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Symmetry analysis of coupled scalar systems under time delay

Fatihcan M Atay[1] **and Haibo Ruan**[2]

[1] Max Planck Institute for Mathematics in the Sciences, D-04103 Leipzig, Germany
[2] Universität Hamburg, Fachbereich Mathematik, D-20146 Hamburg, Germany

E-mail: fatay@mis.mpg.de and haibo.ruan@math.uni-hamburg.de

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Abstract
We study systems of coupled units in a general network configuration with a coupling delay. We show that the destabilizing bifurcations from an equilibrium are governed by the extreme eigenvalues of the coupling matrix of the network. Based on the equivariant degree method and its computational packages, we perform a symmetry classification of destabilizing bifurcations in bidirectional rings of coupled units. Both stationary and oscillatory bifurcations are discussed. We also introduce the concept of secondary dominating orbit types to capture bifurcating solutions of submaximal nature.

Keywords: symmetries, bifurcation, coupled network, coupling delay, equivariant degree, spatiotemporal pattern
Mathematics Subject Classification: 34C14, 34C23, 34C15, 47H11, 34C25

(Some figures may appear in colour only in the online journal)

1. Introduction

Networks of coupled systems are known to be capable of a wide range of interesting dynamics, especially in the presence of time delays [1]. One of the most well-studied types of behaviour involves synchronization of oscillations in various forms [2], and recent work has revealed more complicated activity patterns related to the synchronization. As an example, which has also been a motivation for the present paper, we mention the so-called chimera states: in a network of identical phase oscillators arranged on a ring, with each oscillator coupled to a fixed number of its spatial neighbours, appropriate conditions can lead to oscillators splitting into two contiguous groups, one group oscillating synchronously while the other one incoherently [3], a behaviour which has also been reported in the presence of time delays [4]. An important feature is that such states are observed for identical oscillators and under homogeneous coupling conditions, i.e. in highly symmetric situations. Thus, a natural question arises as to the relation of system's symmetries to its possible dynamical states. The aim of this paper is to present a systematic investigation of the types of dynamics that can be deduced from the symmetries and bifurcations of coupled scalar systems under a time delay.

We consider n identical dynamical systems governed by the equation $\dot{x}_i = f(x_i)$ for $i = 1, \ldots, n$, coupled in a general network configuration:

$$\dot{x}_i(t) = f(x_i(t)) + \kappa g_i(x_1(t-\tau), x_2(t-\tau), \ldots, x_n(t-\tau)), \quad i = 1, 2, \ldots, n. \tag{1}$$

Here $x_i \in \mathbb{R}$, the function g_i describes the interaction among the coupled units, and $\tau \geqslant 0$ is the time delay. The scalar $\kappa > 0$ denotes the coupling strength; it can of course be subsumed into the definition of g_i, but it is sometimes used as a bifurcation parameter when one studies the effects of coupling in comparison to the intrinsic dynamics f, or for distinguishing excitatory from inhibitory coupling by simply changing its sign. The functions $f : \mathbb{R} \to \mathbb{R}$ and $g_i : \mathbb{R}^n \to \mathbb{R}$ are assumed to be continuously differentiable; in addition, g_i will be assumed to be equivariant when we consider symmetry. We also assume that f and the g_i vanish at the origin; hence the zero solution is an equilibrium solution of (1). The local stability of the zero solution is given by the linear variational equation

$$\dot{y}(t) = f'(0)y(t) + \kappa C y(t-\tau), \quad y \in \mathbb{R}^n, \tag{2}$$

where the *coupling matrix* $C = [c_{ij}] := [\partial g_i(0)/\partial x_j]$ is assumed to be a symmetric matrix.

Systems of the form (1) include many well-known examples as special cases. For instance, the *neural network* model

$$\dot{x}_i(t) = -x_i(t) + g\left(\sum_{j=1}^{n} a_{ij} x_j(t-\tau)\right), \tag{3}$$

where g is a sigmoidal function and $a_{ij} \in \mathbb{R}$ are entries of the (weighted and directed) adjacency matrix A that describes the coupling among the neurons, has the form of (1). The component a_{ij} describes how strongly the jth neuron influences the ith one; the influence being excitatory if $a_{ij} > 0$ and inhibitory if $a_{ij} < 0$. Often one excludes self-coupling, taking $a_{ii} = 0\ \forall i$. Linearization of (3) about the zero solution yields the form (2) with $\kappa = g'(0)$ and $C = A$. There are also some variant models which are not strictly in the form (1), for instance *pulse-coupled* systems,

$$\dot{x}_i(t) = f(x_i(t)) + h(x_i(t)) \cdot g\left(\sum_{j=1}^{n} a_{ij} x_j(t-\tau)\right), \tag{4}$$

where the influence of the network on the ith unit may be different depending on the state of the ith unit at that particular time instant. Although (4) is not of form (1) for non-zero τ, its linearization is still given by (2) with $\kappa = h(0)g'(0)$ and $C = A$. This is a crucial observation since many of our bifurcation results will depend only on the linear part (2) of the model at hand, and thus will also apply to (4) in particular. Also, the term 'coupling matrix' is used in a general way that can take other familiar forms in applications. For example, models of synchronization typically involve *diffusive-type interactions*, e.g.,

$$\dot{x}_i(t) = f(x_i(t)) + \sum_{j=1}^{n} a_{ij} g(x_j(t-\tau) - x_i(t-\tau)). \tag{5}$$

Sometimes the order of summation and the function g are interchanged; in this case, (5) becomes a special case of (3) obtained by setting $a_{ii} = -\sum_{j \neq i} a_{ij}\ \forall i$ in (3). The linear variational equation corresponding to (5) has the form (2) with C given by the negative of the Laplacian matrix: $C = -L = A - D$, where $D = \mathrm{diag}\{k_1, \dots, k_n\}$ is the diagonal matrix of vertex in-degrees $k_i = \sum_{j\neq i} a_{ij}$. If the delay pertains only to the interaction between different units (so that there are no self-delays), then one obtains a slightly variant system

$$\dot{x}_i(t) = f(x_i(t)) + \sum_{j=1}^{n} a_{ij}\, g(x_j(t-\tau) - x_i(t)), \tag{6}$$

whose linearization can be put into the form (2) (with the identification $f(x_i) \to f(x_i) + g'(0)k_i x_i$) provided that all vertices have the same degree $k_i = k$. In all cases, the matrix C being symmetric reflects the assumption of bi-directional interactions in the network.

We are interested in the effects of the time delay and the spectrum of C in causing the zero equilibrium to lose its stability as the system bifurcates into other dynamical states. We show that, among all the eigenvalues of C, only the extreme eigenvalues (i.e. the smallest and largest ones) play a role in destabilizing the zero equilibrium. Networks having the same extreme eigenvalues will exhibit the same destabilizing behaviour, independently of the precise network configuration. In the presence of ring (dihedral) symmetry, we give a complete classification of bifurcating states using equivariant degree methods. The ring configuration is motivated by the setting of chimera states mentioned in the first paragraph, although we do not focus on chimeras in this paper but aim to capture all emergent dynamics that can bifurcate from the equilibrium. To illustrate the theoretical results, we use throughout a ring of size 12 with various coupling configurations, since on the one hand it gives rise to a large variety of dynamical patterns, and on the other hand, it can be presented in a manageable size.

Equivariant bifurcations have been extensively studied using various techniques such as those based on singularity theory, as developed by Golubitsky *et al* (see [5, 6]), geometric techniques developed by Field and Richardson (see [7–9]), constructive methods using algebraic geometry developed by Bierstone and Milman (see [10]) and, last but not least, equivariant degree methods developed by Ize and Vignoli, and Krawcewicz and co-workers (see [11, 12]). While geometric methods are based on generic approximations, topological methods rely on deformations subject to homotopy invariance. Since homotopy generally allows 'larger' changes of maps compared to approximations, topological methods tend to overlook finer properties of solutions such as stability, but rather they catch the existence. This is also why topological methods are commonly believed to produce 'weaker' results. On the other hand, they can deal with non-generic situations just as well as generic cases. Results hold without generic assumptions since homotopy makes essentially no distinction between generic and non-generic maps. An example of using topological indices to predict non-generic global equivariant bifurcations with least symmetry can be found in [13]. To compare, the index in [13] is defined for a subgroup pair (K, H) of the symmetry group Γ, where K is normal in H and H/K is finite cyclic, while the invariant we use later is defined for Γ and captures every subgroup; the index in [13] is used to predict global bifurcation with at least symmetry K, while the invariant we use predicts local bifurcation with precise symmetry for all adequate subgroups in Γ (see corollary 4.4 and proposition 4.7); last but not least, while computations of the index in [13] can be technically involved, the invariant we use can be computed instantly.

In this paper we use equivariant degree methods to give a complete classification of bifurcating branches of solutions according to their symmetry properties. This includes dealing with non-simple critical eigenvalues with non-simple representations in the kernel of linearization. An additional advantage of using equivariant degree is that it can be effectively calculated.

The computational tool we use for exact calculations of equivariant degrees is the 'Equivariant Degree Maple© Library Package'[3] (EDML). More precisely, to a given bifurcating equilibrium, one associates a *bifurcation invariant* in form of an equivariant degree. Here, the term 'invariant' refers to the fact that the bifurcation invariant remains constant against all adequate homotopies. The precise value of the bifurcation invariant, once calculated, carries the full topological information about the bifurcating solutions and gives rise to a complete classification of bifurcating branches by their symmetry properties. The calculation task of bifurcation invariants for a given symmetry group Γ is taken over by the EDML package. Examples of this computational approach using EDML can be found in [14–17]. In addition to the symmetry group Γ, the software package takes several parameters as input, which are solely determined by the critical spectrum of the linearized operator. In other words, the exact value of the bifurcation invariant associated to the zero solution of (1) depends only on the characteristic operator of (2). In fact, all results that follow from the bifurcation invariant of (1) remain valid for any Γ-symmetric system whose linearization has the form (2), in particular for systems of form (4)–(6).

Another advantage of using equivariant degree is that its basic degrees can be easily programmed and calculated in other computer programming languages such as GAP[4], MATLAB, C++, Java, and so on. An existing extension of the EDML package is the 'Dihedral Calculator', which is programmed using the non-commercial language GAP. It is currently available for dihedral symmetry D_n for $n \leqslant 200$[5]. Other symmetry groups that are supported by EDML are the quaternion group Q_8, the alternating groups A_4, A_5, and the symmetric group S_4.

Our main results are theorem 4.3, corollary 4.4, theorem 4.5 and proposition 4.7. The classification results for dihedral symmetry D_{12} are summarized in tables 2–5. Theorem 4.3 gives an existence result of steady-state bifurcations with their least symmetry. Corollary 4.4 using the implicit function theorem sharpens this to exact symmetry. For Hopf bifurcations, existence result is stated in theorem 4.5 with the least symmetry. To obtain the precise symmetry as well as for submaximal isotropies, we introduce the concept of *secondary dominating orbit types* (see definition 4.6) to complement dominating orbit types. In proposition 4.7 bifurcating branches of maximal or submaximal isotropies are predicted with their precise symmetry.

The paper is organized as follows. In section 2, we provide the basic definition of equivariant degrees and introduce the necessary notation and preliminary calculations for D_{12}. In section 3 we give a brief account of the stability analysis and the derivation of the basic bifurcation diagram for the linear system (2). We take two quantities $\alpha := \tau f'(0)$ and $\beta := \tau\kappa\xi$ as bifurcation parameters, where ξ is an eigenvalue of C. As we shall see, bifurcations, either of stationary or oscillatory nature, that destabilize the zero equilibrium, are related only to the extreme eigenvalues of the coupling matrix. The main equivariant bifurcation results are given in section 4. In section 5 we apply our results to bidirectional rings of 12 and obtain classification results listed in tables 2–5. For rings of larger size, we refer to the 'Dihedral Calculator' mentioned earlier for calculations of degrees and the method can be applied systematically. In section 6 we connect the extreme eigenvalues to coupling strengths by enumerating all possible first-and second-nearest-neighbour coupling configurations of the

[3] The Equivariant Degree Maple© Library Package was created by Biglands and Krawcewicz at the University of Alberta in 2006 supported by NSERC research grant. It is open source and can be freely downloaded, for example, from www.math.uni-hamburg.de/home/ruan/download.

[4] *GAP* ('Groups, Algorithms, Programming') is a non-commercial system for computational discrete algebra. It provides a programming language and large data libraries of algebraic objects. The system is distributed freely at www.gap-system.org

[5] See *Dihedral Calculator* from MuchLearning http://dihedral.muchlearning.org

12-cell ring. We conclude by giving simulation examples in a concrete nonlinear system, namely the neural network model (3).

In closing this Introduction, we note that since the bifurcation invariant remains invariant against all (admissible, equivariant) continuous deformations on the system, the classification results we obtain using the bifurcation invariant remain valid under modelling variations within the framework of symmetry. They may also be useful for systems encountered in real-world applications that are only 'approximately symmetric'. For modelling issues on systems with imperfect symmetry, we refer to [6].

2. Preliminaries

2.1. Groups and group representations

Throughout we consider groups that are either finite or of form $\Gamma \times S^1$, where Γ is a finite group and S^1 is the group of complex numbers of unit length.

Let G be a group and H be a closed subgroup of G, written as $H \subset G$. Let $N(H) = \{g \in G : gHg^{-1} = H\}$ be the *normalizer* of H and $W(H) = N(H)/H$ the *Weyl group* of H. The set of all closed subgroups of G can be partially ordered by set inclusion. For subgroups $H, K \subset G$, we write $H \leqslant K$ if $H \subseteq K$; $H < K$ if $H \subsetneq K$. The symbol (H) stands for the conjugacy class of the subgroup H in G; that is $(H) = \{gHg^{-1} : g \in G\}$. The set of all conjugacy classes of closed subgroups of G affords a partial order given by: $(H) \leqslant (K)$ if $H \subseteq gKg^{-1}$ for some $g \in G$; similarly, $(H) < (K)$ if $H \subsetneq gKg^{-1}$ for some $g \in G$.

Example 2.1. *(see [12])* Let $\Gamma = D_{12}$ be the dihedral group of order 24, which is represented as the group of 12 rotations: $1, \eta, \eta^2, \ldots, \eta^{11}$ and 12 reflections: $\varsigma, \varsigma\eta, \varsigma\eta^2, \ldots, \varsigma\eta^{11}$ of the complex plane $\mathbb{C}$, where η stands for the complex multiplication by $e^{\frac{i\pi}{6}}$ and ς denotes the complex conjugation. There are two kinds of subgroups in D_{12}: cyclic and dihedral. The cyclic subgroups are $\mathbb{Z}_1, \mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_4, \mathbb{Z}_6, \mathbb{Z}_{12}$, where $\mathbb{Z}_k$ denotes the cyclic subgroup generated by η^l with $l = \frac{12}{k}$. The dihedral subgroups are

$$D_{k,j} = \{1, \eta^l, \eta^{2l}, \ldots, \eta^{(k-1)l}, \varsigma\eta^j, \varsigma\eta^{j+l}, \varsigma\eta^{j+2l}, \ldots, \varsigma\eta^{j+(k-1)l}\}, \quad \text{for } 0 \leqslant j < l = \frac{12}{k},$$

where $k \in \{1, 2, 3, 4, 6, 12\}$. If l is odd, then all subgroups $D_{k,j}$ for $0 \leqslant j < l$ are conjugate to $D_{k,0} := D_k$. If l is even, then all subgroups $D_{k,j}$ with j being even are conjugate to $D_{k,0} = D_k$; all subgroups $D_{k,j}$ with j being odd are conjugate to $D_{k,1} := \tilde{D}_k$. Thus, up to conjugacy relation, we have the dihedral subgroups: D_1, $\tilde{D}_1$, D_2, $\tilde{D}_2$, D_3, $\tilde{D}_3$, D_4, D_6 $\tilde{D}_6$, D_{12}. ◇

A *real* (respectively *complex*) *representation* of G is a finite-dimensional real (respectively, complex) vector space X with a continuous map, or *action*, $\psi : G \times X \to X$ such that the map $\psi(g, \cdot) : X \to X$ is linear, for every $g \in G$. Banach representations are similarly defined for Banach spaces with an action for which $\psi(g, \cdot)$ is linear and bounded. We abbreviate $\psi(g, x)$ with gx.

