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Applied Equivariant Degree. Part III: Global Symmetric Hopf Bifurcation for Functional Diferential Equations

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In this paper we apply the equivariant degree method to a global Hopf bifurcation problem for a system of symmetric functional differential equations to analyse a continuation of symmetric branches of non-constant periodic solutions. As examples, we show that for symmetric configurations of identical oscillators, with dihedral or tetrahedral symmetries, there are periodic solutions with the indicated symmetries for all sufficiently large values of a parameter.

Key words: *equivariant degree, global Hopf bifurcation, symmetric configuration of oscillators.*

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1. INTRODUCTION

The present paper is the third in a series devoted to the equivariant degree theory and its applications to nonlinear problems admitting a certain (in general, non-abelian) compact Lie group of symmetries.

Let $h : \mathbb{R} \to \mathbb{R}$ be a continuously differentiable function, $h(t) \neq 0$ for all $t \in \mathbb{R}$, $h(0) > 0$, and $g : \mathbb{R} \to \mathbb{R}$ a continuously differentiable map such that $g(0) = 0$ and $g'(0) > 0$. Consider a system of FDEs of the type

$$\frac{d}{dt}x(t) = -\alpha x(t) + \alpha H(x(t)) \cdot C(G(x(t-1))), \tag{1}$$

where

$$x = \begin{bmatrix} x^1 \\ x^2 \\ \vdots \\ x^n \end{bmatrix}, G(x) = \begin{bmatrix} g(x^1) \\ g(x^2) \\ \vdots \\ g(x^n) \end{bmatrix}, H(x) = \begin{bmatrix} h(x^1) \\ h(x^2) \\ \vdots \\ h(x^n) \end{bmatrix},$$

$$x \cdot y = \begin{bmatrix} x^1 \\ x^2 \\ \vdots \\ x^n \end{bmatrix} \cdot \begin{bmatrix} y^1 \\ y^2 \\ \vdots \\ y^n \end{bmatrix} = \begin{bmatrix} x^1 y^1 \\ x^2 y^2 \\ \vdots \\ x^n y^n \end{bmatrix}$$

and C is a symmetric $n \times n$ matrix (such a system describes n identical cells coupled by diffusion between certain selected cells (see (Alexander and Auchmuty, 1986; Ashwin *et al.*, 1990; Balanov *et al.* 2005a; 2005b; Krawcewicz and Wu, 1999; Krawcewicz *et al.*, 1993; Turing, 1952; Wu, 1998) and references therein for more physical details)).

Suppose, in addition, that the geometrical configuration of these cells has a symmetry group Γ. The group Γ permutes the vertices of the related polygon or polyhedron, which means it acts on $\mathbb{R}^n$ by permuting the coordinates of the vectors $x \in \mathbb{R}^n$. Clearly, system (1) is symmetric with respect to the corresponding Γ representation on $V := (\mathbb{R}^n, \Gamma)$ (C commutes with it). Put $S^1 := \mathbb{R}/2\pi\,\mathbb{Z}$ and let $W := \{S^1 \to V\}$ be a "reasonable" functional space where periodic solutions to system (1) "live". Consider the natural $\Gamma \times S^1$ action on functions $x \in W$ defined by $(\gamma, e^{i\tau})x(t) := \gamma(x(t + \tau))$, $\forall \gamma \in \Gamma$ and $e^{i\tau} \in S^1$, and take a function $x_0 \in W$ being a periodic solution to (1). A subgroup $H = \{h \in \Gamma \times S^1\}$ is called a *symmetry* of x_o.

The goal of our paper is to study the *global* behaviour of branches of nonconstant periodic solutions to (1) bifurcating from the trivial one according to their symmetries. Being restricted in size, we consider here only the case where Γ is a dihedral or tetrahedral group (see Theorems 3.1 and 3.2). Our approach to study system (1) is based on the equivariant degree theory (a short exposition to this concept is given in Section 2). For the historical comments and bibliography (including the symmetric *local* bifurcation phenomenon in system (1)) we refer to (Balanov *et al.*, 2002; Balanov *et al.*, 2005a; Chow and Mallet-Paret, 1978; Chow *et al.*, 1978; Erbe *et al.*, 1992; Krawcewicz *et al.*, 1997; Krawcewicz and Wu, 1997; Krawcewicz and Wu, 1999;

Krawcewicz *et al.*, 1993; Mallet-Paret, 1997; Mallet-Paret and Yorke, 1982; Wu, 1998).

2. METHOD

2.1. Equivariant Jargon and Notations

To give an exposition to the equivariant degree theory we need some preliminaries.

