$. The *Euler ring* of $G$ is the set $A(G)$, together with the following ring multiplication

$$(H) * (K) = \sum_{(L)\in\Phi(G)} n_L(L), \qquad \text{for } (H), (K) \in \Phi(G), \tag{8}$$

where

$$n_L := \chi_c((G/H \times G/K)_{(L)}/G), \tag{9}$$

for $\chi_c$ being the Euler characteristic taken in Alexander–Spanier cohomology with compact support (cf [22]). The set $G/H \times G/K$ in (9) is considered as a $G$-space under the action given by (7). $\diamondsuit$

**Example 2.5.** Let $G = D_3$, where $D_3 = \mathbb{Z}_3 \cup \kappa\mathbb{Z}_3$ and $\mathbb{Z}_3 = \langle \xi \rangle$. Then,

$$A(D_3) = \{(D_3), (D_1), (\mathbb{Z}_3), (\mathbb{Z}_1)\},$$

where $D_1 = \langle \kappa \rangle$. The Euler ring multiplication is listed in table 1 (see [6]). The geometric meaning of $(H) * (K)$ is that it counts the $G$-orbits in the product $G$-space $G/H \times G/K$ according to their orbit types. For example, consider

$$(D_1) * (D_1) = (D_1) + (\mathbb{Z}_1).$$

Note that $D_3/D_1 = \{D_1, \xi D_1, \xi^2 D_1\}$. Thus, the product space $D_3/D_1 \times D_3/D_1$ consists of 9 elements, which are represented by hollow squares and triangles in figure 1. As shown

**Table 1.** Multiplication table for the Euler ring $A(D_3)$.

| $*$ | $(D_3)$ | $(D_1)$ | $(\mathbb{Z}_3)$ | $(\mathbb{Z}_1)$ |
|---|---|---|---|---|
| $(D_3)$ | $(D_3)$ | $(D_1)$ | $(\mathbb{Z}_3)$ | $(\mathbb{Z}_1)$ |
| $(D_1)$ | $(D_1)$ | $(D_1)+(\mathbb{Z}_1)$ | $(\mathbb{Z}_1)$ | $3(\mathbb{Z}_1)$ |
| $(\mathbb{Z}_3)$ | $(\mathbb{Z}_3)$ | $(\mathbb{Z}_1)$ | $2(\mathbb{Z}_3)$ | $2(\mathbb{Z}_1)$ |
| $(\mathbb{Z}_1)$ | $(\mathbb{Z}_1)$ | $3(\mathbb{Z}_1)$ | $2(\mathbb{Z}_1)$ | $6(\mathbb{Z}_1)$ |

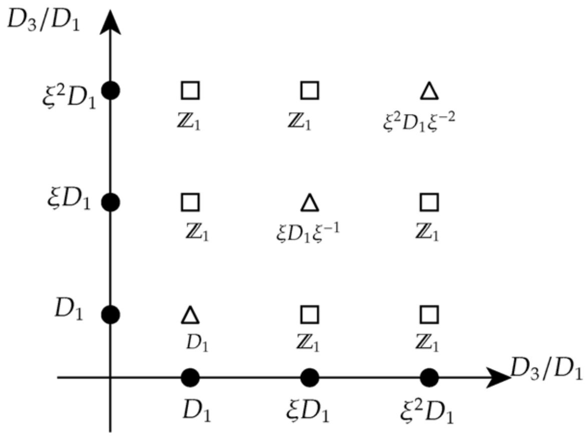


**Figure 1.** Interpretation of $(D_1) * (D_1)$.

in example 2.2, the isotropy of the element $(\xi^a D_1, \xi^b D_1) \in D_3/D_1 \times D_3/D_1$ is given by $\xi^a D_1 \xi^{-a} \cap \xi^b D_1 \xi^{-b}$, for $a, b \in \{0, 1, 2\}$, as indicated in figure 1. These isotropies give rise to two orbit types in $D_3/D_1 \times D_3/D_1$:

$$(D_1) = \{D_1, \xi D_1 \xi^{-1}, \xi^2 D_1 \xi^{-2}\},$$
$$(\mathbb{Z}_1) = \{\mathbb{Z}_1\},$$

corresponding to the hollow triangles and squares in figure 1, respectively. Moreover, all the hollow triangles (respectively, all the hollow squares) consist of 1 orbit under the $D_3$-action on $D_3/D_1 \times D_3/D_1$. Therefore, $D_3/D_1 \times D_3/D_1$ consists of 1 orbit of orbit type $(D_1)$ and 1 orbit of orbit type $(\mathbb{Z}_1)$, or equivalently written as

$$(D_1) * (D_1) = (D_1) + (\mathbb{Z}_1). \qquad \Diamond$$

Given two compact Lie groups $G_1, G_2$ and a group homomorphism $\mathsf{h} : G_2 \to G_1$, one can define a ring homomorphism $\mathsf{H} : A(G_1) \to A(G_2)$ induced by $\mathsf{h}$.

**Definition 2.6.** Let $G_1, G_2$ be compact Lie groups, $X$ be a $G_1$-space and $\mathsf{h} : G_2 \to G_1$ be a group homomorphism. Define a $G_2$-action on $X$ by

$$g_2 x := \mathsf{h}(g_2)x, \qquad \text{for } g_2 \in G_2,\ x \in X,$$

and call it the *induced action* of $G_2$ on $X$ through $\mathsf{h}$. Then, $X$ is also a $G_2$-space.

In particular, consider $G_1/K$ as a $G_1$-space, for any closed subgroup $K \subset G_1$. Then, there is an induced action of $G_2$ on $G_1/K$ through $\mathsf{h}$, where the isotropy of $gK$ under this $G_2$-action is given by

$$\mathsf{h}^{-1}(gKg^{-1}).$$

Define a map

$$\begin{aligned} \mathsf{H} : A(G_1) &\to A(G_2) \\ (K) &\mapsto \sum_{(\tilde{K}) \in \Phi(G_2)} \chi_c((G_1/K)_{(\tilde{K})}/G_2)(\tilde{K}), \end{aligned} \tag{10}$$

where $G_1/K$ is considered as a $G_2$-space with the induced action through $\mathsf{h}$. In a sense, $\mathsf{H}$ 'lifts' a $G_1$-orbit of orbit type $(K)$ to several $G_2$-orbits of orbit types $(\tilde{K})$, where $\tilde{K} = \mathsf{h}^{-1}(gKg^{-1})$ for some $g \in G_1$.

Based on the fact that the Euler ring is a universal additive invariant, we have:

**Theorem 2.7 (see [5, 9]).** *Let $G_i$ be a compact Lie group for $i = 1, 2, 3$ and $\mathsf{h}_i : G_i \to G_{i+1}$ a group homomorphism for $i = 1, 2$. Let $\mathsf{H}_i$ be defined by (10) for $i = 1, 2$. Then, we have*

*(i) $\mathsf{H}_i$ is an Euler ring homomorphism, for $i = 1, 2$.*
*(ii) $\mathsf{H}_2 \circ \mathsf{H}_1$ is precisely the Euler ring homomorphism induced by $\mathsf{h}_2 \circ \mathsf{h}_1$.*

*We call $\mathsf{H}$ defined by (10) the the Euler ring homomorphism induced by $\mathsf{h}$.*

**Example 2.8.** Let $G_1 = D_3$, $G_2 = D_1$. Consider the inclusion homomorphism

$$\mathsf{h} : D_1 \hookrightarrow D_3.$$

Let $\mathsf{H} : A(D_3) \to A(D_1)$ be the induced Euler ring homomorphism. Then, we have

$$\begin{aligned} \mathsf{H} : (D_3) &\mapsto (D_1), \qquad (\mathbb{Z}_3) \mapsto (\mathbb{Z}_1) \\ (D_1) &\mapsto (D_1) + (\mathbb{Z}_1), \qquad (\mathbb{Z}_1) \mapsto 3(\mathbb{Z}_1). \end{aligned}$$

The geometric meaning of $\mathsf{H}((K))$ is that it counts the $G_2$-orbits in the $G_2$-space $G_1/K$ according to their orbit types. For example, consider $K = D_1$. Then, the space $D_3/D_1$ consists of 3 elements: $D_1$, $\xi D_1$, $\xi^2 D_1$ (following the notations in example 2.2). Note that all of them have the same orbit type $(D_1)$, under the $D_3$-action. Now consider the $D_1$-action (induced through $\mathsf{h}$). Then, they have the following isotropies

$$\mathsf{h}^{-1}(D_1) = D_1, \qquad \mathsf{h}^{-1}(\xi D_1 \xi^{-1}) = \mathbb{Z}_1, \quad \mathsf{h}^{-1}(\xi^2 D_1 \xi^{-2}) = \mathbb{Z}_1,$$

respectively (cf figure 2). Moreover, the elements $\xi D_1$, $\xi^2 D_1$ belong to the same orbit, since

$$\mathsf{h}(\kappa)\xi D_1 = \kappa \xi D_1 = \xi^2 \kappa D_1 = \xi^2 D_1.$$

Therefore, the space $D_3/D_1$ (with respect to the induced $D_1$-action) consists of 1 orbit of orbit type $(D_1)$ and 1 orbit of orbit type $(\mathbb{Z}_1)$, i.e.

$$\mathsf{H}((D_1)) = (D_1) + (\mathbb{Z}_1).$$

### *2.2. Representations of compact lie groups*

Representations of a group $G$ are vector spaces that are also $G$-spaces, in which every group element acts linearly.

**Definition 2.9.** Let $V$ be a finite-dimensional real (respectively complex) vector space and $G$ be a Lie group. A *representation* of $G$ on $V$ is a continuous action

$$\varphi : G \times V \to V$$

such that for every $g \in G$, the map $\varphi(g, \cdot) : V \to V$ is linear. The pair $(V, \varphi)$ is called a *real (respectively complex) representation*. We usually denote $(V, \varphi)$ just by the underlying representation space $V$.

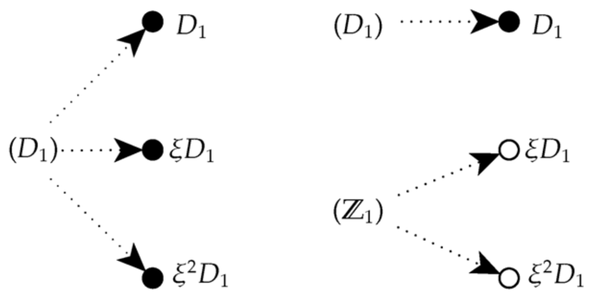


**Figure 2.** Left: the space $D_3/D_1$ considered as $D_3$-space; Right: the space $D_3/D_1$ considered as $D_1$-space.

**Definition 2.10.** Let $W$ be a Banach space over reals (respectively complex numbers) and $G$ be a Lie group. Let $(W, \varphi)$ be a representation of $G$ on $W$. If for every $g \in G$, the map $\varphi(g, \cdot) : W \to W$ is bounded linear, then the pair $(W, \varphi)$ is called a *real (respectively complex) Banach representation* of $G$. A Banach representation $(W, \varphi)$ is called *isometric*, if $\|\varphi(g, w)\| = \|w\|$, for all $g \in G$, $w \in W$. We also denote $(W, \varphi)$ just by $W$.

Using the Haar measure of $G$, one can show that every Banach representation is equivalent to an isometric Banach representation. Throughout the paper, every Banach representation is assumed to be isometric.

**Example 2.11.** Let $n \in \mathbb{N}$ be a positive integer and $C([0, T]; \mathbb{R}^n)$ be the set of all continuous $T$-periodic functions defined on $[0, T]$ and valued in $\mathbb{R}^n$. Then, $C([0, T]; \mathbb{R}^n)$ is a vector space over reals. Moreover, it is a real Banach space with respect to the supremum norm $\|\cdot\|$, which is defined by

$$\|f\| := \sup_{x\in[0,T]}|f(x)|, \qquad \forall\ f \in C([0, T]; \mathbb{R}^n).$$

Let $G = S^1$ be the set of all complex numbers of length 1. Then, $S^1$ is a Lie group. Define an action of $S^1$ on $C([0, T]; \mathbb{R}^n)$ by

$$(\mathrm{e}^{\mathrm{i}\theta} f)(t) := f\left(t + \frac{\theta T}{2\pi}\right), \qquad \forall\ f \in C([0, T]; \mathbb{R}^n), \quad \mathrm{e}^{\mathrm{i}\theta} \in S^1,\ t \in [0, T], \tag{11}$$

which is clearly continuous. Note that $\|\mathrm{e}^{\mathrm{i}\theta} f\| = \|f\|$ for all $\mathrm{e}^{\mathrm{i}\theta} \in S^1$, $f \in C([0, T]; \mathbb{R}^n)$. Thus, $C([0, T]; \mathbb{R}^n)$ is a real isometric Banach representation of $S^1$ with respect to the action (11).

### *2.3. Twisted subgroups in* $\Gamma \times S^1$

Let $\Gamma$ be a finite group and $S^1$ be the group of complex numbers of unit length. The *twisted subgroups* of $\Gamma \times S^1$ are in short, closed subgroups that are not of form $K \times S^1$ for some subgroup $K$ of $\Gamma$. We follow the definition in [6].

**Definition 2.12.** A subgroup $H \subset \Gamma \times S^1$ is called a *twisted l-folded subgroup*, if there exists a subgroup $K \subset \Gamma$, an integer $l \geqslant 0$ and a group homomorphism $\varphi : K \to S^1$ such that

$$H = K^{\varphi,l} := \{(\gamma, z) : \varphi(\gamma) = z^l\}.$$

It can be verified that every closed subgroup $H \subset \Gamma \times S^1$ is either twisted or of form $K \times S^1$ for some subgroup $K$ of $\Gamma$. In the context of applications, where $\Gamma$ stands for symmetry of a system in the phase space and $S^1$ describes the temporal symmetry of possible periodic states, twisted subgroups are precisely the symmetry of nontrivial periodic states.

Let $X$ be a $G$-space and $x \in X$ of orbit type $(H)$. It is known that the orbit of $x$ is diffeomorphic to $G/H$. Thus, to study elements of twisted orbit types, it is sufficient to describe the structure of $\Gamma \times S^1/H$ for twisted subgroups $H$.

**Lemma 2.13.** *Let $\Gamma$ be a finite group and $H = K^{\varphi,l}$ be a twisted $l$-folded subgroup in $\Gamma \times S^1$. Then, $\Gamma \times S^1/H$ is diffeomorphic to a disjoint union of $|\Gamma/K|$ copies of circles, whose isotropy types (under the action of $\Gamma \times S^1$) are $\gamma H \gamma^{-1}$, for $\gamma \in \Gamma/K$.*

**Proof.** It is clear that $\Gamma \times S^1/H$ is a disjoint union of finite circles, since it is a one-dimensional compact manifold. Also every element of $\Gamma \times S^1/H$ has an isotropy type $\gamma H \gamma^{-1}$, for some $\gamma \in \Gamma$. What we need to show is that these circles are precisely indexed by their isotropy types $\gamma H \gamma^{-1}$, as $\gamma$ runs through $\Gamma/K$.

Let $X = \Gamma \times S^1/H$. Consider an $S^1/\mathbb{Z}_l$-action on $X$ given by

$$\psi : S^1/\mathbb{Z}_l \times X \to X, \qquad (\omega, (\gamma, z)H) \mapsto (\gamma, \omega z)H,$$

for $\omega \in S^1/\mathbb{Z}_l$, $\gamma \in \Gamma$ and $z \in S^1$. The action is a well-defined, since the multiplication on the second component of $H$ is abelian. Also, if $(\gamma, \omega z)H = (\gamma, z)H$, then $(1, \omega) \in H$, which implies that $\omega^l = 1$, i.e. $\omega \in \mathbb{Z}_l$. Thus, $\psi$ is a free action. Consequently, $X$ is a one-dimensional compact manifold with a free $S := S^1/\mathbb{Z}_l$-action. Therefore, $S \hookrightarrow X \to X/S$ is a principal bundle (cf [18]) over a finite set, that is,

$$X \simeq S \times X/S.$$

For $x \in X$, write $[x]$ as the $S$-orbit of $x$. Then, we have

$$X/S = \{[(\gamma, 1)H] : \gamma \in \Gamma\} = \{[(\gamma, 1)H] : \gamma \in \Gamma/K\},$$

where the last equality used the fact that $[(\gamma, 1)H] = [H]$ if and only if $\gamma \in K$. Finally, the isotropy type of $[(\gamma, 1)H] \in X/S$ is $\gamma H \gamma^{-1}$, for all $\gamma \in \Gamma/K$. □

**Example 2.14.** Let $\Gamma = D_3$, where $D_3 = \mathbb{Z}_3 \cup \kappa\mathbb{Z}_3$ and $\mathbb{Z}_3 = \langle \xi \rangle$. Then, up to conjugacy, $D_3 \times S^1$ has the following twisted 1-folded subgroups: $\mathbb{Z}_1$, $\mathbb{Z}_3$, $D_1$, $D_3$ and (cf [6])

$$\mathbb{Z}_3^t = \{(1, 1), (\xi, \xi), (\xi^2, \xi^2)\}, \quad D_1^z = \{(1, 1), (\kappa, -1)\},$$
$$D_3^z = \{(1, 1), (\xi, 1), (\xi^2, 1), (\kappa, -1), (\kappa\xi, -1), (\kappa\xi^2, -1)\}.$$

Let $D_3$ act on $\mathbb{R}^3$ as the permutation group $S_3 \simeq D_3$. Let $C([0, T]; \mathbb{R}^3)$ be given by example 2.11. Define a $D_3 \times S^1$-action on $C([0, T]; \mathbb{R}^3)$ by

$$\big((\gamma, e^{i\theta}) f\big)(t) := \gamma . f\left(t + \frac{\theta T}{2\pi}\right),$$

where '.' stands for the $D_3$-action on $\mathbb{R}^3$. Then, a function $u \in C([0, T]; \mathbb{R}^3)$ has an isotropy type $\mathbb{Z}_3^t$ under this action if and only if

$$(\xi, \xi)u(t) := (\xi, \xi)\begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix} = \begin{pmatrix} z\left(t + \frac{T}{3}\right) \\ x\left(t + \frac{T}{3}\right) \\ y\left(t + \frac{T}{3}\right) \end{pmatrix} = \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix}, \qquad \forall t \in [0, T].$$

Thus, $x(t) = y(t + \frac{2T}{3}) = z(t + \frac{T}{3})$ and $u$ is of form

$$u(t) = \left(x(t), x\left(t + \frac{T}{3}\right), x\left(t + \frac{2T}{3}\right)\right), \qquad \forall t \in [0, T].$$

That is, knowing the (twisted) isotropy types of a periodic function $u$ helps determine the form of $u$. More examples of this kind can be found in table 2.