A subset $\Omega \subset X$ is called *invariant* if $gx \in \Omega$ whenever $x \in \Omega$ for all $g \in G$. An action on an invariant subset $\Omega \subset X$ is called *free* if the existence of an $x \in \Omega$ with $gx = x$ implies $g = e$ is the neutral element. A representation X of G is called *irreducible* if $\{0\}$ and X are the only invariant subspaces in X.

Example 2.2. *(see [12])* The dihedral group D_n, for $n \in \mathbb{N}$ even, has the following real irreducible representations:

(i) The trivial representation $\mathcal{V}_0 \simeq \mathbb{R}$, where every element acts as the identity map.
(ii) For $1 \leqslant i \leqslant \frac{n}{2} - 1$, there is the representation $\mathcal{V}_i \simeq \mathbb{R}^2 \simeq \mathbb{C}$ given by the following actions:

$$\eta z = \eta^i \cdot z, \quad \varsigma z = \bar{z},$$

where '$\cdot$' is the complex multiplication and '$\bar{\ }$' is the complex conjugation.
(iii) The representation $\mathcal{V}_{\frac{n}{2}} \simeq \mathbb{R}$ given by: $\eta x = x$ and $\varsigma x = -x$.
(iv) The representation $\mathcal{V}_{\frac{n}{2}+1} \simeq \mathbb{R}$ given by: $\eta x = -x$ and $\varsigma x = x$.
(v) The representation $\mathcal{V}_{\frac{n}{2}+2} \simeq \mathbb{R}$ given by: $\eta x = -x$ and $\varsigma x = -x$.

It has the following complex irreducible representations:

(i) The trivial representation $\mathcal{U}_0 \simeq \mathbb{C}$, where every element acts as the identity map.
(ii) For $1 \leqslant j \leqslant \frac{n}{2} - 1$, there is the representation $\mathcal{U}_j \simeq \mathbb{C} \times \mathbb{C}$ given by the following actions:

$$\eta(z_1, z_2) = (\eta^j \cdot z_1, \eta^{-j} \cdot z_2), \quad \varsigma(z_1, z_2) = (z_2, z_1),$$

where '$\cdot$' is the complex multiplication.
(iii) The representation $\mathcal{U}_{\frac{n}{2}} \simeq \mathbb{C}$ given by: $\eta z = z$ and $\varsigma z = -z$.
(iv) The representation $\mathcal{U}_{\frac{n}{2}+1} \simeq \mathbb{C}$ given by: $\eta z = -z$ and $\varsigma z = z$.
(v) The representation $\mathcal{U}_{\frac{n}{2}+2} \simeq \mathbb{C}$ given by: $\eta z = -z$ and $\varsigma z = -z$.

For $n \in \mathbb{N}$ odd, the dihedral group D_n has the above listed irreducible representations (i)–(iii), where n is replaced with $(n + 1)$. ◇

Let $x \in X$. By the *symmetry* of x, we mean the *isotropy* subgroup of x given by $\mathrm{Iso}(x) := \{g \in G : gx = x\}$ with respect to the group action on X. The set $\mathrm{Orb}(x) := \{gx : g \in G\}$ is called the *orbit* of x and the *symmetry* of the orbit is defined by the *orbit type* of x, which is the conjugacy class $(\mathrm{Iso}(x))$ of $\mathrm{Iso}(x)$. Note that $\mathrm{Iso}(gx) = g\mathrm{Iso}(x)g^{-1}$ for $g \in G$; thus, the symmetry of the orbit is independent of the choice of x from the orbit.

Let $\Omega \subset X$ be a subset and $H \subset G$ be a closed subgroup. Define $\Omega_H = \{x \in X : \mathrm{Iso}(x) = H\}$. It can be verified that the Weyl group $W(H)$ acts freely on Ω_H. Denote the H*-fixed point subspace* in Ω by $\Omega^H = \{x \in X : gx = x, \forall g \in H\}$. In sections 3–5 we use $\mathrm{Fix}(H)$ to denote the H-fixed point subspace, since the space on which G acts will be clear from context. Note that $\Omega_H \subset \Omega^H$. Moreover, Ω^H is the disjoint union of $\Omega_{\tilde{H}}$ for all $\tilde{H} \supseteq H$.

Example 2.3. Let $\Gamma = D_{12}$ and $X = \mathcal{V}_1$ be the real irreducible representation of D_{12} given in example 2.2. Then, orbit types that occur in X are (D_{12}), (D_1), $(\tilde{D}_1)$ and $(\mathbb{Z}_1)$ (refer to example 2.1 for notations), with the corresponding fixed point subspaces:

$$X^{D_{12}} = \{(0, 0)\}, \quad X^{D_1} = \{(x, 0) : x \in \mathbb{R}\}, \quad X^{\tilde{D}_1} = \{re^{-\frac{i\pi}{12}} : r \in \mathbb{R}\}, \quad X^{\mathbb{Z}_1} = X.$$

Note that X^{D_1} is the disjoint union of subsets $X_{D_1} = \{(x, 0) : x \in \mathbb{R}, x \neq 0\}$ and $X_{D_{12}} = \{(0, 0)\}$. On the subset X_{D_1}, the Weyl group $W(D_1) = D_2/D_1 \simeq \mathbb{Z}_2$ acts freely by the reflection. On the subset X_{D_12}, the Weyl group $W(D_{12}) = D_{12}/D_{12} \simeq \mathbb{Z}_1$ acts freely by the neutral element. ◇

Finally, we remark that there is a natural way of 'converting' a complex Γ-representation into a real $\Gamma \times S^1$-representation. Let U be a complex Γ-representation. Define a $\Gamma \times S^1$-action on U by

$$(\gamma, z)u = z \cdot (\gamma u), \quad \text{for } (\gamma, z) \in \Gamma \times S^1, u \in U, \tag{7}$$

where $\cdot$ stands for the complex multiplication. The obtained representation is denoted by $\bar{U}$ and called the $\Gamma \times S^1$*-representation induced* from U. Note that $\bar{U}$ is irreducible as a real $\Gamma \times S^1$-representation if U is irreducible as a complex Γ-representation.

2.2. Equivariant maps and equivariant degree

Let X, Y be two Banach representations of G. A continuous map $f : X \to Y$ is called *equivariant* if $f(g_\circ x) = g_* f(x)$, for all $x \in X$ and $g \in G$, where $_\circ$ and $_*$ stand for the G-actions on X and Y, respectively. In equivariant nonlinear analysis, one is interested in finding zeros of an equivariant map f in an invariant domain $\Omega \subset X$. Note that by equivariance, the set of all zeros of f in Ω is composed of disjoint group orbits; thus one speaks of *zero orbits*, instead of zeros, of f.

A map f is called *admissible* on Ω if $f(x) \neq 0$ for all $x \in \partial\Omega$. A homotopy $h : [0, 1] \times X \to Y$ is called *admissible* if $h(t, \cdot)$ is admissible for all $t \in [0, 1]$. An equivariant degree, intuitively speaking, is an algebraic count of zero orbits of an admissible f in Ω with respect to orbit types, which remains unchanged against all admissible (equivariant) homotopies from f.

In the next two subsections we review from [12] two types of equivariant degrees that will be used in section 4 for bifurcation analysis. In both cases, the equivariant degree is first defined in finite-dimensional representations for continuous maps and then extended to infinite-dimensional Banach representations for compact vector fields.

2.2.1. Equivariant degree without parameters. Let $G = \Gamma$ be a finite group acting on a finite-dimensional Γ-representation X. Let Φ be the set of all orbit types that appear in X. That is, every element of Φ is a conjugacy class of a finite subgroup of Γ. Consider a continuous equivariant map $f : X \to X$ on an open bounded invariant domain $\Omega \subset X$ such that f is admissible on Ω. Define an *equivariant degree (without parameter)* of f in Ω by a finite sum of integer-indexed orbit types:

$$\Gamma\text{−Deg}\,(f, \Omega) = \sum\nolimits_{(K)\in\Phi} n_K \cdot (K), \tag{8}$$

where $n_K \in \mathbb{Z}$ is an integer counting zero orbits of orbit type (K). One can also think of Γ−Deg as associating to every such pair (f, Ω) an integer sequence indexed by the set Φ of conjugacy classes. Depending on the value of f on $\overline{\Omega}$ (with $\overline{\Omega} = \Omega \cup \partial\Omega$), the degree associates different integer values to different conjugacy classes. The precise definition of n_K can be given by the following *recurrence formula*:

$$n_K = \frac{\deg\,(f|_{\Omega^K}, \Omega^K) - \sum_{(\tilde{K})>(K)} n_{\tilde{K}} \cdot |W(\tilde{K})| \cdot n(K, \tilde{K})}{|W(K)|}. \tag{9}$$

We explain the notations used in (9) and their geometric meaning. Recall that Ω^K denotes the fixed point subspace of K in Ω. By restricting f on Ω^K, one obtains an (admissible) map $f|_{\Omega^K} : \Omega^K \to \Omega^K$. Using the classical Brouwer degree 'deg', the integer 'deg $(f|_{\Omega^K}, \Omega^K)$' counts the zeros of f in Ω^K. Since not every element in Ω^K has the precise isotropy K, one needs to subtract those zeros of larger isotropies. This is done by subtracting the summands in (9). Within each summand, $n_{\tilde{K}}$ is the integer counting zero orbits of orbit type $(\tilde{K})$. Since the Weyl group $W(\tilde{K})$ acts freely on $\Omega_{\tilde{K}}$, the integer $n_{\tilde{K}} \cdot |W(\tilde{K})|$ then counts the zeros of isotropy $\tilde{K}$. The number $n(K, \tilde{K})$ is defined as the number of distinct conjugate copies of $\tilde{K}$ that contain K, formally by

$$n(K, \tilde{K}) = \left| \frac{\{g \in \Gamma : K \subset g\tilde{K}g^{-1}\}}{N(\tilde{K})} \right|. \tag{10}$$

Thus, the number $n_{\tilde{K}} \cdot |W(\tilde{K})| \cdot n(K, \tilde{K})$ counts the zeros of isotropy K' for all K' with $(K') = (\tilde{K})$. It follows that the expression of the numerator in (9) gives the count of zeros of f having precise isotropy K. Again, since $W(K)$ acts freely on Ω_K, we have then the total expression on the right-hand side of (9) giving the count of zero orbits of f having orbit type (K).

Example 2.4. Let $\Gamma = D_{12}$ and $X = \mathcal{V}_1$ be the real irreducible representation of D_{12} given in example 2.2. Consider the antipodal map $f = -\mathrm{Id} : X \to X$ on the unit disc $B \subset X$, which is D_{12}-equivariant and B-admissible. As mentioned in example 2.3, orbit types that occur in $\mathcal{V}_1$ are (D_{12}), (D_1), $(\tilde{D}_1)$ and $(\mathbb{Z}_1)$. Thus,

$$\Gamma\text{−Deg}\,(-\mathrm{Id}, B) = n_{D_{12}} \cdot (D_{12}) + n_{D_1} \cdot (D_1) + n_{\tilde{D}_1} \cdot (\tilde{D}_1) + n_{\mathbb{Z}_1} \cdot (\mathbb{Z}_1).$$

We compute n_{D_1} using (9). To do so, we first need to compute $n_{D_{12}}$:

$$n_{D_{12}} = \frac{\deg\,(-\mathrm{Id}, B^{D_{12}})}{|W(D_{12})|} = \frac{1}{1} = 1,$$

where we used the fact $B^{D_{12}} = X^{D_{12}} \cap B = \{(0, 0)\}$, $W(D_{12}) = \mathbb{Z}_1$ from example 2.3 and $\deg\,(-\mathrm{Id}, \mathbb{R}^m) = (-1)^m$ for $m \in \{0\} \cup \mathbb{N}$. Thus, we have

$$n_{D_1} = \frac{\deg\,(-\mathrm{Id}, B^{D_1}) - 1 \cdot 1 \cdot 1}{|W(D_1)|} = \frac{-1-1}{2} = -1,$$

where we used the fact $n(D_1, D_{12}) = \left|\frac{D_{12}}{D_{12}}\right| = 1$ and $W(D_1) = \mathbb{Z}_2$. Following (9) further, one shows that

$$\Gamma\text{−Deg}\,(-\mathrm{Id}, B) = (D_{12}) - (D_1) - (\tilde{D}_1) + (\mathbb{Z}_1).$$

◇

The definition of equivariant degree can be extended, in a standard way, to infinite-dimensional Banach representations for *compact equivariant fields*, namely, equivariant maps of the form $f = \mathrm{Id} - F : D \subset X \to X$ that are admissible on a bounded domain D such that $\overline{F(D)}$ is compact. It was shown in [18] that the equivariant degree defined by (8)–(9), as well as its infinite-dimensional extension, satisfies the usual properties of degree theory, such as the *existence property* which states that

$$n_K \neq 0 \text{ in } (8) \;\Rightarrow\; f^{-1}(0) \cap \Omega^K \neq \emptyset,$$

which can be useful for predicting zero orbits of orbit type at least (K).

2.2.2. Equivariant degree with one parameter. Let $G = \Gamma \times S^1$ be the product of a finite group Γ and the circle group S^1. There are two types of closed subgroups in G: those subgroups that are of the form $K \times S^1$ for some subgroup $K \subset \Gamma$, or the *twisted subgroups* of G, defined as follows.

Definition 2.5. A subgroup $H \subset \Gamma \times S^1$ is called a *twisted l-folded subgroup*, if there exists a subgroup $K \subset \Gamma$, an integer $l \geqslant 0$ and a group homomorphism $\phi : K \to S^1$ such that

$$H = K^{\phi,l} := \{(\gamma, z) \,:\, \phi(\gamma) = z^l\}.$$

For $l = 1$, H is called a *twisted subgroup* for simplicity. Conjugacy classes of twisted subgroups are called *twisted orbit types*. ◇

Remark 2.6. Note that the subgroups of the form $K \times S^1$ (for some $K \subset \Gamma$) and the twisted (l-folded) subgroups can also be distinguished using the dimension of their Weyl groups. While the former have 0-dimensional Weyl groups, the latter have 1-dimensional Weyl groups in $\Gamma \times S^1$. Thus, the Weyl group of a twisted (l-folded) subgroup is homeomorphic to a number of finitely many disjoint circles.

Example 2.7. Let $G = D_{12} \times S^1$ be the product group of the dihedral D_{12} and the unit circle $S^1 \subset \mathbb{C}$. We describe its twisted subgroups $H = K^\phi$. Clearly, all subgroups of D_{12} are twisted subgroups with $\phi \equiv 1 \in S^1$. Besides that, there are twisted subgroups that are not contained in D_{12}. These can be classified into two categories: those for which $K = \mathbb{Z}_k$ and those for which $K = D_{k,j}$ (refer to example 2.1 for notation).