Hereafter, $G = \Gamma \times S^1$, where Γ is a finite group. For a closed subgroup H of G we denote by (H) the conjugacy class of H in G, $N(H)$ is the normalizer of H in G, $W(H) = N(H)/H$ is the Weyl group of H in G and $\mathcal{J}(G)$ is the set of all conjugacy classes in G (this set admits the natural partial order: $(K) \le (H)$ if K is conjugate to a subgroup of H).

Let W be an isometric Banach Grepresentation. For $x \in W$ denote by $G_x = \{g \in G : gx = x\}$ the *isotropy group* of x and we will call the conjugacy class (G_x) the *orbit type* of x in W. Let $X \subset W$ be an invariant subset. We will denote by $\mathcal{J}(G, X)$ the set of all orbit types (G_x) for $x \in X$ and put $X^H := \{x \in X : G_x \supset H\}$, $X_H := \{x \in X : G_x = H\}$, $X_{(H)} := G(X_H)$.

Let Ω be an open bounded G-invariant subset of $\mathbb{R} \oplus W$ (G acts trivially on $\mathbb{R}$) and let $f : \mathbb{R} \oplus W \to W$ be a continuous equivariant map (i.e. $f(gx) = gf(x)$ for all $g \in G$ and $x \in \overline{\Omega}$). Then f is called Ω-*admissible* if $f(x) \ne 0$ for all $x \in \partial\Omega$. The pair (f, Ω) is then called an admissible pair.

We refer to (Bredon, 1972) for the equivariant jargon frequently used throughout the present paper.

2.2. Twisted Subgroups and Dominating Orbit Types

Hereafter, V stands for an *orthogonal* G-representation. Consider the set

$$\Phi_1(G) := \{(H) \in \mathcal{J}(G) : \dim W(H) = 1\}.$$

It is easy to check that the elements of $\Phi_1(G)$ are the conjugacy classes (H) of the socalled φ-*twisted l-folded* subgroups of $\Gamma \times S^1$ with $l = 0, 1, 2, \ldots$, i.e.

$$H = K^{\varphi,l} := \{(\gamma, z) \in K\ S^1 : \varphi(\gamma) = z^l\},$$

where K is a subgroup of Γ and $\varphi : K \to S^1$ is a homomorphism. Notice that $N(K^{\varphi,l}) = N_0 \times S^1$, where $N_o = \{\gamma \in N(K): \forall_{k \in K} \varphi(\gamma k \gamma^{-1}) = \varphi(k)\}$. Therefore, for every $(H) \in \Phi_1(G)$ the Weyl group $W(H)$ has a natural orientation invariant with respect to the right and left translations. We will also assume that, for every orbit type (H) in $\mathcal{J}(G, V)$ there is a fixed orientation of the subspace V^H (and, consequently, also of the space $R \oplus V^H$).

We denote by $A_1(G)$ the free $\mathbb{Z}$-module generated by the symbols $(H) \in \Phi_1(G)$. Then any element $\alpha \in A_1(G)$ can be written as a finite sum

$$\alpha = n_{H_1}(H_1) + n_{H_2}(H_2) + \ldots + n_{H_r}(H_r), \quad n_{H_i} \in \mathbb{Z}.$$

Definition 2.1. Let W be a Banach G-representation and let $\Phi_1(G, W)$ be the set of all orbit types $(H) \in \mathcal{J}(G, W)$ with $(H) \in \Phi_1(G)$. The maximal orbit types in $\Phi_1(G, W)$ are called *dominating* orbit types in W.

Remark 2.1. Dominating orbit types occuring in W can be easily recognised from isotropy lattices of irreducible components of W.

2.3. Primary Equivariant Degree with One Free Parameter

The following Theorem (cf. Balanov *et al.*, 2005c), provides the standard properties of the socalled *primary* G-equivariant degree $G\text{-Deg}(f, \Omega) \in A_1(G)$ of an admissible pair (f, Ω).

Theorem 2.1. *There exists a unique function, denoted by* G-Deg, *assigning to each admissible pair* (f, Ω), $\Omega \subset \mathbb{R} \oplus V$, *an element* $G\text{-Deg}(f, \Omega) \in A_1(G)$ *satisfying the following properties:*

(P1) (Existence) *If* $G\text{-Deg}(f, \Omega) = \sum_{(H)} nH(H)$ *is such that* $n_{H_o} \ne 0$ *for some* $(H_o) \in \Phi_1(G)$, *then there exists* $x_0 \in \Omega$ *with* $f(x_0) = 0$ *and* $G_{x_0} \supset H_0$.

(P2) (Additivity) *Assume that* Ω_1 *and* Ω_2 *are two G-invariant open disjoint subsets of* Ω *such that* $f^{-1}(0) \cap \Omega \subset \Omega_1 \cup \Omega_2$. Then $G\text{-Deg}(f, \Omega) = G\text{-Deg}(f, \Omega_1) + G\text{-Deg}(f, \Omega_2)$.