## 3. Representation lattices

In this section, we give the definition of representation lattices. Besides the basic properties derived from the lattice structure, we discuss algebraic properties of representation lattices that are related to the *inclusion* and the *product* of lattices.

**Definition 3.1.** Let $X$ be a real (respectively complex) Banach space and $\mathcal{L}$ be a finite set of closed linear subspaces of $X$. We say that $\mathcal{L}$ is a *lattice* in $X$, if $X \in \mathcal{L}$ and

$$U_1 \cap U_2 \in \mathcal{L}, \qquad \forall\, U_1, U_2 \in \mathcal{L}.$$

We write $U_1 \leqslant U_2$ (respectively $U_1 < U_2$), if $U_1 \subset U_2$ (respectively $U_1 \subsetneq U_2$). A subset $\mathcal{S} \subset \mathcal{L}$ is called a *sublattice* of $\mathcal{L}$, if it is a lattice on its own right.

**Definition 3.2.** Let $X$ be a real (respectively complex) Banach space and $\mathcal{L}$ be a lattice in $X$. Assume that

(i) (REPRESENTATION) $U$ is a real (respectively complex) isometric Banach representation of a compact Lie group $G_U$, for every $U \in \mathcal{L}$;

(ii) (COMPATIBILITY) there exists a group homomorphism

$$\mathsf{h}_{U_1,U_2} : G_{U_2} \to G_{U_1},$$

for every $U_1, U_2$ with $U_1 \leqslant U_2$ such that

$$g_2 x = \mathsf{h}_{1,2}(g_2)x, \qquad \forall\, g_2 \in G_{U_2},\ x \in U_1;$$

(iii) (CONSISTENCE) $\mathsf{h}_{U_1,U_2} \circ \mathsf{h}_{U_2,U_3} = \mathsf{h}_{U_1,U_3}$ for every $U_1 \leqslant U_2 \leqslant U_3$.

Then, $\mathcal{L}$ is called a *real (respectively complex) representation lattice* in $X$. The collection

$$\{(U_i, G_{U_i}, \mathsf{h}_{U_i,U_j}) \ :\ U_i, U_j \in \mathcal{L},\ U_i \leqslant U_j\}$$

is called a *structure of representation lattice* of $\mathcal{L}$. ◊

Note that a sublattice $\mathcal{S}$ of a representation lattice $\mathcal{L}$ is again a representation lattice, which we call a *representation sublattice* of $\mathcal{L}$.

**Example 3.3.** Let $X = \mathbb{R}^5$ and $\mathcal{L}$ be a lattice of 10 elements in $X$ given in figure 3, where the pair $(\Delta_*, \Gamma_*)$ indicates that $\Delta_*$ is a representation of $\Gamma_*$ and $\Delta_*$ is the linear subspace composed of vectors of form indicated below the pair $(\Delta_*, \Gamma_*)$. The arrows give the direction of homomorphisms between $\Gamma_*$. The structure of the representation lattice $\mathcal{L}$ is specified as follows:

*Representations.* $\mathbb{Z}_1$ acts on $\Delta, \Delta_{21}, \mathbb{R}^5$ trivially; $\mathbb{Z}_2 = \langle\kappa\rangle$ acts on $\Delta_{00}, \Delta_{02}, \Delta_{03}, \Delta_2$ by $\kappa : (a, b, c, d, e) \mapsto (a, b, e, d, c)$; and $D_3 \simeq S_3$ acts on $\Delta_4, \Delta_1, \Delta_{01}$ by the natural action of $S_3$ on symbols $a, b, c$.

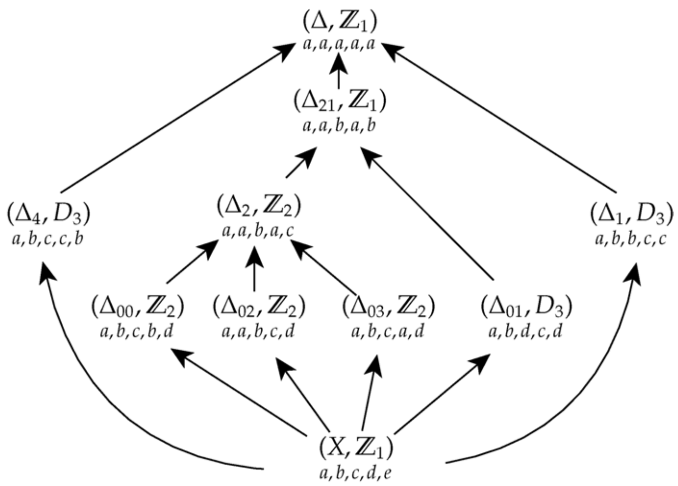


**Figure 3.** A representation lattice $\mathcal{L}$ in $\mathbb{R}^5$.

*Homomorphisms.* $\mathbb{Z}_1 \to \Gamma_x$ are given by the inclusion; homomorphisms $\Gamma_x \to \mathbb{Z}_1$ are given by the projection; and homomorphisms $\mathbb{Z}_2 \to \mathbb{Z}_2$ are given by the identity homomorphism.

It can be verified that $\mathcal{L}$ is a real representation lattice in $X$, with respect to this structure. ♢

### 3.1. Basic properties of lattices

We discuss some basic properties of representation lattices, when viewed as lattices (without representation structure).

Let $\mathcal{L}$ be a lattice in a Banach space $X$ and $U_1, U_2 \in \mathcal{L}$. If $U_1 < U_2$, then $U_2$ is called a *descendant* of $U_1$. A minimal descendant is called an *immediate descendant*. Denote by

$$\mathcal{L}^{\top} := \{U \in \mathcal{L} \,:\, U \text{ has a unique immediate descendant in } \mathcal{L}\}.$$

**Lemma 3.4.** *Let $X$ be a Banach space and $\mathcal{L}$ be a lattice in $X$. Then,*

*(i) $\mathcal{L}$ has a unique minimal element;*
*(ii) for $U \in \mathcal{L}$, the set $\mathcal{L} \setminus \{U\}$ is a sublattice of $\mathcal{L}$ if and only if $U \in \mathcal{L}^{\top}$;*
*(iii) Let $\mathcal{S} \subset \mathcal{L}$ be a sublattice and set $k := |\mathcal{L} \setminus \mathcal{S}|$, where $|\cdot|$ is the count of elements. Then there exists a flag of lattices of length $k$*

$$\mathcal{L} = \mathcal{L}_0 \supset \mathcal{L}_1 \supset \cdots \supset \mathcal{L}_k = \mathcal{S}$$

*such that $\mathcal{L}_{i+1} = \mathcal{L}_i \setminus \{U_i\}$ for certain $U_i \in \mathcal{L}_i$, $i = 0, 1, \ldots, k-1$;*
*(iv) Let $X'$ be another Banach space and $\mathcal{M}$ be a lattice in $X'$. Then,*

$$\mathcal{L} \times \mathcal{M} := \{U \times M \,:\, U \in \mathcal{L},\ M \in \mathcal{M}\}$$

*is a lattice in $X \times X'$.*

**Proof.**

(i) Since $\mathcal{L}$ is closed under set intersections, the minimal element is given by the intersection of all elements of $\mathcal{L}$.

(ii) Let $\mathcal{S} = \mathcal{L} \setminus \{U\}$. If $U \in \mathcal{L}^\top$, then $U \neq X$ and $U = U_1 \cap U_2$ for some $U_1, U_2 \in \mathcal{L}$ only if $U \in \{U_1, U_2\}$. It follows that $X \in \mathcal{S}$ and $U \neq U_1 \cap U_2$ for any $U_1, U_2 \in \mathcal{S}$. Thus, $\mathcal{S}$ is a sublattice. If $U \notin \mathcal{L}^\top$, then $U$ has more than one immediate descendants. Let $U_1, U_2$ be two distinct immediate descendants of $U$ in $\mathcal{L}$. Then, $U = U_1 \cap U_2$ for $U_1, U_2 \in \mathcal{S}$. But $U \notin \mathcal{S}$, which implies that $\mathcal{S}$ is not a sublattice.

(iii) We claim that

$$\mathcal{P}^\top \setminus \mathcal{S} \neq \emptyset, \qquad \text{for every lattice } \mathcal{P} \text{ s.t. } \mathcal{L} \supset \mathcal{P} \supsetneq \mathcal{S}. \tag{12}$$

Assume to the contrary and let $U$ be a maximal element of $\mathcal{P} \setminus \mathcal{S}$. In particular, since $U \neq X$, $U$ has descendants. By assumption, $U$ has at least two distinct immediate descendants in $\mathcal{P}$, say $U_1, U_2$. Then, $U = U_1 \cap U_2$. Moreover, since $U$ is a maximal element of $\mathcal{P} \setminus \mathcal{S}$, we have $U_1, U_2 \in \mathcal{S}$. It follows that $U = U_1 \cap U_2 \in \mathcal{S}$, which is a contradiction to the fact that $U \notin \mathcal{S}$. Thus, (12) holds.

It follows from (12) that there exists $U_0 \in \mathcal{L}^\top \setminus \mathcal{S}$. By (ii), $\mathcal{L}_1 := \mathcal{L} \setminus \{U_0\}$ is a sublattice. By applying (12) inductively to $\mathcal{L}_{i+1} = \mathcal{L}_i \setminus \{U_i\}$, for $U_i \in \mathcal{L}_i^\top \setminus \mathcal{S}$, $i = 1, \ldots, k-1$, we obtain the desired flag of lattices.

(iv) It follows from the fact that

$$(U_1 \times Q_1) \cap (U_2 \times Q_2) = (U_1 \cap U_2) \times (Q_1 \cap Q_2)$$

for $U_i \in \mathcal{L}$, $Q_i \in \mathcal{M}$, $i = 1, 2$. □

In analogue, we have

**Corollary 3.5.** *The properties (i)–(iv) in lemma 3.4 hold for representation lattices.*

**Proof.** Let $\mathcal{L}$ be a representation lattice. Then, (i) clearly holds. Moreover, since representation sublattices are precisely sublattices of representation lattices, (ii) and (iii) also hold.

Let $X'$ be another Banach space and $\mathcal{M}$ be a lattice in $X'$. Then,

$$\{(U \times Q, G_U \times G_Q, \mathsf{h}_{U,V} \times \mathsf{h}_{Q,P}) \;:\; U, V \in \mathcal{L},\; Q, P \in \mathcal{M}, U \subset V, Q \subset P\} \tag{13}$$

gives $\mathcal{L} \times \mathcal{M}$ a structure of representation lattice. □

We call $\mathcal{L} \times \mathcal{M}$ together with (13) *the product representation lattice* of $\mathcal{L}$ and $\mathcal{M}$.

## *3.2. Algebraic properties of representation lattices*

We associate to a representation lattice an algebraic structure based on the Euler ring of compact Lie groups, which will be the range of the lattice degree introduced in the next section. Our goal in this subsection is to extend the usual lattice operation such as the inclusion and product, to the representation lattices with respect to this algebraic structure.

**Definition 3.6.** Let $\mathcal{L}$ be a representation lattice with structure $\{U, G_U, \mathsf{h}_{U,V}\}$. For $U \in \mathcal{L}$, denote by $A(G_U)$ the Euler ring of $G_U$ (see definition 2.4). Let

$$R(\mathcal{L}) := \left\{ \sum_{U \in \mathcal{L}} (U, a_U) \;:\; a_U \in A(G_U) \right\}, \tag{14}$$

which is a $\mathbb{Z}$-module with respect to

$$\sum_{U \in \mathcal{L}} (U, a_U) + \sum_{U \in \mathcal{L}} (U, b_U) := \sum_{U \in \mathcal{L}} (U, a_U + b_U), \qquad a_U, b_U \in A(G_U),$$

$$k \sum_{U \in \mathcal{L}} (U, a_U) := \sum_{U \in \mathcal{L}} (U, k a_U), \qquad k \in \mathbb{Z}.$$

Define a ring multiplication on $R(\mathcal{L})$ by

$$\sum_{U\in\mathcal{L}}(U,a_U)\cdot\sum_{U\in\mathcal{L}}(U,b_U):=\sum_{U\in\mathcal{L}}(U,a_U*b_U),\qquad a_U,b_U\in A(G_U),\tag{15}$$

where '$*$' stands for the Euler ring multiplication in $A(G_U)$. The $\mathbb{Z}$-module $R(\mathcal{L})$ together with (15) is called the *associated ring* of $\mathcal{L}$.

*3.2.1. Reduction map.* Let $\mathcal{L}$ be a representation lattice and $\mathcal{S}\subset\mathcal{L}$ be a representation sublattice. Then, every $U\in\mathcal{L}\setminus\mathcal{S}$ has a unique minimal descendant in $\mathcal{S}$, which is given by the intersection of all the descendants of $U$ in $\mathcal{S}$.

**Definition 3.7.** Let $\mathcal{L}$ be a representation lattice with structure $\{U,G_U,h_{U,V}\}$ and $\mathcal{S}\subset\mathcal{L}$ be a representation sublattice. Define the *reduction map* from $R(\mathcal{L})$ to $R(\mathcal{S})$ by

$$\Phi^{\mathcal{L}}_{\mathcal{S}}:R(\mathcal{L})\to R(\mathcal{S})$$
$$(U,a)\mapsto\begin{cases}(U_d,\mathsf{H}_{U,U_d}(a)), & \text{if } U\in\mathcal{L}\setminus\mathcal{S},\\ (U,a), & \text{if } U\in\mathcal{S},\end{cases}\tag{16}$$

where $U\in\mathcal{L}$, $a\in A(G_U)$, $U_d$ stands for the unique minimal descendant of $U$ in $\mathcal{S}$ and $\mathsf{H}_{U,U_d}$ is the Euler ring homomorphism induced by $\mathsf{h}_{U,U_d}$ (see (10)).

We show that the reduction map is compatible with the inclusion of lattices.

**Lemma 3.8.** *Let $\mathcal{L}$ be a representation lattice, $\mathcal{S},\mathcal{P}\subset\mathcal{L}$ be representation sublattices such that $\mathcal{L}\supset\mathcal{P}\supset\mathcal{S}$. Then, we have $\Phi^{\mathcal{L}}_{\mathcal{S}}=\Phi^{\mathcal{P}}_{\mathcal{S}}\circ\Phi^{\mathcal{L}}_{\mathcal{P}}$.*

**Proof.** Let $U\in\mathcal{L}$ and $a\in A(G_U)$. If $U\in\mathcal{S}$, then $(U,a)$ is a fixed point of $\Phi^{\mathcal{L}}_{\mathcal{S}}$, $\Phi^{\mathcal{P}}_{\mathcal{S}}$ and $\Phi^{\mathcal{L}}_{\mathcal{P}}$. Thus, the statement holds.

Let $U\in\mathcal{L}\setminus\mathcal{S}$ and $U_d$ be the unique minimal descendant of $U$ in $\mathcal{S}$. Then,

$$\Phi^{\mathcal{L}}_{\mathcal{S}}\big((U,a)\big)=\big(U_d,\mathsf{H}_{U,U_d}(a)\big).$$

If $U\in\mathcal{P}$, then $(U,a)$ is a fixed point of $\Phi^{\mathcal{L}}_{\mathcal{P}}$ and $\Phi^{\mathcal{P}}_{\mathcal{S}}((U,a))=(U_d,\mathsf{H}_{U,U_d}(a))$. So $\Phi^{\mathcal{P}}_{\mathcal{S}}\circ\Phi^{\mathcal{L}}_{\mathcal{P}}\big(U,a)\big)$ agrees with $\Phi^{\mathcal{L}}_{\mathcal{S}}\big((U,a)\big)$.

Otherwise if $U\in\mathcal{L}\setminus\mathcal{P}$, then $\Phi^{\mathcal{L}}_{\mathcal{P}}((U,a))=(U_c,\mathsf{H}_{U,U_c}(a))$, where $U_c$ is the unique minimal descendant of $U$ in $\mathcal{P}$. In the case $U_c\in\mathcal{S}$, we have $U_c=U_d$, by the uniqueness of minimal descendant. In the case $U_c\in\mathcal{P}\setminus\mathcal{S}$, $U_d$ is the unique minimal descendant of $U_c$ in $\mathcal{S}$. Consequently, in both cases, we have

$$\Phi^{\mathcal{P}}_{\mathcal{S}}\circ\Phi^{\mathcal{L}}_{\mathcal{P}}\big((U,a)\big)=\Phi^{\mathcal{P}}_{\mathcal{S}}\big((U_c,\mathsf{H}_{U,U_c}(a))\big)=\big(U_d,\mathsf{H}_{U,U_d}(a)\big).\qquad\square$$

**Example 3.9.** Let $\mathcal{L}$ be the representation lattice given in example 3.3 and $\mathcal{S}$ be a representation sublattice given by $\mathcal{S}=\mathcal{L}\setminus\{\Delta_{01}\}$ (see figure 4). Let $\Phi^{\mathcal{L}}_{\mathcal{S}}$ be the reduction map defined by (16). Let $\mathsf{H}_{\Delta_{01},\mathbb{R}_5}$ be the ring homomorphism induced by the inclusion homomorphism $\mathsf{h}_{\Delta_{01},\mathbb{R}_5}:\mathbb{Z}_1\to\mathbb{Z}_2$.

