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APPLIED EQUIVARIANT DEGREE, PART I: AN AXIOMATIC APPROACH TO PRIMARY DEGREE

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Abstract. An axiomatic approach to the primary equivariant degree is discussed and a construction of the primary equivariant degree via fundamental domains is presented. For a class of equivariant maps, which naturally appear in one-parameter equivariant Hopf bifurcation, effective computational primary degree formulae are established.

1. **Introduction.** Many mathematical models of natural phenomena exhibit symmetric properties related to some physical or geometric regularities. These models have been studied using different topological techniques: variational methods (minimax theory, Conley index, Morse-Floer complex) (cf. [29, 6, 8, 32, 27, 5]), singularity theory (cf. [14, 30]), reduction to the fixed-point spaces (cf. [12]), to mention a few. The equivariant degree introduced in [17] is an important alternative to the above approaches. To be more specific, given a compact Lie group G, orthogonal G-representations V and W, open bounded invariant subset $\Omega \subset W$ and continuous equivariant map $f : (\overline{\Omega}, \partial\Omega) \to (V, V \setminus \{0\})$, one can assign the equivariant degree $\deg_G(f, \Omega)$ taking its value in the equivariant homotopy group $\Pi^G_{S^W}(S^V)$ of maps

$$S^W = \partial([0,1] \times B) \to (\mathbb{R} \times V) \setminus \{0\} = S^V, \tag{1}$$

where B is a large ball in W centered at the origin. It is known that $\deg_G(f, \Omega)$ satisfies all the natural properties expected from any reasonable "degree theory," like existence, homotopy invariance, excision, suspension, additivity (up to one suspension), etc. Roughly speaking, the equivariant degree "measures" (equivariant) homotopy obstructions for $f_{|\partial\Omega}$ to have an equivariant extension without zeros over $\overline{\Omega}$ (composed of several orbit types).

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Observe that, in general, the equivariant homotopy group of spheres $\Pi^G_{S^W}(S^V)$ is not stable even under suspensions by G-trivial summands, which makes the practical computation of $\deg_G(f,\Omega)$ very complicated. At the same time, for the most important (from the application point of view) case $W = \mathbb{R}^n \oplus V$ it is possible to define the equivariant degree (using a slight modification of the original construction from [17]) in such a way that its value belongs to the stable limit of $\Pi^G_{S^W}(S^V)$, denoted by $\pi_n^{G,\mathrm{st}}$ (see [1, 2, 3]). For the sake of simplicity, in what follows we will use the same symbol for this modified degree. Then (cf. [1]) the group $\pi_n^{G,\mathrm{st}}$ admits a splitting $\pi_n^{G,\mathrm{st}} = \bigoplus_{\dim W(H)\leq n} \Pi(H)$, where $\Pi(H)$ stands for the (stable) equivariant homotopy group of maps satisfying the normality condition (see Definition 3) and having zeros of the orbit type (H), and $W(H) = N(H)/H$ denotes the Weyl group. Therefore,

$$\deg_G(f,\Omega) = \sum_{\dim W(H)\leq n} a_{(H)}, \tag{2}$$

where $a_{(H)}$ stands for the $\Pi(H)$-component of $\deg_G(f,\Omega)$. Denote by $\Phi_n^+(G,\Omega)$ the set of orbit types (H) occurring in Ω such that $\dim W(H) = n$ and $W(H)$ is bi-orientable (see Definition 1). Since $\Pi(H) \simeq \mathbb{Z}$ for $(H) \in \Phi_n^+(G,\Omega)$ (see [18] for G abelian and [13] for the general case), choosing an invariant orientation on $W(H)$ is equivalent to choosing a generator in $\Pi(H)$. Thus, for each $(H) \in \Phi_n^+(G,\Omega)$, the element $a_{(H)}$ from (2) can be written as $n_H \cdot (H)$ with $n_H \in \mathbb{Z}$. The projection of $\deg_G(f,\Omega)$ onto $\bigoplus_{(H)\in\Phi_n^+(G,\Omega)} \Pi(H)$ is called the *primary degree of f in* Ω. This is the main *object* of our paper.

It should be pointed out that the primary degree was introduced in [13] independently of [17], using the so-called regular normal approximations and winding numbers of their restrictions to normal slices around the orbits of zeros (cf. [10, 11, 23], where the case $G = S^1$ was considered). However, it is well-known (see, for instance, [21, 36]) that the winding number admits an axiomatic definition as an integer-valued function satisfying homotopy, additivity and normalization properties. Developing an axiomatic approach to the primary degree is one of the *goals* of our paper (cf. Proposition 9). Of course, the existence part of Proposition 9 follows from the results of [17] and [13]. However, we give here an alternative proof (cf. Proposition 8) based on the use of the so-called fundamental domains (see Definitions 6 and 7, Theorem 2 and formulae (4)—(7)) — the notion having a tie with different mathematical disciplines: fundamental polygon for isometry groups of Riemannian manifolds, Weierstrass section in invariant theory, Poincaré section in ODE's, to mention a few (for a detailed exposition of this concept we refer to [25]; for abelian group actions see [18]).

Observe that the construction of the primary degree via formulae (4)—(7) is essentially based on the existence of (regular) normal approximations. However, the normality property (being of great theoretical importance) is easy to formulate but difficult to achieve in practice. Therefore, the constructive definition of the primary degree via (4)—(7), as well as the axiomatic one, provided by Proposition 9 (cf. *normalization* and *elimination* properties) do not contain practical hints for its computation, in general. Moreover, the use of a kind of normality condition seems to be unavoidable under any axiomatic approach to the (primary) equivariant degree.