(P3) (Homotopy) *Suppose that* $h : [0, 1] \times \mathbb{R} \oplus V \to V$ *is an* Ω-*admissible G-equivariant homotopy. Then* $G\text{-Deg}(h_t, \Omega)$ = constant, *where* $h_t := h(t, \cdot)$.

(P4) (Suspension) *Suppose* V' *is another orthogonal G-representation and let U be an open, bounded G-invariant neighborhood of 0 in* V'. *Then* $G\text{-Deg}(f \times \mathrm{Id}, \Omega \times U) = G\text{-Deg}(f, \Omega)$.

(P5) (Normalisation) *Suppose f is normal* (cf. Balanov *et al.*, 2002) *and* $f(x_0) = 0$ *for some* $x_0 \in \Omega$ *with* $(G_{x_0}) = (H) \in \Phi_1(G)$. Let $U_{G(x_0)}$ *be an invariant tube around the orbit* $G(x_0)$, S_{x_0} *a positively oriented slice to* $W(H)(x_0)$ *in* $\mathbb{R} \oplus V^H$ *and* $f^{-1}(0) \cap U_{G(x_0)} = G(x_0)$. *Then*

$$G\text{-Deg}(f, \mathcal{U}_{G(x_0)}) = (\text{sign}\ \det(D f(x_0)|_{S_{x_0}})) \cdot (H).$$

(P6) (Elimination) *Suppose f is normal in* Ω *and* $\Omega_H \cap f^{-1}(0) = \emptyset$ *for every* $(0) \in \Phi_1(G)$. *Then* $G\text{-Deg}(f, \Omega) = 0$.

(P7) (Multiplication) *Let* $A(\Gamma)$ *be the Burnside ring of* Γ. *There exists a multiplication* $\cdot : A(\Gamma) \times A_1(G) \to A_1(G)$ *such that for any orthogonal* Γ-*representation* V' *and continuous equivariant map* $g : V' \to V'$ *one has*

$$G\text{-Deg}(f \times g, \Omega \times B) = \Gamma\text{-Deg}(g, B) \cdot G\text{-Deg}(f, \Omega),$$

where $B \subset V'$ *is the unit ball,* $g(x) \ne 0$ *for* $x \in \partial B$ *and* Γ-Deg *stands for the equivariant degree without free parameters* (see, for instance, Krawcewicz and Wu, 1997).

Remark 2.2. (i) The simplest equivariantly homotopically nontrivial maps for which G-Deg can be easily evaluated are defined as follows. Let ν be an orthogonal irreducible G-representation, $\mathcal{O} := \{(t,v) \in \mathbb{R}\oplus V : -1 < t < 1, \|v\| < 2\}$. Define $\mathfrak{b} : \overline{\mathcal{O}} \to V$ (the socalled *basic map associated to* $\mathcal{V}$) by $\mathfrak{b}(t, v) := (1 - \|v\| + it)v$ and put $\deg_{\mathcal{V}} := G\text{-Deg}(\mathfrak{b}, \mathcal{O})$ (cf. Balanov *et al.*, 2005b; 2005c).

(ii) In a standard way (cf. Ize and Vijnoli, 2003; Krawcewicz and Wu, 1997) the concept of the (primary) equivariant degree can be extended to admissible pairs (f, Ω) with $\Omega \subset \mathbb{R}\oplus W$ and f being a *compact vector field* on the Banach G-representation $\mathbb{R}\oplus W$.

2.4. Application of Equivariant Degree: Abstract Setting for Symmetric Global Bifurcation

Let $F : \mathbb{R}^2 \oplus W \to W$ be a G-equivariant map. Assume:

(H1) F is a compact vector field of class C^1 and $F(\lambda, 0) = 0$ for all $(\lambda, 0) \in \mathbb{R}^2 \oplus W$;

(H2) The set $\Lambda := \{\lambda \in \mathbb{R}^2 : D_w F(\lambda, 0) : W \to W$ is not an isomorphism $\}$ is discrete in $\mathbb{R}^2$;

(H3) $D_{w_0} F\big|_{\mathbb{R}^2 \oplus W^{S^1}} (\lambda, 0)$ is an isomorphism from W^{S^1} to W^{S^1} for all $\lambda \in \mathbb{R}^2$ and $w_0 \in W^{S^1}$.

