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Interior Symmetries and Multiple Eigenvalues for Homogeneous Networks*

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Abstract. We analyze the impact of interior symmetries on the multiplicity of the eigenvalues of the Jacobian matrix at a fully synchronous equilibrium for the coupled cell systems associated to homogeneous networks. We consider also the special cases of regular and uniform networks. It follows from our results that the interior symmetries, as well as the *reverse interior symmetries* and *quotient interior symmetries*, of the network force the existence of eigenvalues with algebraic multiplicity greater than one. The proofs are based on the special form of the adjacency matrices of the networks induced by these interior symmetries.

Key words. coupled systems, interior symmetry, multiple eigenvalues

AMS subject classifications. 34C15, 34C23, 20B35, 15B36

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1. Introduction.

A coupled cell system is a finite collection of individual dynamical systems (or *cells*) that are coupled together through mutual interactions. Coupled cell systems can be used to model a wide variety of phenomena in many scientific fields, ranging from physics, biology, and chemistry to engineering, social science, and climatology.

As one of the most prevailing and studied phenomena in dynamical systems, bifurcation describes the sudden change of properties of systems subject to variation of a parameter. In the case of change of stability of an equilibrium, a bifurcation can usually be foreseen by a critical eigenvalue associated to the linearization at the equilibrium. While the bifurcation analysis for simple critical eigenvalues is straightforward, multiple eigenvalues can lead to complicated bifurcating behavior of the system such as multiple bifurcations and secondary bifurcations (cf. Iooss and Joseph [12] for general systems, Golubitsky, Stewart, and Schaeffer [9] for symmetric systems, Leite and Golubitsky [13] and Elmhirst and Golubitsky [4] for coupled systems, and Aguiar et al. [1] for coupled systems with quotient symmetry). However, knowing the cause for existence of multiple eigenvalues can help develop effective techniques for bifurcation analysis. A standard example is the appearance of multiple eigenvalues in equivariant dynamical systems due to the presence of symmetry; then generic behavior of bifurcating branches related to multiple eigenvalues can be analyzed using equivariant bifur-

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cation theory (cf. Golubitsky, Stewart, and Schaeffer [9]).

Topological configuration of a coupled cell system can be described by a directed graph, a *coupled cell network*, whose nodes correspond to the cells and whose edges represent the interactions. Network structure of a coupled cell network of n cells can be represented by *adjacency matrices* $A_1, \ldots, A_s$, where A_l is an $n \times n$ matrix, whose (i, j)-entry equals the number of lth-type edges directing from cell j to cell i. Two cells of a coupled cell network are called *identical* if they have the same phase space and the same internal dynamics (cf. Golubitsky, Pivato, and Stewart [6]). A coupled cell network is called *homogeneous* if it consists of identical cells having identical input couplings. A homogeneous network is called *regular* if all the couplings (*arrows* or *edges*) are of the same type. An important feature of homogeneous networks is that every admissible coupled cell system admits the diagonal subspace Δ, formed by setting all cell coordinates equal in the total phase space, as flow-invariant subspace. Moreover, the restriction of these systems to Δ gives the set of all vector fields on Δ (cf. Theorem 5.2 of Golubitsky, Stewart, and Török [10]). Assume that a homogeneous cell system admits a fully synchronous equilibrium in Δ. We say that the system undergoes a *local synchrony-breaking steady-state bifurcation* if the synchronous equilibrium loses its stability and bifurcates to a steady state with less synchrony as a bifurcation parameter crosses a certain critical value. If it bifurcates to a periodic state with less synchrony, we call it a *local synchrony-breaking Hopf bifurcation*.

Parallel properties exist between synchrony-breaking bifurcations of coupled cell systems and symmetry-breaking bifurcations of equivariant systems, where the stringent symmetry is replaced by a general network structure, and fixed point subspaces of isotropy subgroups are replaced with synchrony subspaces. In this direction, linear theory of regular coupled cell networks was presented in Golubitsky and Lauterbach [5], where it was shown that the linearized normal form at the bifurcation is generically isomorphic to the adjacency matrix restricted to one of its generalized eigenspaces if the dimension of the internal dynamics is at least 2; however, in the case of 1-dimensional internal dynamics, additional degeneracies may occur. Moreover, an analogue of the equivariant branching lemma and the equivariant Hopf theorem has been established in Golubitsky, Pivato, and Stewart [6] and Antoneli, Dias, and Paiva [2] for systems admitting interior symmetry. Following Antoneli, Dias, and Paiva [2], a network $\mathcal{G}$ has an *interior symmetry* on a subset $\mathcal{S}$ of cells if $\mathcal{S}$ together with all the arrows directed to it forms a subnetwork that has a nontrivial symmetry. In the case where $\mathcal{S}$ is the total set of cells, the interior symmetry becomes a symmetry.

However, bifurcation theory for coupled cell systems differs from equivariant bifurcation theory (cf. Golubitsky, Pivato, and Stewart [6], Leite and Golubitsky [13], and Golubitsky and Lauterbach [5]), and this is the case even when the critical eigenvalue is real and simple (cf. Stewart and Golubitsky [17]). It is known that in general (nonsymmetric noncoupled) systems, steady-state or Hopf bifurcations occur at simple eigenvalues under generic conditions (cf. Golubitsky and Schaeffer [7]). It is also known that in symmetric systems, although multiple eigenvalues can occur generically, they only appear generically as a result of a real absolutely irreducible action by the symmetry group for steady-state bifurcations; for Hopf bifurcations, they are related to a complex irreducible action (cf. Golubitsky, Stewart, and Schaeffer [9]). In coupled cell systems, the underlying network structure (which is generally nonsymmetric) can also force multiple eigenvalues in a generic manner, and it determines, even at linear level, the kind of generic transitions from a synchronous equilibrium that can occur as the parameter is varied (cf. Leite and Golubitsky [13]). In fact, it was observed in [13] and Aguiar et al. [1] that most multiple eigenvalues arise as a result of interior symmetry, while the remaining are multiple zero eigenvalues that come from colinear connectivity of two different cells to the other cells of the network.

In this paper, we show that there is a definite relation between interior symmetry and the occurrence of multiple eigenvalues. As an example, in homogeneous networks, an interior symmetry $\Sigma_{\mathcal{S}}$ on a subset $\mathcal{S}$ of k cells such that $\mathbf{D}_k \subseteq \Sigma_{\mathcal{S}} \subseteq \mathbf{S}_k$ always forces multiple eigenvalues (cf. Theorems 4.2 and 4.3). The main reason why interior symmetry may lead to multiple eigenvalues is that it imposes restrictions on the network structure and thus on the form of adjacency matrices of the network. For example, an interior symmetry (ij) on the set of cells $\mathcal{C} = \{1, \ldots, n\}$ of a regular network $\mathcal{G}$ given by the permutation of cells i and j corresponds to the following constraints on the entries of the adjacency matrix $A_{\mathcal{G}}$:

$$a_{ii} = a_{jj}, \quad a_{ij} = a_{ji} \quad \text{and} \quad a_{ik} = a_{jk} \quad \forall k \in \mathcal{C} \setminus \{i, j\}.$$

Moreover, as shown in Golubitsky, Pivato, and Stewart [6], interior symmetry induces additional structure on the form of the linearization at synchrony-breaking bifurcations. Here, we go further and show explicitly how interior symmetry forces additional constraints on the linearization. Consider an n-cell homogeneous network $\mathcal{G}$ with s types of arrows, whose cell internal dynamics is r-dimensional. Assume without loss of generality that the synchronous equilibrium is at the origin. As shown for the case of regular networks in Leite and Golubitsky [13], the Jacobian of a homogeneous coupled cell system at a fully synchronized equilibrium at the origin is determined by the cell internal dynamics and the adjacency matrices of different types of arrows. Let A_l, $l = 1, 2, \ldots, s$, be the adjacency matrix of the lth type of arrows in $\mathcal{G}$. Let α be the linearized internal dynamics at the origin, and let β_l be the linearized internal coupling at the origin with the lth type of input for $l = 1, 2, \ldots, s$. Note that α and β_l are $r \times r$ matrices. Then, the Jacobian at the origin is of the form

$$J_{\mathcal{G}} = \alpha \otimes I_n + \beta_1 \otimes A_1 + \cdots + \beta_m \otimes A_s.$$

Results in Leite and Golubitsky [13] and Aguiar et al. [1] showed that when $\mathcal{G}$ is a regular network (the case $s = 1$), the eigenvalues of $J_{\mathcal{G}}$ are the union of the eigenvalues of the $r \times r$ matrices $\alpha + \mu_j \beta$ for $j = 1, \ldots, n$, including algebraic multiplicity, where $\mu_1, \ldots, \mu_n$ denote the eigenvalues of the adjacency matrix $A_{\mathcal{G}} := A_1$. Thus, the problem of multiple eigenvalues of the Jacobian is reduced to that of the adjacency matrix. On the contrary, if $s > 1$, it is unclear how the spectrum of $J_{\mathcal{G}}$ and that of the A_l's are related (cf. Golubitsky and Lauterbach [5] for product networks of two regular networks). However, as we will see, interior symmetry imposes a "universal" constraint on the form of the A_l's so that multiple eigenvalues of $J_{\mathcal{G}}$ can be related with those of the A_l's if the interior symmetry $\Sigma_{\mathcal{S}}$ is at least $\mathbf{D}_k$ (cf. Theorem 4.3).

In the case of "smaller" interior symmetry, that is, $\mathbf{D}_k \not\subset \Sigma_{\mathcal{S}}$, we obtain partial results for *regular uniform networks* (cf. Corollaries 3.14 and 3.18). Following Stewart [16], we say that a network is *uniform* if it has no multiple arrows or self-couplings. In other words, the adjacency matrix $A_{\mathcal{G}}$ of a uniform network is composed of 0's and 1's. On the other hand, interior symmetry forces integer eigenvalues of adjacency matrices (due to the integer entries of adjacency matrices) of regular networks (cf. Theorems 3.13 and 3.17), and it is known that all eigenvalues λ satisfy $||\lambda|| \leq v$, where v is the number of input arrows of each cell in the regular network. Consequently, interior symmetry exerts an even stronger influence on the multiplicity of eigenvalues of adjacency matrices for regular uniform networks.

We define variations of interior symmetry in a network such as *reverse interior symmetry* and *quotient interior symmetry*, which may also result in multiple eigenvalues for the Jacobian at the origin of the corresponding coupled cell systems. A reverse interior symmetry is an interior symmetry of the *reverse network*, where the direction of arrows of $\mathcal{G}$ is reversed. A quotient interior symmetry is a short-hand notion of an interior symmetry of a *quotient network* of $\mathcal{G}$, which is obtained by restricting $\mathcal{G}$ to a balanced equivalence relation on the cells. If a quotient network has a reverse interior symmetry, then we call this symmetry a *quotient reverse interior symmetry*. All results obtained in this paper about interior symmetry can be easily extended to the above-mentioned variations of interior symmetry (cf. Remark 3.1).

The paper is organized as follows. Section 2 collects preliminary definitions and results from coupled cell networks including definitions of various interior symmetries and some results from linear algebra. In section 3, we discuss the case of regular networks for several important interior symmetries, such as the cyclic group $\mathbb{Z}_k$, the dihedral group $\mathbf{D}_k$, the alternating group $\mathbf{A}_k$, and the symmetric group $\mathbf{S}_k$. Using Theorem 3.3 and Corollary 3.4, we can get results on multiplicity of eigenvalues for interior symmetry groups given by products of these groups. The case of regular uniform networks is discussed in subsection 3.6 for $\mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2$- and V_4-interior symmetry. In section 4, we extend the results obtained in section 3 to homogeneous networks. We give some concluding remarks in section 5. Throughout the paper, numerous examples will be used to illustrate the results.

2. Preliminaries.

In this section, we summarize necessary concepts from coupled cell networks. We restrict our attention to homogeneous coupled cell networks since they are our main case of study. For more general definitions and results on coupled cell networks, we refer to Golubitsky and Stewart [8] and Golubitsky, Stewart, and Török [10] and the references therein.

Definition 2.1. *A* coupled cell network *consists of a finite set $\mathcal{C} = \{1, \ldots, n\}$ of* nodes *or* cells *and a finite set $\mathcal{E} = \{(c, d) : \ c, d \in \mathcal{C}\}$ of* edges *or* arrows *and two equivalence relations, $\sim_C$ on cells in $\mathcal{C}$ and $\sim_E$ on edges in $\mathcal{E}$, with the following consistency condition: if $e_1 \sim_E e_2$ for $e_1 = (c_1, d_1) \in E$ and $e_2 = (c_2, d_2) \in E$, then $c_1 \sim_C c_2$ and $d_1 \sim_C d_2$. We write $\mathcal{G} = (\mathcal{C}, \mathcal{E}, \sim_C, \sim_E)$.*

For an edge $e = (c, d) \in \mathcal{E}$, c is called the *head cell* and d is called the *tail cell* and e is called an *input edge* of c. The set of all input edges of c is called the *input set* of c and is denoted by $I(c)$. Two cells c and d in a network are said to be *input-equivalent* if there is an edge-type preserving isomorphism $\beta : I(c) \to I(d)$ between their input sets. Note that the relation of input-equivalence refines the relation of cell-equivalence.

Definition 2.2. *A* homogeneous *network is a coupled cell network with only one input-equivalence class. A* regular *network is a homogeneous network with only one edge-equivalence class. It follows that in a homogeneous network all cells are of identical type and receive the same number (per type) of input edges. This number, which is the cardinality of the input set, is called the* valency *of the network.*

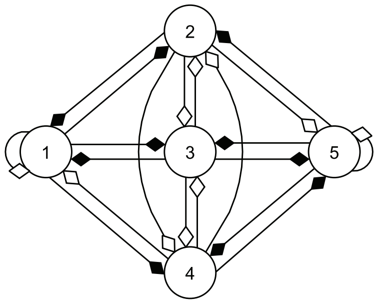

Figure 1. *A homogeneous network $\mathcal{G}$ with valency 4.*

Example 2.3. Consider a five-cell homogeneous network $\mathcal{G}$ with two types of arrows and valency 4, which is shown in Figure 1. This network will be repeatedly referred to by later examples (cf. Examples 2.4, 2.8, 2.11, 2.14, 2.18). Let A_1 (resp., A_2) be the adjacency matrix of the arrows with solid (resp., hollow) arrow head. Then,

$$A_1 = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix}. \tag{2.1}$$

Example 2.4. Consider the two subnetworks $\mathcal{G}_1, \mathcal{G}_2$ obtained from the network $\mathcal{G}$ in Example 2.3 by keeping only all arrows with solid (resp., hollow) arrow head. Then, $\mathcal{G}_1, \mathcal{G}_2$ are regular networks, as shown in Figure 2, with the adjacency matrix given by A_1, A_2, respectively (cf. (2.1)). Notice that $\mathcal{G}_1$ is an example of a uniform network, while $\mathcal{G}_2$ is not.

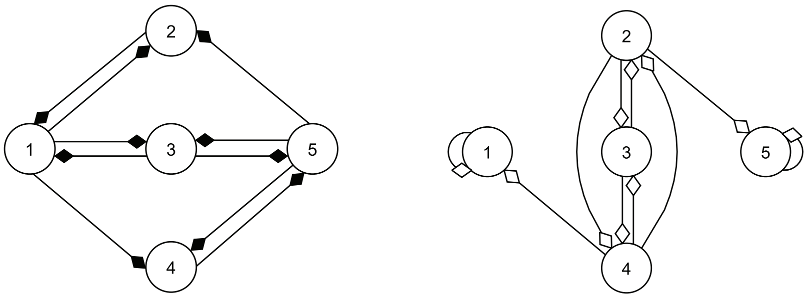

Figure 2. *Regular networks $\mathcal{G}_1, \mathcal{G}_2$ obtained from $\mathcal{G}$ in Figure 1.*

We follow the multiarrow formalism in Golubitsky, Stewart, and Török [10] and thus allow multiple arrows of the same type between two cells and self-coupling arrows. We call the networks without multiple arrows or self-coupling arrows *uniform networks* (cf. Stewart [16]).

2.1. Symmetry and symmetric groups. We adapt and simplify the definition of a symmetry of a general coupled cell network in Antoneli and Stewart [3] to a symmetry of a homogeneous network.

Definition 2.5. *Let $\mathcal{G} = (\mathcal{C}, \mathcal{E}, \sim_C, \sim_E)$ be a homogeneous network. A* symmetry *of $\mathcal{G}$ is a permutation σ on $\mathcal{C}$ such that there is a bijection between the edges $(\sigma(a), \sigma(b))$ and (a, b), which preserves the edge-equivalence relation $\sim_E$ for all $a, b \in \mathcal{C}$.*

Let $\mathcal{G}$ be an n-cell homogeneous network with s edge-equivalence classes, whose adjacency matrices are given by $A_1, A_2, \ldots, A_s$. Write $A_l = [a_{ij}^{(l)}]_{n\times n}$ for $l = 1, 2, \ldots, s$. Then, a permutation σ is a symmetry of $\mathcal{G}$ if and only if

$$a_{ij}^{(l)} = a_{\sigma(i)\sigma(j)}^{(l)} \quad \forall i, j = 1, 2, \ldots, n,\ l = 1, 2, \ldots, s.$$

It is clear that the set of all symmetries of an n-cell homogeneous network $\mathcal{G}$ forms a group, which can be identified canonically with a subgroup of the *symmetric group* $\mathbf{S}_n$, that is defined as the group of all permutations of n symbols. Let $i_1, \ldots, i_k \in \mathbb{N}$ be distinct positive integers. We use the standard notation $(i_1\ \ldots\ i_k)$ to denote a *k-cycle* in $\mathbf{S}_n$, which is a permutation σ defined by

$$\begin{aligned} \sigma:\quad & i_j \mapsto i_{j+1} && \text{for } j = 1, \ldots k-1 \\ & i_k \mapsto i_1 && \\ & l \mapsto l && \text{for } l \notin \{i_1, \ldots, i_k\}. \end{aligned}$$

A 2-cycle is called a *transposition*. Every permutation can be written as a product of simple transpositions. A permutation is called *even* (resp., *odd*) if it can be expressed as a product of an even (resp., odd) number of transpositions. The subset of $\mathbf{S}_n$ consisting of all even permutations is a subgroup called the *alternating group* $\mathbf{A}_n$. A group generated by permutations $\sigma_1, \sigma_2, \ldots, \sigma_m$ will be denoted by $\langle \sigma_1, \sigma_2, \ldots, \sigma_m \rangle$.

Example 2.6. Consider the k-cycle $(1\ 2\ \ldots\ k)$ in $\mathbf{S}_k$ and the cyclic group

$$\mathbb{Z}_k = \langle (1\ 2\ \ldots\ k) \rangle$$

generated by the k-cycle. Let $\mathcal{G}$ be a $\mathbb{Z}_k$-symmetric homogeneous network of k cells and $A_1, A_2, \ldots, A_s$ be the adjacency matrices of $\mathcal{G}$. Then, every A_l is of the form

$$\begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1,k-1} & a_{1k} \\ a_{1k} & a_{11} & a_{12} & \ldots & a_{1,k-1} \\ a_{1,k-1} & a_{1k} & a_{11} & \ddots & a_{1,k-2} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ a_{12} & a_{13} & \ldots & a_{1k} & a_{11} \end{bmatrix}, \tag{2.2}$$

where every row vector is obtained by shifting the preceding row vector to the right by one element.