Let $K = \mathbb{Z}_k$ for some $k \in \{1, 2, 3, 4, 6, 12\}$ and $\phi : K \to S^1$ be given by $\phi(\eta^l) = \eta^{jl}$ for some j with $1 \leqslant j < k$. Then,

$$K^\phi = \{(1, 1), (\eta^l, \eta^{jl}), (\eta^{2l}, \eta^{2jl}), \dots, (\eta^{(k-1)l}, \eta^{j(k-1)l})\} := \mathbb{Z}_k^{t_j}, \quad \text{for } 1 \leqslant j < k.$$

Among these subgroups, $\mathbb{Z}_k^{t_j}$ and $\mathbb{Z}_k^{t_{k-j}}$ are conjugate to each other, for $1 \leqslant j < k$. Thus, for k even, up to conjugacy relation, we have the twisted subgroups $\mathbb{Z}_k^{t_1}, \mathbb{Z}_k^{t_2}, \dots, \mathbb{Z}_k^{t_{\frac{k}{2}}} := \mathbb{Z}_k^d$, and for k odd, $\mathbb{Z}_k^{t_1}, \mathbb{Z}_k^{t_2}, \dots, \mathbb{Z}_k^{t_{\frac{k-1}{2}}}$. That is, we have $\mathbb{Z}_2^d$, $\mathbb{Z}_3^{t_1}$, $\mathbb{Z}_4^{t_1}$, $\mathbb{Z}_4^d$, $\mathbb{Z}_6^{t_1}$, $\mathbb{Z}_6^{t_2}$, $\mathbb{Z}_6^d$, $\mathbb{Z}_{12}^{t_1}$, $\mathbb{Z}_{12}^{t_2}$, $\mathbb{Z}_{12}^{t_3}$, $\mathbb{Z}_{12}^{t_4}$, $\mathbb{Z}_{12}^{t_5}$, $\mathbb{Z}_{12}^d$.

Let $K = D_{k,j}$ for some $k \in \{1, 2, 3, 4, 6, 12\}$ and $0 \leqslant j < l = \frac{12}{k}$. Up to conjugacy, it is sufficient to consider $K = D_k$ in case l is odd, and $K = D_k$, $K = \tilde{D}_k$ in case l is even (see example 2.1). Let $\phi : K \to S^1$ be the group homomorphism such that $\ker \phi = \mathbb{Z}_k$. Then,

$$D_k^\phi = \{(1, 1), (\eta^l, 1), \dots, (\eta^{(k-1)l}, 1), (\varsigma, -1), (\varsigma\eta^l, -1), \dots, (\varsigma\eta^{(k-1)l}, -1)\} := D_k^z,$$

and

$$\tilde{D}_k^\phi = \{(1, 1), (\eta^l, 1), \dots, (\eta^{(k-1)l}, 1), (\varsigma\eta, -1), (\varsigma\eta^{1+l}, -1), \dots, (\varsigma\eta^{1+(k-1)l}, -1)\} := \tilde{D}_k^z, \quad \text{if } l \text{ is even}.$$

Thus, we have D_1^z, $\tilde{D}_1^z$, D_2^z, $\tilde{D}_2^z$, D_3^z, $\tilde{D}_3^z$, D_4^z, D_6^z, $\tilde{D}_6^z$, D_{12}^z.

In the case k is even, there is a group homomorphism $\phi : K \to S^1$ for which $\ker \phi = D_{\frac{k}{2}}$. Then,

$$D_k^\phi = \{(1, 1), (\eta^l, -1), (\eta^{2l}, 1), \dots, (\eta^{(k-1)l}, -1), (\varsigma, 1), (\varsigma\eta^l, -1), \dots, (\varsigma\eta^{(k-1)l}, -1)\} := D_k^d,$$

and

$$\tilde{D}_k^\phi = \{(1, 1), (\eta^l, -1), (\eta^{2l}, 1), \dots, (\eta^{(k-1)l}, -1), (\varsigma\eta, 1), (\varsigma\eta^{1+l}, -1), \dots, (\varsigma\eta^{1+(k-1)l}, -1)\} := \tilde{D}_k^d, \quad \text{if } l \text{ is even}.$$

Also, there is a group homomorphism $\phi : K \to S^1$ for which $\ker \phi = \tilde{D}_{\frac{k}{2}}$. Then,

$$D_k^\phi = \{(1, 1), (\eta^l, -1), (\eta^{2l}, 1), \dots, (\eta^{(k-1)l}, -1), (\varsigma, -1), (\varsigma\eta^l, 1), \dots, (\varsigma\eta^{(k-1)l}, 1)\} := D_k^{\hat{d}},$$

and

$$\tilde{D}_k^\phi = \{(1, 1), (\eta^l, -1), (\eta^{2l}, 1), \dots, (\eta^{(k-1)l}, -1), (\varsigma\eta, -1), (\varsigma\eta^{1+l}, 1), \dots, (\varsigma\eta^{1+(k-1)l}, 1)\} := \tilde{D}_k^{\hat{d}}, \quad \text{if } l \text{ is even}.$$

One shows that for l even, D_k^d and $D_k^{\hat{d}}$ are conjugate, and $\tilde{D}_k^d$ and $\tilde{D}_k^{\hat{d}}$ are conjugate. Thus, in the case k is even, up to conjugacy relation, we have the twisted subgroups D_k^d and $D_k^{\hat{d}}$ if l is odd, and D_k^d and $\tilde{D}_k^d$ if l is even. That is, for D_{12}, we have D_1^d, $\tilde{D}_1^d$, D_2^d, $\tilde{D}_2^d$, D_3^d, $\tilde{D}_3^d$, D_4^d, $D_4^{\hat{d}}$, D_6^d, $\tilde{D}_6^d$, D_{12}^d, $D_{12}^{\hat{d}}$. ◇

Let X be a finite-dimensional representation of G and $\mathbb{R}$ be the one-dimensional *parameter space* on which G acts trivially. Let Φ_1 be the set of all twisted orbit types that appear in $\mathbb{R} \times X$. Consider a continuous equivariant map $f : \mathbb{R} \times X \to X$ on an open bounded invariant domain $\Omega \subset \mathbb{R} \times X$ such that f is admissible on Ω. Define an *equivariant degree (with one parameter)* of f in Ω by a finite sum of integer-indexed twisted orbit types:

$$\Gamma \times S^1\text{–Deg}\,(f, \Omega) = \sum\nolimits_{(H)\in\Phi_1} n_H \cdot (H), \tag{11}$$

where $n_H \in \mathbb{Z}$ is an integer counting zero orbits of the twisted orbit type (H). More precisely, n_H can be computed by the following recurrence formula:

$$n_H = \frac{\sum_k \deg_k(f|_{\Omega^H}, \Omega^H) - \sum_{(\tilde{H})>(H)} n_{\tilde{H}} \cdot n(H, \tilde{H}) \cdot |W(\tilde{H})/S^1|}{|W(H)/S^1|}$$

in the same spirit of (9). We explain the notation in detail. Again, n_H is supposed to count the zero orbits of orbit type (H) in Ω, or equivalently, the zero orbits of isotropy H in Ω^H. Restricting the map f on Ω^H, we consider $f|_{\Omega^H} : \Omega^H \to \Omega^H$. Since $W(H)$ is homeomorphic to $|W(H)/S^1|$ copies of finitely many disjoint circles (see remark 2.6) and $|W(H)|$ acts freely on Ω_H, the number $n_H|W(H)/S^1|$ counts the copies of circles in the zeros having isotropy H in Ω^H. Using the classical S^1-degree (e.g. see [11]), the integer $\deg_k(f|_{\Omega^H}, \Omega^H)$ counts the number of circles in the zeros of f having isotropy $\mathbb{Z}_k$ in Ω^H. The first sum then counts the total number of circles in the zeros of f in Ω^H. The summand in the second sum counts the copies of circles in the zeros of f which have isotopy $\tilde{H}$, where the number $n(H, \tilde{H})$ is given by (10). Thus, we obtain from the numerator the number of circles in the zeros of f in Ω^H with precise isotropy H. Divided by the number $|W(H)/S^1|$ of copies in $W(H)$, this gives the count of zero orbits with precise isotropy H.

This degree can be extended to infinite-dimensional Banach representations for compact equivariant fields. The resulting degree satisfies all classical properties of an equivariant degree theory, among which the *existence property* plays an important role for our purpose:

$$n_H \neq 0 \text{ in } (11) \;\Rightarrow\; f^{-1}(0) \cap \Omega^H \neq \emptyset.$$

3. Stability analysis and the bifurcation diagram

We now consider the coupled system (1) and the corresponding linear variational equation (2) about the zero solution. For $\tau > 0$, it is convenient to rescale the time $t \to t/\tau$ so that the linearized equation takes the form

$$\dot{y}(t) = \tau f'(0)y(t) + \tau\kappa C y(t-1). \tag{12}$$

The characteristic operator $\Delta(\lambda) : \mathbb{C}^n \to \mathbb{C}^n$ for (12) is

$$\Delta(\lambda) = (\lambda - \tau f'(0))I_n - \tau\kappa \mathrm{e}^{-\lambda} C, \tag{13}$$

and the corresponding characteristic equation is

$$\det \Delta(\lambda) = \prod_{\xi\in\sigma(C)} \left(\lambda - \tau f'(0) - \tau\kappa e^{-\lambda}\xi\right) = 0, \tag{14}$$

where $\sigma(C)$ denotes the spectrum of C.

Since C is assumed to be a symmetric matrix, we have $\sigma(C) \subset \mathbb{R}$. In this case, each of the factors on the right side of (14) can be analysed using well-known methods for scalar delay equations with real coefficients [19, 20]. Thus, let $\xi \in \sigma(C) \subset \mathbb{R}$ and consider the

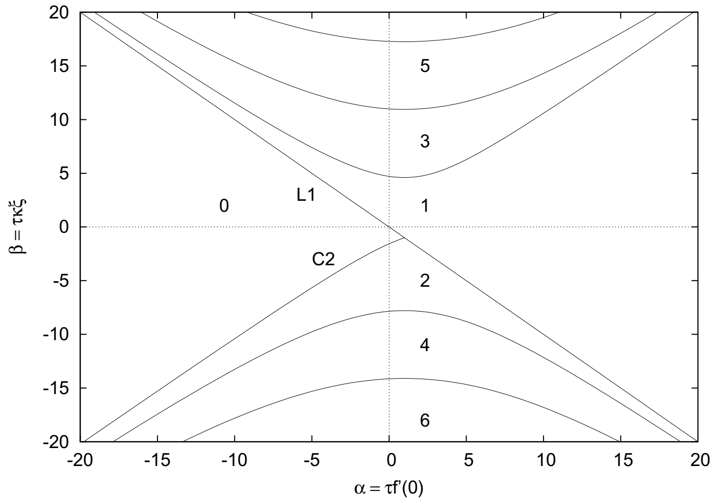


Figure 1. Bifurcation diagram of the characteristic equation (14). The curves indicate the parameter values for which the characteristic equation has a root on the imaginary axis. The curves separate the α–β parameter plane into regions in which the number of characteristic roots with positive real part is a constant, the value of which is indicated in the figure. Hence '0' indicates the region where the origin is stable, which is bounded from above by the straight line L1 and from below by the curve C2.

corresponding factor in (14). If $\lambda = u + iv$ is a characteristic root, then separating real and imaginary parts leads to

$$\begin{cases} u - \alpha - \beta \mathrm{e}^{-u}\cos v = 0 \\ v + \beta \mathrm{e}^{-u}\sin v = 0, \end{cases} \tag{15}$$

where $\alpha = \tau f'(0)$ and $\beta = \tau\kappa\xi$. For purely imaginary roots, we have $u = 0$, giving

$$\begin{cases} -\alpha - \beta\cos v = 0, \\ v + \beta\sin v = 0. \end{cases} \tag{16}$$

For $v = 0$ the solution is the line L1 defined by $\beta = -\alpha$, which corresponds to parameter values for which $\lambda = 0$ is a characteristic root. Over the intervals $v \in (k\pi, (k+1)\pi)$, $k \in \mathbb{Z}$, the solution can be expressed in the parametric form $(\alpha(v), \beta(v)) = (v/\tan(v), -v/\sin(v))$, which gives parametric curves for which there exists a pair of purely imaginary characteristic roots of the form $\lambda = \pm iv$. These bifurcation curves are depicted in figure 1. Knowing that the zero solution is stable for $\beta = 0$ and $\alpha < 0$, and because characteristic roots can cross the imaginary axis only for parameter values belonging to the bifurcation curves, one can then move vertically in the parameter plane, increasing the number of roots with positive real parts appropriately each time a bifurcation curve is crossed. Implicit differentiation on bifurcation curves shows that the characteristic roots on the imaginary axis move to the right as $|\beta|$ increases, yielding the picture shown in figure 1.

The region of stability is indicated in figure 1 by the label '0'. It is bounded from above by the straight line L1 and from below by the curve C2. The latter is given by the parametric branch $(\alpha, \beta) = (v/\tan(v), -v/\sin(v))$, $v \in (0, \pi)$, and meets the line L1 at the point $(1, -1)$. This is for one particular spatial mode corresponding to the eigenvalue ξ. One can then repeat the same argument for all eigenmodes $\xi \in \sigma(C)$. If a parameter pair (α, β) is varied to leave the stable region by crossing the line L1, a bifurcation of steady states may occur, whereas crossing the curve C2 may lead to a bifurcation of periodic solutions. The codimension of these bifurcations is related to the multiplicity of the eigenvalue ξ given by the critical value of $\beta = \tau\kappa\xi$.

3.1. Effects of network structure

Suppose we start with stable systems ($f'(0) < 0$) without coupling, so we are initially on the negative α-axis. As the coupling strength κ is increased from zero, stability may be lost via a stationary or an oscillatory bifurcation through the first eigenmode ξ to hit L1 or C2. The important observation is that this first bifurcation depends only on the extremal eigenvalues ξ of the coupling matrix C. Hence, the number of relevant parameters is greatly reduced and one needs to check only the two extremal eigenvalues of the coupling matrix regardless of the network size. In this way it is possible to classify networks by defining equivalence classes according to the extreme eigenvalues: networks having the same smallest and largest eigenvalues will have identical stability properties with regard to the class.

In special cases it is possible to give more precise statements. For diffusively coupled systems such as (5) or (6), the coupling matrix C equals the negative of the Laplacian matrix. Therefore, in case the connection weights a_{ij} are non-negative, all the eigenvalues of C are non-positive, the largest one always being zero. In fact, for connected networks, the eigenvalues are strictly negative, except for a single zero eigenvalue (see e.g. [21]). In this case, it is the smallest eigenvalue of C (i.e. the largest Laplacian eigenvalue) that determines the first bifurcation. As far as the network structure is concerned, this is the only relevant quantity.

For systems of the form (3) or (4), C is given by the adjacency matrix A, which can have both negative and positive eigenvalues even when all a_{ij} have the same sign. Thus both $\xi_{\min}$ and $\xi_{\max}$ should be considered for the first bifurcation. For sufficiently small τ, the bifurcation occurs in the vicinity of the origin of the α-β parameter plane of figure 1. Since the line L1 intersects the origin where the curve C2 has a gap, the likely bifurcation is a stationary one and the eigenvalue responsible for the bifurcation is the largest positive eigenvalue of A. This agrees with the observation of section 3.2 below that oscillatory bifurcations arise from sufficiently large delays, for the class of scalar systems studied in this paper.