We are interested in solutions to the equation

$$F(\lambda, w) = 0, \quad (\lambda, w) \in \mathbb{R}^2 \oplus W. \tag{2}$$

By obvious reasons, the points $(\lambda, 0)$ are called trivial solutions to (2) (cf. (H1)). All other solutions will be called nontrivial. A trivial solution $(\lambda_0, 0)$ is called a bifurcation point of (2) if in every neighbourhood of $(\lambda_0, 0)$ in $\mathbb{R}^2 \oplus W$ there exists a nontrivial solution to (2). Since, by implicit function theorem, we have that $\lambda_0 \in \Lambda$ if $(\lambda_0, 0)$ is a bifurcation point, we obtain that the set of bifurcation points is discrete in $\mathbb{R}^2$ (cf. (H2)). By the same reason, condition (H3) excludes the possibility of a bifurcation of S^1-fixed points from $(\lambda_0, 0)$.

Observe, first, that the primary equivariant degree can be easily applied as a local bifurcation invariant for the equation (2). For this purpose, consider a point $\lambda_0 \in \Lambda$ and choose $\varepsilon, r > 0$ such that $\overline{D(\lambda_0, \varepsilon)} \cap \Lambda = \{\lambda_0\}$, where $D(\lambda_0, \varepsilon) := \{\lambda : |\lambda - \lambda_0| < \varepsilon\}$, $F(\lambda, w) \neq 0$ for $|\lambda - \lambda_0| = \varepsilon$ and $0 < \|w\| < r$. Then we can define the set

$$\Omega(\lambda_0) := \{(\lambda, w) \in \mathbb{R}^2 \oplus W : |\lambda - \lambda_0| < \varepsilon, \|w\| < r\}, \tag{3}$$

and choose a G-invariant function $\theta_0 : \overline{\Omega(\lambda_0)} \to \mathbb{R}$ such that

$$\begin{cases} \theta_0(\lambda, w) < 0 & \text{if } w = 0; \\ \theta_0(\lambda, w) > 0 & \text{if } \|w\| = r \end{cases} \tag{4}$$

It is clear that the map $F_{\theta_0} : \overline{\Omega(\lambda_0)} \to \mathbb{R}\oplus W$, defined by

$$F_{\theta_0}(\lambda, w) = (\theta_0(\lambda, w), F(\lambda, w)),$$

is $\Omega(\lambda_0)$-admissible, therefore (cf. (H1)) we can define the element $\omega(\lambda_0) \in A_1(G)$ by

$$\omega(\lambda_0) := G\text{-Deg}(F_{\theta_0}, \Omega(\lambda_0)). \tag{5}$$

It can be verified that the element $\omega(\lambda_0)$ is independent of a choice of $\varepsilon, r > 0$. By applying the properties of the primary equivariant degree we immediately obtain the following *local bifurcation theorem*:

Theorem 2.2. *If $F : \mathbb{R}^2 \oplus W \to W$ satisfies the assumptions (H1)–(H3), and $\omega(\lambda_0)$ defined by (3)–(5) satisfies $\omega(\lambda_0) \neq 0$, then $(\lambda_0, 0)$ is a bifurcation point of (2). More precisely, if*

$$\omega(\lambda_0) = \sum_{(H)} n_H (H) \quad \text{with} \quad n_{H_o} \neq 0,$$

then there exists a branch of nontrivial solutions (λ, u) bifurcating from $(\lambda_0, 0)$ such that $G_{(\lambda,w)} \supset H_o$. If, in addition, (H_o) is a dominating orbit type, then $G_{(\lambda,w)} = H_o$.

Below we will discuss the global bifurcation phenomenon for the equation (2). Let $\mathcal{S}$ be the closure of the set of all nontrivial solutions to (2). Notice that $(\lambda_0, 0)$ is a bifurcation point of (2) if $(\lambda_0, 0) \in \mathcal{S}$. Take a connected compontent $\mathcal{C} \subset \mathcal{S}$. If $\mathcal{C}$ contains a bifurcation point $(\lambda, 0)$, $\mathcal{C}$ is clearly G-invariant. Notice that, in general, $\mathcal{C}$ may be composed of several orbit types, i.e. $\mathcal{C} = \cup_{(H)} \mathcal{C}_{(H)}$, and the global behaviour of $\mathcal{C}_{(H)}$ can be different for different orbit types (H), for example, some of the branches $\mathcal{C}_{(H)}$ may be bounded, while the others are unbounded.

The following result can be proved in a standard way and considered as a *global bifurcation theorem*.