However, it turns out that in the case $n = 1$ it is possible "to go around the normality problem," and the primary degree is completely computable. The idea behind this is very simple: (i) for $G = S^1$ it is possible to define the primary degree by a list of axioms (of course, equivalent to those presented in Proposition 8 and Remark 3) with the normality property not being addressed whatsoever (see Theorem 3); (ii) the case of an arbitrary G can be canonically reduced to the computations of the S^1-degree using the so-called Recurrence Formula (see Proposition 13). In turn, the axiomatic approach to the S^1-degree combined with specific one-parameter techniques (see Section 6) allows us to obtain computational formulae for the G-degree of equivariant maps related to G-symmetric Hopf bifurcation. An exposition of this stream of ideas is the *main goal* of our paper.

A similar method works in the case $n = 2$; to this end one should develop the axiomatic approach to the primary $S^1 \times S^1$-degree and establish a suitable recurrence formula. In the case $n > 2$, the situation is much more complicated; possible connected components corresponding to $W(H)$-orbits may be different from tori (for instance, if $n = 3$, the component may be diffeomorphic to the non-abelian group $SU(2)$). Therefore, the techniques needed for possible reductions are more complex. This and related topics, together with applications to symmetric Hopf bifurcations in functional differential equations, constitute the subject of the second part of this paper.

After the Introduction, the paper is organized as follows. In section 2 we recall several notions from equivariant topology and discuss the known facts related to the bi-orientability, normality and the purely group-theoretic quantity $n(L, H)$. In section 3 we develop an axiomatic approach to the primary degree in the case of n free parameters. The main result (see Proposition 9) is preceded by a general discussion of (regular) fundamental domains in the context relevant to equivariant extensions (we believe that the existence result (see Theorem 2) is interesting in its own).

Sections 4 and 5 contain an exposition of the axiomatic approach to the primary S^1-degree in the case of one free parameter. Here the concept of a *basic map* (see formulae (10) and (11)) plays a central role; in a certain sense basic maps are the simplest equivariant ones being close to the "identity" map and having the S^1-degree different from zero. In section 6 we show how the computation of the S^1-degree of several maps related to the equivariant Hopf bifurcation can be reduced to the basic maps (see Theorem 4). Among the developed techniques, the so-called Splitting Lemma is most important. Section 7 is devoted to the Recurrence Formula, which concludes this paper.

2. Preliminaries.

2.1. Equivariant Jargon.

We will recall the equivariant jargon frequently used throughout this paper.

Hereafter, G stands for a compact Lie group. Two closed subgroups H and K of G are *conjugate* if there exists $g \in G$ such that $K = gHg^{-1}$. Obviously, the conjugation relation is an equivalence relation. The equivalence class of H is called a *conjugacy class* of H in G and will be denoted by (H). The set of all conjugacy classes of closed subgroups of G admits a partial order given by $(H) \geq (K)$ if K is conjugate to a subgroup of H. For a closed subgroup H of G, we use $N(H)$ to denote the *normalizer* of H in G, and $W(H)$ to denote the *Weyl group* $N(H)/H$ in G.

Let G act on a topological space M and $x \in M$. We denote by $G_x := \{g \in G : gx = x\}$ the *isotropy group* of x and by $G(x)$ the *orbit* of x. The conjugacy class (G_x) will be called the *orbit type* of x. The symbol $\mathcal{J}(M)$ stands for the set of all orbit types occurring in M. For an invariant subset $X \subset M$ and a closed subgroup H of G we put $X^H := \{x \in X : G_x \supset H\}$, $X_H := \{x \in X : G_x = H\}$, $X_{(H)} := \{x \in X : (G_x) = (H)\}$. Obviously, $W(H)$ acts on X^H and this action is free on X_H.

Assume, in addition, M is a smooth finite-dimensional G-manifold. Then (see, for instance, [19, 33]) for every orbit type $(H) \in \mathcal{J}(M)$, the set $M_{(H)}$ is an invariant smooth submanifold of M. Also, M^H is a smooth submanifold of M, M_H is a smooth submanifold of M^H and the orbit space $M_H/W(H)$ is a smooth manifold (see [19]). We will denote by $\tau(M)$ the *tangent bundle* to M. If M is a Riemannian manifold (equipped with an invariant metric) and N is a smooth G-submanifold of M, then we denote by $\nu(N)$ (resp. $\nu_x(N)$) the *normal vector bundle* of N in M (resp. *normal slice* at x to N).

Hereafter, V denotes an orthogonal G-representation. Let $\Omega \subset \mathbb{R}^n \oplus V$ be an open bounded G-invariant subset ($n \geq 0$ and G acts trivially on $\mathbb{R}^n$). A continuous equivariant map $f : \mathbb{R}^n \oplus V \to V$ (resp. a pair (f, Ω)) is called Ω-*admissible* (resp. *an admissible pair*) if $f(x) \neq 0$ for all $x \in \partial\Omega$. An equivariant homotopy $h : [0,1] \times (\mathbb{R}^n \oplus V) \to V$ is called Ω-*admissible* if $h_t := h(t, \cdot)$ is Ω-admissible for all $t \in [0,1]$. An orbit type (H) in $\mathbb{R}^n \oplus V$ is called *primary* if $\dim W(H) = n$.

For the background of the equivariant topology, we refer to [7, 19, 33].