3.2. Effects of delay

In the absence of delays ($\tau = 0$), the characteristic equation for (2) is

$$\prod_{\xi\in\sigma(C)} (\lambda - f'(0) - \kappa\xi) = 0. \tag{17}$$

from which the characteristic roots can be directly read off as $\lambda = f'(0) + \kappa\xi$, $\xi \in \sigma(C)$. The roots are real for real network eigenvalues ξ; hence the only critical root is $\lambda = 0$, which occurs when $f'(0) = -\kappa\xi$. The corresponding critical curve is a straight line on the parameter plane of $f'(0)$ versus $\kappa\xi$, which is identical with the line L1 of figure 1. Thus, one has stability below this line and one real positive characteristic root above, for a given spatial mode corresponding to ξ. In particular, Hopf bifurcations do not occur.

To see the effects of the delay, we fix the other quantities κ, ξ and $f'(0)$ and notice that the values of α, β in figure 1 then change only along the ray emanating from the origin with slope $m = \kappa\xi/f'(0)$. The delay τ parametrizes the distance of points along the ray to the origin. Hence, to use the delay as a bifurcation parameter, one goes along the ray starting from the origin and obtain bifurcations as the curves given in figure 1 are intersected. Such rays only intersect with L1 at the origin or else completely coincide with L1; moreover, they intersect the other curves if the slope m is sufficiently large. Since the latter curves correspond to pairs of purely imaginary characteristic roots, large values of the delay cause oscillatory bifurcations.

To summarize, stationary bifurcations given by L1 of figure 1 are independent of the delay, whereas the remaining set of curves correspond to oscillatory bifurcation resulting from the delay. In the following sections we will consider both stationary and oscillatory bifurcations in our symmetry analysis. Stationary bifurcations will be relevant for both delayed and undelayed systems, whereas oscillatory bifurcations will be a feature of delayed systems only, in the context of scalar systems that we consider.

4. Symmetries and equivariant bifurcations

By a *symmetry* of a dynamical system, we mean a group of elements acting on the phase space that keep the system invariant. More precisely, given a system of form

$$\frac{\mathrm{d}x}{\mathrm{d}t} = F(x) \tag{18}$$

with $x \in \mathbb{R}^n$ and $F : \mathbb{R}^n \to \mathbb{R}^n$, and an action of a group Γ on the phase space $\mathbb{R}^n$, an element $\gamma \in \Gamma$ is called a *symmetry* of (18) if (18) remains unchanged after applying the action of γ on both sides. Since a group action is linear, it commutes with the linear operator $\frac{d}{dt}$; thus γ is a symmetry of (18) if and only if $\gamma F(x) = F(\gamma x)$ for all $x \in \mathbb{R}^n$.

Let S_n be the group of all permutations of n symbols. For $\Sigma \in S_n$, consider its natural action on $\mathbb{R}^n$ by $(x_1, \ldots, x_n) \mapsto (x_{\Sigma(1)}, \ldots, x_{\Sigma(n)})$. Consider a subgroup $\Gamma \subset S_n$.

Lemma 4.1. Let $\kappa \neq 0$. Then Γ is a symmetry of systems of form (1) if and only if

$$g_{\Sigma(i)}(x_1, x_2, \ldots, x_n) = g_i(x_{\Sigma(1)}, x_{\Sigma(2)}, \ldots, x_{\Sigma(n)}), \tag{19}$$

for all $\Sigma \in \Gamma$ and $(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$.

Proof. Let $\Sigma \in \Gamma$ and apply its action on (1). We obtain

$$\dot{x}_{\Sigma(i)}(t) = f(x_{\Sigma(i)}(t)) + \kappa g_i(x_{\Sigma(1)}(t-\tau), x_{\Sigma(2)}(t-\tau), \ldots, x_{\Sigma(n)}(t-\tau)). \tag{20}$$

Comparing with (1), we see that (20) is the same system as (1) if and only if

$$\begin{aligned}\kappa g_i(x_{\Sigma(1)}&(t-\tau), x_{\Sigma(2)}(t-\tau), \ldots, x_{\Sigma(n)}(t-\tau))\\ &= \kappa g_{\Sigma(i)}(x_1(t-\tau), x_2(t-\tau), \ldots, x_n(t-\tau)).\end{aligned}$$

This leads to (19), since $\kappa \neq 0$. □

Remark 4.2. Note that a necessary condition for (19) to hold is that the coupling matrix C in the linearization (2) satisfies

$$c_{ij} = c_{\Sigma(i)\Sigma(j)}, \quad \forall\, \Sigma \in \Gamma. \tag{21}$$

For the specific systems (3)–(6) it can be checked that (21) is also a sufficient condition, since (19) reduces to $a_{ij} = a_{\Sigma(i)\Sigma(j)}$ $\forall \Sigma \in \Gamma$. ◇

In what follows, we will study the bifurcations that destabilize the zero solution under a group of symmetries $\Gamma \subset S_n$ of the system (1) using the equivariant degree. Exact values of associated bifurcation invariants are calculated using the EDML (Equivariant Degree Maple Library) Package, by calling

$$\texttt{showdegree}[\Gamma](n_0, n_1, \ldots, n_r, m_0, m_1, \ldots, m_s), \quad \text{for } n_i, m_j \in \mathbb{Z}, \tag{22}$$

where the n_i and m_j are integers to be determined by the critical spectrum of the linearized system at the equilibrium. The integers r and s in (22) are predetermined by Γ and are equal to the number of all distinct (non-trivial) irreducible representations of Γ over reals and over complex numbers, respectively. In what follows, we use $\mathcal{V}_0, \mathcal{V}_1, \ldots, \mathcal{V}_r$ for the distinct real irreducible representations and $\mathcal{U}_0, \mathcal{U}_1, \ldots, \mathcal{U}_s$ for the complex ones, where $\mathcal{V}_0$ and $\mathcal{U}_0$ are reserved for the trivial representations.

4.1. Steady-state bifurcations

In reference to figure 1, suppose that the parameters (α, β) are varied to leave the shaded stability region by crossing L1 at some point (α_o, β_o). Then,

$$\alpha_o = -\beta_o = \tau\kappa\xi_o, \tag{23}$$

for an eigenvalue $\xi_o \in \sigma(C)$. For $\tau, \kappa > 0$, ξ_o is the maximal eigenvalue of C. Let $E(\xi_o)$ be the generalized eigenspace of ξ_o. Given the Γ-action on $\mathbb{R}^n$, we decompose $\mathbb{R}^n$ into pieces of $\mathcal{V}_i$'s:

$$\mathbb{R}^n = V_0 \times V_1 \times \cdots \times V_r,$$

where every V_i

$$V_i = \underbrace{\mathcal{V}_i \times \cdots \times \mathcal{V}_i}_{n_i \text{ times}} \tag{24}$$

is a product of n_i copies of $\mathcal{V}_i$ for some integer $n_i \in \mathbb{N} \cup \{0\}$. Also, since $E(\xi_o)$ is a Γ-invariant subspace of $\mathbb{R}^n$, we can decompose $E(\xi_o)$ as:

$$E(\xi_o) = E_0 \times E_1 \times \cdots \times E_r,$$

where every E_i is given by

$$E_i = \underbrace{\mathcal{V}_i \times \cdots \times \mathcal{V}_i}_{e_i \text{ times}} \tag{25}$$

i.e. as a product of e_i copies of $\mathcal{V}_i$ for some integer $e_i \in \mathbb{N} \cup \{0\}$. Using (24)–(25), define

$$u_i := n_i - e_i, \tag{26}$$

for $i = 0, 1, \ldots, r$. Then, the bifurcation invariant around (α_o, β_o) is given by

$$\omega_0 := \texttt{showdegree}[\Gamma](n_0, \ldots, n_r, 1, 0, \ldots, 0) - \texttt{showdegree}[\Gamma](u_0, \ldots, u_r, 1, 0, \ldots, 0). \tag{27}$$

Running the EDML package we obtain the value of ω_0, which is of form

$$c_1(K_1) + c_2(K_2) + \cdots + c_p(K_p),$$

for integers $c_i \in \mathbb{Z}$ and conjugacy classes (K_i) of subgroups K_i in Γ.

Theorem 4.3. *Let (α_o, β_o) be such that $\alpha_o = -\beta_o$ and $\xi_o \in \sigma(C)$ be given by (23). If ω_0 given by (27) is of form*

$$\omega_0 = c_1(K_1) + c_2(K_2) + \cdots + c_p(K_p),$$

for some $c_i \neq 0$, then there exists a bifurcating branch of steady states of symmetry at least (K_i).

Proof. By the existence property of equivariant degree, it is sufficient to prove that the value of the bifurcation invariant for the steady state bifurcations around (α_o, β_o) is indeed given by formula (27), since if this is the case, then any non-zero coefficient in the value indicates a bifurcating branch with the corresponding symmetry. The formula (27) follows from theorem 8.5.2 in [22], but we give an alternative and more straightforward proof in Appendix A for completeness. □

Corollary 4.4. *Assume the hypotheses of theorem 4.3, and suppose furthermore that the subgroup K_i satisfies*

$$\xi_o \notin \sigma(C|_{\mathrm{Fix}(H)}), \quad \forall H \supsetneq K_i. \tag{28}$$

Then there exists a bifurcating branch of steady states of symmetry precisely (K_i).

Proof. By theorem 4.3, there exists a bifurcating branch of steady states of symmetry *at least* (K_i). Let (H) be the symmetry of this branch of solutions. Then, $(H) \geqslant (K_i)$. Up to the group conjugacy, we have $H \supseteq K_i$. We need to show $H = K_i$. Assume to the contrary that $H \supsetneq K_i$. Then by (28) we have that, when restricted to $\mathrm{Fix}(H)$, the characteristic operator $\Delta(0)|_{\mathrm{Fix}(H)} : \mathrm{Fix}(H) \to \mathrm{Fix}(H)$ is an isomorphism for (α, β) in a neighbourhood of (α_o, β_o). By the implicit function theorem, there can be no additional solution in neighbourhood of the trivial solution $x = 0 \in \mathrm{Fix}(H)$, which is a contradiction. □

4.2. Hopf Bifurcations

Assume that (α, β) leaves the shaded stability region of figure 1 by crossing C2 at some point (α_o, β_o). Since C2 bounds the region from below and $\tau, \kappa > 0$, the first parameter pair that crosses C2 must be related to the minimal eigenvalue $\xi_{\min}$ of C. Let $\xi_o \in \sigma(C)$ be the corresponding eigenvalue, i.e.

$$\beta_o = \tau\kappa\xi_o. \tag{29}$$

That is, $\xi_o = \xi_{\min}$ becomes critical. Consider the complexification $\mathbb{C}^n = \mathbb{C} \otimes \mathbb{R}^n$ of the phase space $\mathbb{R}^n$ and extend the Γ-action on $\mathbb{C}^n$ by defining

$$\gamma(z \otimes x) = z \otimes (\gamma x), \quad \text{for} \quad \gamma \in \Gamma,\ x \in \mathbb{R}^n. \tag{30}$$

The (generalized) eigenspace $E(\xi_o)$ remains Γ-invariant as a complex subspace of $\mathbb{C}^n$. Thus, we decompose $E(\xi_o)$ into irreducible representations $\mathcal{U}_0, \mathcal{U}_1, \ldots, \mathcal{U}_s$ as:

$$E(\xi_o) = F_0 \times F_1 \times \cdots \times F_s,$$

where every F_j is given by

$$F_j = \underbrace{\mathcal{U}_j \times \cdots \times \mathcal{U}_j}_{m_j \text{ times}} \tag{31}$$

that is, a product of m_j copies of $\mathcal{U}_j$ for some integer $m_j \in \mathbb{N} \cup \{0\}$. Then, the bifurcation invariant around (α_o, β_o) for Hopf bifurcation is given by

$$\omega_1 := \texttt{showdegree}[\Gamma](0, 0, \ldots, 0, -m_0, -m_1, \ldots, -m_s). \tag{32}$$

Running the EDML package, we obtain the value of ω_1 being of form

$$c_1(H_1) + c_2(H_2) + \cdots + c_q(H_q),$$

for integer coefficients $c_j \in \mathbb{Z}$ and conjugacy classes (H_j) of subgroups $H_j \subset \Gamma \times S^1$.

Theorem 4.5. *Let* (α_o, β_o) *be such that* $(\alpha_o, \beta_o) \in \mathrm{C2}$ *in figure 1 and* ω_1 *be given by (32). If*

$$\omega_1 = c_1(H_1) + c_2(H_2) + \cdots + c_q(H_q),$$

contains a non-zero coefficient $c_j \neq 0$ *for some* (H_j)*, then there exists a bifurcating branch of oscillating states of symmetry at least* (H_j)*.*

Proof. Using equivariant degree theory, the bifurcation invariant is computed by (see [12])

$$\omega_1 = \texttt{showdegree}[\Gamma](k_0, k_1, \ldots, k_r, t_0, t_1, \ldots, t_s),$$

where k_i's are related to the *positive spectrum* of the right-hand side of (12) in the constant function space, and the t_j's are the *crossing numbers* which are equal to either m_j or $-m_j$, depending on the direction of the crossing of the critical characteristic roots.

Consider (12) in the constant function space. Then,

$$\big(\tau f'(0)\mathrm{Id} + \tau\kappa C\big)x = 0, \quad x \in \mathbb{R}^n.$$

The positive spectrum σ_+ of the linear operator $(\tau f'(0)\mathrm{Id} + \tau\kappa C)$ is

$$\sigma_+ = \{\tau f'(0) + \tau\kappa\xi : \tau f'(0) + \tau\kappa\xi > 0,\ \xi \in \sigma(C)\} = \{\alpha + \beta(\xi) : \alpha + \beta(\xi) > 0,\ \xi \in \sigma(C)\},$$

which is an empty set, since the curve C2 lies in the area $\alpha + \beta < 0$. Since the integer k_i is the total number of copies of $\mathcal{V}_i$ in $E(\mu)$ for $\mu \in \sigma_+$, we have $k_i = 0$ for all $i = 0, 1, \ldots, r$.

The crossing numbers are positive if the critical characteristic roots cross from the right to the left of the complex plane; and negative otherwise. As (a, β) crosses C2 at (α_o, β_o) from the shaded region in figure 1, the count of characteristic roots with positive real part increases by 2, thus all non-zero t_j's are negative and equal to $-m_j$. □

Theorem 4.5 gives an existence result of bifurcating branches together with their *least* symmetry. To sharpen to the *precise* symmetry, one can work with orbit types that satisfy certain maximal condition. Here, we recall the concept of *dominating orbit types* from [12] and introduce a new complementing definition of *secondary dominating orbit types*.