Theorem 2.3. *Assume $F : \mathbb{R}^2 \oplus W \to W$ satisfies the assumptions (H1)–(H3) and let $\mathcal{C}_{(H_o)}$ be a bounded connected component of $\mathcal{S}_{(H_o)}$ such that $\mathcal{C}_{(H_o)} \cap \mathbb{R}^2 \times \{0\} = \{(\lambda_1, 0), (\lambda_2, 0), \ldots, (\lambda_N, 0) \neq \emptyset\}$, where (H_o) is a dominating orbit type in W (see Definition 2.1 and Remark 2.1). Suppose that $\omega(\lambda_k) = \sum_N {}_{(H)} n_H^k (H)$, where $\omega(\lambda_k)$ are defined by the formulae similar to (3)–(5). Then $\sum_{k=1}^N n_{H_0}^k = 0$.*

Corollary 2.1. *Under the assumptions of Theorem 2.3 suppose that $N = 1$ and $n_{H_0}^1 \neq 0$. Then $\mathcal{C}_{(Ho)}$ is unbounded.*

2.5. Operator Reformulation for System (1)

The problem of finding periodic solutions to (1) can be reformulated as an equation of type (2), where $W := H^1(S^1; V)$ is the the first Sobolev space of V-valued functions, F is given by

$$F(\alpha, \beta, x)(t) = (L + K)^{-1}[-\frac{\alpha}{\beta}(x(t)) - H(x(t)) \cdot C(G(x(t - \beta)) + Kx], \quad \alpha \in \mathbb{R},\ \beta > 0$$

$Lx = \dot{x}, Kx = \frac{1}{2\pi}\int_0^{2\pi} x(s)ds$ (we identify x with its imbedding into $C(S^1, V)$; see Balanov *et al.*, 2005a for the details). Obviously, F satisfies (H1). To make F satisfy (H3), it is enough to assume

$$\frac{1}{\eta} \notin \sigma(C) \qquad (\eta := h(0)g'(0),\ \sigma(C)) \text{ denotes the spectrum of C}). \quad (6)$$

To take advantage of Theorems 2.2 and 2.3, we need (i) to study the set Λ and (ii) to evaluate $\omega(\lambda_0) \in A_1(G)$ for $\lambda_0 \in \Lambda$ (cf. (H2)).

By taking the linearisation of (1) at zero and using the standard characteristic operator techniques (cf. Balanov *et al.*, 2005a; Krawcewicz and Wu, 1999), one can easily prove

Lemma 2.1. *(i) Take* $0 > \xi_0 \in \sigma(C)$. *Assume* η *from (6) satisfies* $\xi_0\eta < -1$. *Take* $\beta_0\ [\pi/2, \pi]$ *with* $\cos\beta_0 = \frac{1}{\eta\xi_0}$ *and* $\alpha_0 = -\beta_0 \cot\beta_0$. *Then* $(\alpha_0, \beta_0) \in \Lambda$. *(ii) If* $(\alpha_0, \beta) \in \Lambda$ *with* α_0 *as in (i), then there exists a positive integer l such that* $\beta = l\beta_0$. *(iii) If* $\sigma(C)$ *is composed of negative numbers and* η *is big enough, then each element of* Λ *can be obtained in the way described in (i) and (ii).*

To compute $\omega(\lambda_0)$ for $\lambda_0 \in \Lambda$, observe that the S^1action on W induces the S^1-isotypical decomposition: $W = V \oplus \overline{\bigoplus_{l=1}^{\infty}} W_l$, l = 1,2,..., where V = is W^{S^1} is the subspace of constant functions and $W_l = \{e^{ilt}(x_n + iy_n) : x_{n,y_n} \in V\}$. Also, V and W_1 are Γ-invariant and W_1 as a complex Γ-representation is isomorphic to the complexification V^c of V. Thus, the Γ-isotypical decomposition of V together with the G-isotypical decomposition of W_l induce the G-isotypical decomposition $W = V_0 \oplus \ldots \oplus \bigoplus_{j,l} V_r V_{j,l}$, where V_i (resp. $V_{j,l}$) is modelled on $\mathcal{V}_i$ (resp. $\mathcal{V}_{j,1}$), j = 1, ..., s.

Take ξ_0 and η from Lemma 2.1(i) and consider the Γ-isotypical decomposition of the eigenspace $E(\xi_0)$: $E(\xi_0) = E_0(\xi_0)E_1(\xi_0) \oplus \ldots \oplus E_r(\xi_0)$, where $E_i(\xi_0)$ is modelled on $\mathcal{V}_i$. Denote by $\deg^{\mathcal{V}_i}$ the Γ-equivariant degree of the operator $-\mathrm{Id} : V_i \to V_i$ on the unit ball in $\mathcal{V}_i$. Combining the compactness argument with Properties (P1)–(P7) one can easily prove