Then, $\Phi^{\mathcal{L}}_{\mathcal{S}}$ fixes all generators of $R(\mathcal{L})$ except

$$\Phi^{\mathcal{L}}_{\mathcal{S}}\big(\Delta_{01},(\mathbb{Z}_2)\big)=\big(\mathbb{R}^5,\mathsf{H}_{\Delta_{01},\mathbb{R}_5}\big((\mathbb{Z}_2)\big)\big)=\big(\mathbb{R}^5,(\mathbb{Z}_1)\big),$$
$$\Phi^{\mathcal{L}}_{\mathcal{S}}\big(\Delta_{01},(\mathbb{Z}_1)\big)=\big(\mathbb{R}^5,\mathsf{H}_{\Delta_{01},\mathbb{R}_5}\big((\mathbb{Z}_1)\big)\big)=\big(\mathbb{R}^5,2(\mathbb{Z}_1)\big).\qquad\diamond$$

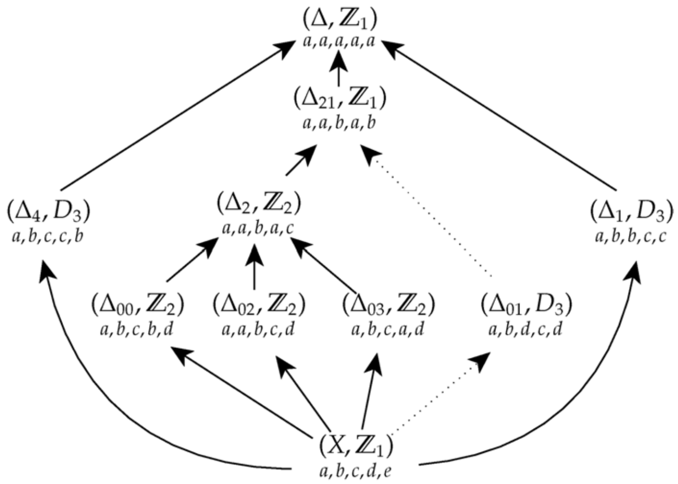


**Figure 4.** A representation sublattice $\mathcal{S} \subset \mathcal{L}$, where $\mathcal{S} = \mathcal{L} \setminus \{\Delta_{01}\}$ and the dashed arrows are removed.

*3.2.2. Product map.* Let $\mathcal{L}$ be a representation lattice with structure $\{U, G_U, \mathsf{h}_{U,V}\}$ and $\mathcal{M}$ be a representation lattice with structure $\{P, G_P, \mathsf{h}_{P,Q}\}$. The projection homomorphisms on groups

$$\mathrm{proj}_U : G_U \times G_P \to G_U, \qquad \mathrm{proj}_P : G_U \times G_P \to G_P$$

induce the inclusion homomorphisms on Euler rings (see (10))

$$\mathrm{inc}_U : A(G_U) \hookrightarrow A(G_U \times G_P),$$
$$\mathrm{inc}_P : A(G_P) \hookrightarrow A(G_U \times G_P).$$

Thus, we can define a product of $a \in A(G_U)$ and $b \in A(G_P)$ through

$$A(G_U) \times A(G_P) \hookrightarrow A(G_U \times G_P) \times A(G_U \times G_P) \xrightarrow{*} A(G_U \times G_P),$$

where $*$ is the ring multiplication in $A(G_U \times G_P)$, i.e.

$$a \star b := \mathrm{inc}_U(a) * \mathrm{inc}_P(b). \tag{17}$$

We show that the product '$\star$' is compatible with the structure of the representation lattices.

**Lemma 3.10.** *Let $\mathcal{L}$ be a representation lattice with structure $\{U, G_U, \mathsf{h}_{U,V}\}$ and $\mathcal{M}$ be a representation lattice with structure $\{P, G_P, \mathsf{h}_{P,Q}\}$. Then, the following diagram commutes*

$$\begin{array}{ccc} A(G_U) \times A(G_P) & \xrightarrow{\quad\star\quad} & A(G_U \times G_P) \\ \downarrow{\scriptstyle \mathsf{H}_{U,V} \times \mathsf{H}_{P,Q}} & & \downarrow{\scriptstyle \mathsf{H}_{U\times P, V\times Q}} \\ A(G_V) \times A(G_Q) & \xrightarrow{\quad\star\quad} & A(G_V \times G_Q) \end{array}$$

*where '$\star$' is defined by (17) and $\mathsf{H}_*$ is the induced homomorphism through $\mathsf{h}_*$ (see (10)).*

**Proof.** By theorem 2.7, $\mathsf{H}_{U\times P, V\times Q}$ is an Euler ring homomorphism. Thus, it suffices to show that the following diagram commutes.

$$\begin{array}{ccc} A(G_U) & \xhookrightarrow{\text{inc}_U} & A(G_U \times G_P) \\ \downarrow \mathsf{H}_{U,V} & & \downarrow \mathsf{H}_{U\times P, V\times Q} \\ A(G_V) & \xhookrightarrow{\text{inc}_V} & A(G_V \times G_Q) \end{array}$$

Let $(K) \in A(G_U)$. It follows from the definition of $\mathsf{H}_*$ that (see (10))

$$\mathsf{H}_{U,V}\big((K)\big) = \sum_{(\tilde{K})\in\Phi(G_V)} \chi_c((G_U/K)_{(\tilde{K})}/G_V)(\tilde{K}),$$

$$\mathsf{H}_{U\times P,V\times Q}\big((K \times G_P)\big) = \sum_{(K')\in\Phi(G_V\times G_Q)} \chi_c((G_U \times G_P/K \times G_P)_{(K')}/G_V \times G_Q)(K').$$

Note that $G_P$ acts trivially on $G_U \times G_P/K \times G_P$, which implies that $G_Q$ also acts trivially on $G_U \times G_P/K \times G_P$ through $\mathsf{h}_{U\times P,V\times Q}$. Therefore, $(G_U \times G_P/K \times G_P)_{(K')} \neq \emptyset$ if and only if $(K') = (\tilde{K} \times G_P)$ for some $\tilde{K}$ such that $(G_U/K)_{(\tilde{K})} \neq \emptyset$. Moreover, $(G_U \times G_P/K \times G_P)_{(K')}/G_V \times G_Q$ is $G_V$-homeomorphic to $(G_U/K)_{(\tilde{K})}/G_V$. Thus, we have

$$\mathsf{H}_{U\times P,V\times Q}\big((K \times G_P)\big) = \mathsf{H}_{U,V}\big((K)\big), \qquad \forall (K) \in A(G_U). \qquad \square$$

**Definition 3.11.** Let $\mathcal{L}$ be a representation lattice with structure $\{U, G_U, \mathsf{h}_{U,V}\}$ and $\mathcal{M}$ be a representation lattice with structure $\{P, G_P, \mathsf{h}_{P,Q}\}$. Consider the product lattice $\mathcal{L} \times \mathcal{M}$ with the product structure (see (13)). Let $R(\mathcal{L})$, $R(\mathcal{M})$ and $R(\mathcal{L} \times \mathcal{M})$ be the associated rings defined by (14). Define a *product map* by

$$\begin{aligned} \cdot : R(\mathcal{L}) \times R(\mathcal{M}) &\to R(\mathcal{L} \times \mathcal{M}) \\ ((U, a), (P, b)) &\mapsto (U \times P, a \star b), \end{aligned} \tag{18}$$

where $U \in \mathcal{L}$, $P \in \mathcal{M}$, $a \in A(G_U)$, $b \in A(G_P)$ and $a \star b$ is defined by (17).

## 4. A degree theory for lattice-equivariant maps

In this section, we give the definition of lattice- equivariant maps and formulate a degree theory for these maps, which we call the *lattice-equivariant degree*. We show that this degree satisfies usual topological properties expected from a degree theory, and moreover, it has algebraic properties compatible with the inclusion and the product of representation lattices.

In what follows, $\mathbb{R}$ stands for a parameter space, in which all groups act trivially.

**Definition 4.1.** Let $\mathcal{L}$ be a representation lattice in $\mathbb{R}^n$ with structure $\{U, G_U, \mathsf{h}_{U,V}\}$. An open bounded subset $\Omega \subset \mathbb{R} \times \mathbb{R}^n$ is called *$\mathcal{L}$-invariant*, if $\Omega \cap (\mathbb{R} \times U)$ is $G_U$-invariant, for every $U \in \mathcal{L}$. A continuous map $f : \overline{\Omega} \to \mathbb{R}^n$ is called *$\mathcal{L}$-equivariant*, if $f(\overline{\Omega} \cap (\mathbb{R} \times U)) \subset U$ and $f|_{\overline{\Omega}\cap(\mathbb{R}\times U)}$ is $G_U$-equivariant for every $U \in \mathcal{L}$. The map $f$ is called *$\Omega$-admissible*, if $f^{-1}(0) \cap \partial\Omega = \emptyset$. In this case, we say that the pair $(f, \Omega)$ is an *admissible pair*. Similarly, one defines *$\Omega$-admissible* and *$\mathcal{L}$-equivariant* homotopies.

Our goal is to define a degree theory for $\Omega$-admissible $\mathcal{L}$-equivariant maps $f$. Motivated by the study of synchrony-breaking Hopf bifurcations in coupled cell networks, we consider representation lattices $\mathcal{L}$ with a structure $\{U, G_U, \mathsf{h}_{U,V}\}$, where $G_U$ is of form $G_U = \Gamma_U \times S^1$ for a finite group $\Gamma_U$ and $S^1$ stands for the group of complex numbers of unit length.

Let $G := \Gamma \times S^1$ for a finite group $\Gamma$ and $V$ be a $G$-representation. Given an open bounded $G$-invariant subset $\mathcal{O} \subset \mathbb{R} \times V$ and a continuous $G$-equivariant map $\mathfrak{f} : \overline{\mathcal{O}} \to V$

such that $\mathfrak{f}^{-1}(0) \cap \partial\mathcal{O} = \emptyset$, the *twisted G-equivariant degree* is a function assigning to $(\mathfrak{f}, \mathcal{O})$ a finite sequence of integers indexed by conjugacy classes of twisted subgroups in $G$ such that it satisfies usual properties of a degree theory (see [4]).

More precisely, let $W(H)$ denote the Weyl group of $H$ in $G$ and

$$\Phi_k(G) = \{(H) : \dim W(H) = k\}, \qquad \text{for } k = 0, 1.$$

It can be verified that $\Phi_0(G)$ consists of all subgroups of form $K \times S^1$ for some subgroups $K \subset \Gamma$ and $\Phi_1(G)$ is composed of twisted subgroups (see definition 2.12). Let

$$A_k(G) := \mathbb{Z}\big[\Phi_k(G)\big] \tag{19}$$

be the free $\mathbb{Z}$-module generated by $\Phi_k(G)$, for $k = 0, 1$. Then, the twisted $G$-equivariant degree, usually denoted by $G\text{-Deg}^t$, is a function assigning to every admissible equivariant pair $(\mathfrak{f}, \mathcal{O})$ an element in $A_1(G)$ such that it satisfies properties like the *existence, homotopy invariance, additivity, normalization, suspension* and *Hopf property* (see [4]). In particular, the twisted equivariant degree $G\text{-Deg}^t(\mathfrak{f}, \mathcal{O})$ has the form of

$$G\text{-Deg}^t(\mathfrak{f}, \mathcal{O}) = \sum_{(H)\in\Phi_1(G)} n_H(H), \qquad n_H \in \mathbb{Z}. \tag{20}$$

### 4.1. Definition and basic properties

Let $\mathcal{L}$ be a representation lattice with structure $\{U, G_U, \mathsf{h}_{U,V}\}$, where $G_U = \Gamma_U \times S^1$ for a finite group $\Gamma_u$. Recall that (see (14))

$$R(\mathcal{L}) := \left\{ \sum_{U\in\mathcal{L}} (U, a_U) \, : \, a_U \in A(G_U) \right\},$$

where $A(G_U)$ is the Euler ring of $G_U$. Denote by

$$R_k(\mathcal{L}) := \left\{ \sum_{U\in\mathcal{L}} (U, a_U) \in R(\mathcal{L}) \, : \, a_U \in A_k(G_U) \right\}, \qquad \text{for } k = 0, 1,$$

where $A_k(G)$ is defined by (19).

**Definition 4.2.** Let $\mathcal{L}$ be a representation lattice in $\mathbb{R}^n$ with the structure given by $\{U, G_U, \mathsf{h}_{U,V}\}$, where $G_U = \Gamma_U \times S^1$ for a finite group $\Gamma_U$. Let $\Omega \subset \mathbb{R} \times \mathbb{R}^n$ be an open bounded $\mathcal{L}$-invariant subset and $f : \overline{\Omega} \to \mathbb{R}^n$ be an $\Omega$-admissible $\mathcal{L}$-equivariant map. Using the twisted equivariant degree, we define for each $U \in \mathcal{L}$, an element $a_U$ in $A_1(G_U)$. Let $U_{\min} \in \mathcal{L}$ be the minimal element and define

$$a_{U_{\min}} := \Gamma_{U_{\min}} \times S^1\text{-Deg}^t(f|_{\Omega\cap(\mathbb{R}\times U_{\min})}, \Omega \cap (\mathbb{R} \times U_{\min})). \tag{21}$$

Suppose that $a_{U'}$ is defined for all $U' < U$. Then, define

$$a_U := \Gamma_U \times S^1\text{-Deg}^t(f|_{\Omega\cap(\mathbb{R}\times U)}, \Omega \cap (\mathbb{R} \times U)) - \sum_{U'<U} \mathsf{H}_{U',U}(a_{U'}), \tag{22}$$

where $\mathsf{H}_{U',U}$ is the Euler ring homomorphism induced by $\mathsf{h}_{U',U}$ (see (10)). The *lattice-equivariant degree* of $f$ in $\Omega$ is then defined by

$$\mathcal{L}\text{-Deg}^t(f, \Omega) := \sum_{U\in\mathcal{L}} (U, a_U) \in R_1(\mathcal{L}). \tag{23}$$

Notice that in the case the representation lattice $\mathcal{L}$ is composed of a single element $\mathbb{R}^n$ as a representation of $\Gamma \times S^1$, the lattice-equivariant degree $\mathcal{L}\text{-Deg}^t$ coincides with the twisted equivariant degree $\Gamma \times S^1\text{-Deg}^t$.

The lattice-equivariant degree satisfies the following basic properties.

**Theorem 4.3.** *Let $\mathcal{L}$ be a representation lattice in $\mathbb{R}^n$ with the structure given by $\{U, G_U, \mathsf{h}_{U,V}\}$, where $G_U = \Gamma_U \times S^1$ for a finite group $\Gamma_U$. Then, the function $\mathcal{L}$-$\mathrm{Deg}^t$ defined by (23) satisfies:*

*(i)* (Existence) *Suppose that $\mathcal{L}\text{-}\mathrm{Deg}^t(f, \Omega) = \sum(U, a_U)$ and $a_U \neq 0$ for some $U \in \mathcal{L}$. Write $a_U = \sum n_H(H)$. If $(H)$ is such that $n_H \neq 0$, then*

$$f^{-1}(0) \cap \left(\Omega^H \cap (\mathbb{R} \times U)\right) \neq \emptyset,$$

*where the meaning of $\Omega^H$ is given in (4).*

*(ii)* (Homotopy Invariance) *If $H : [0, 1] \times \overline{\Omega} \to \mathbb{R}^n$ is an $\Omega$-admissible $\mathcal{L}$-equivariant homotopy, then*

$$\mathcal{L}\text{-}\mathrm{Deg}^t(H(t, \cdot), \Omega) = \text{constant}, \qquad \forall t \in [0, 1].$$

*(iii)* (Additivity) *If $\Omega_1, \Omega_2 \subset \Omega$ are disjoint open bounded $\mathcal{L}$-invariant subsets such that $f^{-1}(0) \cap \Omega \subset \Omega_1 \cup \Omega_2$ and $f$ is $\Omega_i$-admissible for $i = 1, 2$, then*

$$\mathcal{L}\text{-}\mathrm{Deg}^t(f, \Omega) = \mathcal{L}\text{-}\mathrm{Deg}^t(f, \Omega_1) + \mathcal{L}\text{-}\mathrm{Deg}^t(f, \Omega_2).$$

*(iv)* (Suspension) *Let $\mathcal{M}$ be a representation lattice in $\mathbb{R}^m$, $\mathrm{Id} : V' \to V'$ be the identity map and $\Omega' \subset \mathbb{R}^m$ be an open bounded $\mathcal{M}$-invariant neighborhood of 0. Then, $\mathcal{L} \times \mathcal{M}\text{-}\mathrm{Deg}^t(f \times \mathrm{Id}, \Omega \times \Omega')$ is well-defined. Moreover,*

$$\mathcal{L} \times \mathcal{M}\text{-}\mathrm{Deg}^t(f \times \mathrm{Id}, \Omega \times \Omega') = \mathcal{L}\text{-}\mathrm{Deg}^t(f, \Omega),$$

*under the identification: $U \mapsto U \times P_{\min}$ and $(H) \mapsto (H \times \Gamma_{P_{\min}})$ for every $U \in \mathcal{L}$ and $(H) \in \Phi_1(\Gamma_U \times S^1)$, where $P_{\min} \in \mathcal{M}$ is the minimal element.*

**Proof.** (ii) and (iii) follow immediately from the corresponding properties of the twisted equivariant degree (see [4]).

To show (i), assume that $f^{-1}(0) \cap \left(\Omega^H \cap (\mathbb{R} \times U)\right) = \emptyset$. If $U = U_{\min}$, then by the existence property of the twisted equivariant degree, we have that $a_{U_{\min}} = \Gamma_U \times S^1\text{-}\mathrm{Deg}^t(f|_{\Omega \cap (\mathbb{R} \times U_{\min})}, \Omega \cap (\mathbb{R} \times U_{\min}))$ has a zero $(H)$-coefficient, which is a contradiction. Assume that the statement holds for all $U' < U$. By assumption, $a_U$ has a nonzero $(H)$-coefficient. By the existence property of the twisted equivariant degree, $\Gamma_U \times S^1\text{-}\mathrm{Deg}^t(f|_{\Omega \cap (\mathbb{R} \times U)}, \Omega \cap (\mathbb{R} \times U))$ has a zero $(H)$-coefficient. It follows from the definition of $a_U$ that there exists $U' < U$ such that $\mathsf{H}_{U',U}(a_{U'})$ has a nonzero $(H)$-coefficient. Thus, there exists an $H' \subset \Gamma_{U'} \times S^1$ such that $H = \mathsf{h}^{-1}_{U',U}(H')$ and the $(H')$-coefficient in $a_{U'}$ is nonzero. By the induction assumption, we have

$$f^{-1}(0) \cap \left(\Omega^{H'} \cap (\mathbb{R} \times U')\right) \neq \emptyset.$$

Let $x \in f^{-1}(0) \cap \left(\Omega^{H'} \cap (\mathbb{R} \times U')\right)$ and $g \in H$. Then, $\mathsf{h}_{U',U}(g) \in H'$, so $\mathsf{h}_{U',U}(g)x = x$. By the definition of representation lattice (see definition 3.2(ii)), we have then

$$gx = \mathsf{h}_{U',U}(g)x = x.$$

It follows that $x \in f^{-1}(0) \cap \left(\Omega^H \cap (\mathbb{R} \times U)\right)$. In particular, $f^{-1}(0) \cap \left(\Omega^H \cap (\mathbb{R} \times U)\right) \neq \emptyset$, which is a contradiction to our initial assumption.