Definition 4.6. Let $\{\mathcal{U}_1, \mathcal{U}_2, \ldots, \mathcal{U}_m\}$ be the set of irreducible Γ-representations that occur in $\mathbb{C}^n$, where $\mathbb{C}^n$ is the complexification of the phase space $\mathbb{R}^n$ of the system (1). Let $\bar{\mathcal{U}}_j$ be the $\Gamma \times S^1$-representation induced from $\mathcal{U}_j$, for $j = 1, 2, \ldots, m$ (see (7)). Collect maximal orbit types from $\bar{\mathcal{U}}_j$, for $j = 1, 2, \ldots, m$, and denote this collection by $\mathcal{M}$. An orbit type $(H) \in \mathcal{M}$ is called *dominating* if (H) is maximal in $\mathcal{M}$. A non-dominating orbit type $(L) \in \mathcal{M}$ is called *secondary dominating* if all orbit types $(H) \in \mathcal{M}$ satisfying $(L) < (H)$ are dominating. ⋄

Proposition 4.7. *Let* (α_o, β_o) *be such that* $(\alpha_o, \beta_o) \in \mathrm{C2}$ *in figure 1 and* ξ_o *be the corresponding eigenvalue of* C *given by (29). Assume that* ω_1 *defined by (32) contains* (H) *with a non-zero coefficient. Then the following hold:*

(i) If (H) *is a dominating orbit type, then there exists a bifurcating branch of oscillating states of symmetry precisely equal to* (H)*.*

(ii) Suppose that (H) *is a secondary dominating orbit type, and for every dominating orbit type* $(\tilde{H})$ *with* $(H) < (\tilde{H})$ *there exists a flow-invariant subspace* $S \subset \mathbb{R}^n$ *such that*

(a) S *contains every state of symmetry* $\tilde{H}$*; and*

(b) $\xi_o \notin \sigma(C|_S)$*.*

Then there exists a bifurcating branch of oscillating states of symmetry precisely being (H)*.*

Proof. Statement (i) follows from [12], and (ii) follows from the implicit function theorem, in the same spirit as corollary 4.4. More precisely, let (H) be a secondary dominating orbit type with a non-zero coefficient in ω_1. By theorem 4.5, there exists a bifurcating branch of oscillating states of symmetry at least (H). Let $(\tilde{H})$ be the precise symmetry of this branch and suppose that $(H) < (\tilde{H})$. By definition of secondary dominating orbit types, the only orbit types that are strictly larger than (H) are dominating orbit types. Thus $(\tilde{H})$ is dominating, and so there exists a flow-invariant subspace S in $\mathbb{R}^n$ satisfying (a) and b). Consider the restricted flow on S. The bifurcating branch of oscillating states, by condition (a), is contained in S. However, by condition (b) and the implicit function theorem, there can be no bifurcation taking place in S. This leads to a contradiction. □

5. Bidirectional ring configuration

In this section, we study the bifurcations of the system (1) on a particular class of networks, namely bidirectional ring configurations. That is, we assume g_i's satisfy (19) for $\Gamma = D_n$. If the system has one of the specific forms (3)–(6), this assumption can be weakened to (21). In either case, the coupling matrix C in (2) satisfies (21), which in case of dihedral configuration implies that C is a *circulant matrix*[6] with $c_{1j} = c_{1,(n+2-j)}$ for $1 \leqslant j \leqslant n$. In particular, C is a symmetric matrix.

A circulant matrix with first row entries $c_0, c_1, \ldots, c_{n-1}$ has eigenvalues

$$\xi_j = c_0 + c_1\varrho_j + c_2\varrho_j^2 + \cdots + c_{n-1}\varrho_j^{n-1}, \quad j = 0, 1, 2, \ldots, n-1, \tag{33}$$

with corresponding eigenvectors $v_j = (1, \varrho_j, \varrho_j^2, \ldots, \varrho_j^{n-1})^T$, where $\varrho_j = \exp(2\pi i j/n)$ are the nth roots of unity. Moreover, if the circulant matrix is D_n-symmetric, then $\xi_j = \xi_{n-j}$ for $0 < j < n$, which is essentially induced by the D_n-symmetry. In fact, we have

$$\begin{cases} E(\xi_0) = \mathcal{V}_0, \\ E(\xi_j) = E(\xi_{n-j}) = \mathcal{V}_j \quad \text{for } 0 < j < \dfrac{n}{2}, \\ E(\xi_{\frac{n}{2}}) = \mathcal{V}_{(\frac{n}{2}+2)}, \quad \text{if } n \text{ is even} \end{cases} \tag{34}$$

(see example 2.2 for notations $\mathcal{V}_j$). An eigenvalue $\xi \in \sigma(C)$ is called *simple* if $E(\xi)$ is irreducible. To a critical eigenvalue ξ_o, we associate an index set

$$I = \{i \,:\, \xi_i = \xi_o\} \tag{35}$$

(in case n is even and $\xi_{\frac{n}{2}} = \xi_o$, we put $\frac{n}{2} + 2$ into I instead of $\frac{n}{2}$), which collects all indices of irreducible representations that have to do with the critical eigenvalue ξ_o.

5.1. Steady-state bifurcations for bidirectional rings

Recall that D_n acts on the phase space $\mathbb{R}^n$ by

$$\eta(x_1, x_2, \ldots, x_n) = (x_n, x_1, x_2, \ldots, x_{n-1}) \tag{36}$$

$$\varsigma(x_1, x_2, \ldots, x_n) = (x_n, x_{n-1}, \ldots, x_1), \tag{37}$$

for $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$. Using characters of representations, $\mathbb{R}^n$ can be decomposed into irreducible representations of D_n. In case of even n, we have

$$\mathbb{R}^n = \mathcal{V}_0 \times \mathcal{V}_1 \times \mathcal{V}_2 \times \cdots \times \mathcal{V}_{\frac{n}{2}-1} \times \mathcal{V}_{\frac{n}{2}+2} \tag{38}$$

[6] Recall that an $n \times n$ matrix is called *circulant* if every row is the right shift of the previous row (mod n). A circulant matrix $C = (c_{ij})$ is also denoted by $circ[c_{11}, c_{12}, \ldots, c_{1n}]$ using the entries of its first row.

and in case of odd n, we have

$$\mathbb{R}^n = \mathcal{V}_0 \times \mathcal{V}_1 \times \mathcal{V}_2 \times \cdots \times \mathcal{V}_{\frac{n-1}{2}} \tag{39}$$

(see example 2.2 for notations $\mathcal{V}_j$). It follows that the non-zero n_i's in (27) are (see (24))

$$\begin{cases} n_0 = n_1 = n_2 = \cdots = n_{\frac{n}{2}-1} = n_{\frac{n}{2}+2} = 1, & \text{if } n \text{ is even,} \\ n_0 = n_1 = n_2 = \cdots = n_{\frac{n-1}{2}} = 1, & \text{if } n \text{ is odd.} \end{cases} \tag{40}$$

The integers u_i's in (27) are determined by the critical eigenvalue ξ_o and the corresponding I (see (35)). Based on (34) and the definition (26) of u_i, the non-zero u_i's are

$$\begin{cases} u_i = 1, & \text{for } i \in \{0, 1, 2, \ldots, \frac{n}{2} - 1, \frac{n}{2} + 2\} \setminus I, \quad \text{if } n \text{ is even,} \\ u_i = 1, & \text{for } i \in \{0, 1, 2, \ldots, \frac{n-1}{2}\} \setminus I, \quad \text{if } n \text{ is odd.} \end{cases} \tag{41}$$

Thus, the bifurcation invariant ω_0 can be computed using (27), accompanied by (40)–(41).

Example 5.1. *(Simple critical eigenvalues for bidirectional rings.)* Let C be a coupling matrix satisfying (21) for $\Gamma = D_n$. Then C is determined by $(\frac{n}{2}+1)$ or $(\frac{n+1}{2})$ different entries depending on whether n is even or odd, respectively. These entries decide which eigenvalue is maximal. Let $\xi_o \in \sigma(C)$ be the maximal eigenvalue. Assume that ξ_o is simple, i.e. $E(\xi_o)$ is irreducible. Then the index set I is a singleton and there are only possibly $\frac{n}{2}$ or $\frac{n-1}{2}$ different values of ω_0, depending on whether n is even or odd. As an example, for $n = 12$, we have

$$\omega_0 = \begin{cases} -2(D_{12}) + 2(\tilde{D}_6) + 4(D_4) - 2(\tilde{D}_3) + 2(D_3) - 2(\tilde{D}_2) - 2(D_2) & \\ \quad -2(\mathbb{Z}_4) + 2(\mathbb{Z}_2), & \text{if } \xi_o = \xi_0, \\ (D_1) - (\tilde{D}_1), & \text{if } \xi_o = \xi_1, \\ -(D_2) + (\tilde{D}_2) + 2(D_1) - 2(\tilde{D}_1), & \text{if } \xi_o = \xi_2, \\ -(\tilde{D}_3) + (D_3), & \text{if } \xi_o = \xi_3, \\ 2(D_4) - 2(D_2) - (\mathbb{Z}_4) + (\mathbb{Z}_2) - 2(\tilde{D}_1) + 2(D_1), & \text{if } \xi_o = \xi_4, \\ -(\tilde{D}_1) + (D_1), & \text{if } \xi_o = \xi_5, \\ (\tilde{D}_6) - 2(\tilde{D}_3) + (\mathbb{Z}_3), & \text{if } \xi_o = \xi_6. \end{cases} \tag{42}$$

These values, combined with fixed point subspaces of subgroups of D_{12} (see table 1), lead to the classification result summarized in table 2. To illustrate, in case $\xi_o = \xi_1$, we have two orbit types (D_1) and $(\tilde{D}_1)$ with non-zero coefficients in ω_0. Using table 1, we have that $\xi_1 \notin \sigma(C|_{\mathrm{Fix}(H)})$ for all $H > D_1$, thus by corollary 4.4, there exists at least one bifurcating branch of steady states of symmetry precisely (D_1). Since (D_1) consists of 6 isotropy subgroups: $\eta^k D_1 \eta^{-k}$ for $k = 0, 1, \ldots, 5$, we derive the form of the solution for each of these isotropies. The same can be applied to $(\tilde{D}_1)$.

Note that the possible values of ω_0 do not depend on the entries of C directly, but rather on the maximal eigenvalue. For example, if every cell is connected only with its 2 nearest neighbours, then $\xi_o = \xi_0$ if the coupling is excitatory; and $\xi_o = \xi_6$ if it is inhibitory. That is, this configuration does not allow ξ_o to be ξ_i for $i \in \{1, 2, 3, 4, 5\}$. However, if each cell i is connected to its 4 nearest neighbours, with coupling strength d_1 to cells $(i \pm 1)$ and with strength d_2 to $(i \pm 2)$, then every eigenvalue can be maximal for some choices of d_1, d_2. See figure 2 for their precise relation. ◇

Besides those values listed in (42), ω_0 can take other values if ξ_o is non-simple. For example, the coupling configuration with four nearest neighbours allows double critical eigenvalues as shown in figure 2, when the relation between d_1, d_2 follows one of the lines there. In this case, one can work out the index set I and compute ω_0 individually. The same result using theorem 4.3 and corollary 4.4 applies.

Table 1. Fixed point subspaces of $K \subset D_{12}$ and eigenvalues of the coupling matrix $C|_{\mathrm{Fix}(K)} : \mathrm{Fix}(K) \to \mathrm{Fix}(K)$ (up to conjugacy classes of subgroups).

K	$\mathrm{Fix}(K)$	$\sigma(C\vert_{\mathrm{Fix}(K)})$
D_{12}	$\{x_1 = x_2 = \cdots = x_{12}\}$	ξ_0
D_6	$\{x_1 = x_2 = \cdots = x_{12}\}$	ξ_0
$\tilde{D}_6$	$\{x_1 = x_3 = \cdots = x_{11}, x_2 = x_4 = \cdots = x_{12}\}$	ξ_0, ξ_6
$\mathbb{Z}_6$	$\{x_1 = x_3 = \cdots = x_{11}, x_2 = x_4 = \cdots = x_{12}\}$	ξ_0, ξ_6
D_4	$\{x_1 = x_3 = x_4 = x_6 = x_7 = x_9 = x_{10} = x_{12}, x_2 = x_5 = x_8 = x_{11}\}$	ξ_0, ξ_4
$\mathbb{Z}_4$	$\{x_1 = x_4 = x_7 = x_{10}, x_2 = x_5 = x_8 = x_{11}, x_3 = x_6 = x_9 = x_{12}\}$	ξ_0, ξ_4, ξ_4
D_3	$\{x_1 = x_4 = x_5 = x_8 = x_9 = x_{12}, x_2 = x_3 = x_6 = x_7 = x_{10} = x_{11}\}$	ξ_0, ξ_3
$\tilde{D}_3$	$\{x_1 = x_3 = x_5 = x_7 = x_9 = x_{11}, x_2 = x_6 = x_{10}, x_4 = x_8 = x_{12}\}$	ξ_0, ξ_3, ξ_6
$\mathbb{Z}_3$	$\{x_1 = x_5 = x_9, x_2 = x_6 = x_{10}, x_3 = x_7 = x_{11}, x_4 = x_8 = x_{12}\}$	$\xi_0, \xi_3, \xi_3, \xi_6$
D_2	$\{x_1 = x_6 = x_7 = x_{12}, x_2 = x_5 = x_8 = x_{11}, x_3 = x_4 = x_9 = x_{10}\}$	ξ_0, ξ_2, ξ_4
$\tilde{D}_2$	$\{x_1 = x_5 = x_7 = x_{11}, x_2 = x_4 = x_8 = x_{10}, x_3 = x_9, x_6 = x_{12}\}$	$\xi_0, \xi_2, \xi_4, \xi_6$
$\mathbb{Z}_2$	$\{x_1 = x_7, x_2 = x_8, x_3 = x_9, x_4 = x_{10}, x_5 = x_{11}, x_6 = x_{12}\}$	$\xi_0, \xi_2, \xi_2, \xi_4, \xi_4, \xi_6$
D_1	$\{x_1 = x_{12}, x_2 = x_{11}, x_3 = x_{10}, x_4 = x_9, x_5 = x_8, x_6 = x_7\}$	$\xi_0, \xi_1, \xi_2, \xi_3, \xi_4, \xi_5$
$\tilde{D}_1$	$\{x_1 = x_{11}, x_2 = x_{10}, x_3 = x_9, x_4 = x_8, x_5 = x_7\}$	$\xi_0, \xi_1, \xi_2, \xi_3, \xi_4, \xi_5, \xi_6$
$\mathbb{Z}_1$	$\mathbb{R}^{12}$	$\xi_0, \xi_1, \xi_1, \xi_2, \xi_2, \xi_3, \xi_3, \xi_4, \xi_4, \xi_5, \xi_5, \xi_6$

5.2. Hopf bifurcations for bidirectional rings

The complexification of $E(\xi_j)$ for $\xi_j \in \sigma(C)$ satisfies

$$\begin{cases} E^c(\xi_0) = \mathcal{U}_0, \\ E^c(\xi_j) = E^c(\xi_{n-j}) = \mathcal{U}_j \quad \text{for } 0 < j < \frac{n}{2} \\ E^c(\xi_{\frac{n}{2}}) = \mathcal{U}_{(\frac{n}{2}+2)}, \quad \text{if } n \text{ is even} \end{cases} \tag{43}$$

(see example 2.2 for notations $\mathcal{U}_j$). It follows that the non-zero integers m_j's in (32) are

$$m_j = 1, \quad \text{for } j \in I, \tag{44}$$

where I is given by (35). The bifurcation invariant ω_1 can then be computed using (32) together with (44).

