Interior Symmetries and Multiple Eigenvalues for Homogeneous Networks

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Abstract

We analyze the impact of interior symmetries on the multiplicity of the eigenvalues of the Jacobian matrix at a fully synchronous equilibrium for the coupled cell systems associated to homogeneous networks. We consider also the special cases of regular and uniform networks. It follows from our results that the interior symmetries, as well as the reverse interior symmetries and quotient interior symmetries, of the network force the existence of eigenvalues with algebraic multiplicity greater than one. The proofs are based on the special form of the adjacency matrices of the networks induced by those interior symmetries.

Keywords: Coupled systems, interior symmetry, multiple eigenvalues.

1 Introduction

Coupled cell systems model applications and real phenomena consisting of a set of individual dynamical systems (the cells) that are coupled together through interactions. This is transversal to a wide variety of areas. One of the most important aspects from the point of view of the applications is the bifurcation scenario. That is why bifurcation theory is such an important subject of study in dynamical systems. If the critical eigenvalue at bifurcation is simple, then the bifurcation analysis is straightforward. However, when the degenerescency condition is associated to a multiple eigenvalue (in the presence of symmetry, for example), the center subspace has higher dimension and the bifurcation problem turns to be highly degenerate. Thus, the issue of the multiplicity of the critical eigenvalue is crucial in the bifurcation analysis process. Moreover, knowing the causes for the existence of eigenvalues of high multiplicity can help find techniques that facilitate the analysis of the bifurcation problem, as in the case

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Coupled cell systems can be represented by a directed graph, a coupled cell network, whose nodes correspond to cells and whose edges represent couplings. Two cells are called identical, if they have the same phase space and the same internal dynamics (cf. Golubitsky et al. [6]). Here we assume that the internal dynamics of a cell is modeled by a system of ordinary differential equations. In this paper, we mainly discuss homogeneous networks, which are coupled cell networks of identical cells that are identically coupled. A homogeneous network is called regular, if all the couplings (arrows or edges) are of the same type. The valency of a homogeneous network is the number of arrows that input to each cell. Alternatively, the architecture of a homogeneous network with $s$ edge-equivalence classes can be given by $s$ adjacency matrices $A_1, A_2, \ldots, A_s$. More precisely, the $l$-th adjacency matrix $A_l$ of an $n$-cell homogeneous network $G$ is an $n \times n$ matrix, whose $(i, j)$-entry equals to the number of the $l$-th type arrows directing from cell $j$ to cell $i$.

An important feature of homogeneous networks is that the diagonal subspace $\Delta$, formed by setting all cell coordinates equal in the total phase space, is always flow-invariant by admissible coupled cell systems. Moreover, the restriction of these systems to $\Delta$ gives the set of all vector fields on $\Delta$ (cf. Theorem 5.2 of Golubitsky et al. [10]). Assume that a homogeneous cell system admits a fully synchronous equilibrium in $\Delta$. We say that the system undergoes a local synchrony-breaking steady-state bifurcation, if the synchronous equilibrium loses its stability and bifurcates to a steady state with less synchrony, as a bifurcation parameter crosses certain critical value. If it bifurcates to a periodic state with less synchrony, we call it a local synchrony-breaking Hopf bifurcation. As a consequence of implicit function theorem, a steady-state bifurcation only occurs if some real eigenvalues of the Jacobian at the equilibrium become zero, as the bifurcation parameter is varied. In the case of Hopf bifurcation, it is related to a pair of purely imaginary eigenvalues. Synchrony-breaking bifurcations can be considered as a generalization of symmetry-breaking bifurcations in equivariant coupled cell systems. Despite of this analogy, the bifurcation theory for coupled cell systems differs from symmetric bifurcation theory (cf. Golubitsky et al. [6], Leite et al. [13] and Golubitsky et al. [5]). This happens even in the case where the critical eigenvalue is real and simple, as shown in Stewart et al. [17].

It is known that in general (non-symmetric non-coupled) systems, steady-state or Hopf bifurcations occur at simple eigenvalues under generic conditions (cf. Golubitsky et al. [7]). It is also known that in symmetric systems, although multiple eigenvalues can occur generically, they only appear generically as a result of a real absolutely irreducible action by the symmetry group for steady-state bifurcations; for Hopf bifurcations, they are related to a complex irreducible action (cf. Golubitsky et al. [8]). In coupled systems, the underlying network structure (which is generally non-symmetric) can also force multiple eigenvalues in a generic manner and it determines, even at linear level, the kind of generic transitions from a synchronous equilibrium that can occur as the parameter is varied (cf. Leite et al. [13]). It can be observed from the analysis of the coupled cell bifurcations in Leite et al. [13] and Aguiar et al. [1] that most of the cases with multiple eigenvalues correspond to interior symmetric networks. In the remaining cases, the multiple eigenvalue is associated to the eigenvalue zero of the adjacency matrix of the network. Following Antoneli et al. [2], a network $G$ has an interior symmetry on a subset $S$ of cells, if $S$ together with all the arrows directed to it form a subnetwork that has a nontrivial symmetry. In the case $S$ is the total set of cells, the interior symmetry becomes a symmetry. Since interior symmetry seems to be a satisfying explanation to the occurrence
of multiple eigenvalues in coupled systems, we explore this notion in this paper. Of course, multiple eigenvalues can occur for coupled cell systems without any interior symmetry, but that seems to be nongeneric, unless in the special case of an eigenvalue corresponding to the eigenvalue zero of the adjacency matrix. To understand the ‘abundance’ of the eigenvalue zero for the adjacency matrix of a network see Remarks 2.10 (b) and (c) in Aguiar et al. [1] and recall that colinear rows or columns of a matrix imply the eigenvalue zero. We remark that, if two cells in a network receive exactly the same connections from exactly the same cells, then the adjacency matrix of the network has two equal rows. Analogously, if two cells in a network send exactly the same connections to exactly the same cell, then the adjacency matrix of the network has two equal columns.

It is well known that multiple eigenvalues can lead to complicated bifurcating behavior of the system such as multiple bifurcations and secondary bifurcations (cf. Iooss et al. [12] for general systems, Golubitsky et al. [8] for symmetric systems, Leite et al. [13] Elmhirst et al. EG06 for coupled systems, Aguiar et al. [1] for coupled systems with quotient symmetry). In the presence of symmetry, generic behavior of bifurcating branches related to multiple eigenvalues can be analyzed using equivariant bifurcation theory (cf. Golubitsky et al. [8]). In the context of coupled cell systems, analogue of the equivariant branching lemma and the equivariant Hopf theorem has been established in Golubitsky et al. [6] and Antoneli et al. [2] for systems admitting interior symmetry. Linear theory of (regular) coupled cell networks, was discussed in Golubitsky et al. [5]. It was shown that the linearized normal form at the bifurcation, if the dimension of the internal dynamics is at least 2, is generically isomorphic to the adjacency matrix restricted to one of its generalized eigenspaces; in the case of 1-dimensional internal dynamics, additional degeneracies may occur.

In [6], Golubitsky et al. show that interior symmetries induce extra structure on the form of the linearization at synchrony-breaking bifurcations. Here, we go further, and show how interior symmetries force the existence of multiple critical eigenvalues. As an example, in homogeneous networks, an interior symmetry ΣS on a subset S of k cells such that Dk ⊆ ΣS ⊆ Sk always forces multiple eigenvalues (cf. Theorem 4.2 and Theorem 4.3). The main reason why interior symmetry may lead to multiple eigenvalues is that it imposes restrictions on the network structure and thus on the form of adjacency matrices of the network. For example, an interior symmetry (ij) on the set of cells C = {1, ..., n} of a regular network G given by the permutation of cells i and j corresponds to the following constraints on the entries of the adjacency matrix A

\[ a_{ij} = a_{ji}, \quad a_{ij} = a_{ji} \quad \text{and} \quad a_{ik} = a_{jk}, \quad \text{for all } k \in C \setminus \{i, j\}. \]

More precisely, consider an n-cell homogeneous network G with s types of arrows, whose cell internal dynamics is r-dimensional. Assume without loss of generality that the synchronous equilibrium is at the origin. As shown for the case of regular networks in Leite et al. [13], the Jacobian of a homogeneous coupled cell system at a fully-synchronized equilibrium at the origin is determined by the cell internal dynamics and the adjacency matrices of different types of arrows. Let A_l, l = 1, 2, ..., s, be the adjacency matrix of the l-th type of arrows in G, i.e. A_l = [a_{ij}]_{1 \leq i, j \leq n} is a matrix whose entry a_{ij} is the number of the l-th type arrows connecting cell j to i. Let α be the linearized internal dynamics at the origin, β_l be the linearized internal coupling at the origin with the l-th type of input, for l = 1, 2, ..., s. Note that α and β_l are r × r matrices. Then, the Jacobian at the origin is of the form

\[ J_G = \alpha \otimes I_n + \beta_1 \otimes A_1 + \cdots + \beta_s \otimes A_s. \]
Results in Leite et al. [13] and Aguiar et al. [1] showed that when $G$ is a regular network (the case $s = 1$), the eigenvalues of $J_G$ are the union of the eigenvalues of the $r \times r$ matrices $\alpha + \mu_j \beta$, for $j = 1, \ldots, n$, including algebraic multiplicity, where $\mu_1, \ldots, \mu_n$ denote the eigenvalues of the adjacency matrix $A_G := \hat{A}_1$. Thus, the problem of multiple eigenvalues of the Jacobian is reduced to that of the adjacency matrix. On the contrary, if $s > 1$, it is unclear how the spectrum of $J_G$ and that of the $A_i$'s are related (cf. Golubitsky et al. [5] for product networks of two regular networks). However, as we will see, interior symmetry imposes a “universal” constraint on the form of $A_i$'s so that multiple eigenvalues of $J_G$ can be related with those of the $A_i$'s, if the interior symmetry $\Sigma_S$ is at least $D_k$ (cf. Theorem 4.3).

In the case of “smaller” interior symmetry, that is $D_k \not\subset \Sigma_S$, we obtain partial results for regular uniform networks (cf. Corollary 3.14 and Corollary 3.18). Following Stewart [16], we say that a network is uniform, if it has no multiple arrows or self-couplings. That is, adjacency matrices of uniform networks are composed of 0’s and 1’s. On the other hand, interior symmetry forces integer eigenvalues (directly related to the entries) of adjacency matrices of regular networks (cf. Theorem 3.13 and Theorem 3.17) and it is known that all eigenvalues $\lambda$ of $A_G$ of valency $v$ satisfy $||\lambda|| \leq v$. Thus, interior symmetry has an ever stronger impact on the multiplicity of eigenvalues for $A_G$ for regular uniform networks.

We define other variations of interior symmetry in a network such as reverse interior symmetry and quotient interior symmetry, which may also result in multiple eigenvalues for the Jacobian at the origin of the corresponding coupled cell systems. A reverse interior symmetry is an interior symmetry of the reverse network, where the direction of arrows of $G$ is reversed. A quotient interior symmetry is a short-hand notion of an interior symmetry of a quotient network of $G$, which is obtained by restricting $G$ to a balanced equivalence relation on the cells. If a quotient network has a reverse interior symmetry, then we call this symmetry a quotient reverse interior symmetry. All results obtained in this paper about interior symmetry can be easily extended to the above mentioned variations of interior symmetry (cf. Remark 3.1).

The paper is organized as follows. Section 2 collects preliminary definitions and results from coupled cell networks including definitions of various interior symmetries and some results from linear algebra. In Section 3, we discuss the case of regular networks for several important interior symmetries, such as the cyclic group $Z_k$, the dihedral group $D_k$, the alternating group $A_k$ and the symmetric group $S_k$. Using Theorem 3.3 and Corollary 3.4, we can get results on multiplicity of eigenvalues for interior symmetry groups given by products of these groups. The case of regular uniform networks is discussed in Subsection 3.6 for $Z_2 \times \cdots \times Z_2$- and $V_4$-interior symmetry. In Section 4, we extend the results obtained in Section 3 to homogeneous networks. We give some concluding remarks in Section 5. Throughout the paper, numerous examples will be used to illustrate the results.

## 2 Preliminaries

In this section, we summarize necessary concepts from coupled cell networks. We restrict our attention to homogeneous coupled cell networks since they are our main case of study. For more general definitions and results on coupled cell networks, we refer to Golubitsky et al. [9], Golubitsky et al. [10] and references therein.

**Definition 2.1** A **coupled cell network** consists of a finite set $C = \{1, \cdots, n\}$ of nodes or cells and a finite set $E = \{(c, d) : c, d \in C\}$ of edges or arrows and two equivalence relations, $\sim_C$ on cells
in \( \mathcal{C} \) and \( \sim_E \) on edges in \( \mathcal{E} \), with the consistency condition: if \( e_1 \sim_E e_2 \), for \( e_1 = (c_1, d_1) \in E \) and \( e_2 = (c_2, d_2) \in E \), then \( c_1 \sim_C c_2 \) and \( d_1 \sim_C d_2 \). We write \( \mathcal{G} = (\mathcal{C}, \mathcal{E}, \sim_C, \sim_E) \).

For an edge \( e = (c,d) \in \mathcal{E} \), \( c \) is called the head cell and \( d \) is called the tail cell; and \( e \) is called an input edge of \( c \). The set of all input edges of \( c \) is called the input set of \( c \) and denoted by \( I(c) \). Two cells \( c \) and \( d \) in a network are said to be input-equivalent, if there is an edge-type preserving isomorphism \( \beta : I(c) \rightarrow I(d) \) between their input sets. Note that the relation of input-equivalence refines the relation of cell-equivalence.

**Definition 2.2** A homogeneous network is a coupled cell network with only one input-equivalence class. A regular network is a homogeneous network with only one edge-equivalence class. It follows that in a homogeneous network all cells are of identical type and receive the same number (per type) of input edges. This number, which is the cardinality of the input set, is called the valency of the network.

**Example 2.3** Consider a 5-cell homogeneous network \( \mathcal{G} \) with 2 types of arrows and valency 4, which is shown in Figure 1. This network will be repeatedly referred to by later examples (cf. Example 2.4, 2.8, 2.11, 2.14, 2.18). Let \( A_1 \) (resp. \( A_2 \)) be the adjacency matrix of the arrows with solid (resp. hollow) arrow head. Then,

\[
A_1 = \begin{bmatrix}
0 & 1 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 1 \\
0 & 0 & 1 & 1 & 0
\end{bmatrix}, \quad A_2 = \begin{bmatrix}
1 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 1 & 0 \\
0 & 1 & 0 & 1 & 0 \\
0 & 1 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 1
\end{bmatrix}.
\quad (2.1)

**Example 2.4** Consider the two subnetworks \( \mathcal{G}_1, \mathcal{G}_2 \) obtained from the network \( \mathcal{G} \) in Example 2.3, by only keeping all arrows with solid (resp. hollow) arrow head. Then, \( \mathcal{G}_1, \mathcal{G}_2 \) are regular networks, as shown in Figure 2, with the adjacency matrix given by \( A_1, A_2 \) respectively. (cf. (2.1)).

Notice that \( \mathcal{G}_1 \) is an example of a uniform network, while \( \mathcal{G}_2 \) is not.

We follow the multiarrow formalism in Golubitsky et al. [10] and thus allow multiple arrows of the same type between two cells and self-coupling arrows. We call the networks without multiple arrows nor self-coupling arrows uniform networks (cf. Stewart [16]).
2.1 Symmetry and symmetric groups

We adapt and simplify the definition of a symmetry of a general coupled cell network in Antoneli et al. [3] to a symmetry of a homogeneous network.

Definition 2.5 Let \( G = (C, E, \sim_C, \sim_E) \) be a homogeneous network. A symmetry of \( G \) is a permutation \( \sigma \) on \( C \) such that there is a bijection between the edges \((a, b)\) and \((\sigma(a), \sigma(b))\), which preserves the edge-equivalence relation \( \sim_E \), for all \( a, b \in C \).