To show (iv), let

$$\mathcal{L} \times \mathcal{M}\text{-}\mathrm{Deg}^t\,(f \times \mathrm{Id}, \Omega \times \Omega') = \sum(U \times P, b_{U \times P})$$

$$\mathcal{L}\text{-}\mathrm{Deg}^t(f, \Omega) = \sum(U, a_U).$$

Let $U_{\min} \in \mathcal{L}$ and $P_{\min} \in \mathcal{M}$ be the minimal element, respectively. Let $\Gamma_{U_{\min}}$ act trivially on $P_{\min}$ and $\Gamma_{P_{\min}}$ act trivially on $\mathbb{R} \times U_{\min}$. Then, $\Omega \cap (\mathbb{R} \times U_{\min})$ becomes $\Gamma_{U_{\min}} \times \Gamma_{P_{\min}} \times S^1$-invariant and the restricted $f$ is a $\Gamma_{U_{\min}} \times \Gamma_{P_{\min}} \times S^1$-equivariant map. Similarly, $\Omega' \cap P_{\min}$ becomes $\Gamma_{U_{\min}} \times \Gamma_{P_{\min}}$-invariant and the Id is $\Gamma_{U_{\min}} \times \Gamma_{P_{\min}}$-equivariant. By the suspension property of the twisted equivariant degree, we have

$$\begin{aligned} b_{U_{\min}\times P_{\min}} &= \Gamma_{U_{\min}} \times \Gamma_{P_{\min}} \times S^1\text{-Deg}^t(f \times \mathrm{Id}, (\Omega \cap (\mathbb{R} \times U_{\min})) \times (\Omega' \cap P_{\min})) \\ &= \Gamma_{U_{\min}} \times \Gamma_{P_{\min}} \times S^1\text{-Deg}^t(f, \Omega \cap (\mathbb{R} \times U_{\min})) \\ &\simeq a_{U_{\min}}, \end{aligned}$$

where '$\simeq$' means identifying $(H \times \Gamma_{P_{\min}})$ with $(H)$, for $H \subset \Gamma_{U_{\min}} \times S^1$. Using the suspension property of the twisted equivariant degree inductively, one shows

$$b_{U\times P_{\min}} \simeq a_U, \qquad b_{U_{\min}\times P} = 0, \quad \text{for } P > P_{\min}. \tag{24}$$

Let $U > U_{\min}$ and $P > P_{\min}$. We show that $b_{U\times P} = 0$. Assume that $b_{U'\times P'} = 0$ for all $U' \times P' < U \times P$ and $P' > P_{\min}$. Then, we have

$$\begin{aligned} b_{U\times P} &= \Gamma_U \times \Gamma_P \times S^1\text{-Deg}^t(f, (\Omega \cap (\mathbb{R} \times U)) - \sum_{U'\times P' < U\times P} \mathsf{H}_{U'\times P', U\times P}(b_{U'\times P'}) \\ &\overset{(24)}{=} \Gamma_U \times \Gamma_P \times S^1\text{-Deg}^t(f, (\Omega \cap (\mathbb{R} \times U)) - \sum_{U' \lneqq U} \mathsf{H}_{U'\times P_{\min}, U\times P}(b_{U'\times P_{\min}}) \\ &\overset{(24)}{\simeq} \Gamma_U \times S^1\text{-Deg}^t(f, (\Omega \cap (\mathbb{R} \times U)) - \sum_{U' \lneqq U} \mathsf{H}_{U',U}(a_{U'}) = 0. \end{aligned}$$

Thus, (iv) holds. □

**Remark 4.4.** Recall that for a finite group $\Gamma$, an *equivariant degree without parameters* is a function $\Gamma$-Deg assigning to every admissible pair $(g, \Omega')$, where $\Omega' \subset V$ is open bounded $\Gamma$-invariant and $g : \overline{\Omega}' \to V$ is $\Gamma$-equivariant, an element in $A(\Gamma)$ such that it satisfies the usual properties of a degree theory (see [6]). Let $\mathcal{M}$ be a representation lattice in $\mathbb{R}^m$. In a similar way, one can define a *lattice-equivariant degree without parameters* for admissible pairs $(g, \Omega')$, denote by $\mathcal{M}$-Deg$(g, \Omega')$, where $\Omega' \subset \mathbb{R}^m$ is an open bounded $\mathcal{M}$-invariant subset and $g : \overline{\Omega}' \to \mathbb{R}^m$ is an $\mathcal{M}$-equivariant map.

### *4.2. Algebraic properties*

We show that the lattice-equivariant degree is compatible with the reduction homomorphism defined in definition 3.7. Moreover, it has a product property with respect to product lattices.

**Proposition 4.5 (Reduction homomorphism).** *Let $\mathcal{L}$ be a representation lattice in $\mathbb{R}^n$ and $\mathcal{S} \subset \mathcal{L}$ be a representation sublattice. Let $\Phi^{\mathcal{L}}_{\mathcal{S}}$ be the reduction homomorphism from $\mathcal{L}$ to $\mathcal{S}$ defined by (16). Then, we have*

$$\Phi^{\mathcal{L}}_{\mathcal{S}}(\mathcal{L}\text{-Deg}^t(f, \Omega)) = \mathcal{S}\text{-Deg}^t(f, \Omega), \tag{25}$$

*for every admissible $\mathcal{L}$-equivariant pair $(f, \Omega)$.*

**Proof.** Let $(f, \Omega)$ be an admissible $\mathcal{L}$-equivariant pair. By lemma 3.8 and lemma 3.4(iii), we can assume without loss of generality, that $\mathcal{S} = \mathcal{L} \setminus \{U_o\}$ for some $U_o \in \mathcal{L}^\top$. Let

$$\mathcal{L}\text{-Deg}^t(f, \Omega) = \sum_{U\in\mathcal{L}} (U, a_U), \quad \mathcal{S}\text{-Deg}^t(f, \Omega) = \sum_{U\in\mathcal{S}} (U, b_U).$$

By the definition of $a_U$, we have $b_U = a_U$ if $U_o \not< U$. Let $U \in \mathcal{L}$ be such that $U_o < U$ and $U_+$ be the unique immediate descendant of $U_o$. Then, $U_+ \leqslant U$. In case $U = U_+$, we have

$$\begin{aligned} b_{U_+} &= \Gamma_{U_+} \times S^1\text{-Deg}^t(f, \Omega \cap (\mathbb{R} \times U_+)) - \sum_{\substack{U' < U_+ \\ U' \in \mathcal{S}}} \mathsf{H}_{U',U_+}(b_{U'}) \\ &= a_{U_+} + \sum_{\substack{U' < U_+ \\ U' \in \mathcal{L}}} \mathsf{H}_{U',U_+}(a_{U'}) - \sum_{\substack{U' < U_+ \\ U' \in \mathcal{S}}} \mathsf{H}_{U',U_+}(b_{U'}) \\ &= a_{U_+} + \mathsf{H}_{U_o,U_+}(a_{U_o}) + \sum_{\substack{U' < U_+ \\ U' \in \mathcal{S}}} \mathsf{H}_{U',U_+}(a_{U'}) - \sum_{\substack{U' < U_+ \\ U' \in \mathcal{S}}} \mathsf{H}_{U',U_+}(b_{U'}) \\ &= a_{U_+} + \mathsf{H}_{U_o,U_+}(a_{U_o}). \end{aligned} \tag{26}$$

For $U > U_+$, suppose that $a_{U'} = b_{U'}$ for all $U_+ < U' < U$, then we have

$$\begin{aligned} b_U &= \Gamma_U \times S^1\text{-Deg}^t(f, \Omega \cap (\mathbb{R} \times U)) - \sum_{\substack{U' < U \\ U' \in \mathcal{S}}} \mathsf{H}_{U',U}(b_{U'}) \\ &= a_U + \sum_{\substack{U' < U \\ U' \in \mathcal{L}}} \mathsf{H}_{U',U}(a_{U'}) - \sum_{\substack{U' < U \\ U' \in \mathcal{S}}} \mathsf{H}_{U',U}(b_{U'}) \\ &= a_U + \mathsf{H}_{U_+,U}(a_{U_+}) + \mathsf{H}_{U_o,U}(a_{U_o}) - \mathsf{H}_{U_+,U}(b_{U_+}) \\ &\overset{(26)}{=} a_U + \mathsf{H}_{U_o,U}(a_{U_o}) - \mathsf{H}_{U_o,U_+}\mathsf{H}_{U_+,U}(a_{U_o}), \end{aligned}$$

which implies that $b_U = a_U$ by theorem 2.7(ii). □

We show that multiplication properties of the twisted equivariant degree can be extended to the lattice-equivariant degree. Recall that the twisted $\Gamma \times S^1$-equivariant degree has a multiplication property corresponding to the $A(\Gamma)$-module structure on the set $A_1(\Gamma \times S^1)$, which coincides with the Euler ring multiplication in $A(\Gamma \times S^1)$ (see [6, 20]).

**Proposition 4.6 (Product property).** *Let $\mathcal{L}$ be a representation lattice with structure $\{U, \Gamma_U \times S^1, \mathsf{h}_{U,V}\}$ in $\mathbb{R}^n$ and $\mathcal{M}$ be a representation lattice with structure $\{P, \Gamma_P, \mathsf{h}_{P,Q}\}$, where $\Gamma_*$ are finite groups. Suppose that $\Omega \subset \mathbb{R} \times \mathbb{R}^n$ (respectively $\Omega' \subset \mathbb{R}^m$) is an open bounded $\mathcal{L}$-invariant (respectively $\mathcal{M}$-invariant) subset and $f : \overline{\Omega} \to \mathbb{R}^n$ (respectively $g : \overline{\Omega}' \to \mathbb{R}^m$) is an $\Omega$-admissible $\mathcal{L}$-equivariant (respectively $\Omega'$-admissible $\mathcal{M}$-equivariant) map. Then, we have*

$$\mathcal{L} \times \mathcal{M}\text{-Deg}^t(f \times g, \Omega \times \Omega') = \mathcal{L}\text{-Deg}^t(f, \Omega) \cdot \mathcal{M}\text{-Deg}(g, \Omega'), \tag{27}$$

*where '$\cdot$' is defined by (18).*

**Proof.** Let $\mathcal{L} \times \mathcal{M}\text{-Deg}^t(f \times g, \Omega \times \Omega') = \sum (U \times P, a_{U \times P})$, $\mathcal{L}\text{-Deg}^t(f, \Omega) = \sum_{U \in \mathcal{L}} (U, b_U)$ and $\mathcal{M}\text{-Deg}(g, \Omega') = \sum_{P \in \mathcal{M}} (P, c_P)$. It sufficies to show

$$a_{U \times P} = b_U \star c_P.$$

Denote by $U_{\min}$ and $P_{\min}$ the minimal element of $\mathcal{L}$ and $\mathcal{M}$ respectively. Then,

$$\begin{aligned} a_{U_{\min} \times P_{\min}} &= \Gamma_{U_{\min}} \times \Gamma_{P_{\min}} \times S^1\text{-Deg}^t(f \times g, (\Omega \cap (\mathbb{R} \times U_{\min})) \times (\Omega' \cap P_{\min})) \\ &= \Gamma_{U_{\min}} \times \Gamma_{P_{\min}} \times S^1\text{-Deg}^t(f, \Omega \cap (\mathbb{R} \times U_{\min}))) * \Gamma_{U_{\min}} \\ &\quad \times \Gamma_{P_{\min}}\text{-Deg}(g, \Omega' \cap P_{\min}) \\ &= \Gamma_{U_{\min}} \times S^1\text{-Deg}^t(f, \Omega \cap (\mathbb{R} \times U_{\min}))) \star \Gamma_{P_{\min}}\text{-Deg}(g, \Omega' \cap P_{\min}) \\ &= b_{U_{\min}} \star c_{P_{\min}}. \end{aligned}$$

Assume that $a_{U'\times P'} = b_{U'} \star c_{P'}$ for all $U'$, $P'$ such that $U' \times P' < U \times P$. Then, we have

$$\begin{aligned} a_{U\times P} &= \sum_{U'\leqslant U} b_{U'} \star \sum_{P'\leqslant P} c_{P'} - \sum_{U'\times P'<U\times P} \mathsf{H}_{U'\times P',U\times P}(a_{U'\times P'}) \\ &= \sum_{U'\leqslant U} b_{U'} \star \sum_{P'\leqslant P} c_{P'} - \sum_{U'\times P'<U\times P} \mathsf{H}_{U'\times P',U\times P}(b_{U'} \star c_{P'}) \\ &= \sum_{U'\leqslant U} b_{U'} \star \sum_{P'\leqslant P} c_{P'} - \sum_{U'\times P'<U\times P} \mathsf{H}_{U',U}(b_{U'}) \star \mathsf{H}_{P',P}(c_{P'}), \\ &\quad \text{(cf lemma3.10)} \\ &= b_U \star c_P. \end{aligned}$$

□

*4.3. Extension to infinite-dimensional vector spaces*

In this subsection, we extend the lattice-equivariant degree to infinite-dimensional lattice representations for compact lattice-equivariant vector fields. The desired approximation of compact maps by finite-dimensional maps is based on an equivariant version of the Schauder projection.

In what follows, $W$ is an infinite-dimensional real Banach space. Recall that for a bounded subset $X \subset \mathbb{R} \times W$, a continuous map $F : X \to W$ is called *compact*, if $\overline{F(X)}$ is compact in $W$; and $F$ is called *finite dimensional*, if $F(X)$ is contained in a finite dimensional subspace of $W$. Let $\pi : \mathbb{R} \times W \to W$ be the projection on $W$ and $F : X \to W$ be a compact map, then $\pi - F$ is called a *compact vector field*.

**Definition 4.7.** Let $G$ be a compact Lie group and $W$ be a Banach representation of $G$. Let $N = \{c_1, c_2, \ldots, c_n\} \subset W$ be a finite set. For any fixed $\varepsilon > 0$, let

$$U(N, \varepsilon) = \bigcup_{i=1}^{n} \bigcup_{g\in G} gB(c_i, \varepsilon), \tag{28}$$

where the symbol $gA$ means the union of all elements $gx$ for $x \in A$ and $B(c_i, \varepsilon)$ stands for the open $\varepsilon$-disc around $c_i$ in $W$. For $x, y \in W$, define $\rho_\varepsilon(x, y) = \max\{0, \varepsilon - \|x - y\|\}$. We call the map $p_{N,\varepsilon} : U(N, \varepsilon) \to W$ defined by

$$p_{N,\varepsilon}(x) = \frac{\displaystyle\sum_{i=1}^{n} \int_G \rho_\varepsilon(g^{-1}x, c_i) gc_i \, \mathrm{d}\mu(g)}{\displaystyle\sum_{i=1}^{n} \int_G \rho_\varepsilon(g^{-1}x, c_i) \, \mathrm{d}\mu(g)}, \tag{29}$$

for $\mu$ being the Haar measure of $G$, the *equivariant Schauder projection*.

Note that the denominator of $p_{N,\varepsilon}$ is never zero. For every $x \in U(N, \varepsilon)$, there exists $i \in \{1, 2, \ldots, n\}$, $g \in G$ such that $x = gy$ for some $y \in B(c_i, \varepsilon)$. That is, $x \in U(N, \varepsilon)$ if and only if $\|g^{-1}x - c_i\| < \varepsilon$, which implies that $\rho_\varepsilon(g^{-1}x, c_i) > 0$.

The equivariant Schauder projection has the following properties.

**Lemma 4.8.** *Let $G$ be a compact Lie group and $W$ be an isometric Banach representation of $G$. Let $N = \{c_1, c_2, \ldots, c_n\} \subset W$ be a finite set and $\varepsilon > 0$. Let $U(N, \varepsilon)$ be given by (28) and $p_{N,\varepsilon}$ be given by (29). Then,*

*(i) $p_{N,\varepsilon}$ is $G$-equivariant;*
*(ii) $p_{N,\varepsilon}$ is a finite-dimensional map;*
*(iii) $\|x - p_{N,\varepsilon}(x)\| < \varepsilon$, for all $x \in U(N, \varepsilon)$.*

**Proof.**

(i) Let $a(x)$ be the numerator of $p_{N,\varepsilon}(x)$ and $b(x)$ be the denominator of $p_{N,\varepsilon}(x)$. We show that the map $a$ is $G$-equivariant and $b$ is $G$-invariant. Let $g_o \in G$. Then,

$$a(g_o^{-1}x) = \sum_{i=1}^{n} \int_G \rho_\varepsilon(g^{-1}g_o^{-1}x, c_i)gc_i \,\mathrm{d}\mu(g) = \sum_{i=1}^{n} \int_G \rho_\varepsilon((g_o g)^{-1}x, c_i)g_o^{-1}g_o g c_i \,\mathrm{d}\mu(g)$$
$$= g_o^{-1} \sum_{i=1}^{n} \int_G \rho_\varepsilon((g_o g)^{-1}x, c_i)(g_o g)c_i \,\mathrm{d}\mu(g) = g_o^{-1}a(x),$$

and

$$b(g_o^{-1}x) = \sum_{i=1}^{n} \int_G \rho_\varepsilon(g^{-1}g_o^{-1}x, c_i) \,\mathrm{d}\mu(g) = \sum_{i=1}^{n} \int_G \rho_\varepsilon((g_o g)^{-1}x, c_i) \,\mathrm{d}\mu(g) = b(x).$$

Thus, $p_{N,\varepsilon}$ is $G$-equivariant.

(ii) Note that the $G$-orbit of $c_i$ is a finite-dimensional smooth manifold of $W$, thus is contained in a subspace $W_i \subset W$ with $\dim W_i < \infty$, for $i = 1, 2, \ldots, n$. It follows that $\int_G \rho_\varepsilon(g^{-1}x, c_i)gc_i \,\mathrm{d}\mu(g) \in W_i$ and $p_{N,\varepsilon}(x) \in \operatorname{span}\{W_1, W_2, \ldots, W_n\}$ for all $x \in W$.