Example 5.2. *(Simple critical eigenvalues for bidirectional rings).*Following example 5.1, we take C that satisfies (21) with $\Gamma = D_n$. The $(\frac{n}{2}+1)$ or $(\frac{n+1}{2})$ different entries of C decide which eigenvalue is minimal. Let $\xi_o \in \sigma(C)$ be the minimal eigenvalue. Assume that ξ_o is simple. Then the index set I is a singleton and there are only $\frac{n}{2}$ or $\frac{n-1}{2}$ different values of ω_0, depending on whether n is even or odd, respectively. Again for $n = 12$, we have

$$\omega_1 = \begin{cases} -(D_{12}), \quad \text{if } \xi_o = \xi_0 \\ -(\mathbb{Z}_{12}^{t_1}) - (D_2^d) - (\tilde{D}_2^d) + (\mathbb{Z}_2^d), \quad \text{if } \xi_o = \xi_1, \\ -(\mathbb{Z}_{12}^{t_2}) - (D_4^d) - (D_4^{\hat{d}}) + (\mathbb{Z}_4^d), \quad \text{if } \xi_o = \xi_2, \\ -(\mathbb{Z}_{12}^{t_3}) - (D_6^d) - (\tilde{D}_6^d) + (\mathbb{Z}_6^d), \quad \text{if } \xi_o = \xi_3, \\ -(\mathbb{Z}_{12}^{t_4}) - (D_4^z) - (D_4) + (\mathbb{Z}_4), \quad \text{if } \xi_o = \xi_4, \\ -(\mathbb{Z}_{12}^{t_5}) - (D_2^d) - (\tilde{D}_2^d) + (\mathbb{Z}_2^d), \quad \text{if } \xi_o = \xi_5, \\ -(D_{12}^{\hat{d}}), \quad \text{if } \xi_o = \xi_6. \end{cases}$$

Table 2. Summary of distinct forms of steady states bifurcating from the equilibrium $x = 0$ of the system (1) for $n = 12$.

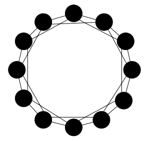
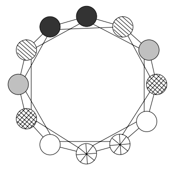
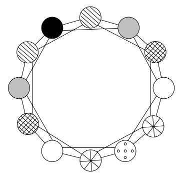
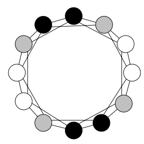
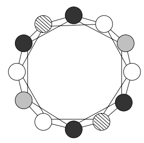
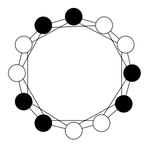
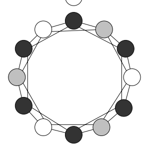
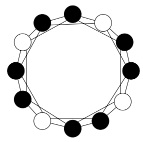
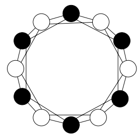
Critical eigenvalue	Symmetry	Form of bifurcating steady-states (for distinct $a, b, c, d, e, f, g \in \mathbb{R}$)	Figure
ξ_0	D_{12}	$(a, a, a, a, a, a, a, a, a, a, a, a)$	
ξ_1 or ξ_5	D_1	$(a, b, c, d, e, f, f, e, d, c, b, a)$	
	$\eta D_1 \eta^{-1}$	$(a, a, b, c, d, e, f, f, e, d, c, b)$	
	$\eta^2 D_1 \eta^{-2}$	$(b, a, a, b, c, d, e, f, f, e, d, c)$	
	$\eta^3 D_1 \eta^{-3}$	$(c, b, a, a, b, c, d, e, f, f, e, d)$	
	$\eta^4 D_1 \eta^{-4}$	$(d, c, b, a, a, b, c, d, e, f, f, e)$	
	$\eta^5 D_1 \eta^{-5}$	$(e, d, c, b, a, a, b, c, d, e, f, f)$	
	$\tilde{D}_1$	$(a, b, c, d, e, f, e, d, c, b, a, g)$	
	$\eta \tilde{D}_1 \eta^{-1}$	$(g, a, b, c, d, e, f, e, d, c, b, a)$	
	$\eta^2 \tilde{D}_1 \eta^{-2}$	$(a, g, a, b, c, d, e, f, e, d, c, b)$	
	$\eta^3 \tilde{D}_1 \eta^{-3}$	$(b, a, g, a, b, c, d, e, f, e, d, c)$	
	$\eta^4 \tilde{D}_1 \eta^{-4}$	$(c, b, a, g, a, b, c, d, e, f, e, d)$	
	$\eta^5 \tilde{D}_1 \eta^{-5}$	$(d, c, b, a, g, a, b, c, d, e, f, e)$	
ξ_2	D_2	$(a, b, c, c, b, a, a, b, c, c, b, a)$	
	$\eta D_2 \eta^{-1}$	$(a, a, b, c, c, b, a, a, b, c, c, b)$	
	$\eta^2 D_2 \eta^{-2}$	$(b, a, a, b, c, c, b, a, a, b, c, c)$	
	$\tilde{D}_2$	$(a, b, c, b, a, d, a, b, c, b, a, d)$	
	$\eta \tilde{D}_2 \eta^{-1}$	$(d, a, b, c, b, a, d, a, b, c, b, a)$	
	$\eta^2 \tilde{D}_2 \eta^{-2}$	$(a, d, a, b, c, b, a, d, a, b, c, b)$	
ξ_3	D_3	$(a, b, b, a, a, b, b, a, a, b, b, a)$	
	$\eta D_3 \eta^{-1}$	$(a, a, b, b, a, a, b, b, a, a, b, b)$	
	$\tilde{D}_3$	$(a, b, a, c, a, b, a, c, a, b, a, c)$	
	$\eta \tilde{D}_3 \eta^{-1}$	$(c, a, b, a, c, a, b, a, c, a, b, a)$	

Table 2. (Continued.)

Critical eigenvalue	Symmetry	Form of bifurcating steady-states (for distinct $a, b, c, d, e, f, g \in \mathbb{R}$)	Figure
ξ_4	D_4 $\eta D_4 \eta^{-1}$ $\eta^2 D_4 \eta^{-2}$	$(a, b, a, a, b, a, a, b, a, a, b, a)$ $(a, a, b, a, a, b, a, a, b, a, a, b)$ $(b, a, a, b, a, a, b, a, a, b, a, a)$	
ξ_6	$\tilde{D}_6$	$(a, b, a, b, a, b, a, b, a, b, a, b)$	

To find dominating and secondary dominating orbit types, consider the maximal orbit types in $\mathcal{U}_i$'s. They are

(D_{12}) in $\mathcal{U}_0$;
$(\mathbb{Z}_{12}^{t_1})$, (D_2^d) $(\tilde{D}_2^d)$ in $\mathcal{U}_1$;
$(\mathbb{Z}_{12}^{t_2})$, (D_4^d) $(D_4^{\hat{d}})$ in $\mathcal{U}_2$;
$(\mathbb{Z}_{12}^{t_3})$, (D_6^d), $(\tilde{D}_6^d)$ in $\mathcal{U}_3$;
$(\mathbb{Z}_{12}^{t_4})$, (D_4^z) (D_4) in $\mathcal{U}_4$;
$(\mathbb{Z}_{12}^{t_5})$, (D_2^d) $(\tilde{D}_2^d)$ in $\mathcal{U}_5$;
$(D_{12}^{\hat{d}})$ in $\mathcal{U}_6$.

Among these orbit types, we find the dominating orbit types: (D_{12}), $(D_{12}^{\hat{d}})$, $(\mathbb{Z}_{12}^{t_1})$, $(\mathbb{Z}_{12}^{t_2})$, $(\mathbb{Z}_{12}^{t_3})$, $(\mathbb{Z}_{12}^{t_4})$, $(\mathbb{Z}_{12}^{t_5})$, (D_6^d), $(\tilde{D}_6^d)$, (D_4^d), (D_4^z) and the secondary dominating orbit types: (D_2^d), $(\tilde{D}_2^d)$, (D_4), $(D_4^{\hat{d}})$. The values of ω_1 together with the dominating and secondary dominating orbit types lead to the classification result summarized in tables 3–5 using proposition 4.7. ◇

Remark 5.3. Depending on the isotropy, different cells may possess different period. As an example, for $\tilde{D}_2^d$ (and its conjugacy copies) in table 3, consider the two diagrams in the bottom row. The cell at 2 o'clock and the cell at 8 o'clock do not change colour (black) as the time $\frac{T}{2}$ elapses. This means they have half the period as other cells. More precisely, for isotropy $\eta^k \tilde{D}_2^d \eta^{-k}$ with $k = 0, 1, 2, 3, 4, 5$, the cells $3 + k$ and $9 + k$ have half the period as other cells. Similarly, in table 4, for isotropy $\eta^k D_4^d \eta^{-k}$ with $k = 0, 1, 2$, the cells $2 + k$, $5 + k$, $8 + k$, $11 + k$ (black in diagram) have period $\frac{T}{2}$; for isotropy $\eta^k \tilde{D}_6^d \eta^{-k}$ with $k = 0, 1$, the cells $1 + k, 3 + k, 5 + k, 7 + k, 9 + k, 11 + k$ (black in diagram) have period $\frac{T}{2}$. In table 5, for isotropy $\eta^k D_4^z \eta^{-k}$ with $k = 0, 1, 2$, the cells $2 + k, 5 + k, 8 + k, 11 + k$ (black in diagram) have period $\frac{T}{2}$. ◇

6. Near-neighbour coupling and simulation examples

In this section we consider the 12-cell ring with all possibilities of first- and second-closest-neighbour couplings. More precisely, each cell i coupled to its two nearest neighbours on

Table 3. Summary of distinct forms of oscillating states bifurcating from the equilibrium $x = 0$ of the system (1), where cells are coupled to their nearest and next nearest neighbours (part I).

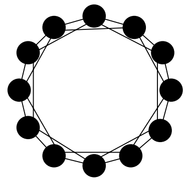
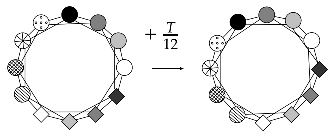
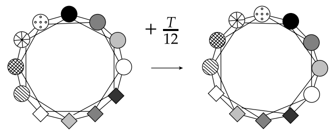
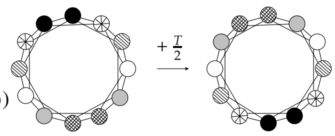
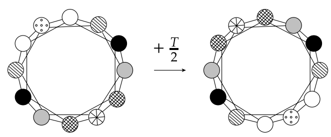
Critical Eigenvalue	Symmetry	Form of oscillating-states (for some period T)	Figure
ξ_0	D_{12}	$(x(t), x(t), x(t), \ldots, x(t))$	
ξ_1	$\mathbb{Z}_{12}^{t_1}$	$(x_1(t), x_1(t+\frac{T}{12}), x_1(t+\frac{2T}{12}) \ldots, x_1(t+\frac{11T}{12}))$	
	$\varsigma\mathbb{Z}_{12}^{t_1}\varsigma^{-1}$	$(x_1(t), x_1(t+\frac{11T}{12}), x_1(t+\frac{10T}{12}) \ldots, x_1(t+\frac{T}{12}))$	
ξ_1 or ξ_5	D_2^d	$(x_1(t), x_2(t), x_3(t), x_3(t+\frac{T}{2}), x_2(t+\frac{T}{2}), x_1(t+\frac{T}{2}), x_1(t+\frac{T}{2}), x_2(t+\frac{T}{2}), x_3(t+\frac{T}{2}), x_3(t), x_2(t), x_1(t))$	
	$\eta D_2^d\eta^{-1}$	$(x_1(t), x_1(t), x_2(t), x_3(t), x_3(t+\frac{T}{2}), x_2(t+\frac{T}{2}), x_1(t+\frac{T}{2}), x_1(t+\frac{T}{2}), x_2(t+\frac{T}{2}), x_3(t+\frac{T}{2}), x_3(t), x_2(t))$	
	$\eta^2 D_2^d\eta^{-2}$	$(x_2(t), x_1(t), x_1(t), x_2(t), x_3(t), x_3(t+\frac{T}{2}), x_2(t+\frac{T}{2}), x_1(t+\frac{T}{2}), x_1(t+\frac{T}{2}), x_2(t+\frac{T}{2}), x_3(t+\frac{T}{2}), x_3(t))$	
	$\eta^3 D_2^d\eta^{-3}$	$(x_3(t), x_2(t), x_1(t), x_1(t), x_2(t), x_3(t), x_3(t+\frac{T}{2}), x_2(t+\frac{T}{2}), x_1(t+\frac{T}{2}), x_1(t+\frac{T}{2}), x_2(t+\frac{T}{2}), x_3(t+\frac{T}{2}))$	
	$\eta^4 D_2^d\eta^{-4}$	$(x_3(t+\frac{T}{2}), x_3(t), x_2(t), x_1(t), x_1(t), x_2(t), x_3(t), x_3(t+\frac{T}{2}), x_2(t+\frac{T}{2}), x_1(t+\frac{T}{2}), x_1(t+\frac{T}{2}), x_2(t+\frac{T}{2}))$	
	$\eta^5 D_2^d\eta^{-5}$	$(x_2(t+\frac{T}{2}), x_3(t+\frac{T}{2}), x_3(t), x_2(t), x_1(t), x_1(t), x_2(t), x_3(t), x_3(t+\frac{T}{2}), x_2(t+\frac{T}{2}), x_1(t+\frac{T}{2}), x_1(t+\frac{T}{2}))$	
	$\tilde{D}_2^d$	$(x_1(t), x_2(t), x_3(t), x_2(t+\frac{T}{2}), x_1(t+\frac{T}{2}), x_4(t), x_1(t+\frac{T}{2}), x_2(t+\frac{T}{2}), x_3(t), x_2(t), x_1(t), x_4(t+\frac{T}{2}))$	
	$\eta\tilde{D}_2^d\eta^{-1}$	$(x_4(t+\frac{T}{2}), x_1(t), x_2(t), x_3(t), x_2(t+\frac{T}{2}), x_1(t+\frac{T}{2}), x_4(t), x_1(t+\frac{T}{2}), x_2(t+\frac{T}{2}), x_3(t), x_2(t), x_1(t))$	
	$\eta^2\tilde{D}_2^d\eta^{-2}$	$(x_1(t), x_4(t+\frac{T}{2}), x_1(t), x_2(t), x_3(t), x_2(t+\frac{T}{2}), x_1(t+\frac{T}{2}), x_4(t), x_1(t+\frac{T}{2}), x_2(t+\frac{T}{2}), x_3(t), x_2(t))$	
	$\eta^3\tilde{D}_2^d\eta^{-3}$	$(x_2(t), x_1(t), x_4(t+\frac{T}{2}), x_1(t), x_2(t), x_3(t), x_2(t+\frac{T}{2}), x_1(t+\frac{T}{2}), x_4(t), x_1(t+\frac{T}{2}), x_2(t+\frac{T}{2}), x_3(t))$	
	$\eta^4\tilde{D}_2^d\eta^{-4}$	$(x_3(t), x_2(t), x_1(t), x_4(t+\frac{T}{2}), x_1(t), x_2(t), x_3(t), x_2(t+\frac{T}{2}), x_1(t+\frac{T}{2}), x_4(t), x_1(t+\frac{T}{2}), x_2(t+\frac{T}{2}))$	
	$\eta^5\tilde{D}_2^d\eta^{-5}$	$(x_2(t+\frac{T}{2}), x_3(t), x_2(t), x_1(t), x_4(t+\frac{T}{2}), x_1(t), x_2(t), x_3(t), x_2(t+\frac{T}{2}), x_1(t+\frac{T}{2}), x_4(t), x_1(t+\frac{T}{2}))$	

Table 4. Summary of distinct forms of oscillating states bifurcating from the equilibrium $x = 0$ of the system (1), where cells are coupled to their nearest and next nearest neighbours (part II).