2.2. **Bi-Orientability.** The notion of *bi-orientability* (originally introduced in [28], also see [13]) is briefly discussed in this subsection, and will play an essential role in our considerations.

For a finite-dimensional smooth orientable G-manifold M, we say that M admits a G-*invariant orientation* if the G-action preserves an orientation of $\tau(M)$. It is easy to show that every compact Lie group G, considered as a G-manifold with the G-action defined by left translations (resp. right translations) admits a G-invariant orientation. In this case we call this G-invariant orientation a *left-invariant orientation* (resp. *right-invariant orientation*) on G.

Definition 1. (cf. [28, 13]). Let G be a compact Lie group. If G admits an orientation which is both, left-invariant and right-invariant, we say that G is *bi-orientable.*

It is not hard to show that G is bi-orientable if it is abelian, finite or has an odd number of connected components (in particular, if G is connected) (cf. [28]). The importance of the notion of bi-orientability rests on the following:

Proposition 1. (cf. [28]). *Let M be a free smooth finite-dimensional G-manifold and let M/G be connected. Assume M admits a G-invariant orientation. Let M_o be a (fixed) connected component of M and put $G_o := \{g \in G : gM_o = M_o\}$. Then $M/G = M_o/G_o$ and moreover, M_o/G_o is an orientable manifold if and only if G_o is bi-orientable.*

Remark and Definition 1. Observe that under the assumptions of Proposition 1, if G_o is bi-orientable, then one can define in a canonical way the orientation on M/G. To this end we need an additional notion. Let X be a smooth finite-dimensional G-manifold and let $(H) \in \mathcal{J}(X)$ be such that $W(H)$ is bi-orientable and X^H admits a $W(H)$-invariant orientation. Take $x \in X^H$ and fix an orientation on $W(H)$ which is invariant with respect to both left and right translations. It induces an orientation of the orbit $W(H)(x) \subset X^H$. Choose an orientation on X^H. Let S_x be a slice (see [19]) to the orbit $W(H)(x)$ in X^H oriented in such a way that the orientation in the slice followed by the orientation of the orbit $W(H)(x)$ gives the orientation of X^H. This orientation on S_x is called *positive*.

Return to M/G from Proposition 1 and assume G_o is bi-orientable. Fix an orientation on G_o, which is invariant with respect to both left and right translations and choose an orientation on M_o. Following the above construction, for any $x \in M_o$ one may consider a slice S_x to the orbit $G_o(x)$ equipped with the positive orientation. Obviously, the positive orientation on slices canonically defines the orientation on $M_o/G_o = M/G$.

We will adopt the following notations: $\Phi_k(G)$ stands for the set of all conjugacy classes (H) in G such that $\dim W(H) = k$; $\Phi_k(G,V)$ denotes the set of all orbit types (H) in $\mathbb{R}^k \oplus V$ such that $(H) \in \Phi_k(G)$; $\Phi_n^+(G) \subset \Phi_n(G)$ stands for the set of all conjugacy classes (H) such that $W(H)$ is bi-orientable (we will also write $\Phi_0^+(G) = \Phi_0(G)$); $\Phi_n^+(G,V) \subset \Phi_n(G,V)$ denotes the set of all orbit types (H) in $\mathbb{R}^n \oplus V$ such that $(H) \in \Phi_n^+(G)$; $A_n^+(G)$ stands for the free $\mathbb{Z}$-module generated by $\Phi_n^+(G)$; $W(H)_o$ is the subgroup of $W(H)$ composed of all g such that $g(R_H)_o = (R_H)_o$, where $(R_H)_o$ stands for some (fixed) connected component of $(\mathbb{R}^n \oplus V)_H$; $\tilde{\Phi}_n^+(G,V) \subset \Phi_n(G,V)$ denotes the set of all orbit types (H) such that $W(H)_o$ is bi-orientable.

Definition 2. An orbit type $(H) \in \Phi_n^+(G,V)$ is called *bi-orientable* in $\Phi_n(G,V)$, and an orbit type $(H) \in \tilde{\Phi}_n^+(G,V) \setminus \Phi_n^+(G,V)$ is called *relatively bi-orientable* in $\Phi_n(G,V)$. All other orbit types in $\Phi_n(G,V)$ are called *non-bi-orientable* and denoted by $\tilde{\Phi}_n^-(G,V)$.

2.3. **Regular Normal Approximations.** Many theoretical problems of the equivariant homotopy classification of Ω-admissible maps can be reduced to the following ones: (i) how to separate zeros having different orbit types? (ii) how to choose representatives of equivariant homotopy classes admitting reasonable transversality/regularity conditions? The first problem gives rise to the so-called *normality* condition. The second problem is more delicate: the equivariance "gets in conflict" with regularity (for instance, due to the restriction requirements on the dimensions of the orbits of zeros). Therefore, one has to look for special transversality requirements which are compatible with such techniques as the induction over orbit types and the suspension operation (for a general discussion related to different G-actions on a domain and target we refer to [25, 18, 6]).