Lemma 2.2. *Assume* $\sigma(C) = \{\xi_1, \ldots, \xi_k\}$ *is composed of negative numbers,* η *from (6) is positive and satisfies* $\eta\xi_q < -1$ *for all* $\xi_q \in \sigma(C)$. *Take* $(\alpha_0, \beta_0) \in \Lambda$. *Then there exist integers* $1 \le l_1 < l_2 < \ldots < l_M$ *such that*

$$\omega(\alpha_0, \beta_0) = \prod_{q=1}^{k}\prod_{i=1}^{r} (\deg^{\mathcal{V}_i})^{m_i(\xi_q)} \cdot \sum_{j=1}^{s} (t_{j,1}(\alpha_0, l_1\beta_0)\deg_{\mathcal{V}_{j,l_1}} + \ldots + t_{j,1}(\alpha_0, l_M\beta_0)\deg_{\mathcal{V}_{j,l_M}}) \quad (7)$$

where $m_i(\ _q)$ = dim $E_i(\ _q)$/dim$\mathcal{V}_i$, *the powers are taken in* $A(\ )$, *" " stands for the multiplication* $A(\ )\quad A_1(G)$ $A_1(G)$ *(see Property (P7)) and* $t_{j,1}(\alpha_0, l_p\beta_0)$ *denotes the (j,* l_p*)-crossing number at* (α_0, β_0) (see Balanov *et al.*, 2005a).

3. RESULTS

3.1. System with Dihedral Symmetries

We consider here the system of equations (1) with the $n\times n$-matrix C (n an even number) of the type

$$C = \begin{bmatrix} -3 & 1 & 0 & \ldots & 0 & 1 \\ 1 & -3 & 1 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 0 & 0 & \ldots & 1 & -3 \end{bmatrix} \quad (8)$$

This system is symmetric with respect to the dihedral group $\Gamma = D_0$ acting on $V = \mathbb{R}_n$ by permuting the coordinates of vectors. Up to equivalence, the D_n isotypical components of V are (see Balanov *et al.*, 2005a; 2005b for notations and details):

(i) the trivial onedimensional representation $\mathcal{V}_0$ with $\deg^{\mathcal{V}^0} = -(D_n)$;

(ii) the twodimensional (irreducible) representations $\mathcal{V}_j$, j = 1, ..., $\frac{n}{2} - 1$, given by $\gamma z = \gamma^j \cdot z$ for $\gamma \in \mathbb{Z}_n$ and $kz = \bar{z} (z \in \mathbb{C})$, where "·" denotes the complex multiplication. Put $h := \gcd(j, n)$ and $m = \frac{n}{h}$. Then $\deg^{\mathcal{V}_j} = (D_n) - 2(D_h) - 2(\bar{D}_h) + 2(\mathbb{Z}_h)$ for m odd and $deg^{\mathcal{V}_j} = (D_n) - (D_h) - (\bar{D}_h) + (\mathbb{Z}_h)$ for m even;

(iii) the onedimensional representation $\mathcal{V}\frac{n}{2}$ given by $d : D_n \to \mathbb{Z}_2 \subset O(1)$ with $\deg^{V^{\frac{n}{2}}} = (D_n) - (D_{\frac{n}{2}})$,

Observe that each subspace $\mathcal{V}_j$, $j = 0, 1, \ldots, \frac{n}{2}$, is an eigenspace of the matrix C corresponding to the eigenvalue $\xi_j := 2\cos\frac{2\pi j}{n} - -3 < 0$.

Further, any $\mathcal{V}_j$, $j = 0, 1, \ldots, \frac{n}{2}$, is of real type. Therefore, any G-isotypical component of W_1 is an irreducible representation $\mathcal{V}_{j,1}$ obtained from the corresponding $\mathcal{V}_j$ by complexification and defining the S^1-action by $zv = z \cdot v$ for $z \in S^1$ and $v \in \mathcal{V}\frac{c}{j}$, where "·" stands for the complex multiplication. We have (cf. Balanov *et al.*, 2005a; 2005b for notations and details):

$$\deg_{\mathcal{V}_{j,1}} = (\mathbb{Z}_n^{t_j}) + (D_h) + (D_h^z) - (\mathbb{Z}_h) \quad \text{if } m \text{ is odd}, 1 \le j < n/2$$
$$\deg_{\mathcal{V}_{j,1}} = (\mathbb{Z}_n^{t_j}) + (D_{2h}^d) + (\tilde{D}_{2h}^d) - (\mathbb{Z}_h) \quad \text{if } m \equiv 2(\text{mod } 4), 1 \le j < n/2$$
$$\deg_{\mathcal{V}_{j,1}} = (\mathbb{Z}_n^{t_j}) + (D_{2h}^d) + (\hat{D}_{2h}^d) - (\mathbb{Z}_h) \quad \text{if } m \equiv 0(\text{mod } 4), 1 \le j < n/2$$
$$\deg_{\mathcal{V}_{j,1}} = (D_n^d) \quad \text{if } j = n/2$$

One can easily observe that $(Z_n^{t_j}, 0 < j < n/2)$ are dominating orbit types in W. Take $(\alpha_n, \beta_n) \in \Lambda$ which is, obviously, related to the orbit type (D_n^d) (cf. Definition 2.1, Remark 2.1

and the list of $deg_{\mathcal{V}_{j,1}}$ given above). As it was proved in Balanov *et al.*, (see also Theorem 2.2), there is a branch $\mathcal{C}(D_n^d)$ of the orbit type (D_n^d) bifurcating from $(\alpha_{\frac{n}{2}}, \beta_{\frac{n}{2}}, 0)$.