A matrix of the form (2.2) is called a *circulant matrix*, which is often written as

$$circ(a_{11}, a_{12}, \ldots, a_{1k})$$

in shorthand form. Circulant matrices and their spectral information are needed for our later discussions [18]. It is known that all circulant matrices of the form (2.2) share the same eigenvectors,

$$v_j = (1, \omega_j, \omega_j^2, \ldots, \omega_j^{k-1}) \quad \text{for } \omega_j = e^{\frac{2\pi i j}{k}},\ j = 0, 1, \ldots, k-1, \tag{2.3}$$

which are eigenvectors of the following eigenvalues:

$$\lambda_j = a_{11} + a_{12}\omega_j + a_{13}\omega_j^2 + \cdots + a_{1k}\omega_j^{k-1}, \quad j = 0, 1, \ldots, k-1. \tag{2.4}$$

Another concept that we will need later is that of a *centrosymmetric matrix*, which is a matrix that is symmetric about its center [14]. More formally, we have the following definition.

Definition 2.7. *A square matrix $A = [a_{ij}]_{n\times n}$ is called* centrosymmetric *if it satisfies the relation*

$$a_{ij} = a_{(n+1-i)(n+1-j)} \quad \forall\, i, j = 1, 2, \ldots, n,$$

which is equivalent to the relation

$$A = JAJ,$$

where $J = [e_{ij}]_{n\times n}$ is the exchange matrix*; that is, $e_{i,n+1-i} = 1$ and $e_{ij} = 0$ for all $j \neq n+1-i$, $i = 1, 2, \ldots, n$. In other words, it has 1 on the antidiagonal and 0 elsewhere.*

Example 2.8. Consider the network $\mathcal{G}$ in Example 2.3. The symmetry group of $\mathcal{G}$ is

$$\mathbb{Z}_2 = \langle (1\ 5)(2\ 4)\rangle.$$

The adjacency matrices A_l's of any five-cell homogeneous network having this symmetry are centrosymmetric matrices of the form

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} \\ a_{31} & a_{32} & a_{33} & a_{32} & a_{31} \\ a_{25} & a_{24} & a_{23} & a_{22} & a_{21} \\ a_{15} & a_{14} & a_{13} & a_{12} & a_{11} \end{bmatrix}.$$

2.2. Interior symmetry. The concept of interior symmetry of a coupled cell network is a generalized notion of a symmetry of a coupled cell network. Roughly speaking, it is a permutation of the cells that preserves a certain amount of input structure. The notion of interior symmetry was first introduced by Golubitsky, Pivato, and Stewart [6]. We adapt and simplify the definition in [6] to define an interior symmetry of a homogeneous network as follows.

Definition 2.9. *Let $\mathcal{G} = (\mathcal{C}, \mathcal{E}, \sim_C, \sim_E)$ be a homogeneous network. Let $\mathcal{S} \subseteq \mathcal{C}$ be a subset. An* interior symmetry *of $\mathcal{G}$ on $\mathcal{S}$ is a permutation σ on $\mathcal{C}$ such that σ fixes every element in $\mathcal{C} \setminus \mathcal{S}$, and there is a bijection between edges $(\sigma(a), \sigma(b))$ and (a, b), which preserves edge-equivalence relation $\sim_E$ for $a \in \mathcal{S}$, $b \in \mathcal{C}$.*

Note that in the case $\mathcal{S} = \mathcal{C}$, an interior symmetry on $\mathcal{C}$ is precisely a symmetry of $\mathcal{G}$. In what follows, when referring to interior symmetry, we also include the case of symmetry.

Let $\mathcal{G}$ be an n-cell homogeneous network with s edge-equivalence classes, whose adjacency matrices are given by $A_1, A_2, \ldots, A_s$. Write $A_l = [a_{ij}^{(l)}]_{n\times n}$ for $l = 1, \ldots, s$. Then, a permutation σ is an interior symmetry of $\mathcal{G}$ on $\mathcal{S}$ if and only if

$$a_{ij}^{(l)} = a_{\sigma(i)\sigma(j)}^{(l)} \quad \forall i \in \mathcal{S},\ j \in \mathcal{C},\ l = 1, \ldots, s. \tag{2.5}$$

Following the formulation in Antoneli, Dias, and Paiva [2], one can characterize the interior symmetry using symmetry of subnetworks. Given a network $\mathcal{G}$ and a subset $\mathcal{S} \subseteq \mathcal{C}$, define

$\mathcal{G}_{\mathcal{S}} = (\mathcal{C}, I(\mathcal{S}), \sim_{\mathcal{C}}, \sim_E)$ to be the subnetwork of $\mathcal{G}$ whose set of cells is $\mathcal{C}$ (together with its cell-equivalence relation $\sim_C$) and whose set of arrows is the input set $I(\mathcal{S})$ of $\mathcal{S}$. The edge-equivalence relation on $I(\mathcal{S})$ is given by the restriction of the edge-equivalence $\sim_E$ of $\mathcal{E}$ to $I(\mathcal{S})$.

Proposition 2.10 (cf. [2]). *Let $\mathcal{G}$ be a coupled cell network and $\mathcal{S} \subseteq \mathcal{C}$ be a subset of cells of the set of cells of $\mathcal{G}$. Consider the network $\mathcal{G}_{\mathcal{S}}$ as defined above. Then the group of interior symmetries of the network $\mathcal{G}$ on $\mathcal{S}$ can be canonically identified with the group of symmetries of the network $\mathcal{G}_{\mathcal{S}}$.*

Example 2.11. Consider the homogeneous network $\mathcal{G}$ in Example 2.3. Let $\mathcal{S} = \{2, 3, 4\}$. Then, the network $\mathcal{G}_{\mathcal{S}}$ has an $\mathbf{S}_3$-symmetry, as shown in Figure 3. Thus, $\mathcal{G}$ has an interior symmetry $\mathbf{S}_3$ on $\mathcal{S}$.

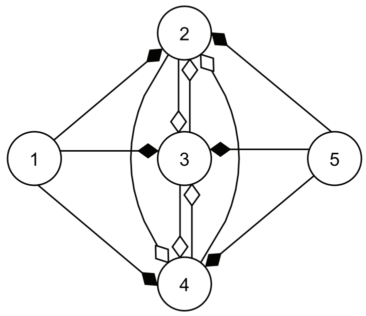

Figure 3. *An $\mathbf{S}_3$-symmetric network $\mathcal{G}_{\mathcal{S}}$ for $\mathcal{S} = \{2, 3, 4\}$.*

Indeed, adjacency matrices A_l's of any five-cell homogeneous networks with $\mathbf{S}_3$ interior symmetry on $\mathcal{S} = \{2, 3, 4\}$ are of the form

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ a_{21} & a_{22} & a_{23} & a_{23} & a_{25} \\ a_{21} & a_{23} & a_{22} & a_{23} & a_{25} \\ a_{21} & a_{23} & a_{23} & a_{22} & a_{25} \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} \end{bmatrix}.$$

2.3. Reverse interior symmetry. We introduce a new concept of symmetry for coupled cell networks, the *reverse interior symmetry*. To do so, we need the notion of the *reverse network* $\mathcal{G}^R$ of a coupled cell network $\mathcal{G}$, which is a network defined on the same set of cells but with all the edges in the reversed direction.

Definition 2.12. *Let $\mathcal{G} = (\mathcal{C}, \mathcal{E}, \sim_C, \sim_E)$ be a coupled cell network. Define*

$$\mathcal{E}^R := \{(d, c) \,:\, (c, d) \in \mathcal{E}\}$$

and an equivalence relation $\sim_{E^R}$ on $\mathcal{E}^R$ by

$$(b, a) \sim_{E^R} (d, c) \Leftrightarrow (a, b) \sim_E (c, d).$$

The reverse network *$\mathcal{G}^R$ of $\mathcal{G}$ is the network given by $\mathcal{G}^R = (\mathcal{C}, \mathcal{E}^R, \sim_C, \sim_{E^R})$.*

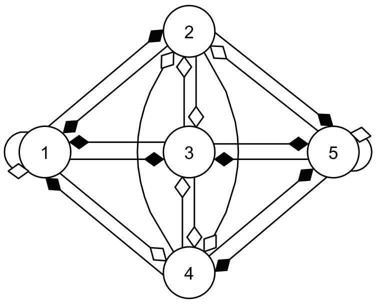

Figure 4. *The reverse network of the homogeneous network $\mathcal{G}$ in Figure 1.*

Note that the adjacency matrices of $\mathcal{G}^R$ are given by the transpose of the adjacency matrices of $\mathcal{G}$. Also, a reverse network of a homogeneous (resp., regular) network may not be homogeneous (resp., regular) again.

Definition 2.13. *Let $\mathcal{G} = (\mathcal{C}, \mathcal{E}, \sim_C, \sim_E)$ be a coupled cell network and $\mathcal{G}^R$ be its reverse network. Let $\mathcal{S} \subseteq \mathcal{C}$ be a subset. A permutation σ is called a* reverse interior symmetry *of $\mathcal{G}$ on $\mathcal{S}$ if σ is an interior symmetry of $\mathcal{G}^R$ on $\mathcal{S}$.*

That is, the group of reverse interior symmetries of $\mathcal{G}$ on $\mathcal{S}$ can be canonically identified with the group of interior symmetries of $\mathcal{G}^R$ on $\mathcal{S}$. Roughly speaking, a reverse interior symmetry is a permutation of the cells that preserves a certain amount of output structure.

Let $\mathcal{G}$ be a homogeneous network with s type arrows whose adjacency matrices are $A_1, A_2, \ldots, A_s$. Then, a permutation σ is a reverse interior symmetry of $\mathcal{G}$ on $\mathcal{S}$ if and only if

$$a_{ij} = a_{\sigma(i)\sigma(j)} \quad \forall i \in \mathcal{C} \text{ and } \forall j \in \mathcal{S},$$

for $l = 1, \ldots, s$.

Example 2.14. Consider the homogeneous network $\mathcal{G}$ in Example 2.3. Then, the reverse network $\mathcal{G}^R$ is as shown in Figure 4. It can be verified that $\mathcal{G}^R$ has an interior symmetry (15) on $\mathcal{S} = \{1, 5\}$. Thus, (1 5) is a reverse interior symmetry of $\mathcal{G}$.

Note that a symmetry of a coupled cell network $\mathcal{G}$ is both an interior symmetry and a reverse interior symmetry of $\mathcal{G}$, but the reverse may not be true.

Example 2.15. Consider the two networks in Figure 5, which are reverse to each other. Both networks have $\mathbf{S}_3$ as an interior symmetry on $\mathcal{S} = \{1, 2, 3\}$; thus $\mathbf{S}_3$ is a reverse interior symmetry of both networks on $\mathcal{S}$. However, neither network has an $\mathbf{S}_3$-symmetry.

2.4. Balanced equivalence relation. Given an equivalence relation $\bowtie$ on the set of cells of a coupled cell network, we can color the nodes of the network in the following way: two cells i, j receive the same color precisely when they belong to the same $\bowtie$-equivalence class. The coloring is called *balanced*, or equivalently, $\bowtie$ is called a *balanced equivalence relation* if any pair of cells with the same color have the same number and type of input arrows from cells of color b for every b.

More formally, we have the following definition.

Definition 2.16 (cf. [10]). *Given a coupled cell network $\mathcal{G} = (\mathcal{C}, \mathcal{E}, \sim_C, \sim_E)$, an equivalence relation $\bowtie$ on the set $\mathcal{C}$ is called* balanced *if for every $c, d \in \mathcal{C}$ with $c \bowtie d$ there exists a bijection*

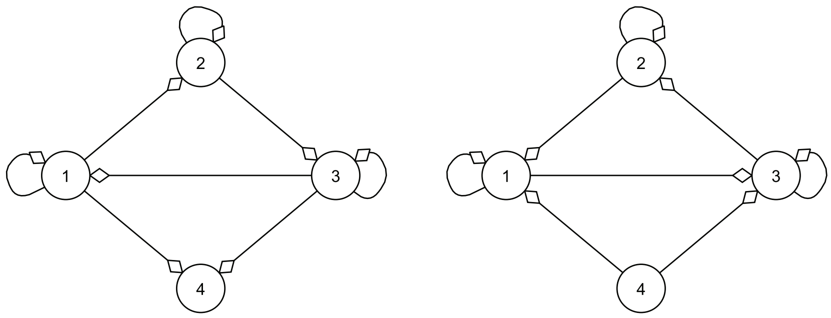

Figure 5. *Two networks that are reverse to each other.*

$\beta: I(c) \to I(d)$ *between their input sets, which preserves the edge-equivalence relation* $\sim_E$ *and is such that for all* $i \in I(c)$*, the tail cells of* i *and* $\beta(i)$ *are in the same* $\bowtie$*-class.*

The next proposition states that every interior symmetry permutation determines a balanced equivalence relation.

Proposition 2.17. *Let* $\mathcal{G}$ *be an* n*-cell homogeneous network and* σ *be an interior symmetry of* $\mathcal{G}$ *on a subset* $\mathcal{S} \subseteq \mathcal{C}$*. If* $\bowtie$ *is an equivalence relation on the cells* $\mathcal{C}$ *of* $\mathcal{G}$ *such that*

$$c \bowtie d \Leftrightarrow c, d \text{ belong to the same orbit under } \sigma,$$

then $\bowtie$ *is balanced.*

Proof. Let c, d be such that $c \bowtie d$. Then, $\sigma^m(c) = d$ for some $m \in \mathbb{N}$. Note that σ^m is an interior symmetry of $\mathcal{G}$ on $\mathcal{S}$ for all $m \in \mathbb{N}$. Thus, by the definition of interior symmetry, there exists an edge-equivalence preserving bijection between the edges $(\sigma^m(c), \sigma^m(x))$ and (c, x) for every input arrow (c, x). Thus, there exists a bijection between the input sets of $d = \sigma^m(c)$ and c, which preserves the edge-equivalence relation. On the other hand, the tail cells x and $\sigma^m(x)$ are in the same orbit by σ, and thus are in the same $\bowtie$-class. Therefore, $\bowtie$ is a balanced equivalence relation. ∎

Let $\Sigma_{\mathcal{S}}$ be the group of all interior symmetries of $\mathcal{G}$ on a subset $\mathcal{S} \subseteq \mathcal{C}$. Let $K \subseteq \Sigma_{\mathcal{S}}$ be a subgroup. By Proposition 2.17, every permutation in K determines a balanced equivalence relation on $\mathcal{G}$. In fact, the set of all these equivalence relations forms a sublattice of the total lattice of balanced equivalence relations on $\mathcal{G}$ (cf. Stewart [15]). Moreover, the balanced equivalence relation $\bowtie_K$ determined by the subgroup K is given by the join of all the equivalence relations determined by permutations in K and corresponds to the top element of this sublattice.

2.5. Quotient networks and quotient interior symmetry. Given a balanced equivalence relation $\bowtie$ on a coupled cell network $\mathcal{G}$, a *quotient network* $\mathcal{G}_{\bowtie} = (\mathcal{C}_{\bowtie}, \mathcal{E}_{\bowtie}, \sim_{C_{\bowtie}}, \sim_{E_{\bowtie}})$ can be defined naturally as follows: the cells in $\mathcal{C}_{\bowtie}$ are the $\bowtie$-equivalence classes of the cells of $\mathcal{G}$ and the edges in $\mathcal{E}_{\bowtie}$ from quotient cell $[c]_{\bowtie}$ to quotient cell $[d]_{\bowtie}$, where $[c]_{\bowtie}$ denotes the $\bowtie$-equivalence class of c, are in correspondence with the edges (c', d') of $\mathcal{G}$ for all $c' \bowtie c$, $d' \bowtie d$. The cell-equivalence $\sim_{C_{\bowtie}}$ and edge-equivalence $\sim_{E_{\bowtie}}$ relations for $\mathcal{G}_{\bowtie}$ are induced from those of $\mathcal{G}$. Since $\bowtie$ is balanced, the quotient network $\mathcal{G}_{\bowtie}$ is well defined. See Golubitsky, Stewart, and Török [10].

Let $\mathcal{G}$ be a homogeneous network of n-cells with s edge-equivalence classes whose adjacency matrices are $A_1, A_2, \ldots, A_s$. Let $\bowtie$ be a balanced equivalence relation which divides the cells of $\mathcal{G}$ into p equivalence classes. Then, $\mathcal{G}_{\bowtie}$ is a homogeneous network of p-cells with s edge-equivalence classes. Denote the adjacency matrices of $\mathcal{G}_{\bowtie}$ by $A_{1_{\bowtie}}, A_{2_{\bowtie}}, \ldots, A_{s_{\bowtie}}$. Let $A_{l_{\bowtie}} = [\bar{a}^{(l)}_{\alpha\beta}]_{p\times p}$. Then, for $\alpha = [i]_{\bowtie}$, $\beta = [j]_{\bowtie}$ in $\mathcal{C}_{\bowtie}$, we have (cf. Proposition 2.3 of [1])

$$\bar{a}^{(l)}_{\alpha\beta} = \sum_{k\in[j]_{\bowtie}} a^{(l)}_{ik}. \tag{2.6}$$

Example 2.18. Let $\mathcal{G}$ be the homogeneous network in Example 2.3. As shown in Examples 2.8 and 2.11, $\mathcal{G}$ has a symmetry $\mathbb{Z}_2 = \langle(15)(24)\rangle$ and an interior symmetry $\mathbf{S}_3$ on $\mathcal{S} = \{2,3,4\}$. Consider $\bowtie_1 = \{\{1\},\{2,3,4\},\{5\}\}$ and $\bowtie_2 = \{\{1,5\},\{2,4\},\{3\}\}$. As seen in subsection 2.4, both $\bowtie_1, \bowtie_2$ are balanced equivalence relations on $\mathcal{G}$. Let $\mathcal{G}_1$ (resp., $\mathcal{G}_2$) be the quotient network induced by $\bowtie_1$ (resp., $\bowtie_2$). Then, the adjacency matrices of $\mathcal{G}_1$ are

$$A_{1_{\bowtie_1}} = \begin{bmatrix} 0 & 2 & 0 \\ 1 & 0 & 1 \\ 0 & 2 & 0 \end{bmatrix}, \quad A_{2_{\bowtie_1}} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 1 & 1 \end{bmatrix},$$

and the adjacency matrices of $\mathcal{G}_2$ are

$$A_{1_{\bowtie_2}} = \begin{bmatrix} 0 & 1 & 1 \\ 2 & 0 & 0 \\ 2 & 0 & 0 \end{bmatrix}, \quad A_{2_{\bowtie_2}} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 2 & 0 \end{bmatrix}.$$

The networks $\mathcal{G}_1, \mathcal{G}_2$ are shown in Figure 6.