Definition 3. (cf. [13, 24, 25]). Let V be an orthogonal G-representation, $\Omega \subset \mathbb{R}^n \oplus V$ an open bounded invariant set and $f : \mathbb{R}^n \oplus V \to V$ an Ω-admissible G-equivariant map. We say that f is *normal* in Ω, if for every $\alpha = (H) \in \mathcal{J}(\Omega)$ and every $x \in f^{-1}(0) \cap \Omega_H$, the following *$\alpha$-normality* condition at x is satisfied: *There exists $\delta_x > 0$ such that for all $w \in \nu_x(\Omega_\alpha)$ with $\|w\| < \delta_x$,*

$$f(x+w) = f(x) + w = w.$$

Similarly, an Ω-admissible G-homotopy $h : [0,1] \times (\mathbb{R}^n \oplus V) \to V$ is called a *normal homotopy* in Ω, if for every $\alpha = (H) \in \mathcal{J}(\Omega)$ and for every $(t,x) \in h^{-1}(0) \cap ([0,1] \times \Omega_H)$, the following *$\alpha$-normality* condition at (t,x) is satisfied: *There exists $\delta_{(t,x)} > 0$ such that for all $w \in \nu_{(t,x)}([0,1] \times \Omega_\alpha)$ with $\|w\| < \delta_{(t,x)}$,*

$$h(t, x+w) = h(t,x) + w = w.$$

Definition 4. (cf. [13, 24, 25]). Let $\Omega \subset \mathbb{R}^n \oplus V$ be an open bounded invariant set and $f : \mathbb{R}^n \oplus V \to V$ an Ω-admissible G-equivariant map. We say that f is a *regular normal* map in Ω if

(i) f is of class C^1;
(ii) f is normal in Ω;
(iii) for every $(H) \in \mathcal{J}(f^{-1}(0) \cap \Omega)$, zero is a regular value of $f_H := f|_{\Omega_H} : \Omega_H \to V^H$.

Similarly, an Ω-admissible G-equivariant homotopy $h : [0,1] \times (\mathbb{R}^n \oplus V) \to V$ is called *a regular normal* homotopy in Ω if

(i) h is of class C^1;
(ii) h is a normal homotopy in Ω;
(iii) for every $(H) \in \mathcal{J}(h^{-1}(0) \cap [0,1] \times \Omega)$, zero is a regular value of the maps h_H, $(h_0)_H$ and $(h_1)_H$, where $h_H := h|_{[0,1]\times\Omega_H}, (h_0)_H := h_0|_{\Omega_H}, (h_1)_H := h_1|_{\Omega_H}$.

We complete this section with an important property of regular normal maps. We first start with the following simple observation:

Proposition 2. (cf. [1], [25]) *Let $\Omega \subset \mathbb{R}^n \oplus V$ be an open bounded invariant set, and $f : \mathbb{R}^n \oplus V \to V$ an Ω-admissible G-equivariant map being regular and normal. Then for every $x \in f^{-1}(0) \cap \Omega$ we have* $\dim(W(G_x)) \leq n$.

We have the following regular normal approximation property:

Proposition 3. (cf. [24], also see [25, 35, 26]). *Let $\Omega \subset \mathbb{R}^n \oplus V$ be an open bounded invariant set and $f : \mathbb{R}^n \oplus V \to V$ an Ω-admissible G-equivariant map. Then for every $\eta > 0$ there exists a regular normal (in Ω) G-equivariant map $\tilde{f} : \mathbb{R}^n \oplus V \to V$ such that* $\sup_{x\in\Omega} \|\tilde{f}(x) - f(x)\| < \eta$. *Similarly, if $h : [0,1] \times (\mathbb{R}^n \oplus V) \to V$ is an Ω-admissible G-equivariant homotopy, then for every $\eta > 0$ there exists a regular normal (in Ω) G-equivariant homotopy $\tilde{h} : [0,1] \times \mathbb{R}^n \oplus V \to V$ such that* $\sup_{(t,x)\in[0,1]\times\Omega} \|\tilde{h}(t,x) - h(t,x)\| < \eta$. *In addition, if h_0 and h_1 are regular normal in Ω, then $\tilde{h}_0 = h_0$ and $\tilde{h}_1 = h_1$.*

2.4. **Numbers $n(L,H)$.** To compute the primary G-degree via a reduction to the S^1-degree, the following quantity $n(L,H)$ is needed for the Recurrence Formula (see Proposition 13):

Definition 5. (cf. [15, 25]) Given two closed subgroups $L \subset H$ of a compact Lie group G, we define the set

$$N(L,H) = \Big\{g \in G : gLg^{-1} \subset H\Big\}.$$

and we put

$$n(L,H) = \left|\frac{N(L,H)}{N(H)}\right|, \tag{3}$$

where the symbol $|X|$ stands for the cardinality of the set X.

Remark 1. It is easy to check that $N(L,H)$ is a compact subset of G, but it is not a subgroup of G in general. Also, the space $N(L,H)/H := \{Ha : a \in N(L,H)\}$ is a right $W(L)$-space. Indeed, since $a \in N(L,H)$ implies $aLa^{-1} \subset H$, for every $l \in L$ there exists $h \in H$ such that $al = ha$. Then for $g \in N(L)$ and $g' = lgl'^{-1}$, l and $l' \in L$, we have

$$(ag)L(ag)^{-1} = a(gLg^{-1})a^{-1} = aLa^{-1} \subset H,$$
$$Hag' = Halgl'^{-1} = Hhagl'^{-1} = Hagl'^{-1} = Hh'ag = Hag,$$

where $h' \in H$ is such that $(ag)l'^{-1} = h'(ag)$. Consequently, the right action of $W(L)$ on $N(L,H)/H$ is well-defined. Note that the correspondence $Ha \mapsto a^{-1}H$defines a $W(L)$-equivariant homeomorphism from $N(L,H)/H$ to $(G/H)^L$(cf. Cor. 1.68 in [19]).