(iii) Let $x \in U(N, \varepsilon)$. Assume that $\rho_\varepsilon(g^{-1}x, c_i) \neq 0$. Then, $\|g^{-1}x - c_i\| < \varepsilon$. Thus, $\|x - gc_i\| < \varepsilon$, since $G$ acts isometrically on $W$. Therefore,

$$\|x - p_{N,\varepsilon}(x)\| = \left\| \frac{\sum_{i=1}^{n} \int_G \rho_\varepsilon(g^{-1}x, c_i)(x - gc_i) \,\mathrm{d}\mu(g)}{\sum_{i=1}^{n} \int_G \rho_\varepsilon(g^{-1}x, c_i) \,\mathrm{d}\mu(g)} \right\|$$
$$\leqslant \frac{\sum_{i=1}^{n} \int_G \rho_\varepsilon(g^{-1}x, c_i) \,\|x - gc_i\| \,\mathrm{d}\mu(g)}{\sum_{i=1}^{n} \int_G \rho_\varepsilon(g^{-1}x, c_i) \,\mathrm{d}\mu(g)} < \varepsilon.$$

□

We have the following approximation theorem.

**Proposition 4.9.** *Let $W$ be an infinite-dimensional real Banach space and $\mathcal{T}$ be representation lattice with structure $\{Y, G_Y, \mathsf{h}_{Y,Y'}\}$ in $W$. Let $X \subset \mathbb{R} \times W$ be a bounded $\mathcal{T}$-invariant subset and $F : X \to W$ be a $\mathcal{T}$-equivariant compact map. Then, for every $\varepsilon > 0$, there exists a $\mathcal{T}$-equivariant finite-dimensional map $F_\varepsilon : X \to W$ such that*

$$\|F(x) - F_\varepsilon(x)\| < \varepsilon, \qquad \text{for all } x \in X.$$

**Proof.** For convenience, we numerate the elements of $\mathcal{T}$ as $Y_1, Y_2, \ldots, Y_m$ such that

$$Y_i \subset Y_j \implies i \leqslant j.$$

Based on lemma 4.8, we define $F_\varepsilon$ inductively on $Y = Y_i$ using the equivariant Schauder projection. Set $\varepsilon = \varepsilon_1$.

For $Y = Y_1$, since $F$ is a compact map, $\overline{F(X \cap (\mathbb{R} \times Y_1))}$ is a compact set in $W$. Thus, there exists a finite set $N_1 = \{c_1, c_2, \ldots, c_{n_1}\} \subset Y_1$ such that the set $U(N_1, \varepsilon_1)$ defined by (28) covers $\overline{F(X \cap (\mathbb{R} \times Y_1))}$. Let $p_{N_1,\varepsilon_1}$ be given by (29) and define

$$F_{\varepsilon_1}(x) = p_{N_1,\varepsilon_1}(F(x)), \qquad \forall x \in X \cap (\mathbb{R} \times Y_1).$$

For $Y = Y_2$, choose $\varepsilon_2 > 0$ such that $\varepsilon_2 < \varepsilon_1$ and

$$\{y \in Y_2 \,:\, \operatorname{dist}(y, F(X \cap (\mathbb{R} \times Y_1)) < \varepsilon_2\} \subset U(N_1, \varepsilon_1).$$

Since $\overline{F(X \cap (\mathbb{R} \times Y_2))}$ is compact, there exists a finite set $N_2 = \{c_{n_1+1}, c_{n_1+2}, \ldots, c_{n_1+n_2}\} \subset Y_2 \setminus U(N_1, \varepsilon_1)$ such that $U(N_2, \varepsilon_2)$ defined by (28) covers $\overline{F(X \cap (\mathbb{R} \times Y_2))} \setminus U(N_1, \varepsilon_1)$. Note that by the choice of $\varepsilon_2$, we have

$$\operatorname{dist}(c_{n_1+j}, F(X \cap (\mathbb{R} \times Y_1)) \geqslant \varepsilon_2, \qquad \forall\, j = 1, 2, \ldots, n_2. \tag{30}$$

Define $p_{N_2,\varepsilon_2} : U(N_1, \varepsilon_1) \cup U(N_2, \varepsilon_2) \to W$ by

$$p_{N_2,\varepsilon_2}(x) = \frac{\displaystyle\sum_{i=1}^{n_1} \int_{G_{Y_1}} \rho_{\varepsilon_1}(g^{-1}x, c_i) g c_i \,\mathrm{d}\mu(g) + \sum_{j=1}^{n_2} \int_{G_{Y_2}} \rho_{\varepsilon_2}(g^{-1}x, c_{n_1+j}) g c_{n_1+j} \,\mathrm{d}\mu(g)}{\displaystyle\sum_{i=1}^{n_1} \int_{G_{Y_1}} \rho_{\varepsilon_1}(g^{-1}x, c_i) \,\mathrm{d}\mu(g) + \sum_{j=1}^{n_2} \int_{G_{Y_2}} \rho_{\varepsilon_2}(g^{-1}x, c_{n_1+j}) \,\mathrm{d}\mu(g)}.$$

It can be verified that $p_{N_2,\varepsilon_2}$ is $G_{Y_2}$-equivariant in $Y_2$ (noting the compatibility condition (ii) of definition 3.2), finite-dimensional and satisfies $\|x - p_{N_2,\varepsilon_2}(x)\| < \varepsilon_1 = \varepsilon$. Let

$$F_{\varepsilon_2}(x) = p_{N_2,\varepsilon_2}(F(x)), \qquad \forall\, x \in X \cap (\mathbb{R} \times Y_2).$$

It should be noted that by (30), $F_{\varepsilon_2}$ coincides with $F_{\varepsilon_1}$ on $X \cap (\mathbb{R} \times Y_1)$. Thus, $F_{\varepsilon_2}$ is a finite-dimensional $\varepsilon$-approximation of $F$ such that $F_{\varepsilon_2}$ is lattice equivariant with respect to the representation sublattice $\{Y_1, Y_2\}$ of $\mathcal{T}$.

By iterating the above procedure until $Y = Y_m$, we obtain the desired map $F_\varepsilon$ given by $F_{\varepsilon_m}$. □

Let $\mathcal{T}$ be a representation lattice in $W$. Let $O \subset \mathbb{R} \times W$ be a $\mathcal{T}$-invariant open bounded subset and $F : \overline{O} \to W$ be a $\mathcal{T}$-equivariant compact map. By proposition 4.9, for given $\varepsilon > 0$, $F$ has a $\mathcal{T}$-equivariant finite-dimensional approximation $F_\varepsilon : \overline{O} \to W$ such that $\|F_\varepsilon(x) - F_1(x)\| < \varepsilon$, for $x \in \overline{O}$. Suppose that $F_\varepsilon(\overline{O}) \subset W_*$ for a finite-dimensional subspace $W_* \subset W$. Set

$$\mathcal{T}_* := \{Y \cap W_* \,:\, Y \in \mathcal{T}\}.$$

We define the *lattice-equivariant degree* of $\pi - F$ in $O$ by

$$\mathcal{T}\text{-Deg}^t(\pi - F, O) := \mathcal{T}_*\text{-Deg}^t(\pi - F_\varepsilon|_{O\cap(\mathbb{R}\times W_*)}, O \cap (\mathbb{R} \times W_*)), \tag{31}$$

where the function $\mathcal{T}_*$-$\text{Deg}^t$ on the right-hand side is defined by (23).

By a standard argument, one shows that the definition is independent of the choice of approximation $F_\varepsilon$ and $W_*$. Moreover, the defined lattice-equivariant degree by (31) satisfies similar properties as listed in theorem 4.3 with $f$ replaced by compact vector fields.

## 5. Synchrony-breaking bifurcations in coupled cell systems

In this section, we adopt the standard degree-theoretical approach and use the lattice-equivariant degree to study synchrony-breaking Hopf bifurcations in homogeneous coupled cell systems.

### 5.1. Statement of the problem

Consider a homogeneous coupled cell system given by

$$\begin{aligned} \dot{x}_1 &= f_o(\lambda; x_1; x_{i_1}, \ldots, x_{i_s}) \\ \dot{x}_2 &= f_o(\lambda; x_2; x_{j_1}, \ldots, x_{j_s}), \\ &\cdots \\ \dot{x}_n &= f_o(\lambda; x_n; x_{k_1}, \ldots, x_{k_s}), \end{aligned} \tag{32}$$

where $\lambda \in \mathbb{R}$ is a parameter, $x_i \in \mathbb{R}^k$ and $f_o : \mathbb{R} \times \mathbb{R}^k \times (\mathbb{R}^k)^s \to \mathbb{R}^k$ of class $C^1$. Let $x = (x_1, \ldots, x_n)^T \in (\mathbb{R}^k)^n$ and $f : \mathbb{R} \times (\mathbb{R}^k)^n \to (\mathbb{R}^k)^n$ be the right-hand side of (32). Then, (32) can be written as

$$\dot{x} = f(\lambda, x). \tag{33}$$

Assume that $x = x_o \in (\mathbb{R}^k)^n$ is an equilibrium of (33), i.e.

(E1) $f(\lambda, x_o) = 0, \quad \forall \lambda \in \mathbb{R}$.

Let $J(\lambda) := Df_x(\lambda, x_o)$ be the Jacobian of $f$ at $x_o$. We say that $(\lambda_o, x_o)$ is a *bifurcation centre* of (33), if $J(\lambda_o)$ has a purely imaginary eigenvalue $i\beta_o$. Assume that

(B1) $(\lambda_o, x_o)$ is an *isolated* bifurcation centre, i.e. $(\lambda_o, x_o)$ is the only bifurcation centre in some neighborhood of $(\lambda_o, x_o)$ in $\mathbb{R} \times (\mathbb{R}^k)^n$.

To avoid steady-state bifurcation around $(\lambda_o, x_o)$, we assume

(B2) $J(\lambda_o) : (\mathbb{R}^k)^n \to (\mathbb{R}^k)^n$ is an isomorphism.

In many cases, due to the external couplings among the cells, (33) may admit a number of flow-invariant subspaces given by equalities of cell coordinates. As an example, every homogeneous coupled cell system admits

$$\Delta_0 = \{x \,:\, x_1 = x_2 = \cdots = x_n\}$$

as a flow-invariant subspace (independent of the specific form of $f$). In general, let $\bowtie$ be a partition on the set $\{1, 2, \ldots, n\}$ and $\overset{\bowtie}{\sim}$ be the induced equivalence relation. Then, $\bowtie$ defines a *polydiagonal subspace*

$$\Delta_{\bowtie} = \{x \,:\, x_c = x_d, \quad \text{if } c \overset{\bowtie}{\sim} d\} \subset (\mathbb{R}^k)^n,$$

characterized by the partial synchrony among the cells. A polydiagonal subspace of (33) is called *robust*, if it is invariant for every vector field $f$ of the form (32). We assume that

(L1) $\mathcal{L}$ is a lattice of (robust) polydiagonal subspaces admitted by (33), which is independent of $\lambda \in \mathbb{R}$;

(L2) $\mathcal{L}$ is a representation lattice with structure $\{\Delta, \Gamma_\Delta, \mathfrak{h}_{\Delta,\Delta'}\}$ for finite groups $\Gamma_\Delta$;

(F) $f$ is $\mathcal{L}$-equivariant;

In what follows, we are interested in studying *synchrony-breaking* Hopf bifurcations around $(\lambda_o, x_o)$, where $x_o$ loses its stability and bifurcates to oscillating states of less synchrony. For simplicity[2], we assume that

(E2) $x_o \in \Delta_0$.

**Definition 5.1.** Let $x \in (\mathbb{R}^k)^n$ and $\mathcal{L}$ be a lattice of robust polydiagonal subspaces of (33). If $x \in \Delta$ for some $\Delta \in \mathcal{L}$, then we say that $x$ is *of synchrony type* $\Delta$. If moreover, $\Delta$ is the smallest element in $\mathcal{L}$ that contains $x$, then we say that $x$ is of *proper* synchrony type $\Delta$. Similarly, a function $x : \mathbb{R} \to (\mathbb{R}^k)^n$ is *of (proper) synchrony type* $\Delta$, if $x(t)$ is of (proper) synchrony type $\Delta$, for all $t \in \mathbb{R}$.

[2] It is also possible to consider synchrony-breaking bifurcation around a partial synchronous equilibrium $x_o$. In this case, additional assumptions are needed to prevent synchrony-preserving bifurcations.

*5.2. Functional reformulation*

Let $p > 0$ be the unknown period of the bifurcating solution $x$ of (33). Let $\beta := \frac{2\pi}{p}$ and $u(t) := x(\frac{1}{\beta}t)$. Then, finding a $p$-periodic solution $x$ of (33) is equivalent to solving

$$\begin{cases} \dot{u} = \dfrac{1}{\beta} f(\lambda, u), \\ u(0) = u(2\pi). \end{cases} \tag{34}$$

It is clear that $\mathcal{L}$ is a representation lattice of robust polydiagonal subspaces admitted by (34). Let $W := H^1(S^1; (\mathbb{R}^k)^n)$ be the first Sobolev space of $(\mathbb{R}^k)^n$-valued functions defined on $S^1$. Then, $\mathcal{L}$ induces a representation lattice $\mathcal{T}$ in $W$ as follows. Let

$$\breve{\Delta} := H^1(S^1; \Delta), \tag{35}$$

be the first Sobolev space of $\Delta$-valued functions defined on $S^1$, for $\Delta \in \mathcal{L}$. Let $\Gamma_\Delta$ be the group of action on $\Delta$ (cf (L2)). Define an (isometric) $\Gamma_\Delta \times S^1$-action on $\breve{\Delta}$ by

$$((\gamma, \mathrm{e}^{\mathrm{i}\theta})u)(t) := \gamma u(t+\theta), \qquad \gamma \in \Gamma_\Delta,\ \mathrm{e}^{\mathrm{i}\theta} \in S^1. \tag{36}$$

Let

$$\mathcal{T} = \{\breve{\Delta} : \Delta \in \mathcal{L}\}, \tag{37}$$

where $\breve{\Delta}$ is defined by (35). Then, with respect to the structure

$$\{\breve{\Delta}, \Gamma_\Delta \times S^1, \mathsf{h}_{\Delta,\Delta'} \times \mathrm{Id}_{S^1}\},$$

$\mathcal{T}$ is a representation lattice, which will be called the *induced lattice* from $\mathcal{L}$.

We reformulate (34) as a $\mathcal{T}$-equivariant fixed point problem. Define

$$\begin{aligned} &L : W \to L^2(S^1; (\mathbb{R}^k)^n), && L(u) = \dot{u} \\ &j : W \to C(S^1; (\mathbb{R}^k)^n), && j(u) = \tilde{u} \\ &N_f : \mathbb{R} \times C(S^1; (\mathbb{R}^k)^n) \to L^2(S^1; (\mathbb{R}^k)^n), && (N_f(\lambda, v))(t) = f(\lambda, v(t)). \end{aligned}$$

Then, (34) is equivalent to

$$Lu = \frac{1}{\beta} N_f(\lambda, j(u)).$$

Define $K : W \to L^2(S^1; (\mathbb{R}^k)^n)$ by $Ku := \frac{1}{2\pi}\int_0^{2\pi} u(t)\,\mathrm{d}t$. Then, $L + K$ is invertible and we have

$$u = (L+K)^{-1}\left[\frac{1}{\beta} N_f(\lambda, j(u)) + Ku\right] := F_1(\lambda, \beta, u). \tag{38}$$

Notice that $F_1$ is a $\mathcal{T}$-equivariant compact map.

Let $(\lambda_o, x_o)$ be the isolated bifurcation centre given by (B1) and $i\beta_o$ be the purely imaginary eigenvalue of $J(\lambda_o)$. Define a neighborhood $O \subset \mathbb{R}^2 \times W$ of $(\lambda_o, \beta_o, u_o)$ by

$$O := \{(\lambda, \beta, u) : \sqrt{(\lambda-\lambda_o)^2 + (\beta-\beta_o)^2} < \varepsilon,\ \|u\| < r\} \subset \mathbb{R}^2 \times W, \tag{39}$$

where $\mathbb{R}^2$ is considered as a parameter space (on which all groups act trivially). Note that $O$ is $\mathcal{T}$-invariant, since every group $\Gamma_\Delta \times S^1$ acts isometrically on $\breve{\Delta}$.

Let $\zeta : \overline{O} \to \mathbb{R}$ be an auxiliary function such that $\zeta(\lambda, \beta, u) > 0$ for $\|u\| = r$ and $\zeta(\lambda, \beta, u) < 0$ for $\|u\| = 0$. For example,

$$\zeta(\lambda, \beta, u) := \sqrt{(\lambda-\lambda_o)^2 + (\beta-\beta_o)^2}(\|u\| - r) + \|u\| - \frac{r}{2}.$$

Define a map $F_\zeta : \overline{O} \to \mathbb{R} \times W$ by

$$F_\zeta(\lambda, \beta, u) = (\zeta(\lambda, \beta, u), u - F_1(\lambda, \beta, u)). \tag{40}$$

Then, solutions of (34) around $(\lambda_o, x_o)$ are zeros of $F_\zeta$ in $O$. It can be verified that $F_\zeta$ is a $\mathcal{T}$-equivariant compact vector field. By (B1), $F_\zeta$ is also $O$-admissible. Therefore, $(F_\zeta, O)$ is an admissible pair and the lattice-equivariant degree of $F_\zeta$ in $O$

$$\omega(\lambda_o, \beta_o, x_o) := \mathcal{T}\text{-Deg}^t(F_\zeta, O)$$

is well-defined, which we call the *bifurcation invariant* around $(\lambda_o, x_o)$.

### 5.3. Classification result

Using the bifurcation invariant around the bifurcation centre $(\lambda_o, x_o)$, we can describe the topological structure of bifurcating branches of periodic solutions of (33) from $x_o$ and classify them according to their synchrony types and symmetric properties.

**Definition 5.2 (see [6]).** Let $\Delta \in \mathcal{L}$ and $H \subset \Gamma_\Delta \times S^1$ be a twisted $l$-folded subgroup such that $(H)$ is an orbit type of $W$. Then, $(H)$ is called *dominating*, if it is maximal in the class of all twisted $l$-folded orbit types of $W$.