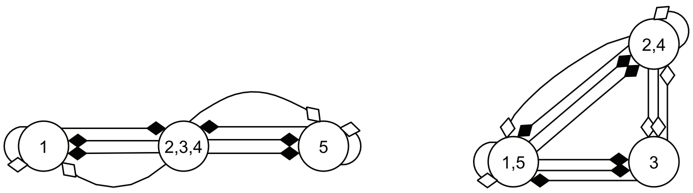

Figure 6. *Quotient networks for $\mathcal{G}$ in Figure 1 given by the $\mathbf{S}_3$-interior symmetry (left) and $\mathbb{Z}_2 = \langle(15)(24)\rangle$-symmetry (right).*

Note that a quotient network of a uniform network is a regular network which may not be uniform in general.

One can also consider interior symmetry and reverse interior symmetry of quotient networks.

Definition 2.19. *Let $\mathcal{G}$ be a coupled cell network. We say that a permutation σ is a* quotient (interior) symmetry *of $\mathcal{G}$ if $\mathcal{G}$ has a quotient network $\mathcal{G}_{\bowtie_1}$ which has σ as an (interior) symmetry for some balanced equivalence relation $\bowtie_1$. Similarly, we say that a permutation γ is a* quotient reverse (interior) symmetry *of $\mathcal{G}$ if $\mathcal{G}$ has a quotient network $\mathcal{G}_{\bowtie_2}$ which has γ as a reverse (interior) symmetry for some balanced equivalence relation $\bowtie_2$.*

Example 2.20. Based on Example 2.18, we conclude that the homogeneous network in Figure 1 has a quotient symmetry $\langle(1\ 5)\rangle$, since $\mathcal{G}_1$ is symmetric with respect to $(1\ 5)$ in Figure 6 (left).

In many cases, symmetric properties of the total network may be inherited by quotient networks. Yet, the following example shows that there is no definite relation between the (interior) symmetry of the total network and the (interior) symmetry of its quotient networks.

Example 2.21. Consider the three-cell bidirectional ring pictured in Figure 7 (left) which is $\mathbf{S}_3$-symmetric and whose quotient networks have no symmetry or interior symmetry.

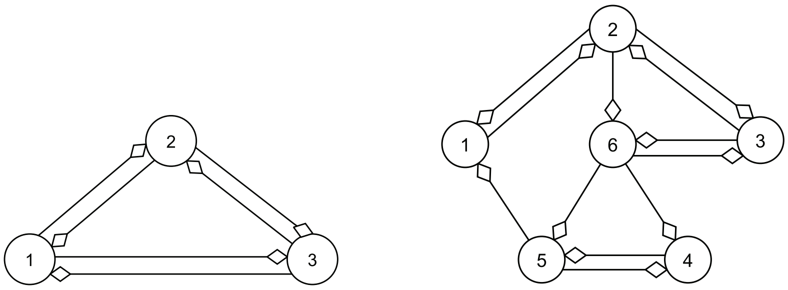

Figure 7. *Left: an $\mathbf{S}_3$-symmetric network which has no quotient (interior) symmetries. Right: a non-(interior)-symmetric network which has an $\mathbf{S}_3$-symmetric quotient network.*

Consider the six-cell regular network in Figure 7 (right). It can be verified that it has no nontrivial symmetry or interior symmetry, but it quotients to the three-cell bidirectional ring for the balanced equivalence relation $\{\{1, 2\}, \{3, 4\}, \{5, 6\}\}$.

However, networks that quotient to (interior) symmetric networks tend to have (interior) symmetry. Examples are five-cell networks given by Figures 8, 9, and 10, all of which have a quotient network which is isomorphic to the $\mathbf{S}_3$-symmetric network in Figure 7 (left), for the balanced equivalence relation $\{\{1\}, \{2, 3\}, \{4, 5\}\}$. At the same time, they all have interior symmetries. More examples of this kind can be found in Aguiar et al. [1], where all the five-cell regular networks admitting the three-cell bidirectional ring as a quotient network are listed.

2.6. Direct sum decomposition of $\mathbb{R}^n$. Let $\mathcal{G}$ be an n-cell homogeneous network with adjacency matrices $A_1, A_2, \ldots, A_s$ and $\bowtie$ be a balanced equivalence relation on $\mathcal{G}$. As seen in the previous subsection, there is an associated quotient network $\mathcal{G}_{\bowtie}$, whose adjacency matrices are given by $A_{1_{\bowtie}}, A_{2_{\bowtie}}, \ldots, A_{s_{\bowtie}}$ (cf. (2.6)). Based on results on regular networks (cf. section 4 of Golubitsky, Pivato, and Stewart [6]), one can show that $\bowtie$ induces a direct sum decomposition of $\mathbb{R}^n$ such that every A_l has a form of block matrix containing $A_{l_{\bowtie}}$ for $l = 1, 2, \ldots, s$ (cf. Theorem 2.9 in Aguiar et al. [1] for regular networks).

More precisely, given a balanced equivalence relation $\bowtie$, define

$$\Delta_{\bowtie}(\mathbb{R}^n) = \{x \in \mathbb{R}^n :\ x_c = x_d \text{ if } c \bowtie d\ \forall c, d \in \mathcal{C}\},$$

which is a linear subspace of $\mathbb{R}^n$. Then, $\Delta_{\bowtie}(\mathbb{R}^n)$ is A_l-invariant for every $l = 1, 2, \ldots, s$, since $\bowtie$ is balanced (cf. Theorem 4.3 in Golubitsky, Stewart, and Török [10]). Let $I_1, \ldots, I_p$ be the

$\bowtie$-equivalence classes of order greater than one and let $I = \bigcup_{l=1}^{p} I_l$. Define

$$W = \left\{ x \in \mathbb{R}^n : \ x_j = 0 \ \forall j \in \mathcal{C} \setminus I \ \text{ and } \sum_{i \in I_l} x_i = 0 \text{ for } 1 \leq l \leq p \right\}, \tag{2.7}$$

$$U = \Delta_{\bowtie}(\mathbb{R}^n). \tag{2.8}$$

Note that if $\bowtie$ is defined by an interior symmetry σ (cf. subsection 2.4), then both W and U are σ-invariant subspaces. Since $W \cap U = \{0\}$, we can decompose $\mathbb{R}^n$ as a direct sum

$$\mathbb{R}^n = W \oplus U. \tag{2.9}$$

Then, with respect to a basis adapted to (2.9), every adjacency matrix A_l has a block form

$$A_l = \left[\begin{array}{cc} A & 0 \\ C & A_{l_{\bowtie}} \end{array} \right],$$

where $A_{l_{\bowtie}}$ is the lth matrix of the quotient network $\mathcal{G}_{\bowtie}$ associated to the balanced equivalence relation $\bowtie$.

3. Interior symmetries and multiple eigenvalues: Regular networks.

In what follows, we analyze how the interior symmetry of a homogeneous network may affect the multiplicity of eigenvalues of the Jacobian at a fully synchronized equilibrium of the associated coupled cell system. In this section, we discuss the case of regular networks and in section 4, we generalize the results to homogeneous networks.

Beyond the notion of interior symmetry introduced by Golubitsky, Pivato, and Stewart [6], we defined in section 2 two further concepts of interior symmetry: the *reverse interior symmetry*, which is the interior symmetry of the reverse network, and the *quotient interior symmetry*, which is the symmetry of a quotient network.

Remark 3.1. The results presented in the following two sections are stated for interior symmetry, but they can be easily extended for reverse interior symmetry and quotient interior symmetry. This follows from the fact that all the arguments we will use are based on the special form of the adjacency matrices of the networks, which is forced by interior symmetry. Since analogous forms of adjacency matrices can be also induced by reverse interior symmetry and quotient interior symmetry, the results also apply to networks with reverse interior symmetry and quotient interior symmetry. More technically, note that each adjacency matrix A_l, for $l = 1, \ldots, s$, of a homogeneous network $\mathcal{G}$ corresponds to the transpose of the adjacency matrix A_l^R of the reverse network $\mathcal{G}^R$. Thus, the eigenvalues of A_l coincide with those of A_l^R. Consequently, multiple eigenvalues of A_l may appear not only due to the interior symmetry of $\mathcal{G}$, but also due to its reverse interior symmetry. As seen in subsection 2.6, for each quotient network $\mathcal{G}_{\bowtie}$ there is a special basis such that each adjacency matrix A_l, for $l = 1, \ldots, s$, of $\mathcal{G}$ has a block lower-triangular form with the adjacency matrix $A_{l_{\bowtie}}$ of the quotient network at one of the diagonal blocks. Thus, the eigenvalues of the adjacency matrix $A_{l_{\bowtie}}$ of a quotient network $\mathcal{G}_{\bowtie}$ are also eigenvalues of A_l. Therefore, multiple eigenvalues of A_l may appear not only due to the interior symmetry of $\mathcal{G}$, but also due to its quotient interior symmetry.

In summary, from the results presented in the following two sections, it follows that the interior symmetries, reverse interior symmetries, and quotient interior symmetries of regular and homogeneous networks favor multiple eigenvalues of the Jacobian matrix at a fully synchronized equilibrium for the associated coupled cell systems.

Let $\mathcal{G}$ be an n-cell regular network with r-dimensional cell internal dynamics. Let $\mu_1, \dots, \mu_n$ be the eigenvalues of the adjacency matrix $A_{\mathcal{G}}$ of $\mathcal{G}$. As shown in Leite and Golubitsky [13] and Aguiar et al. [1], the eigenvalues of the Jacobian $J_{\mathcal{G}}$ of the associated coupled systems at a fully synchronized equilibrium are the union of the eigenvalues of the $r \times r$ matrices

$$\alpha + \mu_j \beta \quad \text{for } j = 1, \dots, n,$$

including algebraic multiplicity.

Remark 3.2. It follows that if $A_{\mathcal{G}}$ has one eigenvalue with multiplicity m_a, then $J_{\mathcal{G}}$ has r eigenvalues with multiplicity at least m_a (note that it can also happen that some of the r eigenvalues are equal).

As mentioned before, interior symmetry imposes restrictions on the network structure and thus on the entries of the adjacency matrix. By Remark 3.2, to analyze the effect of interior symmetries on the multiplicity of the eigenvalues of the Jacobian, it is sufficient to concentrate on the influence of interior symmetries on the multiplicity of the eigenvalues of $A_{\mathcal{G}}$.

As we will see, interior symmetries force the existence of integer eigenvalues for the adjacency matrix $A_{\mathcal{G}}$ of a regular network $\mathcal{G}$. Moreover, all the eigenvalues λ of $A_{\mathcal{G}}$ satisfy $||\lambda|| \leq v$, where v is the valency of $\mathcal{G}$. Thus, for regular networks with valency 2, the eigenvalues -1, 0, and 1 will arise very often in the presence of interior symmetry.

3.1. Product interior symmetry. We show that the case of product interior symmetries can be inferred from their component symmetries.

Let $\mathcal{G}$ be an n-cell regular network having interior symmetry groups $\Sigma_{\mathcal{S}_j}$ for $j = 1, \dots, r$ on disjoint subsets $\mathcal{S}_j$ of cells of $\mathcal{G}$. We say that $\mathcal{G}$ has a *product interior symmetry*

$$\Sigma_{\mathcal{S}} = \Sigma_{\mathcal{S}_1} \times \cdots \times \Sigma_{\mathcal{S}_r},$$

where $\mathcal{S} = \bigcup_{j=1}^{r} \mathcal{S}_j$. Let $\bowtie_j$ be the balanced equivalence relation induced by $\Sigma_{\mathcal{S}_j}$ for $j = 1, \dots, r$. Then, the balanced equivalence relation $\bowtie$ induced by $\Sigma_{\mathcal{S}}$ is given by

$$c \bowtie d \Leftrightarrow c \bowtie_j d \quad \text{for some } j. \tag{3.1}$$

Set $U = \Delta_{\bowtie}(\mathbb{R}^n)$. Let $I_1^j, I_2^j, \dots, I_{p_j}^j$ be the $\bowtie_j$-equivalence classes of order greater than one and let $I^j = \bigcup_{l=1}^{p_j} I_l^j$. Define

$$W_j = \left\{ x \in \mathbb{R}^n : \; x_i = 0 \; \forall i \in \mathcal{C} \setminus I^j \;\; \text{and} \;\; \sum_{i \in I_l^j} x_i = 0 \;\text{ for } 1 \leq l \leq p_j \right\}, \quad j = 1, 2 \dots, r.$$

Let $I = \bigcup_{j=1}^{r} I^j$. Note that $\dim W_j = |\mathcal{S}_j| - p_j$, $\dim U = |\mathcal{C} \setminus I| + \sum_{j=1}^{r} p_j$ and $W_i \cap W_j = \{0\}$, $U \cap W_j = \{0\}$ for $i \neq j$, $j = 1, 2, \dots, r$. Thus, we have

$$\mathbb{R}^n = W_1 \oplus \cdots \oplus W_r \oplus U. \tag{3.2}$$

Theorem 3.3. *Let $\mathcal{G}$ be an n-cell regular network having a product interior symmetry $\Sigma_{\mathcal{S}} = \Sigma_{\mathcal{S}_1} \times \cdots \times \Sigma_{\mathcal{S}_r}$ on disjoint subsets $\mathcal{S}_j$ of cells of $\mathcal{G}$. Then, with respect to the decomposition* (3.2)*, the adjacency matrix $A_{\mathcal{G}}$ of $\mathcal{G}$ takes the form*

$$\begin{bmatrix} A^1 & 0 & \cdots & 0 & 0 \\ 0 & A^2 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & A^r & 0 \\ B_1 & B_2 & \cdots & B_r & A_{\bowtie} \end{bmatrix},$$

where A^j is a matrix of order $(|\mathcal{S}_j| - p_j) \times (|\mathcal{S}_j| - p_j)$ for $j = 1, \dots, r$, and $A_{\bowtie}$ is the adjacency matrix of the quotient network associated with $\bowtie$ (cf. (3.1)*).*

Proof. Let W be the linear subspace induced by $\bowtie$ (cf. (2.7)). Note that $W = W_1 \oplus \cdots \oplus W_r$. Then, as discussed in section 2.6, with respect to the decomposition

$$\mathbb{R}^n = W \oplus U,$$

$A_{\mathcal{G}}$ takes the form

$$\begin{bmatrix} A & 0 \\ C & A_{\bowtie} \end{bmatrix}.$$

It remains to show that A is a block matrix of diagonal form with respect to the dimensions of the W_j's, $j = 1, \dots, r$. Observe that to show that the entries of the jth column of A are all zeros except those on the diagonal block, it is enough to show that $(W_j \oplus U)$ is $A_{\mathcal{G}}$-invariant. Let $x \in W_j$ for a $j \in \{1, 2, \dots, r\}$ and let $y = A_{\mathcal{G}} x$. We need to show that $y \in W_j \oplus U$, i.e.,

$$y_i = y_l \quad \forall i \bowtie_k l, \ \forall k \neq j.$$

Since $x \in W_j$, the ith component x_i of x is zero except when $i \in \mathcal{S}_j$. Thus, the value of y_i (resp., y_l) depends only on the (i, m)th (resp., (l, m)th) entries of $A_{\mathcal{G}}$, where $m \in \mathcal{S}_j$. When $i \bowtie_k l$ and $k \neq j$, we have $i, l \notin \mathcal{S}_j$. Thus, the (i, m)th entry of $A_{\mathcal{G}}$ is equal to the (l, m)th entry of $A_{\mathcal{G}}$ for all $m \in \mathcal{S}_j$. It follows that $y_i = y_l$ for all $i \bowtie_k l$, $k \neq j$.

Therefore, we have $A_{\mathcal{G}} W_j \subseteq W_j \oplus U$. Combined with the fact $A_{\mathcal{G}} U \subseteq U$, we conclude that $(W_j \oplus U)$ is $A_{\mathcal{G}}$-invariant for $j = 1, 2, \dots, r$. ■

Corollary 3.4. *Under the assumptions of Theorem* 3.3*, we have that the set of eigenvalues of the adjacency matrix $A_{\mathcal{G}}$ of $\mathcal{G}$ is given by the disjoint union of the set of eigenvalues of A^j and the set of eigenvalues of $A_{\bowtie}$ for $j = 1, 2, \dots, r$.*

Taking into account Theorem 3.3 and Corollary 3.4, in what follows, we shall concentrate on interior symmetry groups that cannot be written as a product of subgroups. We will certainly not consider here all subgroups of $\mathbf{S}_n$ with this property, as the number of subgroups increases exponentially with n (cf. Holt [11] for an enumeration of subgroups and conjugacy classes of the subgroups of $\mathbf{S}_n$ for $n \leq 18$).

In this paper, we will be primarily interested in the following subgroups of $\mathbf{S}_n$:

(i) the symmetric groups $\mathbf{S}_k = \langle (i_1 \ \dots \ i_k), (i_1 \ i_2) \rangle$, with $2 \leq k \leq n$;
(ii) the alternating groups $\mathbf{A}_k$, with $2 \leq k \leq n$;
(iii) the dihedral groups $\mathbf{D}_k = \langle (i_1 \ \dots \ i_k), (i_2 \ i_k)(i_3 \ i_{k-1}) \dots (i_j \ i_{k+2-j}) \rangle$, with $2 \leq k \leq n$;
(iv) the cyclic groups $\mathbb{Z}_k = \langle (i_1 \ \dots \ i_k) \rangle$, with $2 \leq k \leq n$.

Note that $\mathbf{S}_2 \simeq \mathbf{D}_2 \simeq \mathbb{Z}_2$ and $\mathbf{S}_3 \simeq \mathbf{D}_3$.

3.2. $\mathbf{S}_k$- and $\mathbf{A}_k$-interior symmetry. We show that the following theorem holds.

Theorem 3.5. *Let $\mathcal{G}$ be an n-cell regular network having an interior symmetry group $\mathbf{S}_k$ or $\mathbf{A}_k$ on a subset $\mathcal{S} \subseteq \mathcal{C}$ of k cells of $\mathcal{G}$ for $2 \leq k \leq n$. Let i and j be any two different cells in $\mathcal{S}$. Then, the adjacency matrix $A_{\mathcal{G}} = [a_{\alpha\beta}]_{1\leq\alpha,\beta\leq n}$ of $\mathcal{G}$ has the eigenvalue $a_{ii} - a_{ij}$ with algebraic multiplicity at least $k-1$. As a result, the Jacobian $J_{\mathcal{G}}$ has r eigenvalues with algebraic multiplicity at least $k-1$.*

Proof. Without loss of generality, we can assume $\mathcal{S} = \{1, \ldots, k\}$. First notice that for any $i, j, l, m \in \mathcal{S}$, the product $(i\ j)(l\ m)$ of two transpositions is an element in $\mathbf{A}_k \subset \mathbf{S}_k$. Since $\mathcal{G}$ has an interior symmetry $\mathbf{S}_k$ (resp., $\mathbf{A}_k$), the entries of $A_{\mathcal{G}}$ satisfy (cf. (2.5))

$$\begin{aligned} &a_{ii} = a_{jj} \quad \forall i, j \in \mathcal{S}, \\ &a_{il} = a_{jm} \quad \forall i, j, l, m \in \mathcal{S}, \text{ with } i \neq l \text{ and } j \neq m, \\ &a_{il} = a_{jl} \quad \forall i, j \in \mathcal{S} \text{ and } \forall l \in \mathcal{C} \setminus \mathcal{S}. \end{aligned}$$

Consider the balanced equivalence relation $\bowtie$ induced by $\mathbf{S}_k$ (resp., $\mathbf{A}_k$):

$$\bowtie = \{\{1, 2, \ldots, k\}, \{k+1\}, \ldots, \{n\}\}.$$

Let W, U be given by (2.7)–(2.8). Then, with respect to (2.9), the adjacency matrix $A_{\mathcal{G}}$ takes the form

$$\begin{bmatrix} A & 0 \\ C & A_{\bowtie} \end{bmatrix},$$

where A is a scalar matrix of order $(k-1)$ with the element $(a_{11} - a_{12})$ on the diagonal. Thus, the adjacency matrix $A_{\mathcal{G}}$ has the eigenvalue $(a_{11} - a_{12})$ with algebraic multiplicity at least $(k-1)$. It follows from Remark 3.2 that the Jacobian $J_{\mathcal{G}}$ has r eigenvalues with algebraic multiplicity at least $k-1$. ■

Example 3.6. Let $\mathcal{G}$ be a five-cell regular network that quotients to the three-cell bidirectional ring $\mathcal{R}$ (cf. Figure 7). Examples of $\mathcal{G}$ are networks given in Figures 8, 9, and 10. By Theorem 3.5, the adjacency matrix of $\mathcal{R}$ has -1 as an eigenvalue with algebraic multiplicity 2, as a result of the $\mathbf{S}_3$ (interior) symmetry of $\mathcal{R}$. Thus, due to the $\mathbf{S}_3$ quotient interior symmetry of $\mathcal{G}$, the adjacency matrix of $\mathcal{G}$ has -1 as an eigenvalue with algebraic multiplicity at least 2.