Proposition 4. *Let $L \subset H$ be two closed subgroups of a compact Lie group G such that* $\dim W(L) = \dim W(H) = k$. *Then the number $n(L,H)$ is finite and the set $N(L,H)/H$ is a closed k-dimensional submanifold of G/H.*

Proof: Since the right $W(L)$-space $N(L,H)/H$ is equivariantly homeomorphic to the left $W(L)$-space $(G/H)^L$, which, by Cor. 5.7 in [7], is composed of a finite number of $W(L)$-orbits, it follows that $N(L,H)/H$ consists also of a finite number of $W(L)$-orbits, with each one homeomorphic to $W(L)/L_o$ for a finite collection of subgroups $L_o \subset W(L)$. Since for each of these L_o,

$$\dim \left(\frac{W(L)}{L_o} \right) \leq \dim W(L) = k,$$

we obtain the following estimation of the (covering) dimension of $N(L,H)/H$:

$$\dim \left(\frac{N(L,H)}{H} \right) \leq k.$$

On the other hand, the group $W(H)$ acts freely on the space $N(L,H)/H$. Therefore, by Gleason Lemma, the natural projection

$$\frac{N(L,H)}{H} \longrightarrow \frac{N(L,H)/H}{W(H)} = \frac{N(L,H)}{N(H)}$$

is a locally trivial fiber bundle with the fiber $W(H)$. We note that the action of $W(H)$ on G/H is smooth, hence the action of $W(H)$ on $N(L,H)/H$ is also smooth. Thus, we have

$$\begin{aligned} k \geq \dim \left(\frac{N(L,H)}{H} \right) &= \dim \left(\frac{N(L,H)}{N(H)} \right) + \dim W(H) \\ &= \dim \left(\frac{N(L,H)}{N(H)} \right) + k, \end{aligned}$$

so $\dim \left(\frac{N(L,H)}{N(H)} \right) = 0$. Since $N(L,H)/H$ is composed of a finite number of connected components (notice that $W(L)$ and $W(H)$ have finitely many connected components), $N(L,H)/N(H)$ has also finitely many connected components, and consequently it is finite, which proves that the number $n(L,H)$ is finite. In particular, we obtain that the set $N(L,H)/H$ is composed of a finite number of $W(H)$-orbits, which are all submanifolds of G/H. Therefore the set $N(L,H)/H$ is a closed submanifold of G/H. □

The number $n(L,H)$ defined for two closed subgroups of G with $\dim W(H) = \dim W(L)$ has a very simple geometric interpretation provided by the following:

Lemma 1. *Let L and H be two closed subgroups of a compact Lie group G such that $L \subset H$ and* $\dim W(L) = \dim W(H)$. *Then $n(L,H)$ represents the number of different subgroups $\widetilde{H}$ in the conjugacy class (H) such that $L \subset \widetilde{H}$. In particular, if V is an orthogonal G-representation such that (L), $(H) \in \mathcal{J}(V)$, $L \subset H$, then $V^L \cap V_{(H)}$ is a disjoint union of exactly $m = n(L,H)$ sets of V_{H_j}, $j = 1, 2, \ldots, m$, satisfying $(H_j) = (H)$.*

Proof: Notice

$$N(L,H)=\Big\{g\in G: gLg^{-1}\subset H\Big\}=\Big\{g\in G: L\subset gHg^{-1}\Big\}.$$

Let $\mathcal{H}$ be the set composed of all subgroups H', conjugate to H such that $L\subset H'$, and define a map $b: N(L,H)\to\mathcal{H}$ by $b(g)=gHg^{-1}$, $g\in N(L,H)$. Consider $N(L,H)$ as the left $N(H)$-space. It is clear that b is constant on $N(H)$-orbits, thus there exists a factorization $\overline{b}: N(L,H)/N(H)\to\mathcal{H}$. It is easy to check that $\overline{b}$ is one-to-one and onto. Assume now that V is an orthogonal G-representation, (L), $(H)\in\mathcal{J}(V)$, and $L\subset H$. Then $V_H\subset V^L$ and $gV_H\subset V^L$ if and only if $g\in N(L,H)$. On the other hand, $gV_H=V_H$ if and only if $g\in N(H)$. Therefore, the conclusion follows. □

Notation: In what follows, in the case of two orbit types (L) and (H) such that $(L)\le(H)$, we will assume that the number $n(L,H)$ corresponds to representatives L and H such that $L\subset H$. In the case the orbit types (L) and (H) are not comparable with respect to the partial order relation, we will simply put $n(L,H)=0$.

3. Primary Equivariant Degree in the Case of n Free Parameters: An Axiomatic Approach.

3.1. **Equivariant Extensions and Fundamental Domains.** As mentioned in the Introduction, the equivariant degree "measures" homotopy obstructions for an equivariant map to have equivariant extensions without zeros on a set composed of several orbit types. Therefore, in this subsection we briefly discuss the following problem:

Assume V is a finite-dimensional G-representation, $Y:=V\setminus\{0\}$, X is a G-space and $B\subset X$ is a closed invariant subset in X. Let $f:B\to Y$ be an equivariant map. Under which conditions, does there exist an equivariant extension of f over X?

Using the *induction over orbit types* (see, for instance, [33]), the above problem can be reduced to the following one:

Let X, B, Y and f be as above and assume that G acts freely on $X\setminus B$. Find a G-equivariant extension of f over X.