We show that:

**Theorem 5.3.** *Let $\mathcal{L}$ be a representation lattice of robust synchrony subspaces of (33) satisfying (L1)–(L2) and $f$ such that (F) holds. Consider an equilibrium $x_o \in (\mathbb{R}^k)^n$ of (33) satisfying (E1)–(E2) and a bifurcation centre $(\lambda_o, x_o)$ satisfying (B1)–(B2). Let $\mathcal{T}$ be the induced lattice from $\mathcal{L}$ given by (37) and $O$, $F_\zeta$ be defined by (39) and (40). Assume that*

$$\mathcal{T}\text{-Deg}^t(F_\zeta, O) = \sum(\check{\Delta}, a_{\check{\Delta}}), \qquad \text{for some } a_{\check{\Delta}} \neq 0.$$

*Then,*

*(i) there exists a branch of non-constant periodic solutions of (33) bifurcating from $x_o$ that are of synchrony type $\Delta$;*
*(ii) if moreover, $i\beta_o$ is not an eigenvalue of $J(\lambda_o)|_{\Delta'}$, for any $\Delta' < \Delta$, then this branch of non-constant periodic solutions of (33) is of proper synchrony type $\Delta$;*
*(iii) if $a_{\check{\Delta}}$ contains a nonzero $(H)$-coefficient for a dominating orbit tpye $(H) = (K^{\varphi,l})$, then there exist at least $|\Gamma_\Delta/K|$ different bifurcating branches of non-constant periodic solutions of (33), which have isotropy subgroups $\gamma H \gamma^{-1}$, for $\gamma \in \Gamma_\Delta/K$.*

**Proof.**

(i) follows from the existence property of the lattice-equivariant degree (see theorem 4.3(i)), and a standard argument using parametrized auxiliary functions (see [6]).

(ii) We need to show that the bifurcating solutions given by (i) do not belong to $\check{\Delta}'$, for any $\Delta' < \Delta$. Let $\Delta' \in \mathcal{L}$ be such that $\Delta' < \Delta$. If $i\beta_o$ is not an eigenvalue of $J(\lambda_o)|_{\Delta'}$, then

$$\text{Id} - D_u F_1(\lambda_o, \beta_o, \cdot) : \check{\Delta}' \to \check{\Delta}'$$

is an isomorphism. By the implicit function theorem, $u_o$ is the unique zero of $F_\zeta$ in $O \cap (\mathbb{R}^2 \times \check{\Delta}')$. Thus, the bifurcating solutions can not belong to $\check{\Delta}'$.

(iii) This is a property of twisted equivariant degree (see [6]). For completeness, we give a brief proof here. Since $(H)$ is a dominating orbit type, there is a bifurcating solution $u \in \check{\Delta}$, whose orbit type is precisely $(H)$. The orbit of $u$ is thus diffeomorphic to $\Gamma_\Delta \times S^1/H$. By lemma 2.13, $\Gamma_\Delta \times S^1/H$ is a disjoint union of $|\Gamma_\Delta/K|$ copies of circles, which can be indexed by their isotropy types $\gamma H \gamma^{-1}$ (under the action of $\Gamma_\Delta \times S^1$), for $\gamma \in \Gamma/K$. Since every circle represents a (distinct) periodic solution, thus there exist at least $|\Gamma_\Delta/K|$

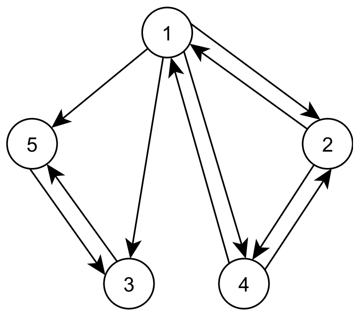


**Figure 5.** A regular coupled cell network of 5 cells.

different bifurcating branches of non-constant periodic solutions of (33), whose isotropy types are $\gamma H\gamma^{-1}$, for $\gamma \in \Gamma_\Delta/K$, respectively. $\square$

*5.4. Example*

We investigate a synchrony-breaking Hopf bifurcation in a regular[3] coupled cell system, which supports a large number of polydiagonal subspaces and nontrivial quotient symmetries on these polydiagonal subspaces. This network has been firstly studied in [1] as one of the twelve 5-cell regular networks which admit a $S_3$-symmetric quotient network, indexed as the network 6. We analyse the synchrony-breaking Hopf bifurcation for systems associated with this network in the case of non-simple eigenvalues) and give a classification of the bifurcating branches of oscillating solutions up to their synchrony types and symmetric properties.

Consider a 5-cell regular coupled cell network given in figure 5. The coupling structure of $\mathcal{N}$ can be described by the following adjacency matrix

$$A = \begin{pmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \end{pmatrix},$$

whose $(i, j)$th element is equal to the number of arrows from the $j$th cell to the $i$th cell. It can be verified that $A$ has the following spectrum

$$\sigma(A) = \{\mu_1 = 2, \mu_2 = 1, \mu_3 = \mu_4 = \mu_5 = -1\}.$$

In what follows, we discuss a synchrony-breaking Hopf bifurcation in the coupled cell system associated to $\mathcal{N}$ that is related to the non-simple eigenvalue $-1$ of the adjacency matrix $A$.

Define $f_o : \mathbb{R} \times \mathbb{R}^2 \times (\mathbb{R}^2)^2 \to \mathbb{R}^2$ by

$$f_o(\lambda, x, y, z) := \alpha(\lambda)x + \beta y + \beta z + xyz, \tag{41}$$

where '$xyz$' stands for the entry-wise multiplication of $x$, $y$, $z$ and

$$\alpha(\lambda) = \begin{pmatrix} 1+\lambda & -2 \\ 2 & 1+\lambda \end{pmatrix}, \qquad \beta = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}.$$

[3] A coupled cell system is called *regular* if it is homogeneous and contains only one type of coupling.

Consider the coupled cell system on $\mathcal{N}$ (with two-dimensional internal dynamics) given by

$$\begin{aligned}\dot{x}_1 &= f_o(\lambda, x_1, x_2, x_4)\\ \dot{x}_2 &= f_o(\lambda, x_2, x_1, x_4)\\ \dot{x}_3 &= f_o(\lambda, x_3, x_1, x_5)\\ \dot{x}_4 &= f_o(\lambda, x_4, x_1, x_2)\\ \dot{x}_5 &= f_o(\lambda, x_5; x_1, x_3),\end{aligned} \tag{42}$$

where $x_i \in \mathbb{R}^2$, $\lambda \in \mathbb{R}$ and $f_o$ is defined by (41). Then, $x = 0$ is an equilibrium.

*5.4.1. The spectrum of the Jacobian.* Let $f : \mathbb{R} \times (\mathbb{R}^2)^5 \to (\mathbb{R}^2)^5$ be the right-hand side of (42). It was shown in [13] that the linearization $J(\lambda) = Df_x(\lambda, 0)$ of $f$ at $(\lambda, 0)$ has the form

$$J(\lambda) = \alpha(\lambda) \otimes I_5 + \beta \otimes A,$$

where $I_5 : \mathbb{R}^5 \to \mathbb{R}^5$ is the identity matrix. Also, the eigenvalues of $J(\lambda)$ are the union of the eigenvalues of the $2 \times 2$-matrices $M_\mu := \alpha(\lambda) + \mu\beta$, for all $\mu \in \sigma(A)$. Moreover, if $v \in \mathbb{C}^5$ is an eigenvector of $A$ and $u \in \mathbb{C}^2$ is an eigenvector of $M_\mu$, then $u \otimes v$ is an eigenvector of $J(\lambda)$ (cf [13]). More precisely, $J(\lambda)$ has the following eigenvalues and eigenvectors

$$\begin{aligned}
&M_2, && \sigma_{1,2} = 3 + \lambda \pm 4i, && \begin{pmatrix}-i\\1\end{pmatrix} \otimes v_1,\ \begin{pmatrix}i\\1\end{pmatrix} \otimes v_1\\
&M_1, && \sigma_{3,4} = 2 + \lambda \pm 3i, && \begin{pmatrix}1\\0\end{pmatrix} \otimes v_2,\ \begin{pmatrix}0\\1\end{pmatrix} \otimes v_2\\
&M_{-1}, && \sigma_{5,6,7,8,9,10} = \lambda \pm i, && \begin{pmatrix}-i\\1\end{pmatrix} \otimes v_j,\ \begin{pmatrix}i\\1\end{pmatrix} \otimes v_j,
\end{aligned}$$

where $v_1, v_2, v_j$ for $3 \leqslant j \leqslant 5$ are eigenvectors of $A$ corresponding to $2, 1, -1$. Consequently, (42) has three isolated bifurcation centres $(-3, 0)$, $(-2, 0)$ and $(0, 0)$. We describe the synchrony-breaking bifurcation around $(0, 0)$, i.e.

$$(\lambda_o, \beta_o, x_o) = (0, 1, 0).$$

*5.4.2. The polydiagonal subspaces.* Invariant polydiagonal subspaces of the adjacency matrix $A$ were listed in [1], which form a lattice in figure 6, where $a, b, c, d, e \in \mathbb{R}$. Note that $\mathbb{R}^2 \otimes U$ is $J(\lambda)$-invariant, for every $A$-invariant linear subspace $U \subset \mathbb{R}^5$. Thus, the lattice in figure 6 is the lattice of polydiagonal subspaces of (42), for $a, b, c, d, e \in \mathbb{R}^2$.

Denote by

$$\breve{U} := H^1(S^1; U),$$

the first Sobolev space of $2\pi$-periodic functions valued in $U$, for $U \in \mathcal{L}$. Let

$$\mathcal{T} := \{\breve{U} : U \in \mathcal{L}\}.$$

Then, $\mathcal{T}$ is a representation lattice with structure $\{\breve{U}, \Gamma_U \times S^1, \mathsf{h}_{U,U''} \times \mathrm{Id}\}$, where $\mathrm{Id} : S^1 \to S^1$ is the identity homomorphism.

*5.4.3. The representation lattice.* Let $\tilde{\mathcal{L}}$ be the lattice given by figure 6. For every $U \in \tilde{\mathcal{L}}$, there is a *quotient network* associated to $U$, whose network structure is given by $A|_U$ (the adjacency matrix $A$ restricted to $U$). A *quotient symmetry* associated to $U$, say $\Gamma_U$, is a symmetry of the quotient network associated to $U$. It follows that $\Gamma_U$ is a symmetry of (42)

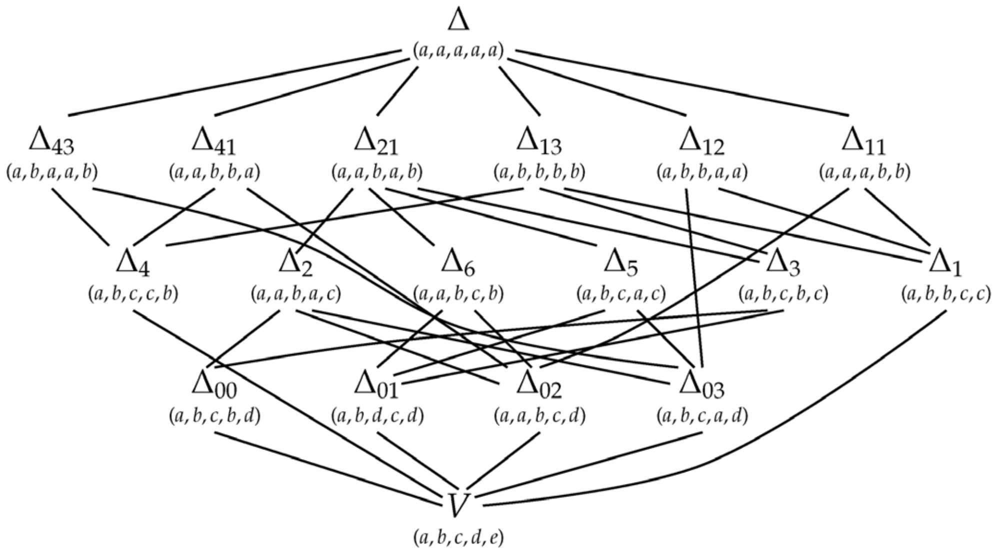


**Figure 6.** The lattice $\tilde{\mathcal{L}}$ of invariant subspaces.

when restricted to $\mathbb{R}^2 \otimes U$. It can be verified that (42) has the following (nontrivial) quotient symmetries:

$$\Gamma_{\Delta_4} = \Gamma_{\Delta_1} = \Gamma_{\Delta_{01}} = S_3 \simeq D_3, \qquad \Gamma_{\Delta_2} = \Gamma_{\Delta_{00}} = \Gamma_{\Delta_{02}} = \Gamma_{\Delta_{03}} = \mathbb{Z}_2,$$

where $D_3$ acts as permutations on symbols $a, b, c$ and $\mathbb{Z}_2 = \langle\kappa\rangle$ acts on $\Delta_{00}, \Delta_{02}, \Delta_{03}$ by $\kappa : (a, b, c, d, e) \mapsto (a, b, e, d, c)$.

However, for any choices of homomorphisms $\mathsf{h}_{U,U'}$'s, $\{U, \Gamma_U, \mathsf{h}_{U,U'}\}_{U\in\tilde{\mathcal{L}}}$ does not give a valid structure of representation lattice to $\tilde{\mathcal{L}}$. Indeed, a necessary condition for $\tilde{\mathcal{L}}$ to be a representation lattice is that $U$ is $\Gamma_{U'}$-invariant subspace of $U'$, for all $U \subset U'$ (see definition 3.2 (ii)). But we have that $\Delta_{43}, \Delta_{41}, \Delta_{13}$ are not $D_3$-invariant in $\Delta_4$; $\Delta_{13}, \Delta_{12}, \Delta_{11}$ are not $D_3$-invariant in $\Delta_1$; and $\Delta_3, \Delta_5, \Delta_6$ are not $D_3$-invariant in $\Delta_{01}$. In fact, in each of these cases, the three subspaces form one orbit under $D_3$-action.

Let $\mathcal{L} = \tilde{\mathcal{L}} \setminus \{\Delta_{43}, \Delta_{41}, \Delta_{13}, \Delta_{12}, \Delta_{11}, \Delta_3, \Delta_5, \Delta_6\}$ and $\Gamma_U$ be the quotient symmetry related to $U$, for $U \in \mathcal{L}$ (see figure 7). The arrows in figure 7 stand for homomorphisms, where $\mathsf{h}_{*,*} : \mathbb{Z}_1 \to \Gamma_x$ are given by the inclusion, $\mathsf{h}_{*,*} : \Gamma_x \to \mathbb{Z}_1$ are given by the projection and $\mathsf{h}_{*,*} : \mathbb{Z}_2 \to \mathbb{Z}_2$ are the identity homomorphism. As shown in example 3.3, $\mathcal{L}$ is a real representation lattice in $V = \mathbb{R}^2 \otimes \mathbb{R}^5$, with respect to this structure.

*5.4.4. The bifurcation invariant.* It can be verified that the assumptions (E1)–(E2), (B1)–(B2), (L1)–(L2) and (F) are satisfied by (42) with $f_o$ given by (41) and $x_o = 0$. Thus, the bifurcation invariant

$$\omega(\lambda_o, \beta_o, x_o) = \mathcal{T}\text{-Deg}^t(F_\zeta, O)$$

is well-defined. We compute this bifurcation invariant around $(\lambda_o, \beta_o, x_o) = (0, 1, 0)$. Write

$$\mathcal{T}\text{-Deg}^t(F_\zeta, O) = \sum\nolimits_{U\in\mathcal{L}} (\breve{U}, a_{\breve{U}}).$$

Let $D_3 = \mathbb{Z}_3 \cup \kappa\mathbb{Z}_3$, where $\mathbb{Z}_3 = \langle\xi\rangle$. Define

$$\mathbb{Z}_3^t = \{(1, 1), (\xi, \xi), (\xi^2, \xi^2)\}, \qquad D_1 = \{(1, 1), (\kappa, 1)\},$$
$$D_1^z = \{(1, 1), (\kappa, -1)\},$$

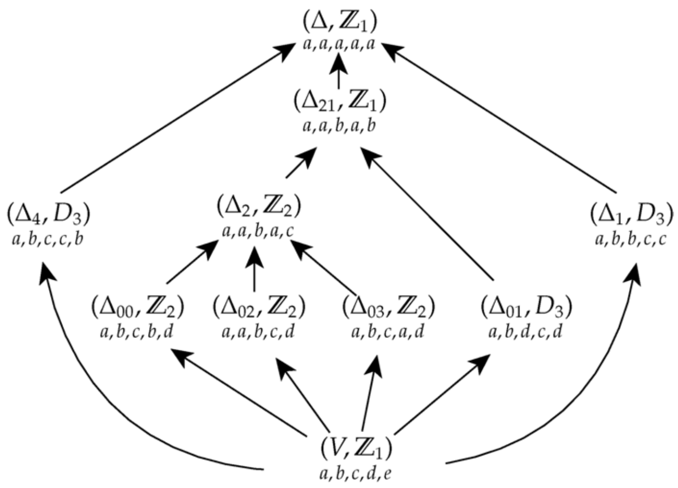


**Figure 7.** A representation lattice $\mathcal{L}$ in $\mathbb{R}^2 \otimes \mathbb{R}^5$.

which are subgroups of $D_3 \times S^1$. Define

$$\mathbb{Z}_2^- := \{(1, 1), (-1, -1)\} \subset \mathbb{Z}_2 \times S^1.$$

Then, we obtain (see the appendix for details of computations)

$$\begin{aligned}\mathcal{T}\text{-Deg}^t(F_\zeta, O) = &\big(\check{\boldsymbol{\Delta}}_{\mathbf{1}}, -2(\boldsymbol{Z}_{\mathbf{3}}^{\boldsymbol{t}}) - 2(\boldsymbol{D}_1) - 2(\boldsymbol{D}_1^z) + 2(\mathbb{Z}_1)\big) + \big(\check{\boldsymbol{\Delta}}_{\mathbf{2}}, -(\boldsymbol{Z}_{\mathbf{2}}^-)\big)\\ &+\big(\check{\boldsymbol{\Delta}}_{\mathbf{4}}, -2(\boldsymbol{Z}_{\mathbf{3}}^{\boldsymbol{t}}) - 2(\boldsymbol{D}_1) - 2(\boldsymbol{D}_1^z) + 2(\mathbb{Z}_1)\big) + \big(\check{\boldsymbol{\Delta}}_{\mathbf{01}}, -2(\boldsymbol{Z}_{\mathbf{3}}^{\boldsymbol{t}})\\ &-2(\boldsymbol{D}_1) - 2(\boldsymbol{D}_1^z) + 2(\mathbb{Z}_1)\big) + \big(\check{\Delta}_{00}, -(\mathbb{Z}_2^-)\big) + \big(\check{\Delta}_{02}, -(\mathbb{Z}_2^-)\big)\\ &+\big(\check{\Delta}_{03}, -(\mathbb{Z}_2^-)\big) + (W, 13(\mathbb{Z}_1)),\end{aligned}$$

where the highlighted $\Delta_1$, $\Delta_2$, $\Delta_4$, $\Delta_{01}$ satisfy the condition of theorem 5.3(ii) and the dominating orbit types in these polydiagonal subspaces are in the bold font.