3.3. $\mathbf{D}_k$-interior symmetry. We prove the following theorem.

Theorem 3.7. *Let $\mathcal{G}$ be an n-cell regular network having an interior symmetry group $\mathbf{D}_k$ for some $k \in \{3, \ldots, n\}$. Set*

$$m = \begin{cases} (k-1)/2 & \text{if } k \text{ is odd}, \\ k/2 & \text{if } k \text{ is even}. \end{cases}$$

Then, the adjacency matrix $A_{\mathcal{G}} = [a_{ij}]_{1\leq i,j\leq n}$ of $\mathcal{G}$ has m eigenvalues with algebraic multiplicity at least 2 if k is odd; $A_{\mathcal{G}}$ has $(m-1)$ eigenvalues with algebraic multiplicity at least 2 if k is even. As a result, if k is odd (resp., even), then the Jacobian $J_{\mathcal{G}}$ has mr (resp., $(m-1)r$) eigenvalues with algebraic multiplicity at least 2.

The following lemma will be needed for the proof of Theorem 3.7.

Lemma 3.8. *Let $m \in \mathbb{N}$. Consider the following two matrices of order $m \times m$:*

(3.3)
$$B_1 = \begin{bmatrix} a_{11}-a_{13} & a_{12}-a_{14} & a_{13}-a_{15} & \cdots & a_{1,m-1}-a_{1,m+1} & a_{1m}-a_{1,m+1} \\ a_{12}-a_{14} & a_{11}-a_{15} & a_{12}-a_{16} & \cdots & a_{1,m-2}-a_{1,m+1} & a_{1,m-1}-a_{1m} \\ a_{13}-a_{15} & a_{12}-a_{16} & a_{11}-a_{17} & \cdots & a_{1,m-3}-a_{1m} & a_{1,m-2}-a_{1,m-1} \\ & \cdots & & \cdots & & \\ a_{1,m-1}-a_{1,m+1} & a_{1,m-2}-a_{1,m+1} & a_{1,m-3}-a_{1m} & \cdots & a_{11}-a_{14} & a_{12}-a_{13} \\ a_{1m}-a_{1,m+1} & a_{1,m-1}-a_{1m} & a_{1,m-2}-a_{1,m-1} & \cdots & a_{12}-a_{13} & a_{11}-a_{12} \end{bmatrix},$$

$$B_2 = \begin{bmatrix} a_{11}+a_{13} & a_{12}+a_{14} & a_{13}+a_{15} & \cdots & a_{1,m-1}+a_{1,m+1} & a_{1m}+a_{1,m+1} \\ a_{12}+a_{14} & a_{11}+a_{15} & a_{12}+a_{16} & \cdots & a_{1,m-2}+a_{1,m+1} & a_{1,m-1}+a_{1m} \\ a_{13}+a_{15} & a_{12}+a_{16} & a_{11}+a_{17} & \cdots & a_{1,m-3}+a_{1m} & a_{1,m-2}+a_{1,m-1} \\ & \cdots & & \cdots & & \\ a_{1,m-1}+a_{1,m+1} & a_{1,m-2}+a_{1,m+1} & a_{1,m-3}+a_{1m} & \cdots & a_{11}+a_{14} & a_{12}+a_{13} \\ a_{1m}+a_{1,m+1} & a_{1,m-1}+a_{1m} & a_{1,m-2}+a_{1,m-1} & \cdots & a_{12}+a_{13} & a_{11}+a_{12} \end{bmatrix},$$

(3.4)
$$+\begin{bmatrix} -2a_{12} & -2a_{12} & \cdots & -2a_{12} & -2a_{12} \\ -2a_{13} & -2a_{13} & \cdots & -2a_{13} & -2a_{13} \\ -2a_{14} & -2a_{14} & \cdots & -2a_{14} & -2a_{14} \\ & \cdots & & & \\ -2a_{1m} & -2a_{1m} & \cdots & -2a_{1m} & -2a_{1m} \\ -2a_{1,m+1} & -2a_{1,m+1} & \cdots & -2a_{1,m+1} & -2a_{1,m+1} \end{bmatrix}.$$

Then, B_1 and B_2 are similar.

Proof. Notice that any matrix $M = (x_{ij})_{m\times m}$ is similar to the matrix $(x_{m-i+1,m-j+1})$ by exchanging rows R_i with R_{m-i+1} and exchanging columns C_i with C_{m-i+1} for $1 \le i \le m$. We will denote by $\tilde{B}_1$ the matrix obtained in this way from B_1.

For $r = 1, 2, \ldots, m-1$, denote by O_r the row operation

$$R_r \quad \rightsquigarrow \quad R_r + R_{r+1} + \cdots + R_m,$$

where the rth row is replaced by the sum of the jth row for $r \le j \le m$. It suffices to show that

$$O_{m-1}O_{m-2}\cdots O_2O_1B_2O_1^{-1}O_2^{-1}\cdots O_{m-2}^{-1}O_{m-1}^{-1} = \tilde{B}_1. \tag{3.5}$$

Write $B_2 = (b_{ij})_{m\times m}$, and denote by $C = (c_{ij})_{m\times m}$ the left-hand side of (3.5). We first show that

$$c_{ij} = \begin{cases} \displaystyle\sum_{p=i}^{m} b_{p1} & \text{if } j = 1, \\ \displaystyle\sum_{p=i}^{m} (b_{pj} - b_{p,j-1}) & \text{if } 1 < j \le m. \end{cases} \tag{3.6}$$

Notice that O_r^{-1} represents the column operations

$$C_{r+1} \rightsquigarrow C_{r+1} - C_r, \quad C_{r+2} \rightsquigarrow C_{r+2} - C_r, \ldots, \quad C_m \rightsquigarrow C_m - C_r.$$

Thus, it is clear that column operations and rth row operations O_r for $r \neq i$ do not change the value of the $(i,1)$th element. Thus, c_{i1} is equal to the $(i,1)$th element of $O_i B_2$, i.e.,

$$c_{i1} = \sum_{p=i}^{m} b_{p1}.$$

Assume $j > 1$. Then, column operations for $i \geq j$ and row operations for $i \neq j$ do not change the value of the (i,j)th element. Thus, c_{ij} is equal to the (i,j)th element of $O_i B_2 O_1^{-1} O_2^{-1} \cdots O_{j-1}^{-1}$. We need to differentiate the cases $i < j$ and $i \geq j$, since this determines the order of the operations.

Case I. $i \geq j$. Let $c_{ij}^{(l)}$ denote the (i,j)th element of $B_2 O_1^{-1} O_2^{-1} \cdots O_{j-l}^{-1}$ for $1 \leq l \leq j-1$. Then,

$$\begin{aligned} c_{ij} &= \sum_{p=i}^{m} c_{pj}^{(1)} = \sum_{p=i}^{m} (c_{pj}^{(2)} - c_{p,j-1}^{(2)}) \\ &= \sum_{p=i}^{m} ((c_{pj}^{(3)} - c_{p,j-2}^{(3)}) - (c_{p,j-1}^{(3)} - c_{p,j-2}^{(3)})) = \sum_{p=i}^{m} (c_{pj}^{(3)} - c_{p,j-1}^{(3)}) = \cdots \\ &= \sum_{p=i}^{m} (c_{pj}^{(j-1)} - c_{p,j-1}^{(j-1)}) = \sum_{p=i}^{m} (b_{pj} - b_{p,j-1}). \end{aligned}$$

Case II. $i < j$. Let $c_{ij}^{(l)}$ denote the (i,j)th element of $O_i B_2 O_1^{-1} O_2^{-1} \cdots O_{j-l-1}^{-1}$ for $1 \leq l \leq j-i$, and let $c_{ij}^{(l)}$ denote the (i,j)th element of $B_2 O_1^{-1} O_2^{-1} \cdots O_{j-l}^{-1}$ for $j-i+1 \leq l \leq j-1$. Then,

$$\begin{aligned} c_{ij} &= c_{ij}^{(1)} - c_{i,j-1}^{(1)} \\ &= (c_{ij}^{(2)} - c_{i,j-2}^{(2)}) - (c_{i,j-1}^{(2)} - c_{i,j-2}^{(2)}) = c_{ij}^{(2)} - c_{i,j-1}^{(2)} = \cdots \\ &= c_{ij}^{(j-i)} - c_{i,j-1}^{(j-i)} \\ &= \sum_{p=i}^{m} (c_{pj}^{(j-i+1)} - c_{p,j-1}^{(j-i+1)}) = \sum_{p=i}^{m} ((c_{pj}^{(j-i+2)} - c_{p,j-2}^{(j-i+2)}) - (c_{p,j-1}^{(j-i+2)} - c_{p,j-2}^{(j-i+2)})) \\ &= \sum_{p=i}^{m} (c_{pj}^{(j-i+2)} - c_{p,j-1}^{(j-i+2)}) = \cdots \\ &= \sum_{p=i}^{m} (c_{pj}^{(j-1)} - c_{p,j-1}^{(j-1)}) = \sum_{p=i}^{m} (b_{pj} - b_{p,j-1}). \end{aligned}$$

Therefore, (3.6) is proved. It remains to show $C = \tilde{B}_1$. Recall that a_{ij} denotes the (i,j)th element of the adjacency matrix $A_{\mathcal{G}}$. Consider the vector

$$v = (a_{11}, a_{12}, a_{13}, \ldots, a_{1m}, a_{1,m+1}, a_{1,m+1}, a_{1m}, \ldots, a_{13}, a_{12})^T$$

and the shifting operator ρ,

$$\rho v = (a_{12}, a_{11}, a_{12}, a_{13}, \dots, a_{1m}, a_{1,m+1}, a_{1,m+1}, a_{1m}, \dots, a_{13})^T.$$

Notation. In the rest of the proof of Lemma 3.8, we use v_p to denote the p'th element of v, with $p' = p \pmod{k}$, for $p \in \mathbb{Z}$ and $v \in \mathbb{R}^k$. Also, we use a short-hand notation for $(\rho v)_p$: ρv_p.

Due to the symmetric form of v, we have

$$v_{m+q} = v_{m-q+3}, \quad q \in \mathbb{Z}, \tag{3.7}$$

and

$$v_p = \rho v_{p+1}, \quad v_p = \rho^{-1} v_{p-1}, \quad p \in \mathbb{Z}. \tag{3.8}$$

In terms of v and ρ, the matrix B_1 consists of the first m rows of the matrix

$$(v - \rho^{-2}v, \rho v - \rho^{-3}v, \rho^2 v - \rho^{-4}v, \dots, \rho^{m-2}v - \rho^{-m}v, \rho^{m-1}v - \rho^{-(m+1)}v),$$

and B_2 consists of the first m rows of the matrix

$$(v + \rho^{-2}v, \rho v + \rho^{-3}v, \rho^2 v + \rho^{-4}v, \dots, \rho^{m-2}v + \rho^{-m}v, \rho^{m-1}v + \rho^{-(m+1)}v) - 2(\rho^{-1}v, \dots, \rho^{-1}v).$$

Assume that $1 \le i \le m$, $1 < j \le m$. By (3.6), we have

$$\begin{aligned} c_{ij} &= \sum_{p=i}^{m} (b_{pj} - b_{p,j-1}) = \sum_{p=i}^{m} \big((\rho^{j-1}v_p + \rho^{-j-1}v_p) - (\rho^{j-2}v_p + \rho^{-j}v_p)\big) \\ &\overset{(3.8)}{=} \rho^{j-1}v_i - \rho^{-j}v_i + \rho^{-j-1}v_m - \rho^{j-2}v_m \\ &\overset{(3.8)}{=} v_{i-j+1} - v_{i+j} + v_{m+j+1} - v_{m-j+2} \\ &\overset{(3.7)}{=} v_{i-j+1} - v_{i+j}. \end{aligned} \tag{3.9}$$

On the other hand, the (i,j)th element of $\tilde{B}_1$ is equal to the $(m-i+1, m-j+1)$th element of B_1, which is equal to

$$\rho^{m-j}v_{m-i+1} - \rho^{-m+j-2}v_{m-i+1} \overset{(3.8)}{=} v_{j-i+1} - v_{2m-i-j+3}. \tag{3.10}$$

By (3.7), we also have

$$v_{i-j+1} = v_{k+i-j+1} = v_{2m+1+i-j+1} = v_{m+(m+i-j+2)} \overset{(3.7)}{=} v_{j-i+1} \tag{3.11}$$

and

$$v_{2m-i-j+3} = v_{m+(m-i-j+3)} \overset{(3.7)}{=} v_{i+j}. \tag{3.12}$$

It follows from (3.9)–(3.12) that the (i,j)th element of C coincides with the (i,j)th element of $\tilde{B}_1$ for $1 \le i \le m$, $1 < j \le m$.

The case of $j=1$ can be similarly proved. By (3.6), we have

$$\begin{aligned} c_{i1} &= \sum_{p=i}^{m} b_{p1} = \sum_{p=i}^{m} (v_p + \rho^{-2} v_p - 2\rho^{-1} v_p) \\ &\overset{(3.8)}{=} v_i - \rho^{-1} v_i + \rho^{-2} v_m - \rho^{-1} v_m \overset{(3.8)}{=} v_i - v_{i+1} + v_{m+2} - v_{m+1} \\ &\overset{(3.7)}{=} v_i - v_{i+1} \overset{(3.11)\text{–}(3.12)}{=} v_{2-i} - v_{2m-i+2} \overset{(3.8)}{=} \rho^{m-1} v_{m-i+1} - \rho^{-m-1} v_{m-i+1}, \end{aligned}$$

which is the $(i,1)$th element of $\tilde{B}_1$.

Consequently, we showed that $C = \tilde{B}_1$ and thus (3.5) holds. ■

Proof of Theorem 3.7. Without loss of generality, assume $\mathcal{G}$ has an interior symmetry $\mathbf{D}_k$ on the cells $\{1,\ldots,k\}$. Due to this interior symmetry, the entries of $A_{\mathcal{G}}$ satisfy

$$\begin{aligned} a_{ij} &= a_{l(j+l-i)(\mathrm{mod}\ k)} && \text{for } i,j,l \in \{1,\ldots,k\}, \\ a_{ij} &= a_{lj} && \text{for } i,l \in \{1,\ldots,k\} \text{ and } j \in \{k+1,\ldots,n\}, \\ a_{1j} &= a_{1(k-j+2)} && \text{for } j \in \{2,\ldots,m,m+1\}. \end{aligned}$$

Thus, $A_{\mathcal{G}}$ has the form

$$A_{\mathcal{G}} = \begin{bmatrix} A & D \\ E & F \end{bmatrix}, \tag{3.13}$$

where D is a $k \times (n-k)$ matrix with all rows equal and A is a (symmetric) circulant matrix

$$A = \begin{cases} circ(a_{11}a_{12}a_{13}\ldots a_{1m+1}a_{1m+1}\ldots a_{13}a_{12}) & \text{if } k \text{ is odd}, \\ circ(a_{11}a_{12}a_{13}\ldots a_{1m}a_{1m+1}a_{1m}\ldots a_{13}a_{12}) & \text{if } k \text{ is even}. \end{cases}$$

It follows from (2.4) that the eigenvalues λ_j, $j=0,\ldots,k-1$, of A are real and satisfy $\lambda_j = \lambda_{k-j}$ for $j=1,\ldots,m$. That is, A has m eigenvalues with algebraic multiplicity at least 2 if k is odd; A has $(m-1)$ eigenvalues with algebraic multiplicity at least 2 if k is even. Our goal is to prove the same property for $A_{\mathcal{G}}$.

Case I. Assume that k is odd. Consider the balanced equivalence relation $\bowtie = \{\{1,2,\ldots,k\}, \{k+1\},\ldots,\{n\}\}$ induced by $\mathbf{D}_k$. Motivated by the direct sum decomposition (2.9), we define a basis $\mathcal{B} = \{b_1,b_2,\ldots,b_n\}$ in $\mathbb{R}^n$ by

$$b_i = \begin{cases} e_{i+1} - e_{k-i+1} & \text{for } 1 \le i \le m, \\ -2e_1 + e_{i-m+1} + e_{k-i+m+1} & \text{for } m+1 \le i \le 2m, \\ e_1 + e_2 + \cdots + e_k & \text{for } i = k, \\ e_i & \text{for } k+1 \le i \le n, \end{cases} \tag{3.14}$$

where $\{e_1,e_2,\ldots,e_n\}$ denote the standard basis in $\mathbb{R}^n$ (cf. Example 3.9 for $k=7$). Then, the adjacency matrix $A_{\mathcal{G}}$ in the basis $\mathcal{B}$ has the form

$$\mathcal{B}^{-1} A_{\mathcal{G}} \mathcal{B} = \left[\begin{array}{cc|c} B_1 & 0 & \multirow{2}{*}{0} \\ 0 & B_2 & \\ \hline \multicolumn{2}{c|}{C} & A_{\bowtie} \end{array}\right],$$

where B_1, B_2 are matrices of order $m \times m$ given by (3.3)–(3.4) and $A_{\bowtie}$ is the adjacency matrix of the quotient network induced by $\bowtie$. By Lemma 3.8, B_1 and B_2 are similar matrices and thus have the same eigenvalues. Consequently, $A_{\mathcal{G}}$ has m eigenvalues of multiplicity at least 2. It follows from Remark 3.2 that the Jacobian $J_{\mathcal{G}}$ has mr eigenvalues with algebraic multiplicity at least 2.