Let \( G \) be an \( n \)-cell homogeneous network with \( s \) edge-equivalence classes, whose adjacency matrices are given by \( A_1, A_2, \ldots, A_s \). Write \( A_l = [a_{ij}^{(l)}]_{n \times n} \), for \( l = 1, 2, \ldots, s \). Then, a permutation \( \sigma \) is a symmetry of \( G \), if and only if

\[
a_{ij}^{(l)} = a_{\sigma(i)\sigma(j)}^{(l)}, \quad \forall i, j = 1, 2, \ldots, n, \ l = 1, 2, \ldots, s.
\]

It is clear that the set of all symmetries of an \( n \)-cell homogeneous network \( G \) forms a group, which can be identified canonically with a subgroup of the symmetric group \( S_n \), that is defined as the group of all permutations of \( n \) symbols. Let \( i_1, \ldots, i_k \in \mathbb{N} \) be distinct positive integers. We use the standard notation \((i_1 \ldots i_k)\) to denote a \( k \)-cycle in \( S_n \), which is a permutation \( \sigma \) defined by

\[
\sigma : \quad i_j \mapsto i_{j+1} \quad \text{for } j = 1, \ldots, k - 1, \\
i_k \mapsto i_1 \\
1 \mapsto l \quad \text{for } l \notin \{i_1, \ldots, i_k\}.
\]

A 2-cycle is called a transposition. Every permutation can be written as a product of simple transpositions. A permutation is called even (resp. odd), if it can be expressed as a product of an even (resp. odd) number of transpositions. The subset of \( S_n \) consisting of all even permutations is a subgroup called the alternating group \( A_n \). A group generated by permutations \( \sigma_1, \sigma_2, \ldots, \sigma_m \) will be denoted by \( \langle \sigma_1, \sigma_2, \ldots, \sigma_m \rangle \).

Example 2.6 Consider the \( k \)-cycle \((1 \ 2 \ \ldots \ k)\) in \( S_k \) and the cyclic group

\[
\mathbb{Z}_k = \langle (1 \ 2 \ \ldots \ k) \rangle
\]

generated by the \( k \)-cycle. Let \( G \) be a \( \mathbb{Z}_k \)-symmetric homogeneous network of \( k \) cells and
\(A_1, A_2, \ldots, A_s\) be the adjacency matrices of \(G\). Then, every \(A_l\) is of the form

\[
\begin{bmatrix}
  a_{11} & a_{12} & \cdots & a_{1,k-1} & a_{1k} \\
  a_{1k} & a_{11} & a_{12} & \cdots & a_{1,k-1} \\
  a_{1,k-1} & a_{1k} & a_{11} & \ddots & a_{1,k-2} \\
  \vdots & \ddots & \ddots & \ddots & \vdots \\
  a_{12} & a_{13} & \cdots & a_{1k} & a_{11}
\end{bmatrix},
\] (2.2)

where every row vector is obtained by shifting the preceding row vector to the right by one element.

A matrix of the form (2.2) is called a circulant matrix, which is often written as

\[
\text{circ}(a_{11}, a_{12}, \ldots, a_{1k})
\]

for a shorthand. Circulant matrices and their spectral information are needed for our later discussions. It is known that all circulant matrices of the form (2.2) share the same eigenvectors

\[
v_j = (1, \omega_j, \omega_j^2, \ldots, \omega_j^{k-1}), \quad \text{for } \omega_j = e^{\frac{2\pi i}{k}}, \ j = 0, 1, \ldots, k - 1,
\] (2.3)

which are eigenvectors of the following eigenvalues

\[
\lambda_j = a_{11} + a_{12}\omega_j + a_{13}\omega_j^2 + \cdots + a_{1k}\omega_j^{k-1}, \ j = 0, 1, \ldots, k - 1.
\] (2.4)

Another concept that we will need later is that of a centrosymmetric matrix, which is a matrix that is symmetric about its center. More formally,

**Definition 2.7** A square matrix \(A = [a_{ij}]_{n \times n}\) is called centrosymmetric, if the following relation is satisfied

\[
a_{ij} = a_{(n+1-i)(n+1-j)}, \quad \forall \ i, j = 1, 2, \ldots, n,
\]

which is equivalent to the relation

\[
A = JAJ,
\]

where \(J = [e_{ij}]_{n \times n}\) is the exchange matrix; that is, \(e_{i,n+1-i} = 1\) and \(e_{ij} = 0\) for all \(j \neq n + 1 - i, i = 1, 2, \ldots, n\), i.e. it has 1 on the anti-diagonal and 0 elsewhere.

**Example 2.8** Consider the network \(G\) in Example 2.3. The symmetry group of \(G\) is

\[Z_2 = \langle(1\ 5)(2\ 4)\rangle.\]

The adjacency matrices \(A_l\)'s of any 5-cell homogeneous network having this symmetry are centrosymmetric matrices of the form

\[
\begin{bmatrix}
  a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\
  a_{21} & a_{22} & a_{23} & a_{24} & a_{25} \\
  a_{31} & a_{32} & a_{33} & a_{34} & a_{35} \\
  a_{41} & a_{42} & a_{43} & a_{44} & a_{45} \\
  a_{51} & a_{52} & a_{53} & a_{54} & a_{55}
\end{bmatrix}.
\]
2.2 Interior symmetry

The concept of interior symmetry of a coupled cell network is a generalized notion of a symmetry of a coupled cell network. Roughly speaking, it is a permutation of the cells that preserves certain amount of input structure. The notion of interior symmetry was first introduced by Golubitsky et al. [6]. We adapt and simplify the definition in [6] to define an interior symmetry of a homogeneous network as follows.

**Definition 2.9** Let $G = (C, \mathcal{E}, \sim_C, \sim_E)$ be a homogeneous network. Let $S \subseteq C$ be a subset. An interior symmetry of $G$ on $S$ is a permutation $\sigma$ on $C$ such that $\sigma$ fixes every element in $C \setminus S$, and there is a bijection between edges $(\sigma(a), \sigma(b))$ and $(a, b)$, which preserves edge-equivalence relation $\sim_E$, for $a \in S$, $b \in C$.

Note that in the case $S = C$, an interior symmetry on $C$ is precisely a symmetry of $G$.

Let $G$ be an $n$-cell homogeneous network with $s$ edge-equivalence classes, whose adjacency matrices are given by $A_1, A_2, \ldots, A_s$. Write $A_l = [a_{ij}^{(l)}]_{n \times n}$, for $l = 1, \ldots, s$. Then, a permutation $\sigma$ is an interior symmetry of $G$ on $S$, if and only if

$$a_{ij}^{(l)} = a_{\sigma(i)\sigma(j)}^{(l)} \quad \forall i \in S, \ j \in C, \ l = 1, \ldots, s. \tag{2.5}$$

Following the formulation in Antoneli et al. [2], one can characterize the interior symmetry using symmetry of subnetworks. Given a network $G$ and a subset $S \subseteq C$, define $G_S = (C, I(S), \sim_C, \sim_E)$ to be the subnetwork of $G$, whose set of cells is $C$ (together with its cell-equivalence relation $\sim_C$) and whose set of arrows is the input set $I(S)$ of $S$. The edge-equivalence relation on $I(S)$ is given by the restriction of the edge-equivalence $\sim_E$ of $\mathcal{E}$ to $I(S)$.

**Proposition 2.10** (cf. [2]) Let $G$ be a coupled cell network and $S \subseteq C$ be a subset of cells of the set of cells of $G$. Consider the network $G_S$ as defined above. Then the group of interior symmetries of the network $G$ on $S$ can be canonically identified with the group of symmetries of the network $G_S$.

**Example 2.11** Consider the homogeneous network $G$ in Example 2.3. Let $S = \{2, 3, 4\}$. Then, the network $G_S$ has an $S_3$-symmetry, as shown in Figure 3. Thus, $G$ has an interior symmetry $S_3$ on $S$.

![Figure 3: An $S_3$-symmetric network $G_S$ for $S = \{2, 3, 4\}$.

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Indeed, adjacency matrices $A_i$’s of any 5-cell homogeneous networks with $S_3$ interior symmetry on $S = \{2, 3, 4\}$ are of form
\[
\begin{bmatrix}
  a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\
  a_{21} & a_{22} & a_{23} & a_{24} & a_{25} \\
  a_{21} & a_{23} & a_{22} & a_{23} & a_{25} \\
  a_{21} & a_{23} & a_{23} & a_{22} & a_{25} \\
  a_{51} & a_{52} & a_{53} & a_{54} & a_{55}
\end{bmatrix}.
\]

\[
\Box
\]

### 2.3 Reverse interior symmetry

We introduce a new concept of symmetry for coupled cell networks, the reverse interior symmetry. To do so, we need the notion of the reverse network $G^R$ of a coupled cell network $G$, which is a network defined on the same set of cells, but with all the edges in the reversed direction.

**Definition 2.12** Let $G = (C, E, \sim_C, \sim_E)$ be a coupled cell network. Define
\[
E^R := \{(d, c) : (c, d) \in E\}.
\]
and an equivalence relation $\sim_{E^R}$ on $E^R$ by
\[
(b, a) \sim_{E^R} (d, c) \iff (a, b) \sim_E (c, d).
\]
The reverse network $G^R$ of $G$ is the network given by $G^R = (C, E^R, \sim_C, \sim_{E^R})$.

Note that the adjacency matrices of $G^R$ are given by the transpose of the adjacency matrices of $G$. Also, a reverse network of a homogeneous (resp. regular) network may not be homogeneous (resp. regular) again.

**Definition 2.13** Let $G = (C, E, \sim_C, \sim_E)$ be a coupled cell network and $G^R$ be its reverse network. Let $S \subseteq C$ be a subset. A permutation $\sigma$ is called a reverse interior symmetry of $G$ on $S$, if $\sigma$ is an interior symmetry of $G^R$ on $S$.

That is, the group of reverse interior symmetries of $G$ on $S$ can be canonically identified with the group of interior symmetries of $G^R$ on $S$. Roughly speaking, a reverse interior symmetry is a permutation of the cells that preserves certain amount of output structure.

Let $G$ be a homogeneous network with $s$ type of arrows whose adjacency matrices are $A_1, A_2, \ldots, A_s$. Then, a permutation $\sigma$ is a reverse interior symmetry of $G$ on $S$ if and only if
\[
a_{ij} = a_{\sigma(i)\sigma(j)} \quad \forall i \in C \text{ and } \forall j \in S,
\]
for $l = 1, \ldots, s$.

**Example 2.14** Consider the homogeneous network $G$ in Example 2.3. Then, the reverse network $G^R$ is as shown in Figure 4. It can be verified that $G^R$ has an interior symmetry (15) on $S = \{1, 5\}$. Thus, (15) is a reverse interior symmetry of $G$.

Note that a symmetry of a coupled cell network $G$ is both an interior symmetry and a reverse interior symmetry of $G$, but the reverse may not be true.

**Example 2.15** Consider the two networks in Figure 5, which are reverse to each other. Both networks have $S_3$ as an interior symmetry on $S = \{1, 2, 3\}$, thus $S_3$ is a reverse interior symmetry of both networks on $S$. However, neither network has an $S_3$-symmetry.
2.4 Balanced equivalence relation

Given an equivalence relation \( \bowtie \) on the set of cells of a coupled cell network, we can color the nodes of the network in the following way: two cells \( i, j \) receive the same color precisely when they belong to the same \( \bowtie \)-equivalence class. The coloring is called balanced, or equivalently \( \bowtie \) is called a balanced equivalence relation, if any pair of cells with the same color have the same number and type of input arrows from cells of color \( b \), for every \( b \).

More formally,

**Definition 2.16** (cf. [10]) Given a coupled cell network \( \mathcal{G} = (C, \mathcal{E}, \sim_C, \sim_E) \), an equivalence relation \( \bowtie \) on the set \( C \) is called balanced, if for every \( c, d \in C \) with \( c \bowtie d \), there exists a bijection \( \beta : I(c) \to I(d) \) between their input sets, which preserves the edge-equivalence relation \( \sim_E \), and such that for all \( i \in I(c) \), the tail cells of \( i \) and \( \beta(i) \) are in the same \( \bowtie \)-class.

The next proposition states that every interior symmetry permutation determines a balanced equivalence relation.

**Proposition 2.17** Let \( \mathcal{G} \) be an \( n \)-cell homogeneous network and \( \sigma \) be an interior symmetry of \( \mathcal{G} \) on a subset \( S \subseteq C \). If \( \bowtie \) is an equivalence relation on the cells \( C \) of \( \mathcal{G} \) such that

\[
c \bowtie d \iff c, d \text{ belong to the same orbit under } \sigma,
\]

then \( \bowtie \) is balanced.
Let \( c, d \) be such that \( c \rightsquigarrow d \). Then, \( \sigma^m(c) = d \) for some \( m \in \mathbb{N} \). Note that \( \sigma^m \) is an interior symmetry of \( G \) on \( S \), for all \( m \in \mathbb{N} \). Thus, by the definition of interior symmetry, there exists an edge-equivalence preserving bijection between the edges \((\sigma^m(c), \sigma^m(x))\) and \((c, x)\), for every input arrows \((c, x)\). Thus, there exists a bijection between the input sets of \( d = \sigma^m(c) \) and \( c \), which preserves the edge-equivalence relation. On the other hand, the tail cells \( x \) and \( \sigma^m(x) \) are in the same orbit by \( \sigma \), thus are in the same \( \rightsquigarrow \)-class. Therefore, \( \rightsquigarrow \) is a balanced equivalence relation. \( \blacksquare \)

Let \( \Sigma_S \) be the group of all interior symmetries of \( G \) on a subset \( S \subseteq C \). Let \( K \subseteq \Sigma_S \) be a subgroup. By Proposition 2.17, every permutation in \( K \) determines a balanced equivalence relation on \( G \). In fact, the set of all these equivalence relations forms a sublattice of the total lattice of balanced equivalence relations on \( G \) (cf. Stewart [15]). Moreover, the balanced equivalence relation \( \rightsquigarrow_K \) determined by the subgroup \( K \) is given by the join of all the equivalence relations determined by permutations in \( K \) and corresponds to the top element of this sublattice.

### 2.5 Quotient networks and quotient interior symmetry

Given a balanced equivalence relation \( \rightsquigarrow \) on a coupled cell network \( G \), a quotient network \( G_{\rightsquigarrow} = (C_{\rightsquigarrow}, E_{\rightsquigarrow}, \sim_{C_{\rightsquigaro}}, \sim_{E_{\rightsquigaro}}) \) can be defined naturally as follows: the cells in \( C_{\rightsquigaro} \) are the \( \rightsquigarrow \)-equivalence classes of the cells of \( G \) and the edges in \( E_{\rightsquigaro} \) from quotient cell \([c]_{\rightsquigaro}\) to quotient cell \([d]_{\rightsquigaro}\), where \([c]_{\rightsquigaro}\) denotes the \( \rightsquigarrow \)-equivalence class of \( c \), are in correspondence with the edges \((c', d')\) of \( G \), for all \( c' \rightsquigarrow c, d' \rightsquigarrow d \). The cell-equivalence \( \sim_{C_{\rightsquigaro}} \) and edge-equivalence \( \sim_{E_{\rightsquigaro}} \) relations for \( G_{\rightsquigaro} \) are induced from those of \( G \). Since \( \rightsquigarrow \) is balanced, the quotient network \( G_{\rightsquigaro} \) is well-defined. See Golubitsky et al. [10].