The following problem arises: *what can one say about the global behaviour of this branch?*

Theorem 3.1. *(i) Consider system (1) with C given by (8) and suppose* $\eta := h(0)g'(0) > 1$. *Assume:*

$(A1)$ $\dfrac{tg(t)}{h(t)} > 0$ *for all* $t \neq 0$; $\lim\limits_{t\to\infty} \dfrac{tg(t)}{h(t)} = \infty.$

Then the branch $C_{(D_n^d)}$ *in of periodic solutions bifurcating from* $(\alpha_{\frac{n}{2}}, \beta_{\frac{n}{2}}, 0)$ *is unbounded in* $\mathbb{R}^2 \oplus W$

(ii) Assume, in addition, the following condition is satisfied:

(A2) There exist constants A, B > 0 and $\delta, \gamma > 0$ *with* $1 > \delta + \gamma$ *such that*

$$|h(t)| \le A + B|t|^{\delta}, \quad |g(t)| \le A + B|t|^{\gamma}. \tag{9}$$

Then

$$[\alpha_{n/2}, \infty) \subset \left\{\alpha : (\alpha, \beta, \chi) \in \mathcal{C}_{(D_n^d)}\right\}$$

Proof: (i) Suppose that (α, β, x) is a solution to (1) belonging to $\mathcal{C}_{(D_n^d)}$. Recall that

$$D_n^d = \left\{(1,1), (\gamma, -1), \ldots, (\gamma^{n-1}, -1), (\kappa, 1), (\kappa\gamma, -1), \ldots, (\kappa\gamma^{n-1}, -1)\right\},$$

where γ is 2×2 matrix representing the complex multiplication by $e^{\frac{2\pi i}{n}}$ and $\kappa = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ is the operator of complex conjugation. Then, the symmetry properties of $x(t)$ can be translated as follows:

$x(t) = \begin{bmatrix} x^0(t) \\ x^1(t) \\ \vdots \\ x^{n-1}(t) \end{bmatrix}$ is a $\dfrac{2\pi}{\beta}$-periodic solution such that

$$x^k(t) = x^{k+1}\left(t - \frac{\pi}{\beta}\right) \qquad (\bmod\ n) \tag{10}$$

and

$$x^k(t) = x^{n-k-1}\left(t - \frac{\pi}{\beta}\right) \quad (\bmod\ n) \tag{11}$$

Combining (10), (11) with condition (A1) and applying the same argument as in Krawcewicz and Wu (1999), one can easily show that the periods $p = \dfrac{2\pi}{\beta}$ of solutions $(\alpha, \beta, x) \in \mathcal{C}_{(D_n^d)}$ satisfy the inequality 2 < p < 4. This fact together with the isotypical structure of V and W_1 described the above yield: (a) in Theorem 2.3, $N = 1$; (b) in Lemma 2.2, $M = 1$; (c) in formula (7) the only nontrivial crossing number is $t_{\frac{n}{2},1} = -1$. Since $\deg_{\mathcal{V}_j} \cdot (D_n^d), j = 0, 1, \ldots, \frac{n}{2}$, contains a nontrivial coefficient related to (D_n^d) (see Balanov *et all.*, 2005a),the application of Corollary 2.1 completes the proof.

(ii) By construction and argument given in (i), $\mathcal{C}_{(D_n^d)} \subset \mathbb{R} \times (\pi/2, \pi) \times W$. Further, using assumption (A2), one can easily show that there exists a constant M > 0 such that for every periodic solution $x(t)$ to (1) we have $\sup\{\|x(t)\| : t \in \mathbb{R}\} \le M$. Indeed, assume that $x(t)$ is a periodic solution of (1) and consider the function $r(t) := \|x(t)\|^2$. Since $r(t)$ is periodic, we have that there exists $t_0 \in \mathbb{R}$ such that

$$r(t_0) = \sup\{r(t) : t \in \mathbb{R}\}, \quad \text{and} \quad r'(t_0) = 0,$$

i.e. we have

$$0 = \left.\frac{dr}{dt}\right|_{t=t_0} = 2x(t_0) \bullet x'(t_0) = 2x(t_0) \bullet (-\alpha x(t_0) + \alpha H(x(t_0)) \cdot$$

$$C(G(x(t_0 - 1))) = -2\alpha\|x(t_0)\|^2 + 2\alpha x(t_0) \bullet (H(x(t_0)) \cdot$$
$$C(G(x(t_0 - 1)))),$$