Notice that we can obtain an "optimal" basis $\tilde{\mathcal{B}}$ by applying the operations specified in the proof of Lemma 3.8 to $\mathcal{B}$, so that $A_{\mathcal{G}}$ has two copies of B_1 lying on the diagonal. More precisely, let $R = O_{m-1}O_{m-2}\cdots O_2O_1$ be the total row operation on B_2 and S the total row switching operation such that $SB_1S^{-1} = \tilde{B}_1$. Then, we have

$$SRB_2R^{-1}S^{-1} = B_1.$$

Set

$$O = \begin{bmatrix} I_m & 0 & 0 \\ 0 & SR & 0 \\ 0 & 0 & I_{n-2m} \end{bmatrix},$$

where I_i stands for the identity matrix of order $i \times i$. Define a new basis by

$$\tilde{\mathcal{B}} = \mathcal{B}O^{-1}.$$

Then, the adjacency matrix $A_{\mathcal{G}}$ has the form

$$\tilde{\mathcal{B}}^{-1}A_{\mathcal{G}}\tilde{\mathcal{B}} = \left[\begin{array}{cc|c} B_1 & 0 & \multirow{2}{*}{0} \\ 0 & B_1 & \\ \hline \multicolumn{2}{c|}{C'} & A_{\bowtie} \end{array}\right].$$

A precise formula of $\tilde{\mathcal{B}} = \{\tilde{b}_1, \tilde{b}_2, \ldots, \tilde{b}_n\}$ is given by

$$\tilde{b}_i = \begin{cases} e_{i+1} - e_{k-i+1} & \text{for } 1 \le i \le m, \\ -e_{k-i} + e_{k-i+1} + e_{i+1} - e_{i+2} & \text{for } m+1 \le i \le 2m-1, \\ -2e_1 + e_2 + e_k & \text{for } i = 2m, \\ e_1 + e_2 + \cdots + e_k & \text{for } i = 2m+1 = k, \\ e_i & \text{for } k+1 \le i \le n \end{cases} \tag{3.15}$$

(cf. Example 3.9 for $k = 7$).

Case II. Assume that k is even. Similar to the case of odd k, we try to find an optimal basis for the diagonal form of $A_{\mathcal{G}}$. Motivated by the direct sum decomposition (2.9), define the following basis $\mathcal{B}$:

$$b_i = \begin{cases} e_{i+1} - e_{k-i+1} & \text{for } 1 \le i \le m-1, \\ e_1 - e_2 + e_3 - e_4 + \cdots + e_{k-1} - e_k & \text{for } i = m, \\ -2e_1 + e_{i-m+1} + e_{k-i+m+1} & \text{for } m+1 \le i \le 2m-1, \\ e_1 + e_2 + \cdots + e_k & \text{for } i = 2m = k, \\ e_i & \text{for } k \le i \le n \end{cases} \tag{3.16}$$

(cf. Example 3.10 for $k = 8$). Then, the adjacency matrix $A_{\mathcal{G}}$ in the basis $\mathcal{B}$ has the form

$$\mathcal{B}^{-1}A_{\mathcal{G}}\mathcal{B} = \left[\begin{array}{c|cc|c} B_1 & 0 & & \\ \cline{1-3} 0 & a & * & 0 \\ & 0 & B_2 & \\ \hline & C & & A_{\bowtie} \end{array}\right],$$

where B_1, B_2 are matrices of order $(m-1)\times(m-1)$,

$$a = a_{11} - 2a_{12} + 2a_{13} - 2a_{1,4} + \cdots + (-1)^{m-1}2a_{1,m} + (-1)^m a_{1,m+1}, \tag{3.17}$$

and $A_{\bowtie}$ is the adjacency matrix of the quotient network. More precisely,

$$B_1 = \begin{bmatrix} a_{11}-a_{13} & a_{12}-a_{14} & a_{13}-a_{15} & \cdots & a_{1,m-1}-a_{1,m+1} \\ a_{12}-a_{14} & a_{11}-a_{15} & a_{12}-a_{16} & \cdots & a_{1,m-2}-a_{1,m} \\ a_{13}-a_{15} & a_{12}-a_{16} & a_{11}-a_{17} & \cdots & a_{1,m-3}-a_{1,m-1} \\ \cdots & \cdots & \cdots & \cdots & \\ a_{1,m-1}-a_{1,m+1} & a_{1,m-2}-a_{1,m} & a_{1,m-3}-a_{1,m-1} & \cdots & a_{11}-a_{13} \end{bmatrix},$$

$$B_2 = \begin{bmatrix} a_{11}+a_{13} & a_{12}+a_{14} & a_{13}+a_{15} & \cdots & a_{1,m-1}+a_{1,m+1} \\ a_{12}+a_{14} & a_{11}+a_{15} & a_{12}+a_{16} & \cdots & a_{1,m-2}+a_{1,m} \\ a_{13}+a_{15} & a_{12}+a_{16} & a_{11}+a_{17} & \cdots & a_{1,m-3}+a_{1,m-1} \\ \cdots & \cdots & \cdots & \cdots & \\ a_{1,m-1}+a_{1,m+1} & a_{1,m-2}+a_{1,m} & a_{1,m-3}+a_{1,m-1} & \cdots & a_{11}+a_{13} \end{bmatrix}$$
$$+ \begin{bmatrix} 2a_{1m}-2a_{1,m+1} & 2a_{1,m-1}-2a_{1,m+1} & \cdots & 2a_{12}-2a_{1,m+1} \\ -2a_{1m}+2a_{1,m+1} & -2a_{1,m-1}+2a_{1,m+1} & \cdots & -2a_{12}+2a_{1,m+1} \\ 2a_{1m}-2a_{1,m+1} & 2a_{1,m-1}-2a_{1,m+1} & \cdots & 2a_{12}-2a_{1,m+1} \\ \cdots & \cdots & \cdots & \\ (-1)^m(2a_{1m}-2a_{1,m+1}) & (-1)^m(2a_{1,m-1}-2a_{1,m+1}) & \cdots & (-1)^m(2a_{12}-2a_{1,m+1}) \end{bmatrix}$$
$$+ \begin{bmatrix} -2a_{12} & -2a_{12} & \cdots & -2a_{12} \\ -2a_{13} & -2a_{13} & \cdots & -2a_{13} \\ -2a_{14} & -2a_{14} & \cdots & -2a_{14} \\ \cdots & \cdots & \cdots & \\ -2a_{1m} & -2a_{1m} & \cdots & -2a_{1m} \end{bmatrix}.$$

Analogously to Lemma 3.8, one can show that B_1 and B_2 are similar. Indeed, denote by O_r the row operation

$$R_r \quad \rightsquigarrow \quad R_r + 2R_{r+1} - 2R_{r+2} + \cdots + (-1)^{m-r}2R_{m-1}, \quad r = 1, 2, \ldots, m-2.$$

Then,

$$O_1O_2\cdots O_{m-3}O_{m-2}B_2O_{m-2}^{-1}O_{m-3}^{-1}\cdots O_2^{-1}O_1^{-1} = B_1. \tag{3.18}$$

By applying the operations specified in (3.18) to $\mathcal{B}$, we can obtain a new basis $\tilde{\mathcal{B}}$. Let $R = O_1O_2\cdots O_{m-3}O_{m-2}$. Define

$$O = \begin{bmatrix} I_m & 0 & 0 \\ 0 & R & 0 \\ 0 & 0 & I_{n-2m+1} \end{bmatrix}, \qquad \tilde{\mathcal{B}} = \mathcal{B}O^{-1}.$$

Then, the adjacency matrix $A_{\mathcal{G}}$ in the basis $\tilde{\mathcal{B}}$ has the form

$$\tilde{\mathcal{B}}^{-1}A_{\mathcal{G}}\tilde{\mathcal{B}} = \left[\begin{array}{c|cc|c} B_1 & \multicolumn{2}{c|}{0} & \\ \cline{1-3} 0 & a & *' & 0 \\ & 0 & B_1 & \\ \hline \multicolumn{3}{c|}{C''} & A_{\bowtie} \end{array}\right].$$

It follows that $A_{\mathcal{G}}$ has $(m-1)$ eigenvalues of multiplicity at least 2 and thus, by Remark 3.2, the Jacobian $J_{\mathcal{G}}$ has $(m-1)r$ eigenvalues with algebraic multiplicity at least 2.

A precise formula of $\tilde{\mathcal{B}} = \{\tilde{b}_1, \ldots, \tilde{b}_n\}$ is given by

$$\tilde{b}_i = \begin{cases} e_{i+1} - e_{k-i+1} & \text{for } 1 \le i \le m-1, \\ e_1 - e_2 + e_3 - e_4 + \cdots + e_{k-1} - e_k & \text{for } i = m, \\ (-1)^{i-m}2(e_1 - e_2 + \cdots + (-1)^{i-m-1}e_{i-m}) + e_{i-m+1} + e_{k-i+m+1} & \\ \qquad\qquad \text{for } m+1 \le i \le 2m-1, & \\ e_1 + e_2 + \cdots + e_k & \text{for } i = 2m = k, \\ e_i & \text{for } k+1 \le i \le n \end{cases} \tag{3.19}$$

(cf. Example 3.10 for $k = 8$). ■

Example 3.9. Let $\mathcal{G}$ be a nine-cell regular network with an interior symmetry $\mathbf{D}_7$ on the cells $\{1, 2, 3, 4, 5, 6, 7\}$. Then, with respect to the basis (cf. (3.14))

$$\mathcal{B} = \begin{bmatrix} 0 & 0 & 0 & -2 & -2 & -2 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

the adjacency matrix $A_{\mathcal{G}}$ has the form

$$\mathcal{B}^{-1}A_{\mathcal{G}}\mathcal{B} = \left[\begin{array}{cc|c} B_1 & 0 & 0 \\ 0 & B_2 & \\ \hline C_1 & C_2 & A_{\bowtie} \end{array}\right],$$

where

$$B_1 = \begin{bmatrix} a_{11}-a_{13} & a_{12}-a_{14} & a_{13}-a_{14} \\ a_{12}-a_{14} & a_{11}-a_{14} & a_{12}-a_{13} \\ a_{13}-a_{14} & a_{12}-a_{13} & a_{11}-a_{12} \end{bmatrix},$$

$$B_2 = \begin{bmatrix} a_{11}-2a_{12}+a_{13} & -a_{12}+a_{14} & a_{13}-2a_{12}+a_{14} \\ a_{12}-2a_{13}+a_{14} & a_{11}-2a_{13}+a_{14} & a_{12}-a_{13} \\ a_{13}-a_{14} & a_{12}-2a_{14}+a_{13} & a_{11}-2a_{14}+a_{12} \end{bmatrix},$$

$$A_{\bowtie} = \begin{bmatrix} a_{11}+2a_{12}+2a_{13}+2a_{14} & a_{18} & a_{19} \\ a_{81}+a_{82}+a_{83}+a_{84}+a_{85}+a_{86}+a_{87} & a_{88} & a_{89} \\ a_{91}+a_{92}+a_{93}+a_{94}+a_{95}+a_{96}+a_{97} & a_{98} & a_{99} \end{bmatrix}$$

and

$$C_1 = \begin{bmatrix} 0 & 0 & 0 \\ a_{82}-a_{87} & a_{83}-a_{86} & a_{84}-a_{85} \\ a_{92}-a_{97} & a_{93}-a_{96} & a_{94}-a_{95} \end{bmatrix},$$

$$C_2 = \begin{bmatrix} 0 & 0 & 0 \\ -2a_{81}+a_{82}+a_{87} & -2a_{81}+a_{83}+a_{86} & -2a_{81}+a_{84}+a_{85} \\ -2a_{91}+a_{92}+a_{97} & -2a_{91}+a_{93}+a_{96} & -2a_{91}+a_{94}+a_{95} \end{bmatrix}.$$

Consider a new basis (cf. (3.15))

$$\tilde{\mathcal{B}} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & -2 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & -1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & -1 & 1 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 & -1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Then, the adjacency matrix $A_{\mathcal{G}}$ is of the form

$$\tilde{\mathcal{B}}^{-1} A_{\mathcal{G}} \tilde{\mathcal{B}} = \left[\begin{array}{cc|c} B_1 & 0 & \multirow{2}{*}{0} \\ 0 & B_1 & \\ \hline C_1' & C_2' & A_{\bowtie} \end{array}\right],$$

where

$$C_1' = \begin{bmatrix} 0 & 0 & 0 \\ a_{82}-a_{87} & a_{83}-a_{86} & a_{84}-a_{85} \\ a_{92}-a_{97} & a_{93}-a_{96} & a_{94}-a_{95} \end{bmatrix},$$

$$C_2' = \begin{bmatrix} 0 & 0 & 0 \\ -a_{83}+a_{84}+a_{85}-a_{86} & -a_{82}+a_{83}+a_{86}-a_{87} & -2a_{81}+a_{82}+a_{87} \\ -a_{93}+a_{94}+a_{95}-a_{96} & -a_{92}+a_{93}+a_{96}-a_{97} & -2a_{91}+a_{92}+a_{97} \end{bmatrix}.$$

Example 3.10. Let $\mathcal{G}$ be a 10-cell regular network with an interior symmetry $\mathbf{D}_8$ on the cells $\{1,2,3,4,5,6,7,8\}$. Then, with respect to the basis (cf. (3.16))

$$\mathcal{B}=\begin{bmatrix} 0&0&0&1&-2&-2&-2&1&0&0\\ 1&0&0&-1&1&0&0&1&0&0\\ 0&1&0&1&0&1&0&1&0&0\\ 0&0&1&-1&0&0&1&1&0&0\\ 0&0&0&1&0&0&0&1&0&0\\ 0&0&-1&-1&0&0&1&1&0&0\\ 0&-1&0&1&0&1&0&1&0&0\\ -1&0&0&-1&1&0&0&1&0&0\\ 0&0&0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&0&0&1 \end{bmatrix},$$

the adjacency matrix $A_{\mathcal{G}}$ has the form

$$\mathcal{B}^{-1}A_{\mathcal{G}}\mathcal{B}=\left[\begin{array}{c|cc|c} B_1 & \multicolumn{2}{c|}{0} & \\ \cline{1-3} 0 & a & a_1 & 0\\ & 0 & B_2 & \\ \hline C_1 & C_2 & C_3 & A_{\bowtie} \end{array}\right],$$

where $a = a_{11} - 2a_{12} + 2a_{13} - 2a_{1,4} + a_{15}$,

$$a_1 = [2a_{14} - 2a_{15}, 2a_{13} - 2a_{15}, 2a_{12} - 2a_{15}],$$

$$B_1 = \begin{bmatrix} a_{11}-a_{13} & a_{12}-a_{14} & a_{13}-a_{15}\\ a_{12}-a_{14} & a_{11}-a_{15} & a_{12}-a_{14}\\ a_{13}-a_{15} & a_{12}-a_{14} & a_{11}-a_{13} \end{bmatrix},$$

$$B_2 = \begin{bmatrix} a_{11}-2a_{12}+a_{13}+2a_{14}-2a_{15} & -a_{12}+2a_{13}+a_{14}-2a_{15} & a_{13}-a_{15}\\ a_{12}-2a_{13}-a_{14}+2a_{15} & a_{11}-4a_{13}+3a_{15} & -a_{12}-2a_{13}+a_{14}+2a_{15}\\ a_{13}-a_{15} & a_{12}+2a_{13}-a_{14}-2a_{15} & a_{11}+2a_{12}+a_{13}-2a_{14}-2a_{15} \end{bmatrix},$$

$$A_{\bowtie} = \begin{bmatrix} a_{11}+2a_{12}+2a_{13}+2a_{14}+a_{15} & a_{19} & a_{1,10}\\ a_{91}+a_{92}+a_{93}+a_{94}+a_{95}+a_{96}+a_{97}+a_{98} & a_{99} & a_{9,10}\\ a_{10,1}+a_{10,2}+a_{10,3}+a_{10,4}+a_{10,5}+a_{10,6}+a_{10,7}+a_{10,8} & a_{10,9} & a_{10,10} \end{bmatrix},$$

and

$$C_1 = \begin{bmatrix} 0 & 0 & 0\\ a_{92}-a_{98} & a_{93}-a_{97} & a_{94}-a_{96}\\ a_{10,2}-a_{10,8} & a_{10,3}-a_{10,7} & a_{10,4}-a_{10,6} \end{bmatrix},$$

$$C_2 = \begin{bmatrix} 0\\ a_{91}-a_{92}+a_{93}-a_{94}+a_{95}-a_{96}+a_{97}-a_{98}\\ a_{10,1}-a_{10,2}+a_{10,3}-a_{10,4}+a_{10,5}-a_{10,6}+a_{10,7}-a_{10,8} \end{bmatrix},$$

$$C_3 = \begin{bmatrix} 0 & 0 & 0\\ -2a_{91}+a_{92}+a_{98} & -2a_{91}+a_{93}+a_{97} & -2a_{91}+a_{94}+a_{96}\\ -2a_{10,1}+a_{10,2}+a_{10,8} & -2a_{10,1}+a_{10,3}+a_{10,7} & -2a_{10,1}+a_{10,4}+a_{10,6} \end{bmatrix}.$$

Consider a new basis (cf. (3.19))

$$\tilde{\mathcal{B}} = \begin{bmatrix} 0 & 0 & 0 & 1 & -2 & 2 & -2 & 1 & 0 & 0 \\ 1 & 0 & 0 & -1 & 1 & -2 & 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & -2 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & -1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 1 & -2 & 1 & 0 & 0 \\ -1 & 0 & 0 & -1 & 1 & -2 & 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Then, the adjacency matrix $A_{\mathcal{G}}$ is of the form

$$\tilde{\mathcal{B}}^{-1} A_{\mathcal{G}} \tilde{\mathcal{B}} = \left[\begin{array}{c|cc|c} B_1 & \multicolumn{2}{c|}{0} & \\ \cline{1-3} 0 & a & a_1' & 0 \\ & 0 & B_1 & \\ \hline C_1' & C_2' & C_3' & A_{\bowtie} \end{array}\right],$$

where $a_1' = [2a_{14} - 2a_{15}, 2a_{13} - 4a_{14} + 2a_{15}, 2a_{12} - 4a_{13} + 4a_{14} - 2a_{15}]$,

$$C_1' = \begin{bmatrix} 0 & 0 & 0 \\ a_{92} - a_{98} & a_{93} - a_{97} & a_{94} - a_{96} \\ a_{10,2} - a_{10,8} & a_{10,3} - a_{10,7} & a_{10,4} - a_{10,6} \end{bmatrix},$$

$$C_2' = \begin{bmatrix} 0 \\ a_{91} - a_{92} + a_{93} - a_{94} + a_{95} - a_{96} + a_{97} - a_{98} \\ a_{10,1} - a_{10,2} + a_{10,3} - a_{10,4} + a_{10,5} - a_{10,6} + a_{10,7} - a_{10,8} \end{bmatrix},$$

$$C_3' = \begin{bmatrix} 0 & 0 & 0 \\ -2a_{91} + a_{92} + a_{98} & 2a_{91} - 2a_{92} + a_{93} + a_{97} - 2a_{98} & -2a_{91} + 2a_{92} - 2a_{93} + a_{94} + a_{96} - 2a_{97} + 2a_{98} \\ -2a_{10,1} + a_{10,2} + a_{10,8} & 2a_{10,1} - 2a_{10,2} + a_{10,3} + a_{10,7} - 2a_{10,8} & -2a_{10,1} + 2a_{10,2} - 2a_{10,3} + a_{10,4} + a_{10,6} - 2a_{10,7} + 2a_{10,8} \end{bmatrix}.$$

3.4. $\Sigma_{\mathcal{S}}$-interior symmetry with $\mathbf{D}_k \subseteq \Sigma_{\mathcal{S}} \subseteq \mathbf{S}_k$. In this subsection, we consider regular networks $\mathcal{G}$ with an interior symmetry group $\Sigma_{\mathcal{S}}$ with $\mathbf{D}_k \subseteq \Sigma_{\mathcal{S}} \subseteq \mathbf{S}_k$. Besides the result of Theorem 3.7 that applies to $\mathcal{G}$, we show that the multiplicity of the eigenvalues of the adjacency matrix $A_{\mathcal{G}}$ can be directly analyzed using the eigenvalues of the circulant part A of $A_{\mathcal{G}}$.