Let \( G \) be a homogeneous network of \( n \)-cells with \( s \) edge-equivalence classes whose adjacency matrices are \( A_1, A_2, \ldots, A_s \). Let \( \rightsquigarrow \) be a balanced equivalence relation, which divides the cells of \( G \) into \( p \) equivalence-classes. Then, \( G_{\rightsquigaro} \) is a homogeneous network of \( p \)-cells with \( s \) edge-equivalence classes. Denote the adjacency matrices of \( G_{\rightsquigaro} \) by \( A_{1_{\rightsquigaro}}, A_{2_{\rightsquigaro}}, \ldots, A_{s_{\rightsquigaro}} \). Let \( A_{a_{\rightsquigaro}} = [a^{(l)}_{ik}]_{p \times p} \). Then, for \( \alpha = [i]_{\rightsquigaro}, \beta = [j]_{\rightsquigaro} \) in \( C_{\rightsquigaro} \), we have (cf. Proposition 2.3, [1])

\[
a^{(l)}_{\alpha \beta} = \sum_{k \in [j]_{\rightsquigaro}} a^{(l)}_{ik}.
\]

**Example 2.18** Let \( G \) be the homogeneous network in Example 2.3. As shown in Example 2.8 and Example 2.11, \( G \) has a symmetry \( Z_2 = \langle (15)(24) \rangle \) and an interior symmetry \( S_3 \) on \( S = \{2, 3, 4\} \). Consider \( \rightsquigarrow_1 = \langle (1), (2, 3, 4), [5] \rangle \) and \( \rightsquigarrow_2 = \langle (1, 5), (2, 4), [3] \rangle \). As seen in Subsection 2.4, both \( \rightsquigarrow_1, \rightsquigarrow_2 \) are balanced equivalence relations on \( G \). Let \( G_1 \) (resp. \( G_2 \)) be the quotient network induced by \( \rightsquigarrow_1 \) (resp. \( \rightsquigarrow_2 \)). Then, the adjacency matrices of \( G_1 \) are

\[
A_{1_{\rightsquigaro}} = \begin{bmatrix} 0 & 2 & 0 \\ 1 & 0 & 1 \\ 0 & 2 & 0 \end{bmatrix}, \quad A_{2_{\rightsquigaro}} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 1 & 1 \end{bmatrix}
\]

and the adjacency matrices of \( G_2 \) are

\[
A_{1_{\rightsquigaro}} = \begin{bmatrix} 0 & 1 & 1 \\ 2 & 0 & 0 \\ 2 & 0 & 0 \end{bmatrix}, \quad A_{2_{\rightsquigaro}} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 2 & 0 \end{bmatrix}
\]

The networks \( G_1, G_2 \) are shown in Figure 6.
Figure 6: Quotient networks for $G$ in Figure 1 given by the $S_3$-interior symmetry (left) and $\mathbb{Z}_2 = \langle (15)(24) \rangle$-symmetry (right).

Note that a quotient network of a uniform network is a regular network which may not be uniform in general.

One can also consider interior symmetry and reverse interior symmetry of quotient networks.

**Definition 2.19** Let $G$ be a coupled cell network. We say that a permutation $\sigma$ is a quotient (interior) symmetry of $G$, if $G$ has a quotient network $G_{\bowtie_1}$ which has $\sigma$ as an (interior) symmetry, for some balanced equivalence relation $\bowtie_1$. Similarly, we say that a permutation $\gamma$ is a quotient reverse (interior) symmetry of $G$, if $G$ has a quotient network $G_{\bowtie_2}$ which has $\gamma$ as a reverse (interior) symmetry, for some balanced equivalence relation $\bowtie_2$.

**Example 2.20** Based on Example 2.18, we conclude that the homogeneous network in Figure 1 has a quotient symmetry $\langle (15) \rangle$, since $G_1$ is symmetric with respect to $(15)$ in Figure 6 (left).

In many cases, symmetric properties of the total network may be inherited by quotient networks. Yet, the following example shows that there is no definite relation between the (interior) symmetry of the total network and the (interior) symmetry of its quotient networks.

**Example 2.21** Consider the three-cell bidirectional ring pictured in Figure 7(left) which is $S_3$-symmetric and whose quotient networks have no symmetry nor interior symmetry.

Consider the six-cell regular network in Figure 7(right). It can be verified that it has no nontrivial symmetry nor interior symmetry, but it quotients to the three-cell bidirectional ring, for the balanced equivalence relation $\{\{1, 2\}, \{3, 4\}, \{5, 6\}\}$.
However, networks that quotient to (interior) symmetric networks tend to have (interior) symmetry. Examples are five-cell networks given by Figures 8, 9, and 10, all of which have a quotient network which is isomorphic to the $S_3$-symmetric network in Figure 7(left), for the balanced equivalence relation $\{\{1\}, \{2, 3\}, \{4, 5\}\}$. At the same time, they all have interior symmetries. More examples of this kind can be found in Aguiar et al. [1], where all the five-cell regular networks admitting the three-cell bidirectional ring as a quotient network are listed.

2.6 Direct sum decomposition of $\mathbb{R}^n$

Let $G$ be an $n$-cell homogeneous network with adjacency matrices $A_1, A_2, \ldots, A_s$ and $\bowtie$ be a balanced equivalence relation on $G$. As seen in the previous subsection, there is an associated quotient network $G_{\bowtie}$ whose adjacency matrices are given by $A_1^{\bowtie}, A_2^{\bowtie}, \ldots, A_s^{\bowtie}$ (cf. (2.6)). Based on results on regular networks (cf. Section 4 of Golubitsky et al. [6]), one can show that $\bowtie$ induces a direct sum decomposition of $\mathbb{R}^n$ such that every $A_l$ has a form of block matrix containing $A_l^{\bowtie}$, for $l = 1, 2, \ldots, s$ (cf. Theorem 2.9 in Aguiar et al. [1] for regular networks).

More precisely, given a balanced equivalence relation $\bowtie$, define

\[ \Delta_{\bowtie}(\mathbb{R}^n) = \{ x \in \mathbb{R}^n : x_c = x_d \text{ if } c \bowtie d, \forall c, d \in C \}, \]

which is a linear subspace of $\mathbb{R}^n$. Then, $\Delta_{\bowtie}(\mathbb{R}^n)$ is $A_l$-invariant, for every $l = 1, 2, \ldots, s$, since $\bowtie$ is balanced (cf. Theorem 4.3 in Golubitsky et al. [10]). Let $I_1, \ldots, I_p$ be the $\bowtie$-equivalence classes of order greater than one and $I = \bigcup_{l=1}^p I_l$. Define

\[ W = \{ x \in \mathbb{R}^n : x_j = 0 \text{ for } j \notin I \text{ and } \sum_{i \in I_l} x_i = 0 \text{ for } 1 \leq l \leq p \} \]

(2.7)

\[ U = \Delta_{\bowtie}(\mathbb{R}^n). \]

(2.8)

Note that if $\bowtie$ is defined by an interior symmetry $\sigma$ (cf. Subsection 2.4), then both $W$ and $U$ are $\sigma$-invariant subspaces. Since $W \cap U = \{0\}$, we can decompose $\mathbb{R}^n$ as a direct sum

\[ \mathbb{R}^n = W \oplus U. \]

(2.9)

Then, with respect to a basis adapted to (2.9), every adjacency matrix $A_l$ has a block form

\[ A_l = \begin{bmatrix} A & 0 \\ C & A_l^{\bowtie} \end{bmatrix}, \]

where $A_l^{\bowtie}$ is the $l$-th matrix of the quotient network $G_{\bowtie}$ associated to the balanced equivalence relation $\bowtie$.

3 Interior symmetries and multiple eigenvalues: regular networks

In what follows, we analyze how the interior symmetry of a homogeneous network may affect the multiplicity of eigenvalues of the Jacobian at a fully-synchronized equilibrium of the associated coupled cell system. In this section, we discuss the case of regular networks and in Section 4, we generalize the results to homogeneous networks.

Beyond the notion of interior symmetry introduced by Golubitsky et al. [6], we defined in Section 2 two further concepts of interior symmetry: the reverse interior symmetry, which is the interior symmetry of the reverse network and the quotient interior symmetry, which is the symmetry of a quotient network.
Remark 3.1 The results presented in the following two sections are stated for interior symmetry, but they can be easily extended for reverse interior symmetry and quotient interior symmetry. This follows from the fact that all the arguments we will use are based on the special form of the adjacency matrices of the networks, which is forced by interior symmetry. Since analogous form of adjacency matrices can be also induced by reverse interior symmetry and quotient interior symmetry, the results also apply to networks with reverse interior symmetry and quotient interior symmetry. More technically, note that each adjacency matrix \( A_l \), for \( l = 1, \ldots, s \), of a homogeneous network \( G \) corresponds to the transpose of the adjacency matrix \( A_l^R \) of the reverse network \( G^R \). Thus, the eigenvalues of \( A_l \) coincide with those of \( A_l^R \). Consequently, multiple eigenvalues of \( A_l \) may appear not only due to the interior symmetry of \( G \), but also due to its reverse interior symmetry. As seen in Subsection 2.6, for each quotient network \( G \triangleleft \bowtie \) there is a special basis such that each adjacency matrix \( A_l \), for \( l = 1, \ldots, s \), of \( G \) has a block lower-triangular form with the adjacency matrix \( A_l \triangleleft \bowtie \) of the quotient network at one of the diagonal blocks. Thus, the eigenvalues of the adjacency matrix \( A_l \triangleleft \bowtie \) of a quotient network \( G \triangleleft \bowtie \) are also eigenvalues of \( A_l \). Therefore, multiple eigenvalues of \( A_l \) may appear not only due to the interior symmetry of \( G \), but also due to its quotient interior symmetry.

In summary, from the results presented in the following two sections, it follows that the interior symmetries, reverse interior symmetries and quotient interior symmetries of regular and homogeneous networks favor multiple eigenvalues of the Jacobian matrix at a fully-synchronized equilibrium for the associated coupled cell systems.

Let \( G \) be an \( n \)-cell regular network with \( r \)-dimensional cell internal dynamics. Let \( \mu_1, \ldots, \mu_n \) be the eigenvalues of the adjacency matrix \( A_G \) of \( G \). As it is shown in Leite et al. [13] and Aguiar et al. [1], the eigenvalues of the Jacobian \( J_G \) of the associated coupled systems at a fully-synchronized equilibrium are the union of the eigenvalues of the \( r \times r \) matrices

\[
\alpha + \mu_j \beta, \quad \text{for } j = 1, \ldots, n
\]

including algebraic multiplicity.

Remark 3.2 It follows that if \( A_G \) has one eigenvalue with multiplicity \( m_a \), then \( J_G \) has \( r \) eigenvalues with multiplicity at least \( m_a \) (note that it can also happen that some of the \( r \) eigenvalues are equal).

As mentioned before, interior symmetry imposes restrictions on the network structure and thus on the entries of the adjacency matrix. By Remark 3.2, to analyze the effect of interior symmetries on the multiplicity of the eigenvalues of the Jacobian, it is sufficient to concentrate on the influence of interior symmetries on the multiplicity of the eigenvalues of \( A_G \).

As we will see, interior symmetries force the existence of integer eigenvalues for the adjacency matrix \( A_G \) of a regular network \( G \). Moreover, all the eigenvalues \( \lambda \) of \( A_G \) satisfy \( ||\lambda|| \leq v \), where \( v \) is the valency of \( G \). Thus, for regular networks with valency 2, the eigenvalues \(-1, 0\) and \( 1 \) will arise very often, in the presence of interior symmetry.

### 3.1 Product interior symmetry

We show that the case of product interior symmetries can be inferred from their component symmetries.
Let \( G \) be an \( n \)-cell regular network having interior symmetry groups \( \Sigma S_j \), for \( j = 1, \ldots, r \), on disjoint subsets \( S_j \) of cells of \( G \). We say that \( G \) has a product interior symmetry

\[
\Sigma S = \Sigma S_1 \times \cdots \times \Sigma S_r,
\]

where \( S = \bigcup_{j=1}^r S_j \). Let \( \triangleright_j \) be the balanced equivalence relation induced by \( \Sigma S_j \), for \( j = 1, \ldots, r \). Then, the balanced equivalence relation \( \triangleright \) induced by \( \Sigma S \) is given by

\[
c \triangleright d \iff c \triangleright_j d \quad \text{for some } j.
\] (3.10)

Set \( U = \Delta_n(\mathbb{R}^n) \). Let \( I_1^j, I_2^j, \ldots, I_p^j \) be the \( \triangleright_j \)-equivalence classes of order greater than one and \( I^j = \bigcup_{i=1}^{p^j} I_i^j \). Define

\[
W_j = \{ x \in \mathbb{R}^n : x_i = 0 \forall i \in C \setminus I^j \quad \text{and} \quad \sum_{i \in I^j} x_i = 0 \text{ for } 1 \leq i \leq p^j \}, \quad j = 1, 2, \ldots, r.
\]

Let \( I = \bigcup_{j=1}^r I^j \). Note that \( \dim W_j = |S_j| - p_j \), \( \dim U = |C| + \sum_{j=1}^r p_j \) and \( W_i \cap W_j = \{0\}, U \cap W_j = \{0\}, \) for \( i \neq j, j = 1, 2, \ldots, r \). Thus, we have

\[
\mathbb{R}^n = W_1 \oplus \cdots \oplus W_r \oplus U. \tag{3.11}
\]

**Theorem 3.3** Let \( G \) be an \( n \)-cell regular network having a product interior symmetry \( \Sigma S = \Sigma S_1 \times \cdots \times \Sigma S_r \) on disjoint subsets \( S_j \) of cells of \( G \). Then, with respect to the decomposition (3.11), the adjacency matrix \( A_G \) of \( G \) takes the form

\[
\begin{bmatrix}
A^1 & 0 & \cdots & 0 & 0 \\
0 & A^2 & \cdots & 0 & 0 \\
0 & 0 & \cdots & A^r & 0 \\
B_1 & B_2 & \cdots & B_r & A_{\Delta_n}
\end{bmatrix},
\]

where \( A^j \) is a matrix of order \( (|S_j| - p_j) \times (|S_j| - p_j) \) for \( j = 1, \ldots, r \), and \( A_{\Delta_n} \) is the adjacency matrix of the quotient network associated with \( \triangleright \) (cf. (3.10)).

**Proof** Let \( W \) be the linear subspace induced by \( \triangleright \) (cf. (2.7)). Note that \( W = W_1 \oplus \cdots \oplus W_r \). Then, as discussed in Section 2.6, with respect to the decomposition

\[
\mathbb{R}^n = W \oplus U,
\]

\( A_G \) takes the form

\[
\begin{bmatrix}
A & 0 \\
C & A_{\Delta_n}
\end{bmatrix}.
\]

It remains to show that \( A \) is a block matrix of diagonal form, with respect to the dimensions of the \( W_j \)'s, \( j = 1, \ldots, r \). Observe that to show that the entries of the \( j \)-th column of \( A \) are all zeros except those on the diagonal block, it is enough to show that \((W_j \oplus U)\) is \( A_G \)-invariant. Let \( x \in W_j \) for a \( j \in \{1, 2, \ldots, r\} \) and \( y = A_G x \). We need to show that \( y \in W_j \oplus U \), i.e.

\[
y_i = y_j, \quad \forall \; i \triangleright_k l, \quad \forall \; k \neq j.
\]
Consider the balanced equivalence relation \( \triangleleft \). Without loss of generality, we can assume \( i \triangleleft k \) for all \( i \neq k \). Thus, the value of \( y_i \) (resp. \( y_j \)) depends only on the \((i, m)\)-th (resp. \((l, m)\)-th) entries of \( A_G \), where \( m \in S_j \). When \( i \triangleleft k \) and \( k \neq j \), we have \( i, l \notin S_j \). Thus, the \((i, m)\)-th entry of \( A_G \) is equal to the \((l, m)\)-th entry of \( A_G \), for all \( m \in S_j \). It follows that \( y_i = y_j \) for all \( i \triangleleft k \), \( k \neq j \).

Therefore, we have \( A_G W_j \subseteq W_j \oplus U \). Combined with the fact \( A_G U \subseteq U \), we conclude that \((W_j \oplus U)\) is \( A_G \)-invariant, for \( j = 1, 2, \ldots, r \).

**Corollary 3.4** Under the assumptions of Theorem 3.3, we have that the set of eigenvalues of the adjacency matrix \( A_G \) of \( G \) is given by the disjoint union of the set of eigenvalues of \( A^* \) and the set of eigenvalues of \( A_{\omega_m} \), for \( j = 1, 2, \ldots, r \).