Consider $\Delta = \boldsymbol{\Delta}_1 \simeq \mathbb{R}^3$ (thus $\Gamma_\Delta = D_3$) and

$$a_{\check{\Delta}_1} = -2(\boldsymbol{Z}_{\mathbf{3}}^{\boldsymbol{t}}) - 2(\boldsymbol{D}_1) - 2(\boldsymbol{D}_1^z) + 2(\mathbb{Z}_1).$$

Since $a_{\check{\Delta}_1}$ contains a nontrivial $(\boldsymbol{Z}_{\mathbf{3}}^{\boldsymbol{t}})$-term, by theorem 5.3(iii), there exist at least $|D_3/\mathbb{Z}_3| = 2$ different branches of non-constant periodic solutions of (42), whose isotropy types are $\mathbb{Z}_3^t$ and $\kappa\mathbb{Z}_3^t\kappa^{-1}$, respectively. More precisely, the branch with the isotropy type $\mathbb{Z}_3^t$ has the form (see example 2.14)

$$u(t) = \left(x(t), x\left(t + \frac{T}{3}\right), x\left(t + \frac{2T}{3}\right)\right),$$

and the branch with the isotropy type $\kappa\mathbb{Z}_3^t\kappa^{-1}$ is of form

$$v(t) = \left(x(t), x\left(t + \frac{2T}{3}\right), x\left(t + \frac{T}{3}\right)\right).$$

By theorem 5.3(ii), these branches of solutions are of proper synchrony type $\Delta_1$. Similarly, since $a_{\check{\Delta}_1}$ contains a nontrivial $(\boldsymbol{D}_1)$-term, there exist at least $|D_3/D_1| = 3$ branches of non-constant periodic solutions of (42), whose isotropy types are $D_1$, $\xi D_1\xi^{-1}$, $\xi^2 D_1\xi^{-2}$, respectively, and they have a proper synchrony type $\Delta_1$. Also, the nontrivial $(\boldsymbol{D}_1^z)$-term indicates the existence of $|D_3/D_1| = 3$ branches of non-constant periodic solutions of (42),

**Table 2.** The summary of synchrony type and symmetric properties of all topological bifurcating branches of solutions from $x_o = 0$ of system (42).

| Synchrony | Symmetry | Form of Periodic Solutions (for some period $T$) |
|---|---|---|
| $\Delta_1$ $(a,b,b,c,c)$ | $\mathbb{Z}_3^t$ | $\left(x(t), x(t+\frac{T}{3}), x(t+\frac{T}{3}), x(t+\frac{2T}{3}), x(t+\frac{2T}{3})\right)$ |
| | $\kappa\mathbb{Z}_3^t\kappa^{-1}$ | $\left(x(t), x(t+\frac{2T}{3}), x(t+\frac{2T}{3}), x(t+\frac{T}{3}), x(t+\frac{T}{3})\right)$ |
| | $D_1$ | $\left(x(t), y(t), y(t), x(t), x(t)\right) \in \Delta_{12}$ |
| | $\xi D_1\xi^{-1}$ | $\left(x(t), x(t), x(t), y(t), y(t)\right) \in \Delta_{11}$ |
| | $\xi^2 D_1\xi^{-2}$ | $\left(x(t), y(t), y(t), y(t), y(t)\right) \in \Delta_{13}$ |
| | $D_1^z$ | $\left(x(t), y(t), y(t), x(t+\frac{T}{2}), x(t+\frac{T}{2})\right)$, for $y(t) = y(t+\frac{T}{2})$ |
| | $\xi D_1^z\xi^{-1}$ | $\left(x(t), x(t+\frac{T}{2}), x(t+\frac{T}{2}), y(t), y(t)\right)$, for $y(t) = y(t+\frac{T}{2})$ |
| | $\xi^2 D_1^z\xi^{-2}$ | $\left(x(t), y(t), y(t), y(t+\frac{T}{2}), y(t+\frac{T}{2})\right)$, for $x(t) = x(t+\frac{T}{2})$ |
| $\Delta_2$ $(a,a,b,a,c)$ | $\mathbb{Z}_2^-$ | $\left(x(t), x(t), y(t), x(t), y(t+\frac{T}{2})\right)$ |
| $\Delta_4$ $(a,b,c,c,b)$ | $\mathbb{Z}_3^t$ | $\left(x(t), x(t+\frac{T}{3}), x(t+\frac{2T}{3}), x(t+\frac{2T}{3}), x(t+\frac{T}{3})\right)$ |
| | $\kappa\mathbb{Z}_3^t\kappa^{-1}$ | $\left(x(t), x(t+\frac{2T}{3}), x(t+\frac{T}{3}), x(t+\frac{T}{3}), x(t+\frac{2T}{3})\right)$ |
| | $D_1$ | $\left(x(t), y(t), x(t), x(t), y(t)\right) \in \Delta_{43}$ |
| | $\xi D_1\xi^{-1}$ | $\left(x(t), x(t), y(t), y(t), x(t)\right) \in \Delta_{41}$ |
| | $\xi^2 D_1\xi^{-2}$ | $\left(x(t), y(t), y(t), y(t), y(t)\right) \in \Delta_{13}$ |
| | $D_1^z$ | $\left(x(t), y(t), x(t+\frac{T}{2}), x(t+\frac{T}{2}), y(t)\right)$, for $y(t) = y(t+\frac{T}{2})$ |
| | $\xi D_1^z\xi^{-1}$ | $\left(x(t), x(t+\frac{T}{2}), y(t), y(t), x(t+\frac{T}{2})\right)$, for $y(t) = y(t+\frac{T}{2})$ |
| | $\xi^2 D_1^z\xi^{-2}$ | $\left(x(t), y(t), y(t+\frac{T}{2}), y(t+\frac{T}{2}), y(t)\right)$, for $x(t) = x(t+\frac{T}{2})$ |
| $\Delta_{01}$ $(a,b,d,c,d)$ | $\mathbb{Z}_3^t$ | $\left(x(t), x(t+\frac{T}{3}), y(t), x(t+\frac{2T}{3}), y(t)\right)$ |
| | $\kappa\mathbb{Z}_3^t\kappa^{-1}$ | $\left(x(t), x(t+\frac{2T}{3}), y(t), x(t+\frac{T}{3}), y(t)\right)$ |
| | $D_1$ | $\left(x(t), y(t), z(t), x(t), z(t)\right) \in \Delta_5$ |
| | $\xi D_1\xi^{-1}$ | $\left(x(t), x(t), z(t), y(t), z(t)\right) \in \Delta_6$ |
| | $\xi^2 D_1\xi^{-2}$ | $\left(x(t), y(t), z(t), y(t), z(t)\right) \in \Delta_3$ |
| | $D_1^z$ | $\left(x(t), y(t), z(t), x(t+\frac{T}{2}), z(t)\right)$, for $y(t) = y(t+\frac{T}{2})$ |
| | $\xi D_1^z\xi^{-1}$ | $\left(x(t), x(t+\frac{T}{2}), z(t), y(t), z(t)\right)$, for $y(t) = y(t+\frac{T}{2})$ |
| | $\xi^2 D_1^z\xi^{-2}$ | $\left(x(t), y(t), z(t), y(t+\frac{T}{2}), z(t)\right)$, for $x(t) = x(t+\frac{T}{2})$ |

whose isotropy types are $D_1^z$, $\xi D_1^z\xi^{-1}$, $\xi^2 D_1^z\xi^{-2}$, respectively (see table 2), and they are of a proper synchrony type $\Delta_1$.

An analogous analysis can be applied to $\mathbf{\Delta_2}$, $\mathbf{\Delta_4}$ and $\mathbf{\Delta_{01}}$. A summary of these bifurcating solutions can be found in table 2. In brief, we predict 8 different branches of non-constant periodic solutions of proper synchrony type $\Delta_1$; 1 branch of non-constant periodic solutions of proper synchrony type $\Delta_2$; 8 different branches of non-constant periodic solutions of proper synchrony type $\Delta_4$; and 8 different branches of non-constant periodic solutions of proper synchrony type $\Delta_{01}$.

Note that we do not exclude the possibility of additional periodic solutions bifurcating from $x_o = 0$, besides those listed in table 2, since as a topological invariant, the lattice degree gives only a lower estimate of the number of solutions. In other words, other non-constant periodic solutions (also possibly symmetric and/or partially synchronized) may bifurcate from $x_o = 0$.

## 6. Conclusions

We introduce *representation lattices* in Banach spaces and maps that are compatible with respect to this structure—*lattice-equivariant maps*. We define a degree theory, called the *lattice-equivariant degree*, for lattice-equivariant maps on lattice invariant domains, using an alternating-sum type of formula and the twisted equivariant degree (see definition 4.2). Based on an equivariant version of the Schauder projection, we extend this degree to infinite-dimensional lattice representations for compact lattice-equivariant vector fields.

We study a *synchrony-breaking Hopf bifurcation* in homogeneous coupled cell systems, using the lattice-equivariant degree. In this case, the representation lattice is given by the lattice of partial polydiagonal subspaces admitted by the system and the action group is given by the symmetry of the corresponding quotient network. We associate a *bifurcation invariant* to bifurcation points using the lattice-equivariant degree, and show that the bifurcation invariant gives a topological classification of all bifurcating branches of oscillating solutions according to their synchrony types and symmetric properties (see theorem 5.3). As an example, we investigate a ten-dimensional coupled system of 5-cells, which has quotient symmetries of $\mathbb{Z}_2$ and $S_3$. We obtain a synchrony- and symmetry-classification of the total 25 bifurcating branches of oscillating solutions (see table 2).

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## Appendix: Computation of the bifurcation invariant

We give the details of the computation of the bifurcation invariant

$$\omega(\lambda_o, \beta_o, x_o) = \mathcal{T}\text{-Deg}^t(F_\zeta, O),$$

associated with the bifurcation centre $(\lambda_o, \beta_o, x_o) = (0, 1, 0)$ of the system (42).

Recall that by definition,

$$\mathcal{T}\text{-Deg}^t(F_\zeta, O) = \sum_{U \in \mathcal{L}} (\breve{U}, a_{\breve{U}}),$$

where

$$a_{\breve{U}} = \Gamma_U \times S^1\text{-Deg}^t(F_\zeta|_{\mathbb{R}^2 \times \breve{U}}, O \cap (\mathbb{R}^2 \times \breve{U})) - \sum_{U' < U} \mathsf{H}_{U',U}(a_{\breve{U}'}).$$

Let $\Gamma = \Gamma_U$, $F = F_\zeta|_{\mathbb{R}^2 \times \breve{U}}$ and $\Omega = O \cap (\mathbb{R}^2 \times \breve{U})$ for some $U \in \mathcal{L}$. The twisted degree $\Gamma \times S^1\text{-Deg}^t(F, \Omega)$ can be computed from the following formula (cf [6])

$$\Gamma \times S^1\text{-Deg}^t(F, \Omega) = \prod_{\mu \in \sigma_+(J(\lambda_o))} \prod_i (\text{deg}_{\mathcal{V}_i})^{m_i(\mu)} \cdot \sum_{j,l} t_{j,l}(\lambda_o, \beta_o) \text{deg}_{\mathcal{V}_{j,l}},$$

where $\sigma_+(J(\lambda_o))$ is the positive spectrum of $J(\lambda_o)$, $\text{deg}_{\mathcal{V}_i}$ is the basic degree of the $i$-th irreducible representation of $\Gamma$ over reals, $m_i(\mu) = \dim(E(\mu) \cap V_i)/\dim \mathcal{V}_i$ is the algebraic multiplicity of $\mu$ when restricted to the $i$-th isotypical component of the eigenspace $E(\mu)$,

$\deg_{\mathcal{V}_{j,l}}$ is the basic degree of the $(j, l)$-th irreducible representation of $\Gamma \times S^1$ over complex numbers, and $t_{j,l}(\lambda_o, \beta_o)$ is the $(j, l)$-th isotypical crossing number of $(\lambda_o, \beta_o)$.

In our example, since $\sigma_+(J(\lambda_o)) = \emptyset$ and $il\beta_o$ is only a critical eigenvalue for $l = 1$, we have

$$\Gamma \times S^1\text{-Deg}^t(F, \Omega) = \sum_j t_{j,1}(\lambda_o, \beta_o)\deg_{\mathcal{V}_{j,1}}.$$

*Computation of* $a_{\check{\Delta}}$. In this case, $\Gamma = \Gamma_\Delta = \mathbb{Z}_1$. Consider $\Delta^c = \mathbb{R}^2 \otimes \mathbb{C} \simeq \mathbb{C}^2$ as a complex $\mathbb{Z}_1$-representation. Then, the $\mathbb{Z}_1$-isotypical decomposition of $\Delta^c$ is

$$\Delta^c = U_0 \oplus U_0,$$

where $U_0$ is the trivial (complex) $\mathbb{Z}_1$-representation. Thus,

$$\Gamma_\Delta \times S^1\text{-Deg}^t(F_\zeta|_{\mathbb{R}^2\times\check{\Delta}}, O \cap (\mathbb{R}^2 \times \check{\Delta})) = t_{0,1}(\lambda_o, \beta_o)\deg_{\mathcal{V}_{0,1}}.$$

Consider $J(\lambda)$ as a complex linear map in $\Delta^c$. Then,

$$J(\lambda) = \alpha(\lambda) + 2\beta, \qquad \text{and} \qquad \sigma(J(\lambda)) = \{\sigma_{1,2}\}.$$

Since $\sigma(J(\lambda_o)) \cap i\mathbb{R} = \emptyset$, there are no eigenvalues crossing the purely imaginary axis, as $\lambda$ crosses $\lambda_o$. Thus, $t_{0,1}(\lambda_o, \beta_o) = 0$ and consequently,

$$a_{\check{\Delta}} = \Gamma_\Delta \times S^1\text{-Deg}^t(F_\zeta|_{\mathbb{R}^2\times\check{\Delta}}, O \cap (\mathbb{R}^2 \times \check{\Delta})) = 0.$$

*Computation of* $a_{\check{\Delta}_{21}}$. In this case, $\Gamma = \Gamma_{\Delta_{21}} = \mathbb{Z}_1$. Similarly, we have

$$\Delta_{21}^c = U_0 \oplus U_0 \oplus U_0 \oplus U_0,$$

where $U_0$ is the trivial (complex) $\mathbb{Z}_1$-representation. Thus,

$$\Gamma_{\Delta_{21}} \times S^1\text{-Deg}^t(F_\zeta|_{\mathbb{R}^2\times\check{\Delta}_{21}}, O \cap (\mathbb{R}^2 \times \check{\Delta}_{21})) = t_{0,1}(\lambda_o, \beta_o)\deg_{\mathcal{V}_{0,1}}.$$

Consider $J(\lambda)$ as a complex linear map in $\Delta_{21}^c$. Then,

$$J(\lambda) = \alpha \otimes \mathbb{R}^2 + \beta \otimes \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix}, \qquad \text{and} \qquad \sigma(J(\lambda)) = \{\sigma_{1,2}, \sigma_{3,4}\}.$$

Since $\sigma(J(\lambda_o)) \cap i\mathbb{R} = \emptyset$, we have $t_{0,1}(\lambda_o, \beta_o) = 0$. Thus,

$$a_{\check{\Delta}_{21}} = \Gamma_{\Delta_{21}} \times S^1\text{-Deg}^t(F_\zeta|_{\mathbb{R}^2\times\check{\Delta}_{21}}, O \cap (\mathbb{R}^2 \times \check{\Delta}_{21})) - 0 = 0.$$

*Computation of* $a_{\check{\Delta}_2}$. In this case, $\Gamma = \Gamma_{\Delta_2} = \mathbb{Z}_2$. Consider $\Delta_2^c = \mathbb{R}^6 \otimes \mathbb{C} \simeq \mathbb{C}^6$ as a complex $\mathbb{Z}_2$-representation. Then, the $\mathbb{Z}_2$-isotypical decomposition of $\Delta_2^c$ is

$$\Delta_2^c = U_0 \oplus U_0 \oplus U_0 \oplus U_0 \oplus U_1 \oplus U_1,$$

where $U_0$ is the trivial (complex) $\mathbb{Z}_2$-representation and $U_1$ is the $\mathbb{Z}_2$-representation given by antipodal action. Thus,

$$\Gamma_{\Delta_2} \times S^1\text{-Deg}^t(F_\zeta|_{\mathbb{R}^2\times\check{\Delta}_2}, O \cap (\mathbb{R}^2 \times \check{\Delta}_2)) = t_{0,1}(\lambda_o, \beta_o)\deg_{\mathcal{V}_{0,1}} + t_{1,1}(\lambda_o, \beta_o)\deg_{\mathcal{V}_{1,1}}.$$

Consider $J(\lambda)$ as a complex linear map in $\Delta_2^c$. Then,

$$J(\lambda) = \alpha \otimes \mathbb{R}^2 + \beta \otimes \begin{pmatrix} 2 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}, \qquad \text{and} \qquad \sigma(J(\lambda)) = \{\sigma_{1,2}, \sigma_{3,4}, \sigma_{5,6}\}.$$

Thus, $t_{0,1}(\lambda_o, \beta_o) = 0$ and $t_{1,1}(\lambda_o, \beta_o) = -1$. Therefore,

$$a_{\check{\Delta}_2} = \Gamma_{\Delta_2} \times S^1\text{-Deg}^t(F_\zeta|_{\mathbb{R}^2\times\check{\Delta}_2}, O \cap (\mathbb{R}^2 \times \check{\Delta}_2)) - 0 - 0 = -\deg_{\mathcal{V}_{1,1}}.$$