As shown in subsection 3.3, the adjacency matrix of a regular network having an interior symmetry at least $\mathbf{D}_k$ is of the form

$$A_{\mathcal{G}} = \begin{bmatrix} A & D \\ E & F \end{bmatrix}, \tag{3.20}$$

where D is a $k \times (n-k)$ matrix with all rows equal and A is a circulant matrix of order $k \times k$ being of the form

$$A = \begin{cases} circ(a_{11}a_{12}a_{13}\dots a_{1m+1}a_{1m+1}\dots a_{13}a_{12}) & \text{if } k \text{ is odd}, \\ circ(a_{11}a_{12}a_{13}\dots a_{1m}a_{1m+1}a_{1m}\dots a_{13}a_{12}) & \text{if } k \text{ is even}. \end{cases} \tag{3.21}$$

It follows from (2.4) that the eigenvalues λ_j, $j = 0, \ldots, k-1$, of A are real and satisfy $\lambda_j = \lambda_{k-j}$ for $j = 1, \ldots, m$. That is, A has m eigenvalues with algebraic multiplicity at least 2 if k is odd; A has $(m-1)$ eigenvalues with algebraic multiplicity at least 2 if k is even. In Theorem 3.7 we proved the same property for $A_{\mathcal{G}}$.

Now, using the proof of Theorem 3.7, we show that the following theorem holds.

Theorem 3.11. *Let $\mathcal{G}$ be an n-cell regular network with an interior symmetry $\Sigma_{\mathcal{S}}$ such that $\mathbf{D}_k \subseteq \Sigma_{\mathcal{S}} \subseteq \mathbf{S}_k$. Let $A_{\mathcal{G}}$ be the adjacency matrix of $\mathcal{G}$, let A be given by (3.21), and let λ_j be eigenvalues of A for $j = 0, \ldots, k-1$ given by (2.4). Then, there exists a basis $\mathcal{B}$ of $\mathbb{R}^n$, which is independent of entries of $A_{\mathcal{G}}$, such that*

$$\mathcal{B}^{-1} A_{\mathcal{G}} \mathcal{B} = \begin{bmatrix} \Lambda & 0 \\ C & A_{\bowtie} \end{bmatrix} \quad \text{for } \Lambda = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_{k-1} \end{bmatrix}, \tag{3.22}$$

where $A_{\bowtie}$ is the adjacency matrix of the quotient network induced by $\Sigma_{\mathcal{S}}$.

Proof. Consider the k-cell regular network $\mathcal{G}_o$ whose adjacency matrix is given by A in (3.20). Since $\mathcal{G}$ is $\Sigma_{\mathcal{S}}$-interior symmetric, $\mathcal{G}_o$ is $\Sigma_{\mathcal{S}}$-symmetric. Let $\tilde{\mathcal{B}}_o$ be a basis in $\mathbb{R}^k$ given by (3.15) for odd k and (3.19) for even k. As shown in the proof of Theorem 3.7, we have

$$\tilde{\mathcal{B}}_o^{-1} A \tilde{\mathcal{B}}_o = \begin{bmatrix} M & 0 \\ 0 & \lambda_0 \end{bmatrix},$$

where M is a matrix of order $(k-1) \times (k-1)$ of the form

$$M = \begin{bmatrix} B_1 & 0 \\ 0 & B_1 \end{bmatrix} \quad \text{or} \quad \left[\begin{array}{c|cc} B_2 & \multicolumn{2}{c}{0} \\ \hline 0 & a & *' \\ & 0 & B_2 \end{array}\right]$$

for odd k or even k, respectively.

Let v_j be the eigenvector of λ_j given by (2.3) for $j = 0, 1, \ldots, k-1$. Set $V = \{v_1, \ldots, v_{k-1}, v_0\}$. Then,

$$V^{-1} A V = \begin{bmatrix} \Lambda & 0 \\ 0 & \lambda_0 \end{bmatrix}.$$

Thus, we have

$$(\tilde{\mathcal{B}}_o^{-1} V)^{-1} \begin{bmatrix} M & 0 \\ 0 & \lambda_0 \end{bmatrix} (\tilde{\mathcal{B}}_o^{-1} V) = \begin{bmatrix} \Lambda & 0 \\ 0 & \lambda_0 \end{bmatrix}.$$

Moreover, it can be verified that

$$\tilde{\mathcal{B}}_o^{-1} V = \begin{bmatrix} X & 0 \\ 0 & 1 \end{bmatrix}$$

for a matrix X of order $(k-1) \times (k-1)$. Consequently, we have

$$X^{-1} M X = \Lambda.$$

On the other hand, let $\tilde{\mathcal{B}}$ be a basis in $\mathbb{R}^n$ given by (3.15) for odd k and by (3.19) for even k. Then,

$$\tilde{\mathcal{B}}^{-1} A_{\mathcal{G}} \tilde{\mathcal{B}} = \begin{bmatrix} M & 0 \\ C' & A_{\bowtie} \end{bmatrix}.$$

Now set

$$\tilde{X} = \begin{bmatrix} X & 0 \\ 0 & I_{n-k+1} \end{bmatrix}, \quad \mathcal{B} := \tilde{\mathcal{B}} \tilde{X}.$$

Then, $\mathcal{B}$ is a basis such that (3.22) holds for $C = C'X$. Moreover, $\mathcal{B}$ is also independent of the entries of $A_{\mathcal{G}}$, since both $\tilde{\mathcal{B}}_o$ and V are independent of the entries of $A_{\mathcal{G}}$. ■

Consequently, the influence of $\Sigma_{\mathcal{S}}$ on the eigenvalues of $A_{\mathcal{G}}$, and thus of $J_{\mathcal{G}}$, can be directly examined by looking at the eigenvalues of A.

Example 3.12. Let $k = 12$, $m = 6$, and $n > 12$. Consider an n-cell regular network $\mathcal{G}$ with an interior symmetry at least $\mathbf{D}_{12}$ on the set of cells $\{1, 2, \ldots, 12\}$. Let $A_{\mathcal{G}}$ be the adjacency matrix, A be the circulant part of $A_{\mathcal{G}}$ (cf. (3.20)), and λ_j be the eigenvalues of A for $j = 0, 1, \ldots, 11$. By (2.4), $\lambda_j = \lambda_{12-j}$ for $j = 1, 2, \ldots, 5$, and, denoting by $\lambda_{i,j}$ both the eigenvalues λ_i and λ_j, we have

$$\begin{aligned}
&\lambda_0 = a_{11} + 2a_{12} + 2a_{13} + 2a_{14} + 2a_{15} + 2a_{16} + a_{17} (= a_{\bowtie}),\\
&\lambda_{1,11} = a_{11} + r_1 a_{12} + r_2 a_{13} - r_2 a_{15} - r_1 a_{16} - a_{17},\\
&\lambda_{2,10} = a_{11} + r_2 a_{12} - r_2 a_{13} - 2a_{14} - r_2 a_{15} + r_2 a_{16} + a_{17},\\
&\lambda_{3,9} = a_{11} - 2a_{13} + 2a_{15} - a_{17},\\
&\lambda_{4,8} = a_{11} - r_2 a_{12} - r_2 a_{13} + 2a_{14} - r_2 a_{15} - r_2 a_{16} + a_{17},\\
&\lambda_{5,7} = a_{11} - r_1 a_{12} + r_2 a_{13} - r_2 a_{15} + r_1 a_{16} - a_{17},\\
&\lambda_6 = a_{11} - 2a_{12} + 2a_{13} - 2a_{14} + 2a_{15} - 2a_{16} + a_{17} (= a),
\end{aligned}$$

where $r_1 = 2\mathrm{Re}\omega_1 = \sqrt{3}$, $r_2 = 2\mathrm{Re}\omega_1^2 = 1$. Note that

(i) if $a_{12} = a_{16}$, then $\lambda_{1,11} = \lambda_{5,7}$;
(ii) if $a_{12} = a_{16}$ and $a_{13} = a_{15}$, then $\lambda_{1,11} = \lambda_{3,9} = \lambda_{5,7}$;
(iii) if $a_{12} = a_{13} = a_{14} = a_{16} = a_{17}$ and $a_{11} = a_{15}$, then $\lambda_{1,11} = \lambda_{2,10} = \lambda_{5,7}$ and $\lambda_{3,9} = \lambda_6$;
(iv) if $a_{11} = a_{12} = a_{13} = a_{15} = a_{16} = a_{17}$, then $\lambda_{1,11} = \lambda_{3,9} = \lambda_{5,7}$ and $\lambda_{2,10} = \lambda_6$;
(v) if $a_{12} = a_{13} = a_{15} = a_{16}$ and $a_{14} = a_{17}$, then $\lambda_{1,11} = \lambda_{2,10} = \lambda_{3,9} = \lambda_{5,7}$;
(vi) if $a_{12} = a_{13} = a_{14} = a_{15} = a_{16} = a_{17}$, then $\lambda_{1,11} = \lambda_{2,10} = \lambda_{3,9} = \lambda_{4,8} = \lambda_{5,7} = \lambda_6$.

Thus, by Theorem 3.11, the following hold for any n-cell regular network $\mathcal{G}$ with $n \geq 12$ having a $\Sigma_{\mathcal{S}}$-interior symmetry:

(i) if $\Sigma_{\mathcal{S}} = \langle \mathbf{D}_{12}, (2\ 6\ 8\ 12)(3\ 11)(4\ 10)(5\ 9) \rangle$, then $A_{\mathcal{G}}$ has 3 eigenvalues of multiplicity at least 2 and 1 eigenvalue of multiplicity at least 4;
(ii) if $\Sigma_{\mathcal{S}} = \langle \mathbf{D}_{12}, (2\ 6\ 8\ 12)(3\ 5\ 9\ 11)(4\ 10) \rangle$, then $A_{\mathcal{G}}$ has 2 eigenvalues of multiplicity at least 2 and 1 eigenvalue of multiplicity at least 6;
(iii) if $\Sigma_{\mathcal{S}} = \langle \mathbf{D}_{12}, (1\ 5\ 9)(2\ 3\ 4\ 6\ 7\ 8\ 10\ 11\ 12) \rangle$, then $A_{\mathcal{G}}$ has 1 eigenvalue of multiplicity at least 2, 1 eigenvalue of multiplicity at least 3, and 1 eigenvalue of multiplicity at least 6;

(iv) if $\Sigma_{\mathcal{S}} = \langle \mathbf{D}_{12}, (4\ 10)(1\ 2\ 3\ 5\ 6\ 7\ 8\ 9\ 11\ 12)\rangle$, then $A_{\mathcal{G}}$ has 1 eigenvalue of multiplicity at least 2, 1 eigenvalue of multiplicity at least 3, and 1 eigenvalue of multiplicity at least 6;

(v) if $\Sigma_{\mathcal{S}} = \langle \mathbf{D}_{12}, (2\ 3\ 5\ 6\ 8\ 9\ 11\ 12)(4\ 7\ 10)\rangle$, then $A_{\mathcal{G}}$ has 1 eigenvalue of multiplicity at least 2 and 1 eigenvalue of multiplicity at least 8;

(vi) if $\Sigma_{\mathcal{S}} = \langle \mathbf{D}_{12}, (2\ 3\ 4\ 5\ 6\ 7\ 8\ 9\ 10\ 11\ 12)\rangle = \mathbf{S}_{12}$, then $A_{\mathcal{G}}$ has 1 eigenvalue of multiplicity at least 11.

3.5. $\mathbb{Z}_k$-interior symmetry. Consider an n-cell regular network $\mathcal{G}$ with adjacency matrix $A_{\mathcal{G}} = [a_{ij}]_{1\le i,j\le n}$, which has an interior symmetry $\mathbb{Z}_k$ for some $3 \le k < n$ on some subset of k cells which, up to a reordering of the cells, we can assume to be the first k cells. Then,

$$A_{\mathcal{G}} = \begin{bmatrix} A & D \\ E & F \end{bmatrix},$$

where D is a $k \times (n-k)$ matrix with all rows equal and A is a circulant matrix,

$$A = circ(a_{11}, a_{12}, a_{13}, \dots, a_{1k}).$$

Examples show that in general, $A_{\mathcal{G}}$ does not have multiple eigenvalues due to $\mathbb{Z}_k$-interior symmetry. In fact, even with additional equalities on $\{a_{12}, a_{13}, \dots a_{1k}\}$, as long as the resulting symmetry is less than $\mathbf{D}_k$, $A_{\mathcal{G}}$ seems to be free of multiple eigenvalues in general.

3.6. Cyclic interior symmetry of regular uniform networks. Despite the fact that cyclic interior symmetries are not sufficient for the adjacency matrix of regular networks to have multiple eigenvalues, this may become different if they are uniform networks.

Recall that *uniform regular networks* are regular networks without multiple arrows or self-coupling arrows (cf. Stewart [16]). In the next two subsections, we analyze two particular types of cyclic interior symmetry groups and show their influence on the multiplicity of eigenvalues of adjacency matrices of uniform networks. As we will see, for regular uniform networks, interior symmetry forces the existence of eigenvalues in $\{-2, -1, 0, 1\}$.

3.6.1. $\mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2$-interior symmetry. We show that the following theorem holds.

Theorem 3.13. *Let $\mathcal{G}$ be an n-cell regular network with a product interior symmetry $\mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2$ on r disjoint subsets $\mathcal{S}_k = \{i_k, j_k\}$ of cells of $\mathcal{G}$ for $k = 1, 2, \dots, r$. Then, the adjacency matrix $A_{\mathcal{G}} = [a_{ij}]_{1\le i,j\le n}$ of $\mathcal{G}$ has r eigenvalues $(a_{i_k i_k} - a_{i_k j_k})$ for $k = 1, 2, \dots, r$. Moreover, if $\mathcal{G}$ is a uniform network, then $(a_{i_k i_k} - a_{i_k j_k}) \in \{-1, 0, 1\}$ for $k = 1, 2, \dots, r$.*

Proof. Without loss of generality, we assume $\mathcal{S}_k = \{2k-1, 2k\}$ for $k = 1, 2, \dots, r$. Then,

$$\mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2 = \langle (1\ 2), \dots, (2r-1\ 2r)\rangle.$$

Due to this interior symmetry, the entries of $A_{\mathcal{G}}$ satisfy

$$a_{ii} = a_{i+1,i+1}, \quad a_{i,i+1} = a_{i+1,i}, \quad \text{and} \quad a_{il} = a_{i+1,l}$$

for all $i = 1, 3, \dots, 2r-1$ and for all $l \ne i, i+1$.

Consider the balanced equivalence relation

$$\bowtie = \{\{1,2\}, \{3,4\}, \dots, \{2r-1, 2r\}, \{2r+1\}, \dots, \{n\}\}$$

induced by $\mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2$ and the basis $\mathcal{B} = \{b_1, b_2, \ldots, b_n\}$ given by

$$b_k = \begin{cases} e_{2k-1} - e_{2k} & \text{if } 1 \leq k \leq r, \\ e_{2(k-r)-1} + e_{2(k-r)} & \text{if } r+1 \leq k \leq 2r, \\ e_k & \text{if } 2r+1 \leq k \leq n, \end{cases}$$

adapted to the decomposition in (3.2). It follows from Theorem 3.3 that

$$\mathcal{B}^{-1} A_{\mathcal{G}} \mathcal{B} = \begin{bmatrix} A & 0 \\ B & A_{\bowtie} \end{bmatrix},$$

where

$$A = \begin{bmatrix} a_{11} - a_{12} & 0 & 0 & \cdots & 0 \\ 0 & a_{33} - a_{34} & 0 & \cdots & 0 \\ 0 & 0 & a_{55} - a_{56} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & a_{2r-1,2r-1} - a_{2r-1,2r} \end{bmatrix}.$$

Thus, $(a_{2k-1,2k-1} - a_{2k-1,2k})$ are eigenvalues of $A_{\mathcal{G}}$ for $k = 1, 2, \ldots, r$.

If $\mathcal{G}$ is a uniform network, then $a_{ij} \in \{0, 1\}$ and, consequently, $(a_{2k-1,2k-1} - a_{2k-1,2k}) \in \{-1, 0, 1\}$ for $k = 1, 2, \ldots, r$. ■

Corollary 3.14. *Let $\mathcal{G}$ be an n-cell uniform network with a product interior symmetry $\mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2$ on r disjoint subsets $\mathcal{S}_k$ of cells of $\mathcal{G}$ for $k = 1, 2, \ldots, r$. Assume that $r \geq 4$. Then, the adjacency matrix $A_{\mathcal{G}}$ of $\mathcal{G}$ has at least one multiple eigenvalue.*

Proof. By Theorem 3.13, $A_{\mathcal{G}} = [a_{ij}]_{1 \leq i,j \leq n}$ has r eigenvalues $\lambda_k := a_{i_k i_k} - a_{i_k j_k} \in \{-1, 0, 1\}$ for $k = 1, 2, \ldots, r$. Thus, if $r \geq 4$, values of λ_k's must be duplicated for some k. ■

Example 3.15. Let $\mathcal{G}$ be the five-cell uniform network given in Figure 8 and let $A_{\mathcal{G}} = [a_{ij}]_{5 \times 5}$ be the adjacency matrix.

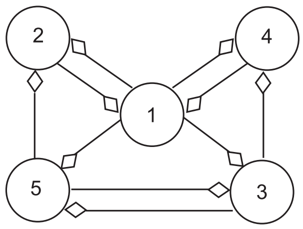

Figure 8. *The five-cell uniform network $\mathcal{G}$ in Example 3.15.*

The network $\mathcal{G}$ has an interior symmetry group $\mathbb{Z}_2 \times \mathbb{Z}_2 = \langle (2\ 3), (4\ 5) \rangle$. It follows from Theorem 3.13 that $a_{22} - a_{23} = 0$ and $a_{44} - a_{45} = 0$ are eigenvalues of $A_{\mathcal{G}}$. Thus, 0 is an eigenvalue of algebraic multiplicity at least 2 for $A_{\mathcal{G}}$.