Taking into account Theorem 3.3 and Corollary 3.4, in what follows, we shall concentrate on interior symmetry groups that cannot be written as a product of subgroups. We will certainly not consider here all subgroups of \( S_n \), with this property, as the number of subgroups increases exponentially with \( n \) (cf. Holt [11] for an enumeration of subgroups and conjugacy classes of the subgroups of \( S_n \), for \( n \leq 18 \)).

In this paper, we will be primarily interested in the following subgroups of \( S_n \):

(i) the symmetric groups \( S_k = \langle (i_1 \ldots i_k), (i_1 i_2) \rangle \), with \( 2 \leq k \leq n \);

(ii) the alternating groups \( A_k \), with \( 2 \leq k \leq n \);

(iii) the dihedral groups \( D_k = \langle (i_1 \ldots i_k), (i_2 i_k)(i_3 i_{k-1}) \ldots (i_j i_{k+2-j}) \rangle \), with \( 2 \leq k \leq n \);

(iv) the cyclic groups \( Z_k = \langle (i_1 \ldots i_k) \rangle \), with \( 2 \leq k \leq n \).

Note that \( S_2 \cong D_2 \cong Z_2 \) and \( S_3 \cong D_3 \).

### 3.2 \( S_k \)- and \( A_k \)-Interior symmetry

We show that

**Theorem 3.5** Let \( G \) be an \( n \)-cell regular network having an interior symmetry group \( S_k \) or \( A_k \) on a subset \( S \subseteq C \) of \( k \) cells of \( G \), for \( 2 \leq k \leq n \). Let \( i \) and \( j \) be any two different cells in \( S \). Then, the adjacency matrix \( A_G = [a_{ij}]_{1 \leq i, j \leq n} \) of \( G \) has the eigenvalue \( a_{ij} - a_{ij} \) with algebraic multiplicity at least \( k - 1 \). As a result, the Jacobian \( J_G \) has \( r \) eigenvalues with algebraic multiplicity at least \( k - 1 \).

**Proof** Without loss of generality, we can assume \( S = \{1, \ldots, k\} \). First notice that for any \( i, j, l, m \in S \), the product \( (i j)(l m) \) of two transpositions is an element in \( A_k \subseteq S_k \). Since \( G \) has an interior symmetry \( S_k \) (resp. \( A_k \)), the entries of \( A_G \) satisfy (cf. (2.5))

\[
\begin{align*}
  a_{ii} &= a_{jj}, & i, j &\in S \\
  a_{ij} &= a_{ji}, & i, j, l, m &\in S \text{ with } i \neq l \text{ and } j \neq m \\
  a_{il} &= a_{jl}, & i, j, l &\in S \text{ and } l \in C \setminus S. 
\end{align*}
\]

Consider the balanced equivalence relation \( \approx \) induced by \( S_k \) (resp. \( A_k \))

\[
\approx = \{[1, 2, \ldots, k], [k + 1], \ldots, [n]\}.
\]
Let $W, U$ be given by (2.7)–(2.8). Then, with respect to (2.9), the adjacency matrix $A_G$ takes the form

$$
\begin{bmatrix}
A & 0 \\
C & A_{k}\end{bmatrix},
$$

where $A$ is a scalar matrix of order $(k-1)$ with the element $(a_{11} - a_{12})$ on the diagonal. Thus, the adjacency matrix $A_G$ has the eigenvalue $(a_{11} - a_{12})$ with algebraic multiplicity at least $(k-1)$. It follows from Remark 3.2 that the Jacobian $J_G$ has $r$ eigenvalues with algebraic multiplicity at least $k-1$.

\[\blacksquare\]

**Example 3.6** Let $G$ be a 5-cell regular network that quotients to the three-cell bidirectional ring $\mathcal{R}$ (cf. Figure 7). Examples of $G$ are networks given in Figures 8, 9 and 10. By Theorem 3.5, the adjacency matrix of $\mathcal{R}$ has $-1$ as an eigenvalue with algebraic multiplicity 2, as a result of the $S_3$ (interior) symmetry of $\mathcal{R}$. Thus, due to the $S_3$ quotient interior symmetry of $G$, the adjacency matrix of $G$ has $-1$ as an eigenvalue with algebraic multiplicity at least 2.

\[\blacksquare\]

### 3.3 D$_k$-Interior symmetry

We prove the following

**Theorem 3.7** Let $G$ be an $n$-cell regular network having an interior symmetry group $D_k$ for some $k \in \{3, \ldots, n\}$. Set

$$m = \begin{cases} (k-1)/2, & \text{if } k \text{ is odd}, \\ k/2, & \text{if } k \text{ is even}. \end{cases}$$

Then, the adjacency matrix $A_G = [a_{ij}]_{1 \leq i, j \leq n}$ of $G$ has $m$ eigenvalues with algebraic multiplicity at least 2, if $k$ is odd; $A_G$ has $(m-1)$ eigenvalues with algebraic multiplicity at least 2, if $k$ is even. As a result, if $k$ is odd (resp. even), then the Jacobian $J_G$ has $mr$ (resp. $(m-1)r$) eigenvalues with algebraic multiplicity at least 2.

The following lemma will be needed for the proof of Theorem 3.7.

**Lemma 3.8** Let $m \in \mathbb{N}$. Consider the following two matrices of order $m \times m$

$$B_1 = \begin{bmatrix}
a_{11} - a_{13} & a_{12} - a_{14} & a_{13} - a_{15} & \cdots & a_{1,m-1} - a_{1,m+1} & a_{1m} - a_{1,m+1} \\
a_{12} - a_{14} & a_{11} - a_{15} & a_{12} - a_{16} & \cdots & a_{1,m-2} - a_{1,m+1} & a_{1,m-1} - a_{1m} \\
a_{13} - a_{15} & a_{12} - a_{16} & a_{11} - a_{17} & \cdots & a_{1,m-3} - a_{1m} & a_{1,m-2} - a_{1,m-1} \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
a_{1,m-1} - a_{1,m+1} & a_{1,m-2} - a_{1,m+1} & a_{1,m-3} - a_{1m} & \cdots & a_{11} - a_{14} & a_{12} - a_{13} \\
a_{1m} - a_{1,m+1} & a_{1,m-1} - a_{1m} & a_{1,m-2} - a_{1,m-1} & \cdots & a_{12} - a_{13} & a_{11} - a_{12}
\end{bmatrix}$$

(3.12)
Then, $B_1$ and $B_2$ are similar.

**Proof** Notice that any matrix $M = (x_{ij})_{m \times n}$ is similar to the matrix $(x_{m-i+1,j})$ by exchanging rows $R_i$ with $R_{m-i+1}$ and exchanging columns $C_i$ with $C_{m-i+1}$ for $1 \leq i \leq m$. We will denote by $\tilde{B}_1$ the matrix obtained in this way from $B_1$.

For $r = 1, 2, \ldots, m - 1$, denote by $O_r$ the row operation

$$R_r \leadsto R_r + R_{r+1} + \cdots + R_m,$$

where the $r$-th row is replaced by the sum of the $j$-th row for $r \leq j \leq m$. It suffices to show that

$$O_{m-1}O_{m-2}\cdots O_2O_1B_2O_1^{-1}O_2^{-1}\cdots O_{m-2}O_{m-1}^{-1} = \tilde{B}_1.$$  \hspace{1cm} (3.14)

Write $B_2 = (b_{ij})_{m \times n}$ and denote by $C = (c_{ij})_{m \times n}$ the left hand side of (3.14). We first show that

$$c_{ij} = \begin{cases} 
\sum_{p=i}^m b_{p1}, & \text{if } j = 1, \\
\sum_{p=i}^m (b_{pj} - b_{p,j-1}), & \text{if } 1 < j \leq m. 
\end{cases}$$  \hspace{1cm} (3.15)

Notice that $O_r^{-1}$ represents the column operations

$$C_{r+1} \leadsto C_{r+1} - C_r, \quad C_{r+2} \leadsto C_{r+2} - C_r, \ldots, \quad C_m \leadsto C_m - C_r.$$  

Thus, it is clear that column operations and $r$-th row operations $O_r$ for $r \neq i$, do not change the value of $(i,1)$-th element. Thus, $c_{i1}$ is equal to the $(i,1)$-th element of $O_iB_2$, i.e.

$$c_{i1} = \sum_{p=i}^m b_{p1}.$$  

Assume $j > 1$. Then, column operations for $i \geq j$ and row operations for $i \neq j$ do not change the value of $(i,j)$-th element. Thus, $c_{ij}$ is equal to the $(i,j)$-th element of $O_iB_2O_1^{-1}O_2^{-1}\cdots O_{j-1}^{-1}$. We need to differentiate the cases $i < j$ and $i \geq j$, since it determines the order of the operations.
Case I. $i \geq j$.
Let $c_{ij}^{(l)}$ denote the $(i, j)$-th element of $B_2^{O_{1}^{-1}}O_2^{-1} \cdots O_{j-l}^{-1}$ for $1 \leq l \leq j - 1$. Then,

$$c_{ij} = \sum_{p=1}^{m} c_{ij}^{(1)} = \sum_{p=1}^{m} (c_{ij}^{(2)} - c_{ij}^{(2)}) = \sum_{p=1}^{m} ((c_{ij}^{(3)} - c_{ij}^{(3)}) - (c_{ij}^{(3)} - c_{ij}^{(3)})) = \sum_{p=1}^{m} (c_{ij}^{(3)} - c_{ij}^{(3)}) = \cdots$$

\[
= \sum_{p=1}^{m} (c_{ij}^{(j-2)} - c_{ij}^{(j-2)}) = \sum_{p=1}^{m} (c_{ij}^{(j-2)} - c_{ij}^{(j-2)}) = \cdots
\]

\[
= \sum_{p=1}^{m} (c_{ij}^{(j-1)} - c_{ij}^{(j-1)}) = \sum_{p=1}^{m} (c_{ij}^{(j-1)} - c_{ij}^{(j-1)}) = \cdots
\]

\[
= \sum_{p=1}^{m} (c_{ij}^{(j-2)} - c_{ij}^{(j-2)}) = \sum_{p=1}^{m} (c_{ij}^{(j-2)} - c_{ij}^{(j-2)}) = \cdots
\]

Therefore, (3.15) is proved. It remains to show $C = \tilde{B}_1$. Recall that $a_{ij}$ denotes the $(i, j)$-th element of the adjacency matrix $A_{G}$. Consider the vector

$$v = (a_{11}, a_{12}, a_{13}, \ldots, a_{1m}, a_{1m+1}, a_{1m+2}, \ldots, a_{13}, a_{14})^T$$

and the shifting operator $\rho$

$$\rho v = (a_{12}, a_{11}, a_{13}, \ldots, a_{1m}, a_{1m+1}, a_{1m+2}, \ldots, a_{13}, a_{14})^T.$$

**Notation:** In the rest of the proof of Lemma 3.8, we use $v_p$ to denote the $p'$-th element of $v$, with $p' = p \pmod{k}$, for $p \in \mathbb{Z}$ and $v \in \mathbb{R}^k$. Also, we use a short-hand notation of $(\rho v)_p$ by $\rho v_p$.

Due to the symmetric form of $v$, we have

$$v_{m+q} = v_{m-q+3}, \quad q \in \mathbb{Z}, \quad (3.16)$$

and

$$v_p = \rho v_{p+1}, \quad v_p = \rho^{-1} v_{p-1}, \quad p \in \mathbb{Z}. \quad (3.17)$$

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Proof of Theorem 3.7

Without loss of generality, assume \( \mathcal{G} \) has an interior symmetry \( \mathbf{D}_k \) on the cells \([1, \ldots, k]\). Due to this interior symmetry, the entries of \( A_\mathcal{G} \) satisfy

\[
\begin{align*}
 a_{ij} &= a_{(i+1-l) \mod k}, & \text{for } i, j, l \in [1, \ldots, k], \\
 a_{ij} &= a_{ij}, & \text{for } i, l \in [1, \ldots, k] \text{ and } j \in [k+1, \ldots, n], \\
 a_{1j} &= a_{1(k-j+2)}, & \text{for } j \in [2, \ldots, m, m+1].
\end{align*}
\]

In terms of \( v \) and \( \rho \), the matrix \( B_1 \) consists of the first \( m \) rows of the matrix

\[
(v - \rho^{-2}v, \rho v - \rho^{-3}v, \rho^2 v - \rho^{-4}v, \ldots, \rho^{m-2}v - \rho^{-m}v, \rho^{m-1}v - \rho^{-(m+1)}v)
\]

and \( B_2 \) consists of the first \( m \) rows of the matrix

\[
(v + \rho^{-2}v, \rho v + \rho^{-3}v, \rho^2 v + \rho^{-4}v, \ldots, \rho^{m-2}v + \rho^{-m}v, \rho^{m-1}v + \rho^{-(m+1)}v) - 2(\rho^{-1}v, \ldots, \rho^{-1}v).
\]

Assume that \( 1 \leq i \leq m, 1 < j \leq m \). By (3.15), we have

\[
c_{ij} = \sum_{p=1}^{m} (b_{pj} - b_{p,j-1}) = \sum_{p=1}^{m} \left( (\rho^{i-1}v_p + \rho^{-j-1}v_p) - (\rho^{i-2}v_p + \rho^{-j}v_p) \right)
\]

\[
= \rho^{i-1}v_i - \rho^{-j}v_i + \rho^{-j-1}v_m - \rho^{i-2}v_m \tag{3.17}
\]

\[
= v_{i-j+1} - v_{i+j} + v_{m+j+1} - v_{m-j+2} \tag{3.16}
\]

\[
= v_{i-j+1} - v_{i+j}. \tag{3.18}
\]

On the other hand, the \((i, j)\)-th element of \( \bar{B}_1 \) is equal to the \((m - i + 1, m - j + 1)\)-th element of \( B_1 \), which equals to

\[
\rho^{m-j}v_{m-i+1} - \rho^{-m-j-2}v_{m-i+1} \tag{3.17' = v_{j-i+1} - v_{2m-i-j+3}.}
\]

By (3.16), we also have

\[
v_{i-j+1} = v_{k+i-j+1} = v_{2m+1+i-j+1} = v_{m+(m+i-j+2)} = v_{j-i+1}, \tag{3.16'}
\]

and

\[
v_{2m-i-j+3} = v_{m+(m-i+j+3)} = v_{i+j}. \tag{3.21'}
\]

It follows from (3.18)–(3.21) that the \((i, j)\)-th element of \( C \) coincides with the \((i, j)\)-th element of \( \bar{B}_1 \), for \( 1 \leq i \leq m, 1 < j \leq m \).

The case of \( j = 1 \) can be similarly proved. By (3.15), we have

\[
c_{ii} = \sum_{p=1}^{m} b_{p1} = \sum_{p=1}^{m} (v_p + \rho^{-2}v_p - 2\rho^{-1}v_p)
\]

\[
= v_i - \rho^{-1}v_i + \rho^{-2}v_m - \rho^{-1}v_m \tag{3.17'' = v_i - v_{i+1} + v_{m+2} - v_{m+1}}
\]

\[
= v_i - v_{i+1} \tag{3.16'' = v_{2-i} - v_{2m-i+2}}
\]

\[
= \rho^{m-1}v_{m-i+1} - \rho^{-m-1}v_{m-i+1}, \tag{3.17'''}
\]

which is the \((i, 1)\)-th element of \( \bar{B}_1 \).