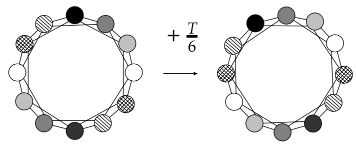
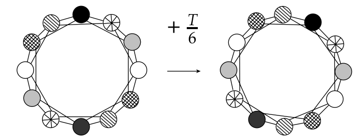
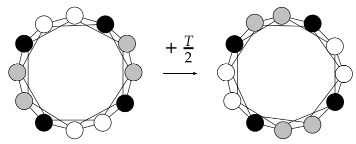
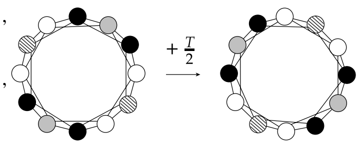
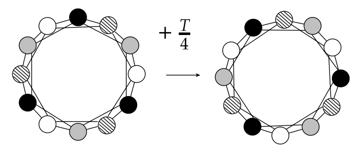
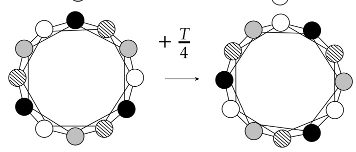
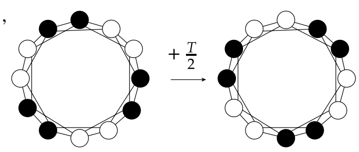
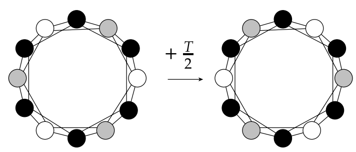
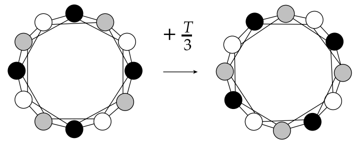
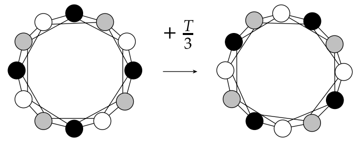
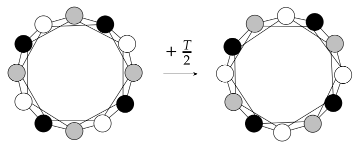
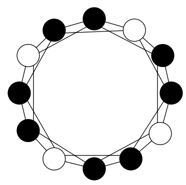
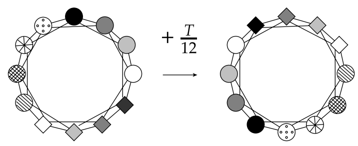
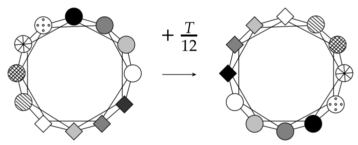
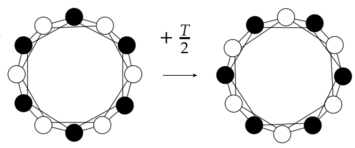
Critical Eigenvalue	Symmetry	Form of oscillating-states (for some period T)	Figure
ξ_2	$\mathbb{Z}_{12}^{t_2}$	$(x_1(t), x_1(t+\frac{T}{6}), x_1(t+\frac{2T}{6}), \dots, x_1(t+\frac{5T}{6}),$ $x_1(t), x_1(t+\frac{T}{6}), x_1(t+\frac{2T}{6}), \dots, x_1(t+\frac{5T}{6}))$	
	$\varsigma\mathbb{Z}_{12}^{t_2}\varsigma^{-1}$	$(x_1(t), x_1(t+\frac{5T}{6}), x_1(t+\frac{4T}{6}) \dots, x_1(t+\frac{T}{6}),$ $x_1(t), x_1(t+\frac{5T}{6}), x_1(t+\frac{4T}{6}) \dots, x_1(t+\frac{T}{6}))$	
	D_4^d	$(x_1(t), x_2(t), x_1(t+\frac{T}{2}), x_1(t+\frac{T}{2}), x_2(t), x_1(t),$ $x_1(t), x_2(t), x_1(t+\frac{T}{2}), x_1(t+\frac{T}{2}), x_2(t), x_1(t))$	
	$\eta D_4^d\eta^{-1}$	$(x_1(t), x_1(t), x_2(t), x_1(t+\frac{T}{2}), x_1(t+\frac{T}{2}), x_2(t),$ $x_1(t), x_1(t), x_2(t), x_1(t+\frac{T}{2}), x_1(t+\frac{T}{2}), x_2(t))$	
	$\eta^2 D_4^d\eta^{-2}$	$(x_2(t), x_1(t), x_1(t), x_2(t), x_1(t+\frac{T}{2}), x_1(t+\frac{T}{2}),$ $x_2(t), x_1(t), x_1(t), x_2(t), x_1(t+\frac{T}{2}), x_1(t+\frac{T}{2}))$	
	$D_4^{\hat{d}}$	$(x_1(t), x_2(t), x_1(t), x_1(t+\frac{T}{2}), x_2(t+\frac{T}{2}), x_1(t+\frac{T}{2}),$ $x_1(t), x_2(t), x_1(t), x_1(t+\frac{T}{2}), x_2(t+\frac{T}{2}), x_1(t+\frac{T}{2}))$	
	$\eta D_4^{\hat{d}}\eta^{-1}$	$(x_1(t+\frac{T}{2}), x_1(t), x_2(t), x_1(t), x_1(t+\frac{T}{2}), x_2(t+\frac{T}{2}),$ $x_1(t+\frac{T}{2}), x_1(t), x_2(t), x_1(t), x_1(t+\frac{T}{2}), x_2(t+\frac{T}{2}))$	
	$\eta^2 D_4^{\hat{d}}\eta^{-2}$	$(x_2(t+\frac{T}{2}), x_1(t+\frac{T}{2}), x_1(t), x_2(t), x_1(t), x_1(t+\frac{T}{2}),$ $x_2(t+\frac{T}{2}), x_1(t+\frac{T}{2}), x_1(t), x_2(t), x_1(t), x_1(t+\frac{T}{2}))$	
ξ_3	$\mathbb{Z}_{12}^{t_3}$	$(x_1(t), x_1(t+\frac{T}{4}), x_1(t+\frac{T}{2}), x_1(t+\frac{3T}{4}),$ $x_1(t), x_1(t+\frac{T}{4}), \dots, x_1(t+\frac{3T}{4}))$	
	$\varsigma\mathbb{Z}_{12}^{t_3}\varsigma^{-1}$	$(x_1(t), x_1(t+\frac{3T}{4}), x_1(t+\frac{T}{2}), x_1(t+\frac{T}{4}),$ $x_1(t), x_1(t+\frac{3T}{4}), \dots, x_1(t+\frac{T}{4}))$	
	D_6^d	$(x_1(t), x_1(t+\frac{T}{2}), x_1(t+\frac{T}{2}), x_1(t), x_1(t), x_1(t+\frac{T}{2}),$ $x_1(t+\frac{T}{2}), x_1(t), x_1(t), x_1(t+\frac{T}{2}), x_1(t+\frac{T}{2}), x_1(t))$	
	$\eta D_6^d\eta^{-1}$	$(x_1(t), x_1(t), x_1(t+\frac{T}{2}), x_1(t+\frac{T}{2}), x_1(t), x_1(t),$ $x_1(t+\frac{T}{2}), x_1(t+\frac{T}{2}), x_1(t), x_1(t), x_1(t+\frac{T}{2}),$ $x_1(t+\frac{T}{2}))$	
	$\tilde{D}_6^d$	$(x_1(t), x_2(t), x_1(t), x_2(t+\frac{T}{2}), x_1(t), x_2(t),$ $x_1(t), x_2(t+\frac{T}{2}), x_1(t), x_2(t), x_1(t), x_2(t+\frac{T}{2}))$	
	$\eta\tilde{D}_6^d\eta^{-1}$	$(x_2(t+\frac{T}{2}), x_1(t), x_2(t), x_1(t), x_2(t+\frac{T}{2}),$ $x_1(t), x_2(t), x_1(t), x_2(t+\frac{T}{2}), x_1(t), x_2(t), x_1(t))$	

Table 5. Summary of distinct forms of oscillating states bifurcating from the equilibrium $x = 0$ of the system (1), where cells are coupled to their nearest and next nearest neighbours (part III).

Critical Eigenvalue	Symmetry	Form of oscillating-states (for some period T)	Figure
ξ_4	$\mathbb{Z}_{12}^{t_4}$	$(x_1(t), x_1(t+\frac{T}{3}), x_1(t+\frac{2T}{3}),$ $x_1(t), x_1(t+\frac{T}{3}), \ldots, x_1(t+\frac{2T}{3}))$	
	$\varsigma\mathbb{Z}_{12}^{t_4}\varsigma^{-1}$	$(x_1(t), x_1(t+\frac{2T}{3}), x_1(t+\frac{T}{3}),$ $x_1(t), x_1(t+\frac{2T}{3}), \ldots, x_1(t+\frac{T}{3}))$	
	D_4^z	$(x_1(t), x_2(t), x_1(t+\frac{T}{2}), x_1(t), x_2(t), x_1(t+\frac{T}{2}),$ $x_1(t), x_2(t), x_1(t+\frac{T}{2}), x_1(t), x_2(t), x_1(t+\frac{T}{2}))$	
	$\eta D_4^z\eta^{-1}$	$(x_1(t+\frac{T}{2}), x_1(t), x_2(t), x_1(t+\frac{T}{2}), x_1(t), x_2(t),$ $x_1(t+\frac{T}{2}), x_1(t), x_2(t), x_1(t+\frac{T}{2}), x_1(t), x_2(t))$	
	$\eta^2 D_4^z\eta^{-2}$	$(x_2(t), x_1(t+\frac{T}{2}), x_1(t), x_2(t), x_1(t+\frac{T}{2}), x_1(t),$ $x_2(t), x_1(t+\frac{T}{2}), x_1(t), x_2(t), x_1(t+\frac{T}{2}), x_1(t))$	
	D_4	$(x_1(t), x_2(t), x_1(t), x_1(t), x_2(t), x_1(t),$ $x_1(t), x_2(t), x_1(t), x_1(t), x_2(t), x_1(t))$	
	$\eta D_4\eta^{-1}$	$(x_1(t), x_1(t), x_2(t), x_1(t), x_1(t), x_2(t),$ $x_1(t), x_1(t), x_2(t), x_1(t), x_1(t), x_2(t))$	
	$\eta^2 D_4\eta^{-2}$	$(x_2(t), x_1(t), x_1(t), x_2(t), x_1(t), x_1(t),$ $x_2(t), x_1(t), x_1(t), x_2(t), x_1(t), x_1(t))$	
ξ_5	$\mathbb{Z}_{12}^{t_5}$	$(x_1(t), x_1(t+\frac{5T}{12}), x_1(t+\frac{10T}{12}), x_1(t+\frac{3T}{12}),$ $x_1(t+\frac{8T}{12}), x_1(t+\frac{T}{12}), x_1(t+\frac{6T}{12}), x_1(t+\frac{11T}{12}),$ $x_1(t+\frac{4T}{12}), x_1(t+\frac{9T}{12}), x_1(t+\frac{2T}{12}), x_1(t+\frac{7T}{12})$	
	$\varsigma\mathbb{Z}_{12}^{t_5}\varsigma^{-1}$	$(x_1(t), x_1(t+\frac{7T}{12}), x_1(t+\frac{2T}{12}), x_1(t+\frac{9T}{12}),$ $x_1(t+\frac{4T}{12}), x_1(t+\frac{11T}{12}), x_1(t+\frac{6T}{12}), x_1(t+\frac{T}{12}),$ $x_1(t+\frac{8T}{12}), x_1(t+\frac{3T}{12}), x_1(t+\frac{10T}{12}), x_1(t+\frac{5T}{12})$	
ξ_6	$D_{12}^{\hat{d}}$	$(x_1(t), x_1(t+\frac{T}{2}), x_1(t), x_1(t+\frac{T}{2}), \ldots, x_1(t+\frac{T}{2}))$	

both sides with coupling strength $c_{i,i\pm1} = d_1$ and to its second-nearest neighbours on both sides with coupling strength $c_{i,i\pm2} = d_2$, as well as possibly having a self-feedback loop with strength $c_{ii} = c_0$. Each coupling strength is allowed to be positive, negative, or zero. We illustrate some of the ensuing dynamics in the context of a concrete model mentioned in the

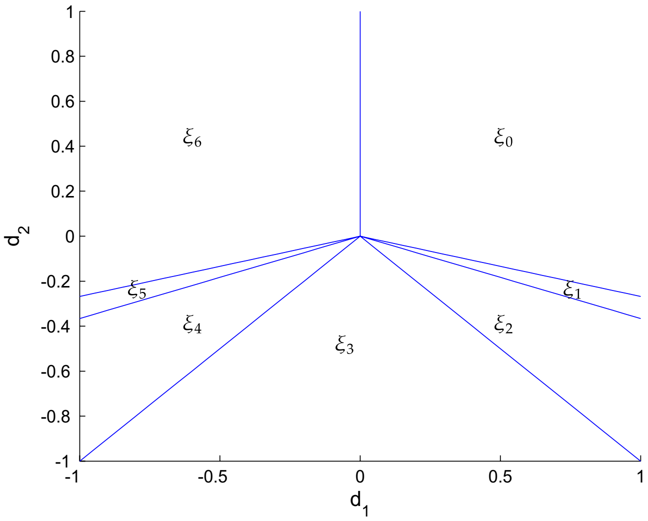


Figure 2. The largest eigenvalue of the coupling matrix C for a circular arrangement of 12 cells where each cell is connected to four others, with coupling strength d_1 to its two immediate neighbours on each side and with coupling strength d_2 to the second-nearest neighbours.

Introduction, namely the nonlinear neural network model (3). We will also make reference to the diffusively coupled system (5) by choosing c_0 appropriately.

Since the coupling matrix C is circulant, we calculate its eigenvalues using (33) as

$$\xi_j = c_0 + 2d_1 \cos(\frac{\pi}{6} j) + 2d_2 \cos(\frac{\pi}{3} j), \quad j = 0, 1, 2, \ldots, n-1. \tag{45}$$

We then determine the largest and smallest eigenvalues of C in terms of the coupling strengths d_1 and d_2. Figures 2 and 3 graphically show which of the ξ_i are the largest and smallest eigenvalues of C for the complete range of coupling strengths d_1 and d_2. The figures are symmetric images of each other with respect to the origin since replacing (d_1, d_2) by $(-d_1, -d_2)$ is equivalent to multiplying C by -1, which reverses the roles of smallest and largest eigenvalues. Note from (45) that the self-coupling coefficient c_0 simply shifts the eigenvalues without altering their magnitude order. Hence, by changing c_0 one can make either the smallest or the largest eigenvalue the dominant one that determines the first bifurcation, in the context of the stability diagram of figure 1. As depicted in figures 2 and 3, every eigenvalue ξ_i of C can arise as the dominant one by appropriate choices of the coupling strengths d_1 and d_2. (Note that for $7 \leqslant j \leqslant 11$, ξ_j is identical to ξ_{12-j}, by (45).) Therefore, the present setting of coupling with two closest neighbour pairs permits a systematic investigation of the whole range of dynamics listed in tables 2–5.

We consider the model (3) with $g(x) = \arctan x$ and $a_{ii} = 0\ \forall i$. Thus $f(x) = -x$, so that $f'(0) = -1$ and $g'(0) = 1$. In the coupling matrix C we have $c_0 = 0$, and we fix the remaining coupling strengths as $d_1 = 0.25$, $d_2 = 0.5$. The eigenvalues ξ_0 to ξ_6 are $\{1.5, 0.933013, -0.25, -1, -0.75, 0.066987, 0.5\}$. We take $\tau = 1$ initially.