Moreover, consider the balanced equivalence relation $\bowtie = \{\{1\}, \{2, 3\}, \{4, 5\}\}$ induced by $\mathbb{Z}_2 \times \mathbb{Z}_2 = \langle (2\ 3), (4\ 5) \rangle$. Then, the quotient network $\mathcal{G}_{\bowtie}$ has an interior symmetry $\mathbf{S}_3$ on the set $\mathcal{C}_{\bowtie} = \{[1]_{\bowtie}, [2]_{\bowtie}, [3]_{\bowtie}\}$ for $[1]_{\bowtie} = \{1\}$, $[2]_{\bowtie} = \{2, 3\}$, and $[3]_{\bowtie} = \{4, 5\}$. Let $A_{\bowtie} = [\bar{a}_{ij}]_{3 \times 3}$ be the adjacency matrix of $\mathcal{G}_{\bowtie}$. By Theorem 3.5, $\bar{a}_{11} - \bar{a}_{12} = -1$ is an eigenvalue of algebraic multiplicity at least 2 for the adjacency matrix $A_{\bowtie}$. Thus, by Theorem 3.3 and Corollary 3.4, -1 is an eigenvalue of algebraic multiplicity 2 for $A_{\mathcal{G}}$.

Last, the remaining eigenvalue of $A_{\mathcal{G}}$ is given by the valency 2 of the network.

Example 3.16. Let $\mathcal{G}$ be the five-cell uniform network given in Figure 9 and let $A_{\mathcal{G}} = [a_{ij}]_{5\times 5}$ be the adjacency matrix. Using Theorem 3.3, Corollary 3.4, and Theorem 3.13, we show that besides the valency 2 of the network, $A_{\mathcal{G}}$ has 0 and -1 as eigenvalues, both with algebraic multiplicity 2.

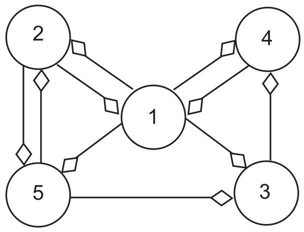

Figure 9. *The five-cell uniform network $\mathcal{G}$ in Example* 3.16.

We first consider the interior symmetry group $\mathbb{Z}_2 = \langle (2\ 3)\rangle$ of $\mathcal{G}$. Then, the eigenvalues of $A_{\mathcal{G}}$ are $a_{22} - a_{23} = 0$ with algebraic multiplicity at least 1 and those of the quotient network $\mathcal{G}_{\bowtie_1}$ induced by the balanced equivalence relation $\bowtie_1 = \{[1]_{\bowtie_1}, [2]_{\bowtie_1}, [3]_{\bowtie_1}, [4]_{\bowtie_1}\}$ for $[1]_{\bowtie_1} = \{1\}$, $[2]_{\bowtie_1} = \{2, 3\}$, $[3]_{\bowtie_1} = \{4\}$, and $[4]_{\bowtie_1} = \{5\}$.

The quotient network $\mathcal{G}_{\bowtie_1}$, in turn, has an interior symmetry $\mathbb{Z}_2 = \langle ([2]_{\bowtie_1}\ [4]_{\bowtie_1})\rangle$. Let $A_{\bowtie_1} = (\bar{a}^1_{ij})_{4\times 4}$ be the adjacency matrix of $\mathcal{G}_{\bowtie_1}$. Then, the eigenvalues of $A_{\bowtie_1}$ are $\bar{a}^1_{22} - \bar{a}^1_{24} = (a_{22} + a_{23}) - a_{25} = -1$ with algebraic multiplicity at least 1 and those of the quotient network $\mathcal{G}_{\bowtie_2}$ induced by the balanced equivalence relation $\bowtie_2 = \{[1]_{\bowtie_2}, [2]_{\bowtie_2}, [3]_{\bowtie_2}\}$ for $[1]_{\bowtie_2} = \{[1]_{\bowtie_1}\}$, $[2]_{\bowtie_2} = \{[2]_{\bowtie_1}, [4]_{\bowtie_1}\}$, and $[3]_{\bowtie_2} = \{[3]_{\bowtie_1}\}$.

Further, the quotient network $\mathcal{G}_{\bowtie_2}$ has an interior symmetry $\mathbb{Z}_2 = \langle ([1]_{\bowtie_2}\ [3]_{\bowtie_2})\rangle$. Let $A_{\bowtie_2} = (\bar{a}^2_{ij})_{3\times 3}$ be the adjacency matrix of $\mathcal{G}_{\bowtie_2}$. Then, the eigenvalues of $A_{\bowtie_2}$ are $\bar{a}^2_{11} - \bar{a}^2_{13} = a_{11} - a_{14} = -1$ with algebraic multiplicity at least 1 and those of the quotient network $\mathcal{G}_{\bowtie_3}$ for the balanced equivalence relation $\bowtie_3 = \{[1]_{\bowtie_3}, [2]_{\bowtie_3}\}$ with $[1]_{\bowtie_3} = \{[1]_{\bowtie_2}, [3]_{\bowtie_2}\}$ and $[2]_{\bowtie_3} = \{[2]_{\bowtie_2}\}$.

The quotient network $\mathcal{G}_{\bowtie_3}$ also has an interior symmetry $\mathbb{Z}_2 = \langle ([1]_{\bowtie_3}\ [2]_{\bowtie_3})\rangle$. Let $A_{\bowtie_3} = (\bar{a}^3_{ij})_{2\times 2}$ be the adjacency matrix of $\mathcal{G}_{\bowtie_3}$. By Theorem 3.13, $\bar{a}^3_{11} - \bar{a}^3_{12} = (a_{11} + a_{14}) - (a_{12} + a_{13} + a_{15}) = 0$ is an eigenvalue of algebraic multiplicity 1 for $A_{\bowtie_3}$.

3.6.2. V_4-interior symmetry. In this subsection we discuss n-cell uniform networks with an interior symmetry group

$$V_4 := \langle (i\ j)(k\ l), (i\ k)(j\ l)\rangle \subset \mathbf{S}_n,$$

where i, j, k, l are distinct cells of $\mathcal{G}$.

Theorem 3.17. *Let $\mathcal{G}$ be an n-cell uniform network having an interior symmetry group $V_4 = \langle (i\ j)(k\ l), (i\ k)(j\ l)\rangle \subset \mathbf{S}_n$ on a subset $\{i, j, k, l\}$ of cells of $\mathcal{G}$. Then, the adjacency* *matrix* $A_{\mathcal{G}} = [a_{\alpha\beta}]_{1\le\alpha,\beta\le n}$ *has the* 3 *eigenvalues*

$$\begin{aligned} -a_{ij} + a_{ik} - a_{il}, \\ -a_{ij} - a_{ik} + a_{il}, \\ a_{ij} - a_{ik} - a_{il}, \end{aligned}$$

which take value in $\{-2, -1, 0, 1\}$.

Proof. Due to the interior symmetry $\mathbb{Z}_2 = \langle (i\ j)(k\ l)\rangle$, the entries of $A_{\mathcal{G}}$ satisfy

$$\begin{array}{llllll} a_{ii} = a_{jj}, & a_{ij} = a_{ji}, & a_{ik} = a_{jl}, & a_{il} = a_{jk}, & \text{and} & a_{im} = a_{jm} \quad \forall m \neq i, j, k, l, \\ a_{kk} = a_{ll}, & a_{kl} = a_{lk}, & a_{ki} = a_{lj}, & a_{kj} = a_{li}, & \text{and} & a_{km} = a_{lm} \quad \forall m \neq i, j, k, l. \end{array}$$

Due to the interior symmetry $\mathbb{Z}_2 = \langle (i\ k)(j\ l)\rangle$, they satisfy

$$\begin{array}{llllll} a_{ii} = a_{kk}, & a_{ik} = a_{ki}, & a_{ij} = a_{kl}, & a_{il} = a_{kj}, & \text{and} & a_{im} = a_{km} \quad \forall m \neq i, j, k, l, \\ a_{jj} = a_{ll}, & a_{jl} = a_{lj}, & a_{ji} = a_{lk}, & a_{jk} = a_{li}, & \text{and} & a_{jm} = a_{lm} \quad \forall m \neq i, j, k, l. \end{array}$$

Thus, due to the interior symmetry V_4, the entries of $A_{\mathcal{G}}$ satisfy

$$\begin{array}{lll} a_{ii} = a_{jj} = a_{kk} = a_{ll}, & a_{ij} = a_{ji} = a_{kl} = a_{lk}, & a_{ik} = a_{jl} = a_{ki} = a_{lj}, \\ a_{il} = a_{jk} = a_{kj} = a_{li}, & a_{im} = a_{jm} = a_{km} = a_{lm} & \forall m \neq i, j, k, l. \end{array} \tag{3.23}$$

Without loss of generality, we assume $i = 1$, $j = 2$, $k = 3$, and $l = 4$. Let $\bowtie$ be the balanced equivalence relation induced by $\mathbb{Z}_2 = \langle (1, 2)(3, 4)\rangle$, i.e.,

$$\bowtie = \{\{1, 2\}, \{3, 4\}, \{5\}, \dots, \{n\}\}.$$

Let W, U be given by (2.7)–(2.8). Then, we have

$$\mathbb{R}^n = W \oplus U.$$

A basis $\mathcal{B} = \{b_1, b_2, \dots, b_n\}$ adapted to this decomposition is given by

$$b_k = \begin{cases} e_{2k-1} - e_{2k} & \text{if } 1 \le k \le 2, \\ e_{2(k-2)-1} + e_{2(k-2)} & \text{if } 3 \le k \le 4, \\ e_k & \text{if } 5 \le k \le n. \end{cases}$$

Then, the adjacency matrix $A_{\mathcal{G}}$ in the basis $\mathcal{B}$ has the form

$$\mathcal{B}^{-1} A_{\mathcal{G}} \mathcal{B} = \begin{bmatrix} A & 0 \\ C & A_{\bowtie} \end{bmatrix},$$

where

$$A = \begin{bmatrix} a_{11} - a_{12} & a_{13} - a_{14} \\ a_{13} - a_{14} & a_{11} - a_{12} \end{bmatrix}.$$

Since $\mathcal{G}$ is uniform, we have $a_{11} = 0$. Thus, A has eigenvalues $(-a_{12} \pm (a_{13} - a_{14}))$, which are also eigenvalues of $A_{\mathcal{G}}$. Similarly, using symmetry $\mathbb{Z}_2 = \langle (13)(24)\rangle$, one can show that $A_{\mathcal{G}}$ has eigenvalues $(-a_{13} \pm (a_{12} - a_{14}))$. Thus, altogether $A_{\mathcal{G}}$ has the following 3 eigenvalues:

$$-a_{12} + a_{13} - a_{14}, \quad -a_{12} - a_{13} + a_{14}, \quad a_{12} - a_{13} - a_{14},$$

which take value in $\{-2,-1,0,1\}$, since $a_{\alpha\beta} \in \{0,1\}$. ■

Corollary 3.18. *Let $\mathcal{G}$ be an n-cell uniform network with adjacency matrix $A_{\mathcal{G}} = [a_{\alpha\beta}]_{1\leq\alpha,\beta\leq n}$ having an interior symmetry group $V_4 \rtimes \mathbb{Z}_2 = \langle (i\ j)(k\ l), (i\ k)(j\ l), (a\ b)\rangle \subset \mathbf{S}_n$ on a subset $\mathcal{S} = \{i,j,k,l\}$ of cells of $\mathcal{G}$, with $a \neq b$ in $\mathcal{S}$. If $(a\ b) = (i\ j)$ or $(a\ b) = (k\ l)$, then $-a_{ij} \in \{-2,-1,0,1\}$ is an eigenvalue of $A_{\mathcal{G}}$ with algebraic multiplicity at least 2. Analogously, if $(a\ b) = (i\ k)$ or $(a\ b) = (j\ l)$, then $-a_{ik} \in \{-2,-1,0,1\}$ is an eigenvalue of $A_{\mathcal{G}}$ with algebraic multiplicity at least 2; if $(a\ b) = (i\ l)$ or $(a\ b) = (j\ k)$, then $-a_{il} \in \{-2,-1,0,1\}$ is an eigenvalue of $A_{\mathcal{G}}$ with algebraic multiplicity at least 2.*

Proof. Consider $A_{\mathcal{G}}$ as a network having V_4 as an interior symmetry group. Then, by Theorem 3.17, $A_{\mathcal{G}}$ has the following 3 eigenvalues:

$$-a_{ij} + a_{ik} - a_{il}, \quad -a_{ij} - a_{ik} + a_{il}, \quad a_{ij} - a_{ik} - a_{il}.$$

We give the proof only for the case of $(a\ b) = (i\ j)$ or $(a\ b) = (k\ l)$. The other two cases can be proved in a similar way. Due to the interior symmetry $(i\ j)$ or $(k\ l)$, we have $a_{ik} = a_{il}$. Thus, $-a_{ij}$ is an eigenvalue of algebraic multiplicity 2 in $\{-2,-1,0,1\}$. ■

Example 3.19. Let $\mathcal{G}$ be the five-cell uniform network given in Figure 10 and $A_{\mathcal{G}} = [a_{ij}]_{1\leq i,j\leq 5}$ the adjacency matrix.

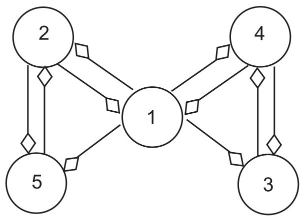

Figure 10. *The five-cell uniform network $\mathcal{G}$ in Example 3.19.*

The network $\mathcal{G}$ has an interior symmetry group $V_4 \rtimes \mathbb{Z}_2 = \langle (2\ 3)(4\ 5), (2\ 4)(3\ 5), (a\ b)\rangle$ on the set of cells $\mathcal{S} = \{2,3,4,5\}$ for $(a\ b) = (2\ 5)$, as well as for $(a\ b) = (3\ 4)$. By Corollary 3.18, the adjacency matrix $A_{\mathcal{G}}$ has the eigenvalue $-a_{25} = -1$ with algebraic multiplicity at least 2.

In fact, using Theorem 3.3, Corollary 3.4, and Theorem 3.5, we can show that the algebraic multiplicity of the eigenvalue -1 is at least 3. Note that $\mathcal{G}$ has an interior symmetry group $\mathbb{Z}_2 = \langle (2\ 3)(4\ 5)\rangle$. Let $\bowtie = \{\{1\},\{2,3\},\{4,5\}\}$. Then, with respect to the basis

$$b = ((0,1,-1,0,0),(0,0,0,1,-1),(1,0,0,0,0),(0,1,1,0,0),(0,0,0,1,1)),$$

$A_{\mathcal{G}}$ is given by

$$\begin{bmatrix} A & 0 \\ C & A_{\bowtie} \end{bmatrix},$$

where $A_{\bowtie}$ is the adjacency matrix of the quotient network associated with $\bowtie$ and

$$A = \begin{bmatrix} a_{22} - a_{23} & a_{24} - a_{25} \\ a_{42} - a_{43} & a_{44} - a_{45} \end{bmatrix}.$$

Since the quotient network is isomorphic to the ($\mathbf{S}_3$-symmetric) three-cell bidirectional ring in Figure 7, by Theorem 3.5, $A_{\bowtie}$ has $a_{11} - a_{12} = -1$ as an eigenvalue with algebraic multiplicity at least 2. On the other hand, due to the fact that $\mathcal{G}$ also has an interior symmetry (2 4)(3 5) and (3 4), we have $a_{23} = a_{24} = a_{42} = a_{45}$ and $a_{25} = a_{43}$. Also, since $\mathcal{G}$ is uniform, $a_{22} = a_{44} = 0$. Thus, the eigenvalues of A are $-a_{23} \pm (a_{23} - a_{25})$, one of which is equal to $-a_{25} = -1$. Therefore, -1 is an eigenvalue of $A_{\mathcal{G}}$ with algebraic multiplicity at least 3.

4. Interior symmetries and multiple eigenvalues: Homogeneous networks. We generalize our results on regular networks to homogeneous networks. Recall that a *homogeneous network* is a coupled cell network in which all cells are identical but which may have multiple types of arrows. Let $\mathcal{G}$ be an n-cell homogeneous network with s types of arrows, whose adjacency matrices are $A_1, \ldots, A_s$. Let r be the dimension of the cell internal dynamics. Then, the Jacobian at a fully synchronized equilibrium has the form

$$J_{\mathcal{G}} = \alpha \otimes I_n + \sum_{l=1}^{s} \beta_l \otimes A_l, \tag{4.1}$$

where α is the linearized internal dynamics and β_l is the linearized internal coupling for the lth-type arrow for $l = 1, \ldots, s$.

4.1. $\mathbf{S}_k$- and $\mathbf{A}_k$-interior symmetry. We show that the following theorem holds.

Theorem 4.1. *Let $\mathcal{G}$ be an n-cell homogeneous network with s types of arrows with adjacency matrices $A_1, \ldots, A_s$. Let $J_{\mathcal{G}}$ be given by (4.1). Assume that all matrices A_l, $l = 1, \ldots, s$, have an interior symmetry $\mathbf{S}_k$ or $\mathbf{A}_k$, on the same subset $\mathcal{S} \subseteq \mathcal{C}$ of k cells of $\mathcal{G}$, for some $k \in \{3, \ldots, n\}$. Then, the Jacobian $J_{\mathcal{G}}$ has r eigenvalues of algebraic multiplicity at least $k-1$.*

Proof. Without loss of generality, we assume $\mathcal{S} = \{1, \ldots, k\}$. Write $A_l = [a_{ij}^{(l)}]_{1 \le i,j \le n}$ for $l = 1, \ldots, s$. It follows from the proof of Theorem 3.5 that there is a basis $\mathcal{B}$ such that

$$\mathcal{B}^{-1} A_l \mathcal{B} = \begin{bmatrix} \bar{A}_l & 0 \\ C_l & A_{\bowtie_l} \end{bmatrix} \qquad \forall l = 1, \ldots, s,$$

where $\bar{A}_l$ is a scalar matrix of order $k-1$, being equal to $(a_{11}^{(l)} - a_{12}^{(l)}) I_{k-1}$. For convenience, we denote $a_l := a_{11}^{(l)} - a_{12}^{(l)}$.

Let $\hat{\mathcal{B}} = I_r \otimes \mathcal{B}$. Then, we have

$$\begin{aligned} \hat{\mathcal{B}}^{-1} J_{\mathcal{G}} \hat{\mathcal{B}} &= \begin{bmatrix} \alpha \otimes I_{k-1} + \sum_{l=1}^{s} \beta_l \otimes \bar{A}_l & 0 \\ \sum_{l=1}^{s} \beta_l \otimes C_l & \alpha \otimes I_{n-k+1} + \sum_{l=1}^{s} \beta_l \otimes A_{\bowtie_l} \end{bmatrix} \\ &= \begin{bmatrix} \left(\alpha + \sum_{l=1}^{s} a_l \beta_l \right) \otimes I_{k-1} & 0 \\ \sum_{l=1}^{s} \beta_l \otimes C_l & \alpha \otimes I_{n-k+1} + \sum_{l=1}^{s} \beta_l \otimes A_{\bowtie_l} \end{bmatrix}. \end{aligned}$$

Thus, every eigenvalue of $\alpha + \sum_{l=1}^{s} a_l \beta_l$ is an eigenvalue of $J_{\mathcal{G}}$ with algebraic multiplicity at least $k-1$. Therefore, $J_{\mathcal{G}}$ has r eigenvalues of multiplicity at least $k-1$. ∎

4.2. D_k-interior symmetry.

We show that the following theorem holds.