Consequently, we showed that \( C = \bar{B}_1 \) and thus (3.14) holds. \( \square \)

Proof of Lemma 3.8.
Thus, $A_G$ has the form
\[
A_G = \begin{bmatrix} A & D \\ E & F \end{bmatrix},
\]
where $D$ is a $k \times (n - k)$ matrix with all rows equal and $A$ is a (symmetric) circulant matrix
\[
A = \begin{cases} \text{circ}(a_{11}a_{12}a_{13} \ldots a_{1m+1}a_{1m+1} \ldots a_{13}a_{12}), & \text{if } k \text{ is odd}, \\ \text{circ}(a_{11}a_{12}a_{13} \ldots a_{1m}a_{1m+1}a_{1m} \ldots a_{13}a_{12}), & \text{if } k \text{ is even}. \end{cases}
\]
It follows from (2.4) that the eigenvalues $\lambda_j$, $j = 0, \ldots, k - 1$ of $A$ are real and satisfy $\lambda_j = \lambda_{k-j}$, for $j = 1, \ldots, m$. That is, $A$ has $m$ eigenvalues with algebraic multiplicity at least 2, if $k$ is odd; $A$ has $(m - 1)$ eigenvalues with algebraic multiplicity at least 2, if $k$ is even. Our goal is to prove the same property for $A_G$.

**Case I:** Assume that $k$ is odd.

Consider the balanced equivalence relation $\mathbin{\bowtie} = \{(1, 2, \ldots, k), (k + 1, \ldots, n)\}$ induced by $D_k$. Motivated by the direct sum decomposition (2.9), we define a basis $B = \{b_1, b_2, \ldots, b_n\}$ in $\mathbb{R}^n$ by
\[
b_i = \begin{cases} e_{i+1} - e_{k-i+1}, & \text{for } 1 \leq i \leq m \\ -2e_1 + e_{i-m+1} + e_{k-i+m+1}, & \text{for } m + 1 \leq i \leq 2m \\ e_1 + e_2 + \cdots + e_k, & \text{for } i = k \\ e_i, & \text{for } k + 1 \leq i \leq n, \end{cases}
\]
where $\{e_1, e_2, \ldots, e_n\}$ denote the standard basis in $\mathbb{R}^n$ (cf. Example 3.9 for $k = 7$). Then, the adjacency matrix $A_G$ in the basis $B$ has the form
\[
B^{-1}A_GB = \begin{bmatrix} B_1 & 0 & 0 \\ 0 & B_2 & 0 \\ C & A_{\mathbin{\bowtie}} \end{bmatrix},
\]
where $B_1, B_2$ are matrices of order $m \times m$ given by (3.12)–(3.13) and $A_{\mathbin{\bowtie}}$ is the adjacency matrix of the quotient network induced by $\mathbin{\bowtie}$. By Lemma 3.8, $B_1$ and $B_2$ are similar matrices, thus have the same eigenvalues. Consequently, $A_G$ has $m$ eigenvalues of multiplicity at least 2. It follows from Remark 3.2 that the Jacobian $J_G$ has $mr$ eigenvalues with algebraic multiplicity at least 2.

Notice that we can obtain an “optimal” basis $\tilde{B}$ by applying the operations specified in the proof of Lemma 3.8 to $B$, so that $A_G$ has two copies of $B_1$ lying on the diagonal. More precisely, let $R = O_{m-1}O_{m-2} \cdots O_2O_1$ be the total row operation on $B_2$ and $S$ the total row switching operation such that $SB_1S^{-1} = \tilde{B}_1$. Then, we have
\[
SRB_2R^{-1}S^{-1} = B_1.
\]
Set
\[
O = \begin{bmatrix} I_m & 0 & 0 \\ 0 & SR & 0 \\ 0 & 0 & I_{n-2m} \end{bmatrix},
\]
where $I_i$ stands for the identity matrix of order $i \times i$. Define a new basis by
\[
\tilde{B} = BO^{-1}.
\]
Motivated by the direct sum decomposition (2.9), define the following basis

Case II: Assume that (cf. Example 3.9 for \( k \))

\[
\begin{align*}
A & = a_{11} - 2a_{12} + 2a_{13} - 2a_{1,4} + \cdots + (-1)^{m-1}2a_{1,m} + (-1)^m a_{1,m+1} \\
\end{align*}
\]

Then, the adjacency matrix \( A_G \) has the form

\[
\begin{bmatrix}
B_1 & 0 & 0 \\
0 & B_1 & 0 \\
C' & A_{\text{sw}} & 0
\end{bmatrix}
\]

A precise formula of \( \tilde{B} = (\tilde{b}_1, \tilde{b}_2, \ldots, \tilde{b}_n) \) is given by

\[
\tilde{b}_i = \begin{cases}
  e_{i+1} - e_{k-i+1}, & \text{for } 1 \leq i \leq m-1 \\
  -e_k + e_{k-i+1} + e_{i+1} - e_{i+2}, & \text{for } m + 1 \leq i \leq 2m - 1 \\
  -2e_1 + e_2 + e_k, & \text{for } i = 2m \\
  e_1 + e_2 + \cdots + e_k, & \text{for } i = 2m + 1 = k \\
  e_i, & \text{for } k + 1 \leq i \leq n,
\end{cases}
\]

(cf. Example 3.9 for \( k = 7 \)).

Case II: Assume that \( k \) is even.

Similar to the case of odd \( k \), we try to find an optimal basis for the diagonal form of \( A_G \). Motivated by the direct sum decomposition (2.9), define the following basis \( B \)

\[
\begin{align*}
B_1 & = \begin{bmatrix}
  a_{11} - a_{13} & a_{12} - a_{14} & a_{13} - a_{15} & \cdots & a_{1,m-1} - a_{1,m+1} \\
  a_{12} - a_{14} & a_{11} - a_{15} & a_{12} - a_{16} & \cdots & a_{1,m-2} - a_{1,m} \\
  a_{13} - a_{15} & a_{12} - a_{16} & a_{11} - a_{17} & \cdots & a_{1,m-3} - a_{1,m-1} \\
  \vdots & \vdots & \vdots & \ddots & \vdots \\
  a_{1,m-1} - a_{1,m+1} & a_{1,m-2} - a_{1,m} & a_{1,m-3} - a_{1,m-1} & \cdots & a_{11} - a_{13}
\end{bmatrix}
\]

where \( B_1, B_2 \) are matrices of order \((m - 1) \times (m - 1), \)

\[
a = a_{11} - 2a_{12} + 2a_{13} - 2a_{1,4} + \cdots + (-1)^{m-1}2a_{1,m} + (-1)^m a_{1,m+1}
\]

and \( A_{\text{sw}} \) is the adjacency matrix of the quotient network. More precisely,
By applying the operations specified in (3.27) to $\tilde{B}$, we can obtain a new basis $\tilde{B}$. Let $R = O_1 O_2 \cdots O_{m-3} O_{m-2}$. Define

$$O = \begin{bmatrix} I_m & 0 & 0 \\ 0 & R & 0 \\ 0 & 0 & I_{r-2m+1} \end{bmatrix}, \quad \tilde{B} = BO^{-1}.$$ 

Then, the adjacency matrix $A_G$ in the basis $\tilde{B}$ has the form

$$\tilde{B}^{-1} A_G \tilde{B} = \begin{bmatrix} B_1 & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & B_1 \end{bmatrix}.$$ 

It follows that $A_G$ has $(m - 1)$ eigenvalues of multiplicity at least 2 and thus, by Remark 3.2, the Jacobian $J_G$ has $(m - 1)r$ eigenvalues with algebraic multiplicity at least 2.

A precise formula of $\tilde{B} = [\tilde{b}_1, \ldots, \tilde{b}_n]$ is given by

$$\tilde{b}_i = \begin{cases} 
    e_{i+1} - e_{k-i+1}, & \text{for } 1 \leq i \leq m - 1 \\
    e_1 - e_2 + e_3 - e_4 + \cdots + e_{k-1} - e_k, & \text{for } i = m \\
    (-1)^{i-m+1}(e_1 - e_2 + \cdots + (-1)^{i-m+1}e_{i-m}) + e_{i-m+1} + e_{k-i+m+1}, & \text{for } m + 1 \leq i \leq 2m - 1 \\
    e_1 + e_2 + \cdots + e_k, & \text{for } i = 2m = k \\
    e_i, & \text{for } k + 1 \leq i \leq n,
\end{cases}$$

(cf. Example 3.10 for $k = 8$).
Example 3.9 Let \( \mathcal{G} \) be a 9-cell regular network with an interior symmetry \( D_7 \) on the cells \([1, 2, 3, 4, 5, 6, 7]\). Then, with respect to the basis (cf. (3.23))

\[
B = \begin{bmatrix}
0 & 0 & 0 & -2 & -2 & 1 & 0 & 0 \\
1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & -1 & 0 & 0 & 1 & 1 & 0 \\
0 & -1 & 0 & 0 & 1 & 0 & 1 & 0 \\
-1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix},
\]

the adjacency matrix \( A_G \) has the form

\[
B^{-1} A_G B = \begin{bmatrix}
B_1 & 0 & 0 \\
0 & B_2 & 0 \\
C_1 & C_2 & A_{\text{ext}}
\end{bmatrix},
\]

where

\[
B_1 = \begin{bmatrix}
a_{11} - a_{13} & a_{12} - a_{14} & a_{13} - a_{14} \\
a_{12} - a_{14} & a_{11} - a_{14} & a_{12} - a_{13} \\
a_{13} - a_{14} & a_{12} - a_{13} & a_{11} - a_{12}
\end{bmatrix},
\]

\[
B_2 = \begin{bmatrix}
a_{11} - 2a_{12} + a_{13} & -a_{12} + a_{14} & a_{13} - 2a_{12} + a_{14} \\
a_{12} - 2a_{13} + a_{14} & a_{11} - 2a_{13} + a_{14} & a_{12} - a_{13} \\
a_{13} - a_{14} & a_{12} - 2a_{14} + a_{13} & a_{11} - 2a_{14} + a_{12}
\end{bmatrix},
\]

\[
A_{\text{ext}} = \begin{bmatrix}
a_{11} + 2a_{12} + 2a_{13} + 2a_{14} & a_{18} & a_{19} \\
a_{81} + a_{82} + a_{83} + a_{84} + a_{85} + a_{86} + a_{87} & a_{88} & a_{89} \\
a_{91} + a_{92} + a_{93} + a_{94} + a_{95} + a_{96} + a_{97} & a_{98} & a_{99}
\end{bmatrix},
\]

and

\[
C_1 = \begin{bmatrix}
0 & 0 & 0 \\
a_{82} - a_{87} & a_{83} - a_{86} & a_{84} - a_{85} \\
a_{92} - a_{97} & a_{93} - a_{96} & a_{94} - a_{95}
\end{bmatrix},
\]

\[
C_2 = \begin{bmatrix}
0 & 0 & 0 \\
-2a_{81} + a_{82} + a_{87} & -2a_{81} + a_{83} + a_{86} & -2a_{81} + a_{84} + a_{85} \\
-2a_{91} + a_{92} + a_{97} & -2a_{91} + a_{93} + a_{96} & -2a_{91} + a_{94} + a_{95}
\end{bmatrix}.
\]

Consider a new basis (cf. (3.24))

\[
\tilde{B} = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & -2 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 & -1 & 1 & 1 & 0 & 0 \\
0 & 1 & 0 & -1 & 1 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & -1 & 1 & 0 & 0 & 1 & 0 & 0 \\
0 & -1 & 0 & -1 & 1 & 0 & 1 & 0 & 0 \\
-1 & 0 & 0 & 0 & -1 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}.
\]
Then, the adjacency matrix $A_G$ is of form

$$\tilde{B}^{-1} A_G \tilde{B} = \begin{bmatrix}
B_1 & 0 & 0 \\
0 & B_1 & 0 \\
C_1' & C_2' & A_{\infty}
\end{bmatrix},$$

where

$$C_1' = \begin{bmatrix}
0 & 0 & 0 \\
a_{82} - a_{87} & a_{83} - a_{86} & a_{84} - a_{85} \\
a_{92} - a_{87} & a_{93} - a_{86} & a_{94} - a_{95}
\end{bmatrix},$$

$$C_2' = \begin{bmatrix}
-a_{83} + a_{84} + a_{85} - a_{86} & -a_{82} + a_{83} + a_{86} - a_{87} & -2a_{81} + a_{82} + a_{87} \\
-a_{93} + a_{94} + a_{95} - a_{96} & -a_{92} + a_{93} + a_{96} - a_{97} & -2a_{91} + a_{92} + a_{97}
\end{bmatrix}.$$

\[\square\]

**Example 3.10** Let $G$ be a 10-cell regular network with an interior symmetry $D_8$ on the cells $\{1, 2, 3, 4, 5, 6, 7, 8\}$. Then, with respect to the basis (cf. (3.25))

$$B = \begin{bmatrix}
0 & 0 & 0 & 1 & -2 & -2 & -2 & 1 & 0 & 0 \\
1 & 0 & 0 & -1 & 1 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & -1 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & -1 & -1 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & -1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\
-1 & 0 & 0 & -1 & 1 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix},$$

the adjacency matrix $A_G$ has the form

$$B^{-1} A_G B = \begin{bmatrix}
B_1 & 0 & 0 \\
0 & a & a_1 \\
0 & 0 & B_2
\end{bmatrix},$$

where $a = a_{11} - 2a_{12} + 2a_{13} - 2a_{14} + a_{15}$,

$$a_1 = [2a_{14} - 2a_{15}, 2a_{13} - 2a_{15}, 2a_{12} - 2a_{15}],$$

$$B_1 = \begin{bmatrix}
a_{11} - a_{13} & a_{12} - a_{14} & a_{13} - a_{15} \\
a_{12} - a_{14} & a_{11} - a_{15} & a_{12} - a_{14} \\
a_{13} - a_{15} & a_{12} - a_{14} & a_{11} - a_{13}
\end{bmatrix},$$

$$B_2 = \begin{bmatrix}
a_{11} - 2a_{12} + a_{13} + 2a_{14} - 2a_{15} & -a_{12} + 2a_{13} + a_{14} - 2a_{15} & a_{13} - a_{15} \\
a_{12} - a_{14} & a_{11} - 4a_{13} + 3a_{15} & -a_{12} - 2a_{13} + a_{14} + 2a_{15} \\
a_{13} - a_{15} & a_{12} + 2a_{13} - a_{14} - 2a_{15} & a_{11} + 2a_{12} + a_{13} - 2a_{14} - 2a_{15}
\end{bmatrix},$$

$$A_{\infty} = \begin{bmatrix}
a_{11} + 2a_{12} + 2a_{13} + 2a_{14} + a_{15} & a_{19} & a_{1,10} \\
a_{91} + a_{92} + a_{93} + a_{94} + a_{95} + a_{96} + a_{97} + a_{98} & a_{99} & a_{9,10} \\
a_{10,1} + a_{10,2} + a_{10,3} + a_{10,4} + a_{10,5} + a_{10,6} + a_{10,7} + a_{10,8} & a_{10,9} & a_{10,10}
\end{bmatrix}.$$
and
\[
C_1 = \begin{bmatrix}
0 & a_{92} - a_{98} & a_{93} - a_{97} & a_{94} - a_{96} \\
\frac{a_{10,2}}{a_{10,1}} & a_{10,3} - a_{10,7} & a_{10,4} - a_{10,6}
\end{bmatrix},
\]
\[
C_2 = \begin{bmatrix}
0 & a_{91} - a_{92} + a_{93} - a_{94} + a_{95} - a_{96} + a_{97} - a_{98} \\
\frac{a_{10,1}}{a_{10,2}} & a_{10,3} - a_{10,4} + a_{10,5} - a_{10,6} + a_{10,7} - a_{10,8}
\end{bmatrix},
\]
\[
C_3 = \begin{bmatrix}
-2a_{91} + a_{92} + a_{98} & -2a_{91} + a_{93} + a_{97} & -2a_{91} + a_{94} + a_{96} \\
-2a_{10,1} + a_{10,2} + a_{10,8} & -2a_{10,1} + a_{10,3} + a_{10,7} & -2a_{10,1} + a_{10,4} + a_{10,6}
\end{bmatrix}.
\]

Consider a new basis (cf. (3.28))
\[
\tilde{B} = \begin{bmatrix}
0 & 0 & 0 & 1 & -2 & 2 & -2 & 1 & 0 & 0 \\
1 & 0 & 0 & -1 & 1 & -2 & 2 & 1 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 1 & -2 & 1 & 0 & 0 \\
0 & 0 & 1 & -1 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & -1 & -1 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & -1 & 0 & 1 & 0 & 1 & -2 & 1 & 0 & 0 \\
-1 & 0 & 0 & -1 & 1 & -2 & 2 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
\]