The key to the extension results is the following notion:

Definition 6. Let a topological group Q act on a finite-dimensional metric space X. Let $D_0\subset X$ be open in its closure D. Then D is said to be a *fundamental domain* of the Q-action on X if the following conditions are satisfied:

(i) $Q(D)=X$;
(ii) $g(D_o)\cap h(D_o)=\emptyset$ for $g,h\in Q$, $g\ne h$;
(iii) $X\setminus Q(D_o)=Q(D\setminus D_o)$;
(iv) $\dim D=\dim X/Q$, $\dim(D\setminus D_o)<\dim D$, $\dim Q(D\setminus D_o)<\dim X$, where "dim" stands for the covering dimension.

Proposition 5. (see [25]) *Let G be a compact Lie group, and let X be a finite-dimensional metric G-space on which G acts freely. Then a fundamental domain $D\subset X$ always exists.*

Let us return to the equivariant extension problem (recall that we assume $X\setminus B$ is a free G-subspace). By Proposition 5, there exists a fundamental domain $D^{(0)}\subset L^{(0)}:=X\setminus B$. Let $D_o^{(0)}$ be the corresponding open subset of $D^{(0)}$ satisfying the conditions (ii)—(iv) of Definition 6, and let $X^{(1)} := B \cup G(D^{(0)} \setminus D_o^{(0)})$, $L^{(1)} := X^{(1)} \setminus B$. Now, by applying Proposition 5 to $X^{(1)} \setminus B$, we obtain $X^{(2)}$ and $L^{(2)}$, etc. Consequently, by following the same steps, we obtain a closed finite G-invariant filtration

$$X = X^{(0)} \supset X^{(1)} \supset X^{(2)} \supset \cdots \supset X^{(r)} = B.$$

Proposition 6. (see [25]) *Under the above assumptions, any G-equivariant map $f : B \to Y$ extends equivariantly over X if for all $i \geq 1$ any equivariant map $X^{(i)} \to Y$ has a (non-equivariant) extension over $X^{(i)} \cup D^{(i-1)}$.*

3.2. **Regular Fundamental Domains.** Proposition 6 reduces the equivariant extension problem to the non-equivariant one. To make this scheme compatible with an appropriate equivariant degree theory (in particular, to have the Hopf property (see statements (P8)* and (P8) from Proposition 8 and Remark 3), a more careful analysis of the geometry of a fundamental domain is needed.

Definition 7. Under the notations of Definition 6, assume there exists an open *contractible* subset $T_0 \subset X/Q$ such that the natural projection $p : X \to X/Q$ induces the homeomorphism $p|_{D_0} : D_0 \to T_0$. Then D is called a *regular fundamental domain.*

Theorem 2. *Let G be a compact Lie group. For any smooth finite-dimensional free G-manifold X such that X/G is connected, there always exists a regular fundamental domain D.*

Proof: Since every smooth connected manifold admits a (smooth) triangulation (cf. [34], p. 124-135), the proof is essentially based on the following:

Lemma 2. *Let M be a smooth connected n-dimensional manifold (in general non-compact), and let $\mathcal{S} := \left\{ s_i^k : i \in J^k, \;\; k = 0, 1, 2, \ldots, n \right\}$ be a smooth triangulation of M, where the sets of indices J^k are countable. Then there always exists a subset T_o of M satisfying the following conditions:*

(i) *T_o is open in M;*
(ii) *T_o is dense in M;*
(iii) *T_o is contractible;*
(iv) *$M \setminus T_o$ is contained in the $n-1$-dimensional skeleton.*

Proof: For a given k-dimensional simplex s^k, we denote by $\mathring{s}^k$ its interior. We call the n-dimensional simplices in $\mathcal{S}$ s_1^n, s_2^n, ... and begin our recursive definition with $T_1 := \mathring{s}_1^n$ and $\mathcal{S}_1 : \mathcal{S} \setminus \{s_1^n\}$.

Assume now that T_m and $\mathcal{S}_m \subset \mathcal{S}$ are already constructed with T_m being open in M and contractible. If $\mathcal{S}_m$ still contains n-dimensional simplices, we choose the minimal $j_{m+1} \in \mathbb{N}$ such that

(a) $s_{j_{m+1}}^n \in \mathcal{S}_m$;
(b) $s_{j_{m+1}}^n \cap \overline{T}_m$ contains an $(n-1)$-dimensional simplex $s_{k_{m+1}}^{n-1} \in \mathcal{S}_m$.

We define $T_{m+1} := T_m \cup \mathring{s}_{k_{m+1}}^{n-1} \cup \mathring{s}_{j_{m+1}}^n$ and $\mathcal{S}_{m+1} := \mathcal{S}_m \setminus \{s_{j_{m+1}}^n, s_{k_{m+1}}^{n-1}\}$. Clearly, T_{m+1} is open in M and contractible.

Let $T_o := \bigcup_m T_m$ and $\mathcal{S}_o := \bigcap_m \mathcal{S}_m$. By construction, T_o is open and (by connectedness of M) dense in M. Also, $S_o = M \setminus T_o$ is a subset of the $n-1$-dimensional skeleton of $\mathcal{S}$.