Theorem 4.2. *Let $\mathcal{G}$ be an n-cell homogeneous network with s types of arrows with adjacency matrices $A_1, \ldots, A_s$. Let $J_{\mathcal{G}}$ be given by (4.1). Assume that all matrices A_l, $l = 1, \ldots, s$, have an interior symmetry $\mathbf{D}_k$, on the same subset $\mathcal{S} \subseteq \mathcal{C}$ of k cells of $\mathcal{G}$, for some $k \in \{3, \ldots, n\}$. Set*

$$m = \begin{cases} (k-1)/2 & \text{if } k \text{ is odd,} \\ k/2 & \text{if } k \text{ is even.} \end{cases}$$

Then, $J_{\mathcal{G}}$ has mr eigenvalues with algebraic multiplicity at least 2 if k is odd; $J_{\mathcal{G}}$ has $(m-1)r$ eigenvalues with algebraic multiplicity at least 2 if k is even.

Proof. For simplicity, we present the proof for $s = 2$. The general case can be proved analogously.

Without loss of generality, we assume that the $\mathbf{D}_k$-interior symmetry is on the cells $\mathcal{S} = \{1, \ldots, k\}$. It follows from the proof of Theorem 3.7 that the adjacency matrices A_1 and A_2 can be diagonalized to a "double-block" form using the same basis $\tilde{\mathcal{B}}$ given by (3.15) for odd k and by (3.19) for even k.

By applying this basis to A_l, $l = 1, 2$, in the case of odd k, we have

$$\tilde{\mathcal{B}}^{-1} A_l \tilde{\mathcal{B}} = \left[\begin{array}{cc|c} B_l & 0 & \multirow{2}{*}{0} \\ 0 & B_l & \\ \hline \multicolumn{2}{c|}{C_l'} & A_{\bowtie_l} \end{array}\right], \quad l = 1, 2,$$

where B_l is a matrix of order $m \times m$. Consider the following basis for $J_{\mathcal{G}}$:

$$\hat{\mathcal{B}} = I_r \otimes \tilde{\mathcal{B}}.$$

Then, we have

$$\hat{\mathcal{B}}^{-1} J_{\mathcal{G}} \hat{\mathcal{B}} = \alpha \otimes I_n + \beta_1 \otimes \left[\begin{array}{cc|c} B_1 & 0 & \multirow{2}{*}{0} \\ 0 & B_1 & \\ \hline \multicolumn{2}{c|}{C_1'} & A_{\bowtie_1} \end{array}\right] + \beta_2 \otimes \left[\begin{array}{cc|c} B_2 & 0 & \multirow{2}{*}{0} \\ 0 & B_2 & \\ \hline \multicolumn{2}{c|}{C_2'} & A_{\bowtie_2} \end{array}\right]$$

$$= \left[\begin{array}{cc|c} \alpha \otimes I_m + \beta_1 \otimes B_1 + \beta_2 \otimes B_2 & 0 & \multirow{2}{*}{0} \\ 0 & \alpha \otimes I_m + \beta_1 \otimes B_1 + \beta_2 \otimes B_2 & \\ \hline \multicolumn{2}{c|}{\beta_1 \otimes C_1' + \beta_2 \otimes C_2'} & \alpha \otimes I_{n-2m} + \beta_1 \otimes A_{\bowtie_1} + \beta_2 \otimes A_{\bowtie_2} \end{array}\right].$$

Thus, every eigenvalue of $(\alpha \otimes I_m + \beta_1 \otimes B_1 + \beta_2 \otimes B_2)$ is also an eigenvalue of $J_{\mathcal{G}}$. Therefore, $J_{\mathcal{G}}$ has mr eigenvalues with algebraic multiplicity at least 2.

In the case k is even, we have

$$\tilde{\mathcal{B}}^{-1} A_i \tilde{\mathcal{B}} = \left[\begin{array}{c|cc|c} B_i & \multicolumn{2}{c|}{0} & \multirow{3}{*}{0} \\ \cline{1-3} \multirow{2}{*}{0} & a_i & *_i & \\ & 0 & B_i & \\ \hline \multicolumn{3}{c|}{C_i''} & A_{\bowtie_i} \end{array}\right], \quad i = 1, 2,$$

where B_i is a matrix of order $(m-1)\times(m-1)$. Consider again the basis $\hat{\mathcal{B}} = I_r \otimes \tilde{\mathcal{B}}$. Then,

$$\hat{\mathcal{B}}^{-1} J_{\mathcal{G}} \hat{\mathcal{B}} = \alpha \otimes I_n + \beta_1 \otimes \left[\begin{array}{c|cc|c} B_1 & \multicolumn{2}{c|}{0} & \\ \cline{1-3} 0 & a_1 & *_1 & 0 \\ & 0 & B_1 & \\ \hline \multicolumn{3}{c|}{C_1''} & A_{\bowtie_1} \end{array}\right] + \beta_2 \otimes \left[\begin{array}{c|cc|c} B_2 & \multicolumn{2}{c|}{0} & \\ \cline{1-3} 0 & a_2 & *_2 & 0 \\ & 0 & B_2 & \\ \hline \multicolumn{3}{c|}{C_2''} & A_{\bowtie_2} \end{array}\right]$$

$$= \left[\begin{array}{c|cc|c} \alpha \otimes I_{m-1} + \beta_1 \otimes B_1 + \beta_2 \otimes B_2 & \multicolumn{2}{c|}{0} & \\ \cline{1-3} 0 & \alpha + \beta_1 \otimes a_1 + \beta_2 \otimes a_2 & \beta_1 \otimes *_1 + \beta_2 \otimes *_2 & 0 \\ & 0 & \alpha \otimes I_{m-1} + \beta_1 \otimes B_1 + \beta_2 \otimes B_2 & \\ \hline \multicolumn{3}{c|}{\beta_1 \otimes C_1'' + \beta_2 \otimes C_2''} & \alpha \otimes I_{n-2m+1} + \beta_1 \otimes A_{\bowtie_1} + \beta_2 \otimes A_{\bowtie_2} \end{array}\right].$$

Thus, every eigenvalue of $(\alpha \otimes I_{m-1} + \beta_1 \otimes B_1 + \beta_2 \otimes B_2)$ is also an eigenvalue of $J_{\mathcal{G}}$. Therefore, $J_{\mathcal{G}}$ has $(m-1)r$ eigenvalues with algebraic multiplicity at least 2. ■

4.3. $\Sigma_{\mathcal{S}}$-interior symmetry with $\mathbf{D}_k \subseteq \Sigma_{\mathcal{S}} \subseteq \mathbf{S}_k$. Let $\mathcal{G}$ be an n-cell homogeneous network with s types of arrows with adjacency matrices $A_1, \ldots, A_s$. Assume that every A_l for $l = 1, \ldots, s$ has an interior symmetry $\Sigma_{\mathcal{S}}^l$ on the same subset $\mathcal{S} \in \mathcal{C}$ such that $\mathbf{D}_k \subseteq \Sigma_{\mathcal{S}}^l \subseteq \mathbf{S}_k$. Let $\bar{A}_l$ denote the upper left $k \times k$ submatrix of A_l, $l = 1, \ldots, s$ (cf. (3.20)). Then, $\bar{A}_l$ is a circulant matrix of the form (3.21). We show that the multiplicity of the eigenvalues of $J_{\mathcal{G}}$ can be directly analyzed by the eigenvalues of $\bar{A}_1, \ldots, \bar{A}_s$.

Theorem 4.3. *Let $\mathcal{G}$ be an n-cell homogeneous network with s types of arrows, where every adjacency matrix A_l has an interior symmetry $\Sigma_{\mathcal{S}}^l$ on the same subset $\mathcal{S} \subseteq \mathcal{C}$ such that $\mathbf{D}_k \subseteq \Sigma_{\mathcal{S}}^l \subseteq \mathbf{S}_k$ for $l = 1, 2, \ldots, s$. Let $\bar{A}_l$ be the upper left $k \times k$ submatrix of A_l for $l = 1, 2, \ldots, s$. Let $\lambda_j^{(l)}$ be the jth eigenvalue of $\bar{A}_l$ for $j = 0, 1, \ldots, k-1$, $l = 1, 2, \ldots, s$ (cf. (2.4)). Then, every eigenvalue of $(\alpha + \sum_{l=1}^s \lambda_j^{(l)} \beta_l)$ is an eigenvalue of $J_{\mathcal{G}}$ for $j = 1, \ldots, k-1$.*

Proof. We present the proof only for the case $s = 2$. The general case can be proved analogously. Without loss of generality we assume $\mathcal{S} = \{1, \ldots, k\}$.

Let $\mathcal{B}$ be the basis given by Theorem 3.11. Then, we have

$$\mathcal{B}^{-1} A_l \mathcal{B} = \begin{bmatrix} \Lambda_l & 0 \\ C_l & Q_l \end{bmatrix} \quad \text{for } \Lambda_l = \begin{bmatrix} \lambda_1^{(l)} & 0 & \cdots & 0 \\ 0 & \lambda_2^{(l)} & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_{k-1}^{(l)} \end{bmatrix}, \quad l = 1, 2.$$

Consider the basis $\hat{\mathcal{B}} = I_r \otimes \mathcal{B}$. Then,

$$\hat{\mathcal{B}}^{-1} J_{\mathcal{G}} \hat{\mathcal{B}} = \begin{bmatrix} \alpha \otimes I_{k-1} + \beta_1 \otimes \Lambda_1 + \beta_2 \otimes \Lambda_2 & 0 \\ \beta_1 \otimes C_1 + \beta_2 \otimes C_2 & \alpha \otimes I_{n-k+1} + \beta_1 \otimes Q_1 + \beta_2 \otimes Q_2 \end{bmatrix}.$$

Let $u \in \mathbb{R}^r$ and let v_j be the eigenvector of $\lambda_j^{(1)}$ and $\lambda_j^{(2)}$ given by (2.3) for some $j \in \{1, 2, \ldots, k-1\}$. Then,

$$\begin{aligned}(\alpha \otimes I_{k-1} + \beta_1 \otimes \Lambda_1 + \beta_2 \otimes \Lambda_2)(u \otimes v_j) &= \alpha u \otimes v_j + \beta_1 u \otimes \Lambda_1 v_j + \beta_2 u \otimes \Lambda_2 v_j \\ &= \alpha u \otimes v_j + \beta_1 u \otimes \lambda_j^{(1)} v_j + \beta_2 u \otimes \lambda_j^{(2)} v_j \\ &= (\alpha + \lambda_j^{(1)} \beta_1 + \lambda_j^{(2)} \beta_2) u \otimes v_j.\end{aligned}$$

Thus, every eigenvalue of $(\alpha+\lambda_j^{(1)}\beta_1+\lambda_j^{(2)}\beta_2)$ is an eigenvalue of $(\alpha\otimes I_{k-1}+\beta_1\otimes\Lambda_1+\beta_2\otimes\Lambda_2)$, which is also an eigenvalue of $J_{\mathcal{G}}$. ■

Example 4.4. Let $n > 12$. Let $\mathcal{G}$ be an n-cell homogeneous network with 2 types of arrows whose adjacency matrices A_1, A_2 have an interior symmetry, respectively, of

$$\Sigma_{\mathcal{S}}^1 = \langle \mathbf{D}_{12}, (1\ 5\ 9)(2\ 3\ 4\ 6\ 7\ 8\ 10\ 11\ 12)\rangle,$$
$$\Sigma_{\mathcal{S}}^2 = \langle \mathbf{D}_{12}, (4\ 10)(1\ 2\ 3\ 5\ 6\ 7\ 8\ 9\ 11\ 12)\rangle$$

on cells $\{1, 2, \ldots, 12\}$. Note that $\mathbf{D}_{12} \subset \Sigma_{\mathcal{S}}^i \subset \mathbf{S}_{12}$ for $i = 1, 2$. An example of A_1, A_2 is

$$A_1 = \begin{pmatrix} 1&0&0&0&1&0&0&0&1&0&0&0&0\\ 0&1&0&0&0&1&0&0&0&1&0&0&0\\ 0&0&1&0&0&0&1&0&0&0&1&0&0\\ 0&0&0&1&0&0&0&1&0&0&0&1&0\\ 1&0&0&0&1&0&0&0&1&0&0&0&0\\ 0&1&0&0&0&1&0&0&0&1&0&0&0\\ 0&0&1&0&0&0&1&0&0&0&1&0&0\\ 0&0&0&1&0&0&0&1&0&0&0&1&0\\ 1&0&0&0&1&0&0&0&1&0&0&0&0\\ 0&1&0&0&0&1&0&0&0&1&0&0&0\\ 0&0&1&0&0&0&1&0&0&0&1&0&0\\ 0&0&0&1&0&0&0&1&0&0&0&1&0\\ 0&0&1&0&0&0&0&0&0&2&0&0&0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0&0&0&1&0&0&0&0&0&1&0&0&0\\ 0&0&0&0&1&0&0&0&0&0&1&0&0\\ 0&0&0&0&0&1&0&0&0&0&0&1&0\\ 1&0&0&0&0&0&1&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&1&0&0&0&0&0\\ 0&0&1&0&0&0&0&0&1&0&0&0&0\\ 0&0&0&1&0&0&0&0&0&1&0&0&0\\ 0&0&0&0&1&0&0&0&0&0&1&0&0\\ 0&0&0&0&0&1&0&0&0&0&0&1&0\\ 1&0&0&0&0&0&1&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&1&0&0&0&0&0\\ 0&0&1&0&0&0&0&0&1&0&0&0&0\\ 0&0&0&0&1&1&0&0&0&0&0&0&0 \end{pmatrix}.$$

The corresponding network is then as shown in Figure 11, where the arrows with solid (resp., hollow) heads depict connections given by A_1 (resp., A_2).

Let $\bar{A}_l$ be the upper left 12×12 submatrix of A_l and let $\lambda_j^{(l)}$ be the jth eigenvalue of $\bar{A}_l$ (cf. (2.4)) for $j = 0, 1, \ldots, 11$, $l = 1, 2$. Then, we have (cf. Example 3.12(iii)–(iv))

$$\lambda_{1,11}^{(1)} = \lambda_{2,10}^{(1)} = \lambda_{5,7}^{(1)}, \quad \lambda_{3,9}^{(1)} = \lambda_6^{(1)}$$

and

$$\lambda_{1,11}^{(2)} = \lambda_{3,9}^{(2)} = \lambda_{5,7}^{(2)}, \quad \lambda_{2,10}^{(2)} = \lambda_6^{(2)}.$$

Thus, by Theorem 4.3, for every homogeneous network $\mathcal{G}$ with interior symmetries $\Sigma_{\mathcal{S}}^1$ and $\Sigma_{\mathcal{S}}^2$, every eigenvalue of $(\alpha + \lambda_{1,5,7,11}^{(1)}\beta_1 + \lambda_{1,5,7,11}^{(2)}\beta_2)$ is an eigenvalue of $J_{\mathcal{G}}$ of multiplicity at least 4; every eigenvalue of $(\alpha + \lambda_{2,10}^{(1)}\beta_1 + \lambda_{2,10}^{(2)}\beta_2)$ is an eigenvalue of $J_{\mathcal{G}}$ of multiplicity at least 2; every eigenvalue of $(\alpha + \lambda_{3,9}^{(1)}\beta_1 + \lambda_{3,9}^{(2)}\beta_2)$ is an eigenvalue of $J_{\mathcal{G}}$ of multiplicity at least 2; every eigenvalue of $(\alpha + \lambda_{4,8}^{(1)}\beta_1 + \lambda_{4,8}^{(2)}\beta_2)$ is an eigenvalue of $J_{\mathcal{G}}$ of multiplicity at least 2.

5. Conclusions. Interior symmetry may be viewed as an appropriate generalization of symmetry for coupled cell networks. Besides the original concept of interior symmetry, we introduced further notions including quotient interior symmetry, reverse interior symmetry, and quotient reverse interior symmetry.

For homogeneous coupled cell systems, we analyzed how multiple eigenvalues of the Jacobian at fully synchronized equilibria may occur due to these different types of interior symme-

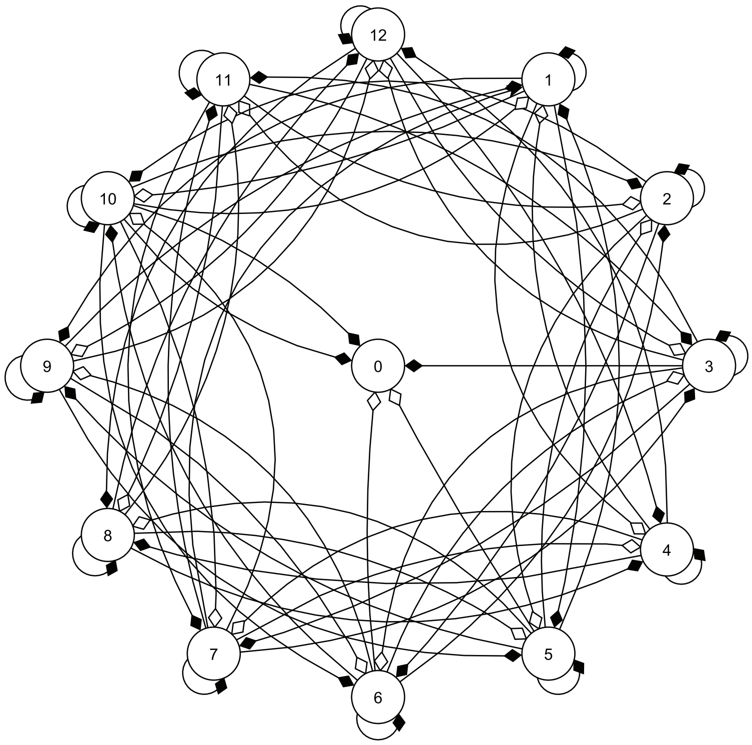

Figure 11. *The* 12*-cell homogeneous network* $\mathcal{G}$ *in Example* 4.4 *with interior symmetries* $\Sigma^1_{\mathcal{S}} = \langle \mathbf{D}_{12}, (1\ 5\ 9)(2\ 3\ 4\ 6\ 7\ 8\ 10\ 11\ 12)\rangle$ *and* $\Sigma^2_{\mathcal{S}} = \langle \mathbf{D}_{12}, (4\ 10)(1\ 2\ 3\ 5\ 6\ 7\ 8\ 9\ 11\ 12)\rangle$ *for* $\mathcal{S} = \{1, \ldots, 12\}$.

try. The groups of interior symmetry that we focused on are symmetric groups, alternating groups, dihedral groups, cyclic groups, and their products.

Based on our analysis, we concluded that the eigenvalue multiplicity of the Jacobian is sensitively dependent on the interior symmetric properties of the underlying network structure, and that symmetry alone is not sufficient to explain this dependence.

Indeed, in the examples we present throughout the paper, all the multiple eigenvalues are a consequence of an interior symmetry, in one form or another. In the case of uniform networks, even a relative weak interior symmetry may be sufficient to give rise to multiple eigenvalues.

Since, very easily, a homogeneous network has some type of interior symmetry, we can say that multiple eigenvalues of the Jacobian at a fully synchronous equilibrium are frequent for homogeneous coupled cell systems.

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