Then, the adjacency matrix \( A_G \) is of form
\[
\tilde{B}^{-1} A_G \tilde{B} = \begin{bmatrix}
B_1 & 0 & 0 \\
0 & a & a' \\
0 & 0 & B_3
\end{bmatrix}
\]

where \( a' = [2a_{14} - 2a_{15}, 2a_{13} - 4a_{14} + 2a_{15}, 2a_{12} - 4a_{13} + 4a_{14} - 2a_{15}] \),
\[
C_1' = \begin{bmatrix}
0 & a_{92} - a_{98} & a_{93} - a_{97} & a_{94} - a_{96} \\
\frac{a_{10,2}}{a_{10,1}} & a_{10,3} - a_{10,7} & a_{10,4} - a_{10,6}
\end{bmatrix},
\]
\[
C_2' = \begin{bmatrix}
0 & a_{91} - a_{92} + a_{93} - a_{94} + a_{95} - a_{96} + a_{97} - a_{98} \\
\frac{a_{10,1}}{a_{10,2}} & a_{10,3} - a_{10,4} + a_{10,5} - a_{10,6} + a_{10,7} - a_{10,8}
\end{bmatrix},
\]
\[
C_3' = \begin{bmatrix}
-2a_{91} + a_{92} + a_{98} & -2a_{91} + a_{93} + a_{97} & -2a_{91} + a_{94} + a_{96} - 2a_{97} + 2a_{98} \\
-2a_{10,1} + a_{10,2} + a_{10,8} & -2a_{10,1} + a_{10,2} + a_{10,3} + a_{10,7} - 2a_{10,8} & -2a_{10,1} + 2a_{10,2} - 2a_{10,3} + a_{10,4} + a_{10,6} - 2a_{10,7} + 2a_{10,8}
\end{bmatrix}.
\]

### 3.4 \( \Sigma_S \)-Interior symmetry with \( D_k \subseteq \Sigma_S \subseteq S_k \)

In this subsection, we consider regular networks \( G \) with an interior symmetry group \( \Sigma_S \) with \( D_k \subseteq \Sigma_S \subseteq S_k \). Besides the result of Theorem 3.7 that applies to \( G \), we show that the multiplicity
of the eigenvalues of the adjacency matrix $A_G$ can be directly analyzed using the eigenvalues of the circulant part $A$ of $A_G$.

As shown in Subsection 3.3, the adjacency matrix of a regular network having an interior symmetry at least $D_k$ is of the form

$$A_G = \begin{bmatrix} A & D \\ E & F \end{bmatrix},$$

(3.29)

where $D$ is a $k \times (n-k)$ matrix with all rows equal and $A$ is a circulant matrix of order $k \times k$ being of the form

$$A = \begin{cases} \text{circ}(a_{11}a_{12}a_{13} \ldots a_{1m+1}a_{1m+2} \ldots a_{13}a_{12}), & \text{if } k \text{ is odd}, \\ \text{circ}(a_{11}a_{12}a_{13} \ldots a_{1m+1}a_{1m+2} \ldots a_{13}a_{12}), & \text{if } k \text{ is even}. \end{cases}$$

(3.30)

It follows from (2.4) that the eigenvalues $\lambda_j$, $j = 0, \ldots, k-1$ of $A$ are real and satisfy $\lambda_j = \lambda_{k-j}$, for $j = 1, \ldots, m$. That is, $A$ has $m$ eigenvalues with algebraic multiplicity at least 2, if $k$ is odd; $A$ has $(m-1)$ eigenvalues with algebraic multiplicity at least 2, if $k$ is even. In Theorem 3.7 we proved the same property for $A_G$.

Now, using the proof of Theorem 3.7, we show that

**Theorem 3.11** Let $G$ be an $n$-cell regular network with an interior symmetry $\Sigma_S$ such that $D_k \subseteq \Sigma_S \subseteq S_k$. Let $A_G$ be the adjacency matrix of $G$, $A$ be given by (3.30) and $\lambda_j$ be eigenvalues of $A$, for $j = 0, \ldots, k-1$ given by (2.4). Then, there exists a basis $B$ of $\mathbb{R}^n$, which is independent of entries of $A_G$, such that

$$B^{-1}A_G B = \begin{bmatrix} \Lambda & 0 \\ C & A_{\text{circ}} \end{bmatrix}, \quad \text{for } \Lambda = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_{k-1} \end{bmatrix},$$

(3.31)

where $A_{\text{circ}}$ is the adjacency matrix of the quotient network induced by $\Sigma_S$.

**Proof** Consider the $k$-cell regular network $G_o$ whose adjacency matrix is given by $A$ in (3.29). Since $G$ is $\Sigma_S$-interior symmetric, $G_o$ is $\Sigma_S$-symmetric. Let $\tilde{B}_0$ be a basis in $\mathbb{R}^k$ given by (3.24) for odd $k$ and (3.28) for even $k$. As shown in the proof of Theorem 3.7, we have

$$\tilde{B}_0^{-1} A \tilde{B}_0 = \begin{bmatrix} M & 0 \\ 0 & \lambda_0 \end{bmatrix},$$

where $M$ is a matrix of order $(k-1) \times (k-1)$ of form

$$M = \begin{bmatrix} B_1 & 0 \\ 0 & B_1 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} B_2 & 0 \\ 0 & a \\ 0 & B_2 \end{bmatrix},$$

for odd $k$ or even $k$, respectively. Let $v_j$ be the eigenvector of $\lambda_j$ given by (2.3), for $j = 0, 1, \ldots, k-1$. Set $V = [v_1, \ldots, v_{k-1}, v_0]$. Then,

$$V^{-1} A V = \begin{bmatrix} \Lambda & 0 \\ 0 & \lambda_0 \end{bmatrix}.$$ 

Thus, we have

$$(\tilde{B}_0^{-1} V)^{-1} \begin{bmatrix} M & 0 \\ 0 & \lambda_0 \end{bmatrix} (\tilde{B}_0^{-1} V) = \begin{bmatrix} \Lambda & 0 \\ 0 & \lambda_0 \end{bmatrix}.$$
Moreover, it can be verified that
\[
\mathcal{B}_0^{-1}V = \begin{bmatrix} X & 0 \\ 0 & 1 \end{bmatrix},
\]
for a matrix \(X\) of order \((k-1) \times (k-1)\). Consequently, we have
\[
X^{-1}MX = \Lambda.
\]
On the other hand, let \(\mathcal{B}\) be a basis in \(\mathbb{R}^n\) given by (3.24) for odd \(k\) and (3.28) for even \(k\). Then,
\[
\mathcal{B}^{-1}A_G\mathcal{B} = \begin{bmatrix} M & 0 \\ C' & A_{\omega} \end{bmatrix}.
\]
Now set
\[
\tilde{X} = \begin{bmatrix} X & 0 \\ 0 & I_{n-k+1} \end{bmatrix}, \quad \mathcal{B} := \mathcal{B}\tilde{X}.
\]
Then, \(\mathcal{B}\) is a basis such that (3.31) holds, for \(C = C'X\). Moreover, \(\mathcal{B}\) is also independent of the entries of \(A_G\); since both \(\mathcal{B}_0\) and \(V\) are independent of the entries of \(A_G\). ■

Consequently, the influence of \(\Sigma_S\) on the eigenvalues of \(A_G\), and thus of \(J_G\) can be directly examined by looking at eigenvalues of \(A\).

**Example 3.12** Let \(k = 12, m = 6\) and \(n > 12\). Consider an \(n\)-cell regular network \(G\) with an interior symmetry at least \(D_{12}\) on the set of cells \(\{1, 2, \ldots, 12\}\). Let \(A_G\) be the adjacency matrix, \(A\) be the circulant part of \(A_G\) (cf. (3.29)) and \(\lambda_j\) be the eigenvalues of \(A\), for \(j = 0, 1, \ldots, 11\). By (2.4), \(\lambda_j = \lambda_{12-j}\), for \(j = 1, 2, \ldots, 5\) and, denoting by \(\lambda_{i,j}\) both the eigenvalues \(\lambda_i\) and \(\lambda_j\), we have

\[
\begin{align*}
\lambda_0 &= a_1 + 2a_2 + 2a_3 + 2a_4 + 2a_5 + 2a_6 + a_7 (= a_\omega) \\
\lambda_{1,1} &= a_1 + r_1a_2 + r_2a_3 - r_2a_1 - r_1a_6 - a_7 \\
\lambda_{2,10} &= a_1 + r_2a_2 - r_2a_1 - 2a_4 - r_2a_5 + r_2a_6 + a_7 \\
\lambda_{3,0} &= a_1 - 2a_3 + 2a_5 - a_7 \\
\lambda_{4,8} &= a_1 - r_2a_2 - r_2a_1 + 2a_4 - r_2a_5 - r_2a_6 + a_7 \\
\lambda_{5,7} &= a_1 - r_1a_2 + r_2a_3 - r_2a_5 + r_1a_6 - a_7 \\
\lambda_6 &= a_1 - 2a_2 + 2a_3 - 2a_4 + 2a_5 - 2a_6 + a_7 (= a)
\end{align*}
\]

where \(r_1 = 2\text{Re} \omega_1 = \sqrt{3}, r_2 = 2\text{Re} \omega_1^2 = 1\). Note that

(i) if \(a_{12} = a_{16}\), then \(\lambda_{1,11} = \lambda_{5,7}\);

(ii) if \(a_{12} = a_{16}\) and \(a_{13} = a_{15}\), then \(\lambda_{1,11} = \lambda_{3,9} = \lambda_{5,7}\);

(iii) if \(a_{12} = a_{13} = a_{14} = a_{15} = a_{17}\) and \(a_{11} = a_{15}\), then \(\lambda_{1,11} = \lambda_{2,10} = \lambda_{5,7}\) and \(\lambda_{3,9} = \lambda_6\);

(iv) if \(a_{11} = a_{12} = a_{13} = a_{15} = a_{16} = a_{17}\), then \(\lambda_{1,11} = \lambda_{3,9} = \lambda_{5,7}\) and \(\lambda_{2,10} = \lambda_6\);

(v) if \(a_{12} = a_{13} = a_{15} = a_{16} = a_{17}\), then \(\lambda_{1,11} = \lambda_{2,10} = \lambda_{3,9} = \lambda_{5,7}\);

(vi) if \(a_{12} = a_{13} = a_{14} = a_{15} = a_{16} = a_{17}\), then \(\lambda_{1,11} = \lambda_{2,10} = \lambda_{3,9} = \lambda_{4,8} = \lambda_{5,7} = \lambda_6\).

Thus, by Theorem 3.11, the following holds for any \(n\)-cell regular network \(G\) with \(n \geq 12\) having a \(\Sigma_S\)-interior symmetry:
(i) if \( \Sigma_S = \langle D_{12}, (2\ 6\ 8\ 12)(3\ 11)(4\ 10)(5\ 9) \rangle \), then \( A_G \) has 3 eigenvalues of multiplicity at least 2 and 1 eigenvalue of multiplicity at least 4;

(ii) if \( \Sigma_S = \langle D_{12}, (2\ 6\ 8\ 12)(3\ 5\ 9\ 11)(4\ 10) \rangle \), then \( A_G \) has 2 eigenvalues of multiplicity at least 2 and 1 eigenvalue of multiplicity at least 6;

(iii) if \( \Sigma_S = \langle D_{12}, (1\ 5\ 9)(2\ 3\ 4\ 6\ 7\ 8\ 10\ 11\ 12) \rangle \), then \( A_G \) has 1 eigenvalue of multiplicity at least 2, 1 eigenvalue of multiplicity at least 3 and 1 eigenvalue of multiplicity at least 6;

(iv) if \( \Sigma_S = \langle D_{12}, (4\ 10)(1\ 2\ 3\ 5\ 6\ 7\ 8\ 9\ 11\ 12) \rangle \), then \( A_G \) has 1 eigenvalue of multiplicity at least 2 and 1 eigenvalue of multiplicity at least 6;

(v) if \( \Sigma_S = \langle D_{12}, (2\ 3\ 5\ 6\ 8\ 9\ 11\ 12)(4\ 7\ 10) \rangle \), then \( A_G \) has 1 eigenvalue of multiplicity at least 2 and 1 eigenvalue of multiplicity at least 8;

(vi) if \( \Sigma_S = \langle D_{12}, (2\ 3\ 4\ 5\ 6\ 7\ 8\ 9\ 10\ 11\ 12) \rangle = S_{12} \), then \( A_G \) has 1 eigenvalue of multiplicity at least 11.

\[ \square \]

3.5 \( Z_k \)-Interior symmetry

Consider an \( n \)-cell regular network \( G \) with adjacency matrix \( A_G = [a_{ij}]_{1 \leq i,j \leq n} \), which has an interior symmetry \( Z_k \) for some \( 3 \leq k < n \) on some subset of \( k \) cells which, up to a reordering of the cells, we can assume to be the first \( k \) cells. Then,

\[
A_G = \begin{bmatrix} A & D \\ E & F \end{bmatrix},
\]

where \( D \) is a \( k \times (n-k) \) matrix with all rows equal and \( A \) is a circulant matrix

\[
A = circ(a_{11}, a_{12}, a_{13}, \ldots, a_{1k}).
\]

Examples show that in general, \( A_G \) does not have multiple eigenvalues due to \( Z_k \)-interior symmetry. In fact, even with additional equalities on \( \{a_{12}, a_{13}, \ldots, a_{1k}\} \), as long as the resulting symmetry is less than \( D_k \), \( A_G \) seems to be free of multiple eigenvalues in general.

3.6 Cyclic interior symmetry of regular uniform networks

Despite of the fact that cyclic interior symmetries are not sufficient for the adjacency matrix of regular networks to have multiple eigenvalues, this may become different if they are uniform networks.

Recall that uniform regular networks are regular networks without multiple arrows nor self-coupling arrows (cf. Stewart [16]). In the next two subsections, we analyze two particular types of cyclic interior symmetry groups and show their influence on the multiplicity of eigenvalues of adjacency matrices of uniform networks. As we will see, for regular uniform networks, interior symmetry forces the existence of eigenvalues in \( \{-2, -1, 0, 1\} \).
3.6.1 $\mathbb{Z}_2 \times \ldots \times \mathbb{Z}_2$-Interior symmetry

We show that

**Theorem 3.13** Let $G$ be an $n$-cell regular network with a product interior symmetry $\mathbb{Z}_2 \times \ldots \times \mathbb{Z}_2$ on $r$ disjoint subsets $S_k = \{i_k, j_k\}$ of cells of $G$, for $k = 1, 2, \ldots, r$. Then, the adjacency matrix $A_G = [a_{ij}]_{1 \leq i, j \leq n}$ of $G$ has $r$ eigenvalues $(a_{ik} - a_{jk})$, for $k = 1, 2, \ldots, r$. Moreover, if $G$ is a uniform network, then $(a_{ik} - a_{jk}) \in [-1, 0, 1]$, for $k = 1, 2, \ldots, r$.

**Proof** Without loss of generality, we assume $S_k = \{2k - 1, 2k\}$ for $k = 1, 2, \ldots, r$. Then, $\mathbb{Z}_2 \times \ldots \times \mathbb{Z}_2 = \langle (1 2), \ldots, (2r - 1, 2r) \rangle$.

Due to this interior symmetry, the entries of $A_G$ satisfy

$$a_{ii} = a_{i+1,i+1}, \quad a_{i,i+1} = a_{i+1,i} \quad \text{and} \quad a_{ii} = a_{i+1,i},$$

for all $i = 1, 3, \ldots, 2r - 1$ and for all $l \neq i, i + 1$.