In order to show that T_o is contractible, notice that T_o is a CW-complex and for every continuous map $\varphi : S^k \to T_o$, $k = 0, 1, 2, \dots$, the image $\varphi(S^k)$ is compact, so it is entirely contained in some of the contractible sets T_m. Consequently, φ is null-homotopic, hence $\pi_k(T_o) = 0$ for all $k = 0, 1, 2, \dots$. Therefore, T_o is contractible (see [31], Cor. 24, Chap. 7, Sec. 6) and Lemma 2 is proved. □

Continuation of the proof of Theorem 2. Let $p : X \to X/G$ be the natural projection. To complete the proof of Theorem 2, we take the set $T_o \subset M := X/G$ provided by Lemma 2 and consider the restriction of p over $p^{-1}(T_o)$. The fiber bundle $p : p^{-1}(T_o) \to T_o$, by contractibility of T_o, is trivial. Fix a trivialization $\psi : p^{-1}(T_o) \to G \times T_o$. We put $D_o := \psi^{-1}(\{1\} \times T_o)$. It is clear (cf. [25]) that $D := \overline{D_o}$ is the regular fundamental domain.

The proof of Theorem 2 is complete. □

3.3. **Primary Equivariant Degree: Construction.** Let G be a compact Lie group, V an orthogonal G-representation, $\Omega \subset \mathbb{R}^n \oplus V$ an open bounded invariant subset. Recall (see Definition 2) that $\tilde{\Phi}_n^+(G,V) \subset \Phi_n(G,V)$ denotes the set of all bi-orientable and relatively bi-orientable orbit types, and $\tilde{\Phi}_n^-(G,V) \subset \Phi_n(G,V)$ denotes the set of all non-bi-orientable orbit types.

Define
$$\widetilde{A}_n(G,V) = \bigoplus_{(H)\in\tilde{\Phi}_n^+(G,V)} \mathbb{Z} \oplus \bigoplus_{(H)\in\tilde{\Phi}_n^-(G,V)} \mathbb{Z}_2.$$

Take an Ω-admissible G-equivariant map $f : \mathbb{R}^n \oplus V \to V$ and assume that it is *regular and normal* (in particular, $\overline{(f^{-1}(0) \cap \Omega_H)} \cap \overline{(f^{-1}(0) \cap \Omega_K)} = \emptyset$ for $(H) \neq (K)$). Take $(H) \in \Phi_n(G,V)$ and put $f_H = f_{|\Omega_H}$.

Assume that $(H) \in \tilde{\Phi}_n^+(G,V)$. Then (see Proposition 1), the manifold $\Omega_H/W(H)$ is orientable. Fix an orientation of $W(H)$ which is both left-invariant and right-invariant. Also, by fixing an orientation on V^H, we obtain the orientation on $\mathbb{R}^n \oplus V^H$ and thus on Ω_H. Take a canonical orientation on $\Omega_H/W(H)$ described in Remark and Definition 1.

Choose a regular fundamental domain D on Ω_H provided by Theorem 2 with $T_o = p(D_o)$ such that $f_H^{-1}(0) \cap (D \setminus D_o) = \emptyset$. Notice that under the assumption that f is regular normal, the set $p(f_H^{-1}(0) \cap D_o)$ is finite (i.e. $f_H^{-1}(0)$ is composed of a finite number of $W(H)$-orbits), therefore, it is always possible to construct T_o in such a way that $p(f_H^{-1}(0)) \subset T_o$. We call the homeomorphism $\xi := (p_{|D_o})^{-1} : T_o \to D_o$ the *lifting homeomorphism.* Then we can define the (H)-*component* of the primary degree by
$$n_H = n_H(f) := \deg(f_H \circ \xi, T_o) \tag{4}$$
(here deg stands for the (local) Brouwer degree with respect to zero).

Similar to the non-equivariant case, if $(H) \in \widetilde{\Phi}_n^-(G,V)$, one defines the (H)-component $n_H(f)$ of the primary equivariant degree as the corresponding residue class modulo 2 (following literally the above construction).

Remark 2. If we choose an orientation on D_o in such a way that ξ preserves it, then $\deg(f_H, D_o)$ is correctly defined and coincides with $\deg(f_H \circ \xi, T_o)$. In this sense one can think of $n_H(f)$ as a "degree of f_H on a fundamental domain D."

Definition 8. We define the *complete primary degree* of an Ω-admissible G-equivariant regular normal map $f : \mathbb{R}^n \oplus V \to V$ to be an element $G\text{-Deg}^*(f,\Omega) \in \widetilde{A}_n(G,V)$ with
$$G\text{-Deg}^*(f,\Omega) = n_{H_1}(H_1) + n_{H_2}(H_2) + \cdots + n_{H_r}(H_r), \tag{5}$$

where n_{H_i} is defined by (4) if $(H_i) \in \widetilde{\Phi}_n^+(G,V)$ (taken modulo 2 if $W(H)_o$ is non-bi-orientable). If $g : \mathbb{R}^n \oplus V \to V$ is a G-equivariant Ω-admissible map (in general, not necessarily normal nor regular in Ω), choose a regular normal Ω-admissible map $f : \mathbb{R}^n \oplus V \to V$ equivariantly homotopic to g by an Ω-admissible homotopy (see Proposition 3) and put

$$G\text{-Deg}^*(g,\Omega) = G\text{-Deg}^*(f,\Omega). \tag{6}$$

Clearly, relatively bi-orientable orbit types a priori depend on the representation V (cf. Definition 2, and note there, the connected component $(R_H)_o$ depends on the G-action on V, thus the subgroup $W(H)_o$ which fixes $(R_H)_o$ depends on the representation V). Therefore, it seems reasonable to exclude them from a more "workable" definition of the primary equivariant degree. Also, we exclude the non-bi-orientable orbit types $\widetilde{\Phi}_n^-(G,V)$ for the computational reason, and we define the *primary equivariant degree* of f to be an element of $A_n^+(G)$ given by

$$G\text{-Deg}\,(f,\Omega) = n_{H_1}(H_1) + n_{H_2}(H_2) + \cdots + n_{H_m}(H_m), \tag{7}$$

where $n_{H_i}(H_i)$ are the components of G-Deg$^*(f,\Omega)$ corresponding to the $(H_i) \in \Phi_n^+(G)$.