where $\bullet$ stands for the inner product in V. Therefore, by (A2) we get

$$\|x(t_0)\|^2 \le \left|x(t_0) \bullet (H(x(t_0)) \cdot C(G(x(t_0 - 1)))\right|$$

$$\le \|x(t_0)\| \, \|C\| \, (A + B\|x(t_0)\|^{\delta}))(A + B\|x(t_0 + 1)\|^{\gamma}))|$$

$$\le c_0 + c_1\|x(t_0)\|^{\delta+1} + c_2\|x(t_0)\|^{\gamma+1} + c_3\|x(t_0)\|^{\delta+\gamma+1},$$

for certain constants c_0, c_1, c_2, $c_3 > 0$. Since $\delta + \gamma + 1 < 2$, it follows that there exists a constant $M > 0$ such that every solution s of the inequality

$$s^2 - c_3|s|^{\delta+\gamma+1} - c_2|s|^{\gamma+1} - c_1|s|^{\delta+1} - c_0 \le 0,$$

satisfies the inequality $|s| \le M$. Consequently,

$$\sup\{\|x(t)\| : t \in \mathbb{R}\} = \|x(t_0)\| \le M.$$

Thus, $C_{(D_n^d)} \subset \mathbb{R} \times (\pi/2, \pi) \times \{x \in W : \|x\| \le M\}$. Finally, system (1) has no nonconstant periodic solution for $\alpha = 0$, from which it follows $C_{(D_n^d)} \subset (0, \infty) \times (\pi/2, \pi) \times \{x \in W : \|x\| \le M\}$. However, by (i), the connected component $C_{(D_n^d)}$ is unbounded, therefore $[\alpha_{n/2}, \infty) \subset \left\{\alpha : (\alpha, \beta, x) \in C_{(D_n^d)}\right\}$. □

3.2. System with Tetrahedral Symmetries

We consider here the system of equations (1) with the matrix C given by

$$C = \begin{bmatrix} -4 & 1 & 1 & 1 \\ 1 & -4 & 1 & 1 \\ 1 & 1 & -4 & 1 \\ 1 & 1 & 1 & -4 \end{bmatrix} \tag{12}$$

This system is symmetric with respect to the tetrahedral group $\Gamma = A_4$ acting on $V = \mathbb{R}^4$ by permuting the coordinates of vectors. We have $\sigma(C) = \{\xi_0 = -1,\ \xi_1 = -5\}$. The isotypical decomposition of V takes the form: $V = V_0 \oplus V_1$, where V_0 (spanned by the vector $\langle 1,1,1,1 \rangle$) is the fixedpoint subspace of the A_4-action, and V_1 is equivalent to the natural three-dimensional representation of A_4. These two subspaces are the eigenspaces of the matrix C: the subspace V_0 corresponds to ξ_0 and V_1 to ξ_1. One can verify that the dominating orbit types in W are $(\mathbb{Z}_3^{t_1})$, $(\mathbb{Z}_3^{t_2})$, and (V_4^-) (see Balanob et all., 2005a; 2005b for the notations and details). Assuming $\eta > 1$, we are interested in the global behaviour of the branch $\mathcal{C}_{(V_4^-)}$ of periodic solutions to (1) bifurcating from $(\alpha_1, \beta_1, 0) \in \lambda \times \{0\}$.

Suppose that (α, β, x) is a solution to (1) belonging to $\mathcal{C}_{(V_4^-)}$. Recall that

$$V_4^- = \{((1),1),((12)(34),1),((13)(24),-1),((14)(23),-1)\}.$$

Then the symmetry properties of x(t) can be translated as follows:

$$x(t) = \begin{bmatrix} x^1(t) \\ x^2(t) \\ x^3(t) \\ x^4(t) \end{bmatrix}$$

with

$$x^2(t) = x^1\left(t - \frac{\pi}{\beta}\right), \quad x^4(t) = x^2\left(t - \frac{\pi}{\beta}\right), \tag{13}$$

$$x^4(t) = x^1\left(t - \frac{\pi}{\beta}\right), \quad x^3(t) = x^2\left(t - \frac{\pi}{\beta}\right) \tag{14}$$

Using (13), (14) and following the same lines as in the case of dihedral symmetries, one can easily establish

Theorem 3.2. *(i) Consider system (1) with C given by (12) and suppose* $\eta := h(0)g'(0) > 1$. *Assume condition (A1) is satisfied. Then the branch* $\mathcal{C}_{(V_4^-)}$ *solutions bifurcating from* $(\alpha_1, \beta_1, 0)$ *is unbounded in* $\mathbb{R}^2 \oplus W$.

(ii) Assume, in addition, condition (A2) is satisfied. Then $[\alpha_1, \infty) \subset \left\{\alpha : (\alpha, \beta, x) \in C_{(V_4^1)}\right\}$.

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