Consider the balanced equivalence relation

$$= = \{(1, 2), (3, 4), \ldots, (2r - 1, 2r), (2r + 1), \ldots, (n)\}$$

induced by $\mathbb{Z}_2 \times \ldots \times \mathbb{Z}_2$ and the basis $B = \{b_1, b_2, \ldots, b_n\}$ given by

$$b_k = \begin{cases} e_{2k-1} - e_{2k}, & \text{if } 1 \leq k \leq r \\ e_{2k-1} - e_{2k}, & \text{if } r + 1 \leq k \leq 2r \\ e_{2k-1} - e_{2k}, & \text{if } 2r + 1 \leq k \leq n, \end{cases}$$

adapted to the decomposition in (3.11). It follows from Theorem 3.3 that

$$B^{-1}A_GB = \begin{bmatrix} A & 0 \\ B & A \end{bmatrix},$$

where

$$A = \begin{bmatrix} a_{11} - a_{12} & 0 & 0 & \ldots & 0 \\ 0 & a_{33} - a_{34} & 0 & \ldots & 0 \\ 0 & 0 & a_{55} - a_{56} & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & a_{2r-1,2r-1} - a_{2r-1,2r} \end{bmatrix}. $$

Thus, $(a_{2k-1,2k-1} - a_{2k-1,2k})$ are eigenvalues of $A_G$, for $k = 1, 2, \ldots, r$.

If $G$ is a uniform network, then $a_{ij} \in [0, 1)$ and consequently, $(a_{2k-1,2k-1} - a_{2k-1,2k}) \in [-1, 0, 1]$, for $k = 1, 2, \ldots, r$.

**Corollary 3.14** Let $G$ be an $n$-cell uniform network with a product interior symmetry $\mathbb{Z}_2 \times \ldots \times \mathbb{Z}_2$ on $r$ disjoint subsets $S_k$ of cells of $G$, for $k = 1, 2, \ldots, r$. Assume that $r \geq 4$. Then, the adjacency matrix $A_G$ of $G$ has at least one multiple eigenvalue.

**Proof** By Theorem 3.13, $A_G = [a_{ij}]_{1 \leq i, j \leq n}$ has $r$ eigenvalues $\lambda_k := a_{ik} - a_{jk} \in [-1, 0, 1]$, for $k = 1, 2, \ldots, r$. Thus, if $r \geq 4$, values of $\lambda_k$'s must be duplicated for some $k$. 

\[\Box\]
Example 3.15 Let \( G \) be the 5-cell uniform network given in Figure 8 and \( A_G = [a_{ij}]_{5 \times 5} \) be the adjacency matrix.

The network \( G \) has an interior symmetry group \( Z_2 \times Z_2 = \langle (2, 3), (4, 5) \rangle \). It follows from Theorem 3.13 that \( a_{22} - a_{23} = 0 \) and \( a_{44} - a_{45} = 0 \) are eigenvalues of \( A_G \). Thus, 0 is an eigenvalue of algebraic multiplicity at least 2 for \( A_G \).

Moreover, consider the balanced equivalence relation \( \trianglerighteq = \{ \{1\}_{\omega_1}, \{2, 3\}_{\omega_1}, \{4, 5\}_{\omega_1} \} \) induced by \( Z_2 \times Z_2 = \langle (2, 3), (4, 5) \rangle \). Then, the quotient network \( G_{\omega_1} \) has an interior symmetry \( S_3 \) on the set \( C_{\omega_1} = \{ \{1\}_{\omega_1}, \{2\}_{\omega_1}, \{3\}_{\omega_1}, \{4\}_{\omega_1} \} \), for \( \{1\}_{\omega_1} = \{1\}, \{2\}_{\omega_1} = \{2, 3\} \) and \( \{3\}_{\omega_1} = \{4, 5\} \). Let \( A_{\omega_1} = [\bar{a}_{ij}]_{3 \times 3} \) be the adjacency matrix of \( G_{\omega_1} \). By Theorem 3.5, \( \bar{a}_{11} - \bar{a}_{12} = -1 \) is an eigenvalue of algebraic multiplicity at least 2 for the adjacency matrix \( A_{\omega_1} \). Thus, by Theorem 3.3 and Corollary 3.4, \(-1\) is an eigenvalue of algebraic multiplicity 2 for \( A_G \).

Lastly, the remaining eigenvalue of \( A_G \) is given by the valency 2 of the network. \( \square \)

Example 3.16 Let \( G \) be the 5-cell uniform network given in Figure 9 and \( A_G = [a_{ij}]_{5 \times 5} \) be the adjacency matrix. Using Theorem 3.3, Corollary 3.4 and Theorem 3.13, we show that besides the valency 2 of the network, \( A_G \) has 0 and \(-1\) as eigenvalues, both with algebraic multiplicity 2.

We first consider the interior symmetry group \( Z_2 = \langle (2, 3) \rangle \) of \( G \). Then, the eigenvalues of \( A_G \) are \( a_{22} - a_{23} = 0 \) with algebraic multiplicity at least 1 and those of the quotient network \( G_{\omega_1} \) induced by the balanced equivalence relation \( \trianglerighteq_{\omega_1} = \{ \{1\}_{\omega_1}, \{2\}_{\omega_1}, \{3\}_{\omega_1}, \{4\}_{\omega_1} \} \), for \( \{1\}_{\omega_1} = \{1\}, \{2\}_{\omega_1} = \{2, 3\}, \{3\}_{\omega_1} = \{4\} \) and \( \{4\}_{\omega_1} = \{5\} \).

The quotient network \( G_{\omega_1} \) in turn, has an interior symmetry \( Z_2 = \langle \{2\}_{\omega_1}, \{4\}_{\omega_1} \rangle \). Let \( A_{\omega_1} = (\bar{a}_{ij}^1)_{4 \times 4} \) be the adjacency matrix of \( G_{\omega_1} \). Then, the eigenvalues of \( A_{\omega_1} \) are \( \bar{a}_{22}^1 - \bar{a}_{24}^1 = (a_{22} + a_{23}) - a_{25} = -1 \) with algebraic multiplicity at least 1 and those of the quotient network \( G_{\omega_2} \).
induced by the balanced equivalence relation $\sim_2 = \{(1)_{\omega_2}, (2)_{\omega_2}, (3)_{\omega_2}\}$, for $[1]_{\omega_2} = \{(1)_{\omega_1}\}$, $[2]_{\omega_2} = \{(2)_{\omega_1}, (4)_{\omega_1}\}$ and $[3]_{\omega_2} = \{(3)_{\omega_1}\}$.

Further, the quotient network $G_{\omega_2}$ has an interior symmetry $Z_2 = \langle (1)_{\omega_2}, (3)_{\omega_2} \rangle$. Let $A_{\omega_2} = (a_{ij})_{3 \times 3}$ be the adjacency matrix of $G_{\omega_2}$. Then, the eigenvalues of $A_{\omega_2}$ are $\lambda_1^2 - \lambda_2^2 = a_{11} - a_{14} = -1$ with algebraic multiplicity at least 1 and those of the quotient network $G_{\omega_2}$ for the balanced equivalence relation $\sim_2 = \{(1)_{\omega_3}, (2)_{\omega_3}\}$, with $[1]_{\omega_3} = \{(1)_{\omega_2}, (3)_{\omega_2}\}$ and $[2]_{\omega_3} = \{(2)_{\omega_2}\}$.

The quotient network $G_{\omega_3}$ also has an interior symmetry $Z_2 = \langle (1)_{\omega_3}, (2)_{\omega_3} \rangle$. Let $A_{\omega_3} = (a_{ij})_{2 \times 2}$ be the adjacency matrix of $G_{\omega_3}$. By Theorem 3.13, $\lambda_1^3 - \lambda_2 = (a_{11} + a_{14}) - (a_{12} + a_{13} + a_{15}) = 0$ is an eigenvalue of algebraic multiplicity 1 for $A_{\omega_3}$.

\[\square\]

### 3.6.2 $V_4$-Interior symmetry

We discuss in this subsection $n$-cell uniform networks with an interior symmetry group

\[V_4 := \langle (i)_{j} (k)_{l} \rangle \subset S_n,\]

where $i,j,k,l$ are distinct cells of $G$.

**Theorem 3.17** Let $G$ be an $n$-cell uniform network having an interior symmetry group $V_4 = \langle (i)_{j} (k)_{l} \rangle \subset S_n$ on a subset $\{i,j,k,l\}$ of cells of $G$. Then, the adjacency matrix $A_{G} = [a_{ij}]_{1 \leq i,j \leq n}$ has the 3 eigenvalues

\[-a_{ij} + a_{ik} - a_{il},\]
\[-a_{ij} - a_{ik} + a_{il},\]
\[a_{ij} - a_{ik} - a_{il},\]

which take value in \{-2, -1, 0, 1\}.

**Proof** Due to the interior symmetry $Z_2 = \langle (i)_{j} (k)_{l} \rangle$, the entries of $A_G$ satisfy

\[a_{ii} = a_{jj}, \quad a_{ij} = a_{ji}, \quad a_{ik} = a_{jk}, \quad a_{il} = a_{lj} \quad \text{and} \quad a_{im} = a_{jm}, \quad \forall m \neq i,j,k,l\]
\[a_{ik} = a_{jk}, \quad a_{jl} = a_{ki}, \quad a_{ki} = a_{ik} \quad \text{and} \quad a_{km} = a_{km}, \quad \forall m \neq i,j,k,l\]

Due to the interior symmetry $Z_2 = \langle (i)_{j} (k)_{l} \rangle$, they satisfy

\[a_{ii} = a_{kk}, \quad a_{jk} = a_{kj}, \quad a_{ij} = a_{kl}, \quad a_{ik} = a_{kj} \quad \text{and} \quad a_{im} = a_{km}, \quad \forall m \neq i,j,k,l\]
\[a_{jj} = a_{ll}, \quad a_{jl} = a_{lj}, \quad a_{ji} = a_{lk}, \quad a_{jk} = a_{li} \quad \text{and} \quad a_{jm} = a_{lm}, \quad \forall m \neq i,j,k,l\]

Thus, due to the interior symmetry $V_4$, the entries of $A_G$ satisfy

\[a_{ii} = a_{ij}, \quad a_{ij} = a_{jk}, \quad a_{ij} = a_{jl} \quad \text{and} \quad a_{im} = a_{jm}, \quad \forall m \neq i,j,k,l\]

Without loss of generality, we assume $i = 1$, $j = 2$, $k = 3$ and $l = 4$. Let $\sim$ be the balanced equivalence relation induced by $Z_2 = \langle (1,2)(3,4) \rangle$, i.e.

\[\sim = \{(1,2), (3,4), (5), \ldots, (n)\}.

Let $W, U$ be given by (2.7)–(2.8). Then, we have

\[\mathbb{I}^n = W \oplus U.

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A basis $B = \{b_1, b_2, \ldots, b_n\}$ adapted to this decomposition is given by

$$b_k = \begin{cases} 
\epsilon_{2k-1} - \epsilon_{2k}, & \text{if } 1 \leq k \leq 2 \\
\epsilon_{2(k-2)-1} + \epsilon_{2(k-2)}, & \text{if } 3 \leq k \leq 4 \\
\epsilon_k, & \text{if } 5 \leq k \leq n.
\end{cases}$$

Then, the adjacency matrix $A_G$ in the basis $B$ has the form

$$B^{-1}A_GB = \begin{bmatrix} A & 0 \\
C & A_{a_b} \end{bmatrix},$$

where

$$A = \begin{bmatrix} a_{11} - a_{12} & a_{13} - a_{14} \\
a_{13} - a_{14} & a_{11} - a_{12} \end{bmatrix}. $$

Since $G$ is uniform, we have $a_{11} = 0$. Thus, $A$ has eigenvalues $(-a_{12} \pm (a_{13} - a_{14}))$, which are also eigenvalues of $A_G$. Similarly, using symmetry $Z_2 = \langle (13) \rangle$, one can show that $A_G$ has eigenvalues $(-a_{13} \pm (a_{12} - a_{14}))$. Thus, altogether $A_G$ has the following 3 eigenvalues

$$-a_{12} + a_{13} - a_{14}, \quad -a_{12} - a_{13} + a_{14}, \quad a_{12} - a_{13} - a_{14},$$

which take value in $\{-2, -1, 0, 1\}$, since $a_{a_b} \in \{0, 1\}$. \hfill \qed

**Corollary 3.18** Let $G$ be an $n$-cell uniform network with adjacency matrix $A_G = [a_{a_b}]_{1 \leq a, b \leq n}$ having an interior symmetry group $V_4 \rtimes Z_2 = \langle (i j) (k l) \rangle$ on a subset $S = \{i, j, k, l\}$ of cells of $G$, with $a \neq b$ in $S$. If $(a b) = (i j)$ or $(a b) = (k l)$ then $-a_{i j} \in \{-2, -1, 0, 1\}$ is an eigenvalue of $A_G$ with algebraic multiplicity at least 2. Analogously, if $(a b) = (i k)$ or $(a b) = (j l)$, then $-a_{i k} \in \{-2, -1, 0, 1\}$ is an eigenvalue of $A_G$ with algebraic multiplicity at least 2; if $(a b) = (i l)$ or $(a b) = (j k)$, then $-a_{i l} \in \{-2, -1, 0, 1\}$ is an eigenvalue of $A_G$ with algebraic multiplicity at least 2.

**Proof** Consider $A_G$ as a network having $V_4$ as an interior symmetry group. Then, by Theorem 3.17, $A_G$ has the following 3 eigenvalues

$$-a_{i j} + a_{i k} - a_{i l}, \quad -a_{i j} - a_{i k} + a_{i l}, \quad a_{i j} - a_{i k} - a_{i l}.$$ We only give the proof for the case of $(a b) = (i j)$ or $(a b) = (k l)$. The other two cases can be proved in a similar way. Due to the interior symmetry $(i j)$ or $(k l)$, we have $a_{i k} = a_{i l}$. Thus, $-a_{i j}$ is an eigenvalue of algebraic multiplicity 2 in $\{-2, -1, 0, 1\}$. \hfill \qed

**Example 3.19** Let $G$ be the 5-cell uniform network given in Figure 10 and $A_G = [a_{i j}]_{1 \leq i, j \leq 5}$ be the adjacency matrix.

The network $G$ has an interior symmetry group $V_4 \rtimes Z_2 = \langle (2 3)(4 5), (2 4)(3 5), (a b) \rangle$, on the set of cells $S = \{2, 3, 4, 5\}$, for $(a b) = (2 5)$, as well as for $(a b) = (3 4)$. By Corollary 3.18, the adjacency matrix $A_G$ has the eigenvalue $-a_{25} = -1$ with algebraic multiplicity at least 2.

In fact, using Theorem 3.3, Corollary 3.4 and Theorem 3.5, we can show that the algebraic multiplicity of the eigenvalue $-1$ is at least 3. Note that $G$ has an interior symmetry group $Z_2 = \langle (2 3)(4 5) \rangle$. Let $\mathcal{S} = \{[1], [2, 3], [4, 5]\}$. Then, with respect to the basis

$$b = ((0, 1, -1, 0, 0), (0, 0, 0, 1, -1), (1, 0, 0, 0, 0), (0, 1, 1, 0, 0), (0, 0, 0, 1, 1),$$
Figure 10: The 5-cell uniform network $G$ in Example 3.19.