Here $\mathcal{V}_{1,1} \simeq U_1$ is a $\mathbb{Z}_2 \times S^1$-representation given by 'complexifying' the $\mathbb{Z}_2$-action on $U_1$, that is $(\xi, z)w := z\xi w$, for $\xi \in \mathbb{Z}_2$, $z \in S^1$, $w \in \mathbb{C}$. The orbit type of $w \neq 0$ is $\mathbb{Z}_2^- := \{(1,1), (-1,-1)\}$. Thus, $\deg_{\mathcal{V}_{1,1}} = (\mathbb{Z}_2^-)$ and so

$$a_{\check{\Delta}_2} = -\deg_{\mathcal{V}_{1,1}} = -(\mathbb{Z}_2^-).$$

*Computation of* $a_{\check{\Delta}_4}$. In this case, $\Gamma = \Gamma_{\Delta_4} = D_3$. Consider $\Delta_4^c = \mathbb{R}^6 \otimes \mathbb{C} \simeq \mathbb{C}^6$ as a complex $D_3$-representation. Then, the $D_3$-isotypical decomposition of $\Delta_4^c$ is

$$\Delta_4^c = U_0 \oplus U_0 \oplus U_1 \oplus U_1,$$

where $U_0$ is the trivial $D_3$-representation, $U_1 \simeq \mathbb{C} \oplus \mathbb{C}$ is the complex $D_3$-representation given by $\xi(z_1, z_2) = (\xi z_1, \xi^{-1} z_2)$, $\kappa(z_1, z_2) = (z_2, z_1)$, for $z_1, z_2 \in \mathbb{C}$. Thus,

$$\Gamma_{\Delta_4} \times S^1\text{-Deg}^t(F_\zeta|_{\mathbb{R}^2 \times \check{\Delta}_4}, O \cap (\mathbb{R}^2 \times \check{\Delta}_4)) = t_{0,1}(\lambda_o, \beta_o)\deg_{\mathcal{V}_{0,1}} + t_{1,1}(\lambda_o, \beta_o)\deg_{\mathcal{V}_{1,1}}.$$

Consider $J(\lambda)$ as a complex linear map in $\Delta_4^c$. Then,

$$J(\lambda) = \alpha \otimes \mathbb{R}^2 + \beta \otimes \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}, \qquad \text{and} \qquad \sigma(J(\lambda)) = \{\sigma_{1,2}, \sigma_{5,6,7,8}\}.$$

Thus, $t_{0,1}(\lambda_o, \beta_o) = 0$ and $t_{1,1}(\lambda_o, \beta_o) = -2$. Therefore,

$$a_{\check{\Delta}_4} = \Gamma_{\Delta_4} \times S^1\text{-Deg}^t(F_\zeta|_{\mathbb{R}^2 \times \check{\Delta}_4}, O \cap (\mathbb{R}^2 \times \check{\Delta}_4)) - 0 = -2\deg_{\mathcal{V}_{1,1}}.$$

It was shown in [6] that (cf example 2.14 for the definition of $\mathbb{Z}_3^t$ and $D_1^z$)

$$\deg_{\mathcal{V}_{1,1}} = (\mathbb{Z}_3^t) + (D_1) + (D_1^z) - (\mathbb{Z}_1).$$

Thus,

$$a_{\check{\Delta}_4} = -2(\mathbb{Z}_3^t) - 2(D_1) - 2(D_1^z) + 2(\mathbb{Z}_1).$$

*Computation of* $a_{\check{\Delta}_1}$. In this case, $\Gamma = \Gamma_{\Delta_1} = D_3$. Consider $\Delta_4^c = \mathbb{R}^6 \otimes \mathbb{C} \simeq \mathbb{C}^6$ as a complex $D_3$-representation. Similar as the case for $\Delta_4$, the $D_3$-isotypical decomposition of $\Delta_1^c$ is

$$\Delta_1^c = U_0 \oplus U_0 \oplus U_1 \oplus U_1.$$

Thus,

$$\Gamma_{\Delta_1} \times S^1\text{-Deg}^t(F_\zeta|_{\mathbb{R}^2 \times \check{\Delta}_1}, O \cap (\mathbb{R}^2 \times \check{\Delta}_1)) = t_{0,1}(\lambda_o, \beta_o)\deg_{\mathcal{V}_{0,1}} + t_{1,1}(\lambda_o, \beta_o)\deg_{\mathcal{V}_{1,1}}.$$

Consider $J(\lambda)$ as a complex linear map in $\Delta_1^c$. Then,

$$J(\lambda) = \alpha \otimes \mathbb{R}^2 + \beta \otimes \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}, \qquad \text{and} \qquad \sigma(J(\lambda)) = \{\sigma_{1,2}, \sigma_{5,6,7,8}\}.$$

Thus, $t_{0,1}(\lambda_o, \beta_o) = 0$ and $t_{1,1}(\lambda_o, \beta_o) = -2$. Therefore,

$$\begin{aligned} a_{\check{\Delta}_1} &= \Gamma_{\Delta_1} \times S^1\text{-Deg}^t(F_\zeta|_{\mathbb{R}^2 \times \check{\Delta}_1}, O \cap (\mathbb{R}^2 \times \check{\Delta}_1)) - 0 = -2\deg_{\mathcal{V}_{1,1}} \\ &= -2(\mathbb{Z}_3^t) - 2(D_1) - 2(D_1^z) + 2(\mathbb{Z}_1). \end{aligned}$$

*Computation of* $a_{\check{\Delta}_{00}}$. In this case, $\Gamma = \Gamma_{\Delta_{00}} = \mathbb{Z}_2$. Consider $\Delta_{00}^c = \mathbb{R}^8 \otimes \mathbb{C} \simeq \mathbb{C}^8$ as a complex $\mathbb{Z}_2$-representation. Then, the $\mathbb{Z}_2$-isotypical decomposition of $\Delta_{00}^c$ is

$$\Delta_{00}^c = U_0 \oplus U_0 \oplus U_0 \oplus U_0 \oplus U_0 \oplus U_0 \oplus U_1 \oplus U_1,$$

where $U_0$ is the trivial (complex) $\mathbb{Z}_2$-representation and $U_1$ is the $\mathbb{Z}_2$-representation given by antipodal action. Thus,

$$\Gamma_{\Delta_{00}} \times S^1\text{-Deg}^t(F_\zeta|_{\mathbb{R}^2\times\check{\Delta}_{00}}, O \cap (\mathbb{R}^2 \times \check{\Delta}_{00})) = t_{0,1}(\lambda_o, \beta_o)\deg_{\mathcal{V}_{0,1}} + t_{1,1}(\lambda_o, \beta_o)\deg_{\mathcal{V}_{1,1}}.$$

Consider $J(\lambda)$ as a complex linear map in $\Delta_{00}^c$. Then,

$$J(\lambda) = \alpha \otimes \mathbb{R}^2 + \beta \otimes \begin{pmatrix} 0 & 2 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}, \qquad \text{and} \qquad \sigma(J(\lambda)) = \{\sigma_{1,2}, \sigma_{3,4}, \sigma_{5,6,7,8}\}.$$

Thus, $t_{0,1}(\lambda_o, \beta_o) = 0$ and $t_{1,1}(\lambda_o, \beta_o) = -2$. Therefore,

$$\Gamma_{\Delta_{00}} \times S^1\text{-Deg}^t(F_\zeta|_{\mathbb{R}^2\times\check{\Delta}_{00}}, O \cap (\mathbb{R}^2 \times \check{\Delta}_{00})) = -2\deg_{\mathcal{V}_{1,1}} = -2(\mathbb{Z}_2^-).$$

On the other hand, $\mathsf{H}_{\Delta_2,\Delta_{00}} = \text{Id}$, since $\mathsf{h}_{\Delta_2,\Delta_{00}} = \text{Id}$. Consequently,

$$a_{\check{\Delta}_{00}} = -2(\mathbb{Z}_2^-) - \mathsf{H}_{\Delta_2,\Delta_{00}}\big(-(\mathbb{Z}_2^-)\big) = -2(\mathbb{Z}_2^-) + (\mathbb{Z}_2^-) = -(\mathbb{Z}_2^-).$$

*Computation of* $a_{\check{\Delta}_{02}}$. This is a similar case as for $\Delta_{00}$. We have $\Gamma = \Gamma_{\Delta_{02}} = \mathbb{Z}_2$ and the $\mathbb{Z}_2$-isotypical decomposition of $\Delta_{02}^c$ is

$$\Delta_{02}^c = U_0 \oplus U_0 \oplus U_0 \oplus U_0 \oplus U_0 \oplus U_0 \oplus U_1 \oplus U_1,$$

where $U_0$ is the trivial (complex) $\mathbb{Z}_2$-representation and $U_1$ is the $\mathbb{Z}_2$-representation given by antipodal action. Thus,

$$\Gamma_{\Delta_{02}} \times S^1\text{-Deg}^t(F_\zeta|_{\mathbb{R}^2\times\check{\Delta}_{02}}, O \cap (\mathbb{R}^2 \times \check{\Delta}_{02})) = t_{0,1}(\lambda_o, \beta_o)\deg_{\mathcal{V}_{0,1}} + t_{1,1}(\lambda_o, \beta_o)\deg_{\mathcal{V}_{1,1}}.$$

Consider $J(\lambda)$ as a complex linear map in $\Delta_{02}^c$. Then,

$$J(\lambda) = \alpha \otimes \mathbb{R}^2 + \beta \otimes \begin{pmatrix} 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 2 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix}, \qquad \text{and} \qquad \sigma(J(\lambda)) = \{\sigma_{1,2}, \sigma_{3,4}, \sigma_{5,6,7,8}\}.$$

Thus, $t_{0,1}(\lambda_o, \beta_o) = 0$ and $t_{1,1}(\lambda_o, \beta_o) = -2$. Therefore,

$$\Gamma_{\Delta_{02}} \times S^1\text{-Deg}^t(F_\zeta|_{\mathbb{R}^2\times\check{\Delta}_{02}}, O \cap (\mathbb{R}^2 \times \check{\Delta}_{02})) = -2\deg_{\mathcal{V}_{1,1}} = -2(\mathbb{Z}_2^-).$$

On the other hand, $\mathsf{H}_{\Delta_2,\Delta_{02}} = \text{Id}$, since $\mathsf{h}_{\Delta_2,\Delta_{02}} = \text{Id}$. Consequently,

$$a_{\check{\Delta}_{02}} = -2(\mathbb{Z}_2^-) - \mathsf{H}_{\Delta_2,\Delta_{02}}\big(-(\mathbb{Z}_2^-)\big) = -2(\mathbb{Z}_2^-) + (\mathbb{Z}_2^-) = -(\mathbb{Z}_2^-).$$

*Computation of* $a_{\check{\Delta}_{03}}$. Similar to $\Delta_{00}$ and $\Delta_{02}$, we have

$$\Gamma_{\Delta_{03}} \times S^1\text{-Deg}^t(F_\zeta|_{\mathbb{R}^2\times\check{\Delta}_{03}}, O \cap (\mathbb{R}^2 \times \check{\Delta}_{03})) = t_{0,1}(\lambda_o, \beta_o)\deg_{\mathcal{V}_{0,1}} + t_{1,1}(\lambda_o, \beta_o)\deg_{\mathcal{V}_{1,1}}.$$

Consider $J(\lambda)$ as a complex linear map in $\Delta_{03}^c$. Then,

$$J(\lambda) = \alpha \otimes \mathbb{R}^2 + \beta \otimes \begin{pmatrix} 1 & 1 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}, \qquad \text{and} \qquad \sigma(J(\lambda)) = \{\sigma_{1,2}, \sigma_{3,4}, \sigma_{5,6,7,8}\}.$$

Thus, $t_{0,1}(\lambda_o, \beta_o) = 0$ and $t_{1,1}(\lambda_o, \beta_o) = -2$. Therefore,

$$\Gamma_{\Delta_{03}} \times S^1\text{-Deg}^t(F_\zeta|_{\mathbb{R}^2\times\check{\Delta}_{03}}, O \cap (\mathbb{R}^2 \times \check{\Delta}_{03})) = -2\deg_{\mathcal{V}_{1,1}} = -2(\mathbb{Z}_2^-),$$

and

$$a_{\check{\Delta}_{03}} = -2(\mathbb{Z}_2^-) - \mathsf{H}_{\Delta_2,\Delta_{03}}\big(-(\mathbb{Z}_2^-)\big) = -2(\mathbb{Z}_2^-) + (\mathbb{Z}_2^-) = -(\mathbb{Z}_2^-).$$

*Computation of* $a_{\check{\Delta}_{01}}$. In this case, $\Gamma = \Gamma_{\Delta_{01}} = D_3$. Similar to the case for $\Delta_1$, $\Delta_4$, one shows that the $D_3$-isotypical decomposition of $\Delta_{01}^c$ is

$$\Delta_{01}^c = U_0 \oplus U_0 \oplus U_0 \oplus U_0 \oplus U_1 \oplus U_1.$$

Thus,

$$\Gamma_{\Delta_{01}} \times S^1\text{-Deg}^t(F_\zeta|_{\mathbb{R}^2\times\check{\Delta}_{01}}, O \cap (\mathbb{R}^2 \times \check{\Delta}_{01})) = t_{0,1}(\lambda_o, \beta_o)\deg_{\mathcal{V}_{0,1}} + t_{1,1}(\lambda_o, \beta_o)\deg_{\mathcal{V}_{1,1}}.$$

Consider $J(\lambda)$ as a complex linear map in $\Delta_{01}^c$. Then,

$$J(\lambda) = \alpha \otimes \mathbb{R}^2 + \beta \otimes \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix}, \qquad \text{and} \qquad \sigma(J(\lambda)) = \{\sigma_{1,2}, \sigma_{3,4}, \sigma_{5,6,7,8}\}.$$

Thus, $t_{0,1}(\lambda_o, \beta_o) = 0$ and $t_{1,1}(\lambda_o, \beta_o) = -2$. Therefore,

$$\begin{aligned}\Gamma_{\Delta_{01}} \times S^1\text{-Deg}^t(F_\zeta|_{\mathbb{R}^2\times\check{\Delta}_{01}}, O \cap (\mathbb{R}^2 \times \check{\Delta}_{01})) &= -2\deg_{\mathcal{V}_{1,1}} \\ &= -2(\mathbb{Z}_3^t) - 2(D_1) - 2(D_1^z) + 2(\mathbb{Z}_1).\end{aligned}$$

Consequently,

$$a_{\check{\Delta}_{01}} = -2(\mathbb{Z}_3^t) - 2(D_1) - 2(D_1^z) + 2(\mathbb{Z}_1).$$

*Computation of* $a_W$. In this case, $\Gamma = \Gamma_V = \mathbb{Z}_1$. Consider $V^c = \mathbb{R}^{10} \otimes \mathbb{C} \simeq \mathbb{C}^{10}$ as a complex $\mathbb{Z}_1$-representation. Then, the $\mathbb{Z}_1$-isotypical decomposition of $V^c$ is

$$V^c = U_0 \oplus U_0 \oplus U_0 \oplus U_0 \oplus U_0 \oplus U_0 \oplus U_0 \oplus U_0 \oplus U_0 \oplus U_0,$$

where $U_0 \simeq \mathbb{C}$ is the trivial $\mathbb{Z}_1$-representation. Thus,

$$\Gamma_V \times S^1\text{-Deg}^t(F_\zeta|_{\mathbb{R}^2\times W}, O \cap (\mathbb{R}^2 \times W)) = t_{0,1}(\lambda_o, \beta_o)\deg_{\mathcal{V}_{0,1}}.$$

Consider $J(\lambda)$ as a complex linear map in $V^c$. Then,

$$J(\lambda) = \alpha \otimes \mathbb{R}^2 + \beta \otimes A, \qquad \text{and} \qquad \sigma(J(\lambda)) = \{\sigma_{1,2}, \sigma_{3,4}, \sigma_{5,6,7,8,9,10}\}.$$

Thus, $t_{0,1}(\lambda_o, \beta_o) = -3$. Therefore,

$$\Gamma_V \times S^1\text{-Deg}^t(F_\zeta|_{\mathbb{R}^2\times W}, O \cap (\mathbb{R}^2 \times W)) = -3\deg_{\mathcal{V}_{0,1}} = -3(\mathbb{Z}_1).$$

Consequently,

$$\begin{aligned} a_W &= -3(\mathbb{Z}_1) - \mathsf{H}_{\Delta_1,V}(a_{\check{\Delta}_1}) - \mathsf{H}_{\Delta_2,V}(a_{\check{\Delta}_2}) - \mathsf{H}_{\Delta_3,V}(a_{\check{\Delta}_3}) - \mathsf{H}_{\Delta_4,V}(a_{\check{\Delta}_4}) \\ &\quad - \mathsf{H}_{\Delta_{00},V}(a_{\check{\Delta}_{00}}) - \mathsf{H}_{\Delta_{01},V}(a_{\check{\Delta}_{01}}) - \mathsf{H}_{\Delta_{02},V}(a_{\check{\Delta}_{02}}) - \mathsf{H}_{\Delta_{03},V}(a_{\check{\Delta}_{03}}) \\ &= -3(\mathbb{Z}_1) - 3\mathsf{H}_{\Delta_1,V}(a_{\check{\Delta}_1}) - 4\mathsf{H}_{\Delta_2,V}(a_{\check{\Delta}_2}) \\ &= (-3 - 3\cdot(-4) - 4\cdot(-1))(\mathbb{Z}_1) = 13(\mathbb{Z}_1),\end{aligned}$$

where the last equality used the fact that if $\mathsf{h} : \mathbb{Z}_1 \to G$ is the inclusion homomorphism, then by definition of $\mathsf{H}$, $\mathsf{H}\big((K)\big) = \chi_c(G/K)\,(\mathbb{Z}_1)$, for $(K) \in \Phi(G)$.

In summary, we have

$$\begin{aligned}\mathcal{T}\text{-Deg}^t(F_\zeta, O) = &\big(\check{\Delta}_1, -2(\mathbb{Z}_3^t) - 2(D_1) - 2(D_1^z) + 2(\mathbb{Z}_1)\big) + \big(\check{\Delta}_2, -(\mathbb{Z}_2^-)\big) \\ &+ \big(\check{\Delta}_4, -2(\mathbb{Z}_3^t) - 2(D_1) - 2(D_1^z) \\ &+ 2(\mathbb{Z}_1)\big) + \big(\check{\Delta}_{01}, -2(\mathbb{Z}_3^t) - 2(D_1) - 2(D_1^z) + 2(\mathbb{Z}_1)\big) \\ &+ \big(\check{\Delta}_{00}, -(\mathbb{Z}_2^-)\big) + \big(\check{\Delta}_{02}, -(\mathbb{Z}_2^-)\big) + \big(\check{\Delta}_{03}, -(\mathbb{Z}_2^-)\big) + (W, 13(\mathbb{Z}_1)).\end{aligned}$$

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