We first consider excitatory coupling by setting $\kappa = 1$. The dominant eigenvalue is $\xi_0 = 1.5$; so the system settles to a non-zero steady-state solution starting from random initial conditions, as shown in figure 4. This is also the behaviour of the undelayed system, which persists under the presence of delays. If we include a self-coupling term $c_0 = -1.5$,

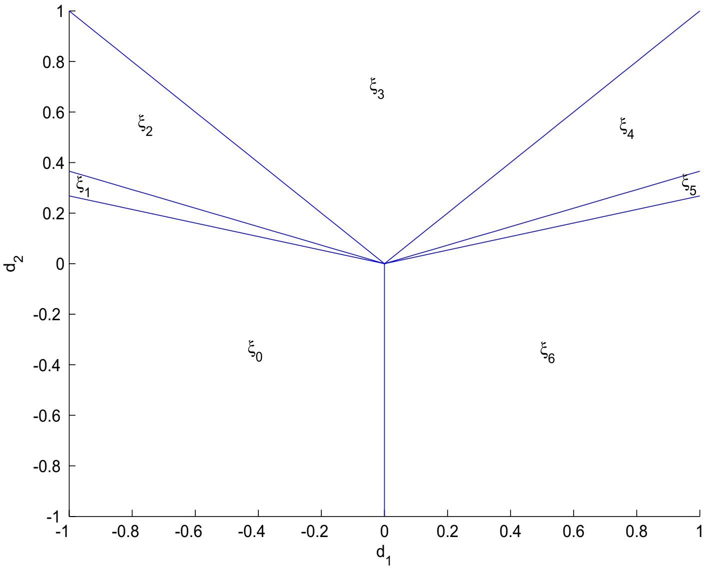


Figure 3. The smallest eigenvalue of the coupling matrix C for a circular ring of size 12, in terms of the coupling strengths. (See the caption of figure 2 for explanation.)

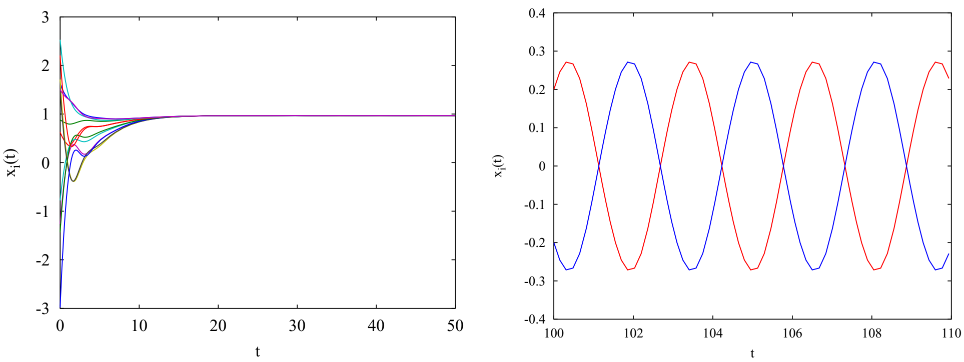


Figure 4. System (3) with excitatory coupling ($\kappa = 1$) approaching a uniform steady-state solution from random initial conditions (left). The negative of the final state is also a stable equilibrium which can be observed for a different choice of initial conditions. Adding self-coupling yields the diffusively coupled system (5), which exhibits a stable oscillatory pattern of two clusters (right): cells $\{2, 3, 6, 7, 10, 11\}$ form a synchronized cluster (blue curve) that oscillate in anti-phase with cells $\{1, 4, 5, 8, 9, 12\}$ (red curve).

as in the diffusively coupled system (5), all eigenvalues of C are shifted by -1.5. Now a negative eigenvalue, namely $\xi_3 = -2.5$, becomes the dominant one responsible for bifurcation. Consequently, the network splits into an oscillatory pattern (figure 4). It is worth noting that, although diffusive coupling is expected to drive the system to a spatially uniform solution, in this example it breaks a uniform equilibrium and replaces it with a non-trivial spatial pattern of two clusters of synchronized oscillators.

We now change the coupling from excitatory to inhibitory by setting $\kappa = -1.2$ (figure 5). We keep $d_1 = 0.25$, $d_2 = 0.5$ and $c_0 = 0$ as before. The extreme eigenvalues of κC are

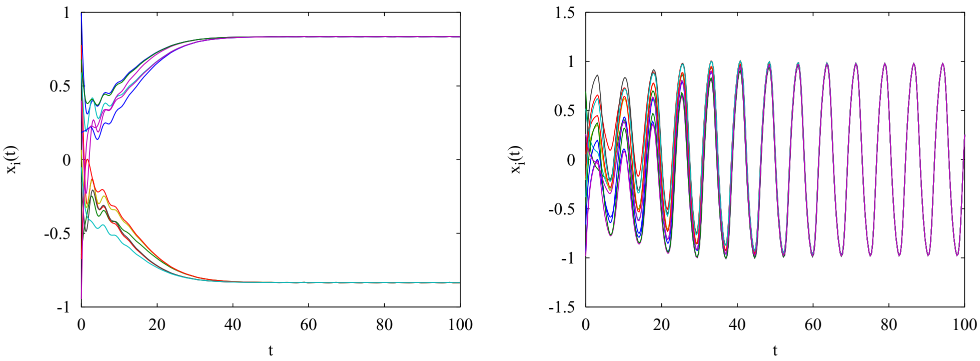


Figure 5. System (3) with inhibitory coupling ($\kappa = -1.2$). When $\tau = 1$, the network approaches a two-cluster steady-state solution from random initial conditions (left). The clusters are the same as in the oscillatory pattern of figure 4. Increasing the delay to $\tau = 3$ yields spatially uniform synchronized oscillations shown on the right.

$-1.2\xi_0 = -1.8$ and $-1.2\xi_3 = 1.2$. For the present value of $\tau = 1$, the positive eigenvalue leaves the stability region first, so the systems settles into a two-cluster steady-state solution in accordance with ξ_3. When we take $\tau = 3$, however, the negative eigenvalue becomes responsible for the bifurcation and the system exhibits spatially uniform periodic oscillations, in accordance with ξ_0. Here, in the absence of diffusive coupling, the delay apparently plays an important role in driving the system to a spatially uniform state, albeit an oscillatory one.

Although a rigorous stability analysis of the emerging spatio-temporal patterns is beyond the scope of the present study, repeated numerical simulations starting from random initial conditions manifest their stability. However, in many cases several stable patterns co-exist, so stability should only be inferred in a local sense.

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Appendix A. Proof of theorem 4.3

Theorem 4.3 *Let (α_o, β_o) be such that $\alpha_o = -\beta_o$, and let $\xi_o \in \sigma(C)$ be given by (23). If ω_0 given by (27) is of form*

$$\omega_0 = c_1(K_1) + c_2(K_2) + \cdots + c_p(K_p)$$

for some $c_i \neq 0$, then there exists a bifurcating branch of steady states of symmetry at least (K_i).

Proof. The parameter pair (α, β) escapes the shaded region in figure 1 by crossing over L1 through (α_o, β_o) (see figure A1).

Let $c : [\lambda_-, \lambda_+] \subset \mathbb{R} \to \mathbb{R}^2$ be a parametrization of the crossing curve such that $c(\lambda_-) = (\alpha_-, \beta_-)$, $c(\lambda_o) = (\alpha_o, \beta_o)$ and $c(\lambda_+) = (\alpha_+, \beta_+)$. Then the initial bifurcation problem becomes a bifurcation problem around λ_o. More precisely, we have a Γ-equivariant map $F : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ such that $F(\lambda, 0) = 0$ for all $\lambda \in [\lambda_-, \lambda_+]$. The spectrum of $D_x F(\lambda, 0)$

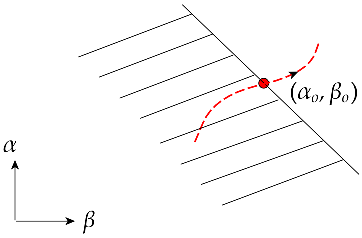

Figure A1. The crossing of (α, β) through $(\alpha_o, \beta_o) \in$ L1.

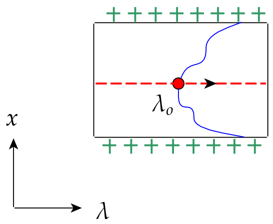

Figure A2. An isolating box Ω_1 around the bifurcating point $\lambda = \lambda_o$, where the red line is the equilibrium, the blue curves are potential bifurcating solutions and the plus signs '+' are the signs of auxiliary function ζ_1.

belongs to $\mathbb{C}_-$ (the left half of the complex plane) for all $\lambda \in [\lambda_-, \lambda_o]$, and as λ crosses λ_o, the spectrum of $D_x F(\lambda_o, 0)$ intersects with $i\mathbb{R}$ at 0.

Without loss of generality, let $\lambda_\pm = \pm 4$ and $\lambda_o = 0$. Define a box around the bifurcation point $(0, 0) \in \mathbb{R} \times \mathbb{R}^n$ (see figure A2):

$$\Omega_1 := \{(\lambda, x) \,:\, |\lambda| < 4, \quad \|x\| < \rho\},$$

where $\rho > 0$ is such that $F(\pm 4, \cdot)$ is a homeomorphism on $\{x \in \mathbb{R}^n \,:\, \|x\| < \rho\}$.

Without loss of generality, let $\rho = 2$. Define $\mathcal{F}_1 : \overline{\Omega}_1 \to \mathbb{R} \times \mathbb{R}^n$ by

$$\mathcal{F}_1(\lambda, x) := \big(|\lambda|(\|x\| - 2) + \|x\| - 1,\, F(\lambda, x)\big) := (\zeta_1(\lambda, x),\, F(\lambda, x)).$$

Note that $\zeta_1 > 0$ for $\|x\| = 2$ and $\zeta_1 < 0$ for $\|x\| = 0$. Functions with this property are called *auxiliary functions* on Ω_1. Thus, by construction, zeros of $\mathcal{F}_1$ in Ω_1 are contained properly in Ω_1, i.e. $\mathcal{F}_1^{-1}(0) \cap \overline{\Omega}_1 \subset \Omega_1$, and if $\mathcal{F}_1(\lambda, x) = 0$, then $x \neq 0$. In other words, zeros of $\mathcal{F}_1$ correspond precisely to non-trivial zeros of F in Ω_1. The bifurcation invariant ω_0 is defined by

$$\omega_0 = \Gamma\text{−Deg}\,(\mathcal{F}_1, \Omega_1).$$

To compute ω_0, we perform several homotopies on $\mathcal{F}_1$. Define $\mathcal{F}_2 : \overline{\Omega}_1 \to \mathbb{R} \times \mathbb{R}^n$ by

$$\mathcal{F}_2(\lambda, x) := \big(|\lambda|(\|x\| - 1) + \|x\| + 1,\, F(\lambda, x)\big) := (\zeta_2(\lambda, x),\, F(\lambda, x)).$$

Since $\zeta_2 > 0$ for $\|x\| = 2$, we have $\mathcal{F}_1$ and $\mathcal{F}_2$ are homotopic on Ω_1 by a linear homotopy. Thus, by homotopy invariance, we have $\Gamma\text{−Deg}\,(\mathcal{F}_1, \Omega_1) = \Gamma\text{−Deg}\,(\mathcal{F}_2, \Omega_1)$. Also, $\zeta_2 > 0$ for $|\lambda| \leqslant \frac{1}{2}$. Thus, all zeros of $\mathcal{F}_2$ in Ω_1 are contained in the following subset of Ω_1:

$$\Omega_2 := \{(\lambda, x) \,:\, \frac{1}{2} < \lambda < 4,\ \|x\| < 2\} \subset \Omega_1.$$

In other words, $\mathcal{F}_2$ does not have zeros in $\Omega_1 \setminus \Omega_2$. By (the double negation of) the existence property, we have

$$\Gamma\text{−Deg}\,(\mathcal{F}_2, \Omega_1 \setminus \Omega_2) = 0.$$

It then follows that

$$\Gamma\text{−Deg}\,(\mathcal{F}_2, \Omega_1) = \Gamma\text{−Deg}\,(\mathcal{F}_2, \Omega_2) + \Gamma\text{−Deg}\,(\mathcal{F}_2, \Omega_1 \setminus \Omega_2) = \Gamma\text{−Deg}\,(\mathcal{F}_2, \Omega_2).$$

Moreover, $\mathcal{F}_2$ is homotopic to $\mathcal{F}_3 : \overline{\Omega}_2 \to \mathbb{R} \times \mathbb{R}^n$ defined by

$$\mathcal{F}_3(\lambda, x) := (\zeta_2(\lambda, x), D_x F(\lambda, 0)).$$

Decompose $\mathbb{R}^n$ into the sum of the critical eigenspace and the eigenspaces of the rest (all negative) eigenvalues of $D_x F(\lambda_o, 0)$, say $\mathbb{R}^n = R_0 \times R_1$. Then, for $x = (x_1, x_2) \in R_0 \times R_1$, the linear map $D_x F(\lambda, 0)(x_1, x_2)$ is homotopic to $(\lambda x_1, -x_2)$. Thus, $\mathcal{F}_3$ is homotopic to $\mathcal{F}_4 : \overline{\Omega}_2 \to \mathbb{R} \times \mathbb{R}^n$ defined by

$$\mathcal{F}_4(\lambda, x) := (\zeta_2(\lambda, x), (\lambda x_1, -x_2)), \quad \text{for } x = (x_1, x_2) \in R_0 \times R_1.$$

Note that $\mathcal{F}_4(\lambda, x) = 0$ only if $x = 0$. Substituting $x = 0$ into $\zeta_2(\lambda, x)$, we have $\zeta_2(\lambda, 0) = 0$ if and only if $\lambda = \pm 1$. That is,

$$\mathcal{F}_4^{-1}(0) \cap \Omega_2 = \{(-1, 0), (1, 0)\}.$$

It follows that

$$\Gamma\text{−Deg}\,(\mathcal{F}_2, \Omega_2) = \Gamma\text{−Deg}\,(\mathcal{F}_4, \Omega_2) = \Gamma\text{−Deg}\,(\mathcal{F}_4, \mathcal{N}_{-1}) + \Gamma\text{−Deg}\,(\mathcal{F}_4, \mathcal{N}_1),$$

where $\mathcal{N}_{-1}$ (respectively $\mathcal{N}_1$) is a small neighbourhood of $(-1, 0)$ (respectively $(1, 0)$). On $\mathcal{N}_{-1}$, we have that $\mathcal{F}_4$ is homotopic to $(1 + \lambda, -x_1, -x_2)$. By suspension, we obtain

$$\Gamma\text{−Deg}\,(\mathcal{F}_4, \mathcal{N}_{-1}) = \Gamma\text{−Deg}\,(-\mathrm{Id}, B_1(\mathbb{R}^n)),$$

where $B_1(\cdot)$ denotes the unit ball. On the other hand, $\mathcal{F}_4$ is homotopic to $(1 - \lambda, x_1, -x_2)$ on $\mathcal{N}_1$, so by multiplication, we have

$$\Gamma\text{−Deg}\,(\mathcal{F}_4, \mathcal{N}_1) = -\Gamma\text{−Deg}\,(-\mathrm{Id}, B_1(R_1)).$$

Therefore,

$$\omega_0 = \Gamma\text{−Deg}\,(-\mathrm{Id}, B_1(\mathbb{R}^n)) - \Gamma\text{−Deg}\,(-\mathrm{Id}, B_1(R_1)).$$

Using `showdegree`, it is expressed as

$$\omega_0 = \texttt{showdegree}[\Gamma](n_0, n_1, \ldots, n_r, 1, 0, \ldots, 0) - \texttt{showdegree}[\Gamma] \\ \times (u_0, u_1, \ldots, u_r, 1, 0, \ldots, 0),$$

where n_i's and u_i's are defined by (24)–(26).

The statement then follows from the existence property of degree. □

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