$A_G$ is given by

$$
\begin{bmatrix}
A & 0 \\
C & A_{\infty}
\end{bmatrix},
$$

where $A_{\infty}$ is the adjacency matrix of the quotient network associated with $\bowtie$ and

$$
A = 
\begin{bmatrix}
a_{22} - a_{23} & a_{24} - a_{25} \\
a_{42} - a_{43} & a_{44} - a_{45}
\end{bmatrix}.
$$

Since the quotient network is isomorphic to the ($S_3$-symmetric) three-cell bidirectional ring in Figure 7, by Theorem 3.5, $A_{\bowtie}$ has $a_{11} - a_{12} = -1$ as an eigenvalue with algebraic multiplicity at least 2. On the other hand, due to the fact that $G$ also has an interior symmetry $(2 \ 4)(3 \ 5)$ and $(3 \ 4)$, we have $a_{23} = a_{24} = a_{43} = a_{45}$ and $a_{25} = a_{25}$. Also, since $G$ is uniform, $a_{22} = a_{44} = 0$. Thus, the eigenvalues of $A$ are $-a_{23} \pm (a_{23} - a_{25})$, one of which is equal to $-a_{25} = -1$. Therefore, $-1$ is an eigenvalue of $A_G$ with algebraic multiplicity at least 3. □

4 Interior symmetries and multiple eigenvalues: homogeneous networks

We generalize our results on regular networks to homogeneous networks. Recall that a homogeneous network is a coupled cell network in which all cells are identical but which may have multiple type of arrows. Let $G$ be an $n$-cell homogeneous network with $s$ types of arrows, whose adjacency matrices are $A_1, \ldots, A_s$. Let $r$ be the dimension of the cell internal dynamics. Then, the Jacobian at a fully-synchronized equilibrium has the form

$$
J_G = \alpha \otimes I_n + \sum_{l=1}^s \beta_l \otimes A_l,
$$

where $\alpha$ is the linearized internal dynamics and $\beta_l$ is the linearized internal coupling for the $l$-th type arrow, for $l = 1, \ldots, s$.

4.1 $S_k$- and $A_k$-Interior symmetry

We show that

**Theorem 4.1** Let $G$ be an $n$-cell homogeneous network with $s$ types of arrows with adjacency matrices $A_1, \ldots, A_s$. Let $J_G$ be given by (4.33). Assume that all matrices $A_l$, $l = 1, \ldots, s$ have an interior symmetry $S_k$ or $A_k$, on the same subset $S \subseteq C$ of $k$ cells of $G$, for some $k \in \{3, \ldots, n\}$. Then, the Jacobian $J_G$ has $r$ eigenvalues of algebraic multiplicity at least $k - 1$.
Proof Without loss of generality, we assume $S = \{1, \ldots, k\}$. Write $A_l = [a_{ij}^{(l)}]_{1 \leq i, j \leq n}$, for $l = 1, \ldots, s$. It follows from the proof of Theorem 3.5 that there is a basis $\mathcal{B}$ such that

$$\mathcal{B}^{-1}A_l\mathcal{B} = \begin{bmatrix} \tilde{A}_l & 0 \\ C_l & A_{\infty} \end{bmatrix}, \quad \forall l = 1, \ldots, s,$$

where $\tilde{A}_l$ is a scalar matrix of order $k - 1$, being equal to $(a_{11}^{(l)} - a_{12}^{(l)})I_{k-1}$. For convenience, we denote $a_l := a_{11}^{(l)} - a_{12}^{(l)}$.

Let $\tilde{\mathcal{B}} = I_r \otimes \mathcal{B}$. Then, we have

$$\tilde{\mathcal{B}}^{-1}J_i\tilde{\mathcal{B}} = \begin{bmatrix} \alpha \otimes I_{k-1} + \sum_{l=1}^{s} \beta_l \otimes \tilde{A}_l & 0 \\ \sum_{l=1}^{s} \beta_l \otimes C_l & \alpha \otimes I_{n-k+1} + \sum_{l=1}^{s} \beta_l \otimes A_{\infty} \end{bmatrix}$$

$$= \begin{bmatrix} \alpha + \sum_{l=1}^{s} a_l \beta_l \otimes I_{k-1} & 0 \\ \sum_{l=1}^{s} \beta_l \otimes C_l & \alpha \otimes I_{n-k+1} + \sum_{l=1}^{s} \beta_l \otimes A_{\infty} \end{bmatrix}.$$

Thus, every eigenvalue of $\alpha + \sum_{l=1}^{s} a_l \beta_l$ is an eigenvalue of $J_i\mathcal{B}$ with algebraic multiplicity at least $k - 1$. Therefore, $J_i\mathcal{B}$ has $r$ eigenvalues of multiplicity at least $k - 1$.

4.2 $D_k$-Interior Symmetry

We show that

**Theorem 4.2** Let $G$ be an $n$-cell homogeneous network with $s$ types of arrows with adjacency matrices $A_1, \ldots, A_s$. Let $J_i\mathcal{B}$ be given by (4.33). Assume that all matrices $A_l$, $l = 1, \ldots, s$ have an interior symmetry $D_k$, on the same subset $S \subseteq C$ of $k$ cells of $G$, for some $k \in \{3, \ldots, n\}$. Set

$$m = \begin{cases} (k - 1)/2, & \text{if } k \text{ is odd}, \\
 k/2, & \text{if } k \text{ is even}. \end{cases}$$

Then, $J_i\mathcal{B}$ has $mr$ eigenvalues with algebraic multiplicity at least 2, if $k$ is odd; $J_i\mathcal{B}$ has $(m - 1)r$ eigenvalues with algebraic multiplicity at least 2, if $k$ is even.

**Proof** For simplicity, we present the proof for $s = 2$. The general case can be proved analogously.

Without loss of generality, we assume that the $D_k$-interior symmetry is on the cells $S = \{1, \ldots, k\}$. It follows from the proof of Theorem 3.7 that the adjacency matrices $A_1$ and $A_2$ can be diagonalized to a “double-block” form using the same basis $\tilde{\mathcal{B}}$ given by (3.24) for odd $k$ and (3.28) for even $k$.

By applying this basis to $A_l$, $l = 1, 2$ in case of odd $k$, we have

$$\tilde{\mathcal{B}}^{-1}A_l\tilde{\mathcal{B}} = \begin{bmatrix} B_l & 0 \\ 0 & B_l \end{bmatrix}, \quad l = 1, 2$$

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where $B_l$ is a matrix of order $m \times m$. Consider the following basis for $I_G$

$$\hat{B} = I_r \otimes \bar{B}.$$ 

Then, we have

$$\hat{B}^{-1} I_G \hat{B} = \alpha \otimes I_n + \beta_1 \otimes \begin{bmatrix}
B_1 & 0 & 0 \\
0 & B_1 & 0 \\
C_1' & 0 & A_{w_1}
\end{bmatrix} + \beta_2 \otimes \begin{bmatrix}
B_2 & 0 & 0 \\
0 & B_2 & 0 \\
C_2' & 0 & A_{w_2}
\end{bmatrix}.$$ 

Thus, every eigenvalue of $(\alpha \otimes I_m + \beta_1 \otimes B_1 + \beta_2 \otimes B_2)$ is also an eigenvalue of $I_G$. Therefore, $I_G$ has $mr$ eigenvalues with algebraic multiplicity at least 2.

In the case $k$ is even, we have

$$\hat{B}^{-1} A_l \hat{B} = \begin{bmatrix}
B_1 & 0 & 0 \\
a_i & a_i & 0 \\
0 & B_i & C_i''
\end{bmatrix}, \quad i = 1, 2,$$

where $B_i$ is a matrix of order $(m - 1) \times (m - 1)$. Consider again the basis $\hat{B} = I_r \otimes \bar{B}$. Then,

$$\hat{B}^{-1} I_G \hat{B} = \alpha \otimes I_n + \beta_1 \otimes \begin{bmatrix}
B_1 & 0 & 0 \\
0 & a_1 & 0 \\
0 & 0 & B_1 \\
C_1'' & 0 & A_{w_1}
\end{bmatrix} + \beta_2 \otimes \begin{bmatrix}
B_2 & 0 & 0 \\
0 & a_2 & 0 \\
0 & 0 & B_2 \\
C_2'' & 0 & A_{w_2}
\end{bmatrix}.$$ 

Thus, every eigenvalue of $(\alpha \otimes I_{m-1} + \beta_1 \otimes B_1 + \beta_2 \otimes B_2)$ is also an eigenvalue of $I_G$. Therefore, $I_G$ has $(m - 1)r$ eigenvalues with algebraic multiplicity at least 2.

### 4.3 $\Sigma_S$-Interior symmetry with $D_k \subseteq \Sigma_S \subseteq S_k$

Let $G$ be an $n$-cell homogeneous network with $s$ types of arrows with adjacency matrices $A_1, \ldots, A_s$. Assume that every $A_l$, for $l = 1, \ldots, s$, has an interior symmetry $\Sigma_S^l$ on the same subset $S \subseteq C$. Assume also that $D_k \subseteq \Sigma_S^l \subseteq S_k$, for $l = 1, 2, \ldots, s$. Let $\hat{A}_l$ denote the upper left $k \times k$-submatrix of $A_l$, $l = 1, \ldots, s$ (cf. (3.29)). Then, $\hat{A}_l$ is a circulant matrix of the form (3.30). We show that the multiplicity of the eigenvalues of $I_G$ can be directly analyzed by the eigenvalues of $\hat{A}_1, \ldots, \hat{A}_s$.

**Theorem 4.3** Let $G$ be an $n$-cell homogeneous network with $s$ types of arrows, where every adjacency matrix $A_l$ has an interior symmetry $\Sigma_S^l$ on the same subset $S \subseteq C$ such that $D_k \subseteq \Sigma_S^l \subseteq S_k$, for $l = 1, 2, \ldots, s$. Let $\hat{A}_l$ be the upper left $k \times k$-submatrix of $A_l$, for $l = 1, 2, \ldots, s$. Let $\lambda_j^{(l)}$ be the $j$-th eigenvalue of $\hat{A}_l$, for $j = 0, 1, \ldots, k-1, l = 1, 2, \ldots, s$ (cf. (2.4)). Then, every eigenvalue of $(\alpha + \sum_{l=1}^{s} \lambda_j^{(l)} \beta_l)$ is an eigenvalue of $I_G$, for $j = 1, \ldots, k-1$. 

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Proof. We only present the proof for the case $s = 2$. The general case can be proved analogously. Without loss of generality we assume $S = \{1, \ldots, k\}$.

Let $B$ be the basis given by Theorem 3.11. Then, we have

$$B^{-1}A_iB = \begin{bmatrix} \Lambda_l & 0 \\ C_l & Q_l \end{bmatrix}, \quad \text{for } \Lambda_l = \begin{bmatrix} \lambda_1^{(l)} & 0 & \cdots & 0 \\ 0 & \lambda_2^{(l)} & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_{k-1}^{(l)} \end{bmatrix}, \quad l = 1, 2.$$

Consider the basis $\hat{B} = I_r \otimes B$. Then,

$$\hat{B}^{-1}I_G \hat{B} = \begin{bmatrix} \alpha \otimes I_{k-1} + \beta_1 \otimes \Lambda_1 + \beta_2 \otimes \Lambda_2 & 0 \\ \beta_1 \otimes C_1 + \beta_2 \otimes C_2 & \alpha \otimes I_{n-k+1} + \beta_1 \otimes Q_1 + \beta_2 \otimes Q_2 \end{bmatrix}$$

Let $u \in \mathbb{R}^r$ and $v_j$ be the eigenvector of $\lambda_j^{(1)}$ and $\lambda_j^{(2)}$ given by (2.3), for some $j \in \{1, 2, \ldots, k-1\}$. Then,

$$(\alpha \otimes I_{k-1} + \beta_1 \otimes \Lambda_1 + \beta_2 \otimes \Lambda_2)(u \otimes v_j) = \alpha u \otimes v_j + \beta_1 u \otimes \Lambda_1 v_j + \beta_2 u \otimes \Lambda_2 v_j = \alpha u \otimes v_j + \beta_1 u \otimes \lambda_j^{(1)} v_j + \beta_2 u \otimes \lambda_j^{(2)} v_j = (\alpha + \lambda_j^{(1)} \beta_1 + \lambda_j^{(2)} \beta_2) u \otimes v_j.$$

Thus, every eigenvalue of $(\alpha + \lambda_j^{(1)} \beta_1 + \lambda_j^{(2)} \beta_2)$ is an eigenvalue of $(\alpha \otimes I_{k-1} + \beta_1 \otimes \Lambda_1 + \beta_2 \otimes \Lambda_2)$, which is also an eigenvalue of $I_G$.

Example 4.4. Let $n > 12$. Let $G$ be an $n$-cell homogeneous network with 2 types of arrows whose adjacency matrices $A_1, A_2$ have an interior symmetry (respectively)

$$\Sigma_i^1 = \langle D_{12}, (1 5 9)(2 3 4 6 7 8 10 11 12) \rangle,$$
$$\Sigma_i^2 = \langle D_{12}, (4 10)(1 2 3 5 6 7 8 9 11 12) \rangle,$$

on cells $\{1, 2, \ldots, 12\}$. Note that $D_{12} \subseteq \Sigma_i^i \subseteq S_{12}$, for $i = 1, 2$. An example of $A_1, A_2$ is

$$A_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$
Figure 11: The 12-cell homogeneous network $G$ in Example 4.4 with interior symmetries $\Sigma_1 = \langle D_{12}, (1\ 5\ 9)(2\ 3\ 4\ 6\ 7\ 8\ 10\ 11\ 12) \rangle$ and $\Sigma_2 = \langle D_{12}, (4\ 10)(1\ 2\ 3\ 5\ 6\ 7\ 8\ 9\ 11\ 12) \rangle$, for $S = \{1, \ldots, 12\}$.

Let $\bar{A}_l$ be the upper left $12 \times 12$-submatrix of $A_l$ and $\lambda^{(l)}_j$ be the $j$-th eigenvalue of $\bar{A}_l$ (cf. (2.4)), for $j = 0, 1, \ldots, 11, l = 1, 2$. Then, we have (cf. Example 3.12 (iii)-(iv))

$$\lambda^{(1)}_{1,11} = \lambda^{(1)}_{2,10} = \lambda^{(1)}_{5,7}, \quad \lambda^{(1)}_{3,9} = \lambda^{(1)}_{6}$$

and

$$\lambda^{(2)}_{1,11} = \lambda^{(2)}_{3,9} = \lambda^{(2)}_{5,7}, \quad \lambda^{(2)}_{2,10} = \lambda^{(2)}_{6}.$$ 

Thus, by Theorem 4.3, for every homogeneous network $G$ with interior symmetries $\Sigma_1$, and $\Sigma_2$, every eigenvalue of $(\alpha + \lambda^{(1)}_{1,5,7,11} \beta_1 + \lambda^{(2)}_{1,5,7,11} \beta_2)$ is an eigenvalue of $J_G$ of multiplicity at least 4; every eigenvalue of $(\alpha + \lambda^{(1)}_{2,10} \beta_1 + \lambda^{(2)}_{2,10} \beta_2)$ is an eigenvalue of $J_G$ of multiplicity at least 2; every eigenvalue of $(\alpha + \lambda^{(1)}_{3,9} \beta_1 + \lambda^{(2)}_{3,9} \beta_2)$ is an eigenvalue of $J_G$ of multiplicity at least 2; every eigenvalue of $(\alpha + \lambda^{(1)}_{4,8} \beta_1 + \lambda^{(2)}_{4,8} \beta_2)$ is an eigenvalue of $J_G$ of multiplicity at least 2. 

5 Conclusions

Interior symmetry may be viewed as an appropriate generalization of symmetry for coupled cell networks. Besides the original concept of interior symmetry, we introduced further notions including quotient interior symmetry, reverse interior symmetry and quotient reverse interior symmetry.
For homogeneous coupled cell systems, we analyzed how multiple eigenvalues of the Jacobian at fully-synchronized equilibria may occur due to these different types of interior symmetry. The groups of interior symmetry that we focused on are symmetric groups, alternating groups, dihedral groups, cyclic groups and their products.

Based on our analysis, we concluded that the eigenvalue multiplicity of the Jacobian is sensitively dependent on the interior symmetric properties of the underlying network structure, and that symmetry alone is not sufficient to explain.

Indeed, in the examples we present throughout the paper, all the multiple eigenvalues are a consequence of an interior symmetry, in one form or another. In the case of uniform networks, already a relative weak interior symmetry may be sufficient to give rise to multiple eigenvalues.

Since, very easily, a homogeneous network has some type of interior symmetry we can say that multiple eigenvalues of the Jacobian at a fully synchronous equilibrium are frequent for homogeneous coupled cell systems.

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References