In other words, the primary equivariant degree G-Deg$\,(f,\Omega)$ is the restriction of the complete primary degree G-Deg$^*(f,\Omega)$ to the components corresponding to the bi-orientable orbit types.

3.4. **Primary Equivariant Degree: Justification.**

Proposition 7. *Let G be a compact Lie group, $\Omega \subset \mathbb{R}^n \oplus V$ an open bounded invariant subset and $f : \mathbb{R}^n \oplus V \to V$ an Ω-admissible G-equivariant map. Then the complete primary degree (see (4)—(6)) (as well as the primary equivariant degree (7)) is well-defined.*

Proof: (i) We first show that formula (4) is independent of a choice of a regular fundamental domain D. Suppose that D' is another regular fundamental domain such that $f_H^{-1}(0) \cap (D' \setminus D'_o) = \emptyset$, $p(D'_o) = T'_o$ with the lifting homeomorphism $\xi' : T'_o \to D'_o$. By applying the additivity property of the Brouwer degree, we can assume, without loss of generality, that $f_H^{-1}(0)$ is composed of a single orbit $W(H)(x_o)$ and put $p(x_o) = y_o$. Suppose that $B_o \subset T_o \cap T'_o$ is a contractible neighborhood of y_o, put $E_o = \xi(B_o)$, $E'_o = \xi'(B_o)$ and we assume $x_o \in E_o$. Then, by excision property of the degree,

$$\deg(f_H \circ \xi, T_o) = \deg(f_H \circ \xi, B_o), \quad \deg(f_H \circ \xi', T'_o) = \deg(f_H \circ \xi', B_o).$$

We will show that

$$\deg(f_H \circ \xi, B_o) = \deg(f_H \circ \xi', B_o). \tag{8}$$

Case 1. $x_o \in E_o \cap E'_o$. Observe that $\xi_{|B_o}$ and $\xi'_{|B_o}$ are sections of the (trivial) bundle $p : p^{-1}(B_o) \to B_o$, thus there exists a continuous map $\mu : E_o \to W(H)$ such that for every $x \in E_o$, we have

$$\Psi(x) := \mu(x)x \in E'_o$$

and $\Psi : E_o \to E'_o$ is a homeomorphism since so are $\xi_{|B_o}$ and $\xi'_{|B_o}$. In particular, $\mu(x_o) = 1$ and E_o is contractible. Therefore, there exists a homotopy μ_t of μ with a constant map $\mu_o(x) \equiv 1$. Put $\Psi_t(x) := \mu_t(x)x$, i.e. Ψ_t is a homotopy between Ψ

and $\mathrm{Id}_{|E_o}$. Observe that $\xi' = \Psi \circ \xi$, therefore, by the homotopy invariance of the degree, we have

$$\deg(f_H \circ \xi', B_o) = \deg(f_H \circ \Psi \circ \xi, B_o) = \deg(f_H \circ \Psi_t \circ \xi, B_o) = \deg(f_H \circ \xi, B_o).$$

Case 2. $x_o \notin E_o \cap E'_o$. In this case, there exists $g \in W(H)_o$ such that $gx_o =: x'_o \in E'_o$. Put $\widetilde{D}_o := g(D_o)$. Clearly, D_o and $\widetilde{D}_o$ have a natural smooth structure and, by the smoothness of the orbit map, $g : D_o \to \widetilde{D}_o$ is also smooth. Since $W(H)_o$ acts freely, $\widetilde{D} := \overline{\widetilde{D}_o}$ is a fundamental domain with a lifting homeomorphism $\tilde{\xi} = g \circ \xi$, and we put $\widetilde{E}_o = g(E_o)$. By the Sard-Brown theorem, we can assume that y_o is a regular point of the map $f_H \circ \xi$. Since f_H is $W(H)$-equivariant, we have

$$f_H \circ \xi = f_H \circ g^{-1} \circ g \circ \xi g^{-1} \circ f_H \circ g \circ \xi = g^{-1} \circ f_H \circ \tilde{\xi},$$

i.e.

$$g \circ f_H \circ \xi = f_H \circ \tilde{\xi},$$

which implies that y_o is also a regular point of $f_H \circ \tilde{\xi}$. Since the action of $W(H)$ preserves the orientation of the slice, we obtain immediately

$$\deg(f_H \circ \xi, B_o) = \deg(f_H \circ \tilde{\xi}, B_o).$$

Since $x'_o \in E'_o \cap \widetilde{E}_o$, the equality (8) follows from the Case 1.

(ii) We show that the formula (4) does not depend on a choice of a representative f. Take two regular normal G-equivariant maps f_0 and $\tilde{f}_1$, which are equivariantly homotopic by an Ω-admissible homotopy $\Psi : [0,1] \times \mathbb{R}^n \oplus V \to V$ with $\Psi_0 = f_0$ and $\Psi_1 = \tilde{f}_1$ (where $\Psi_t := \Psi(t, \cdot)$). Let $(H) \in \Phi_n(G, V)$ and choose D^1 to be a regular fundamental domain for the $W(H)$-action on Ω_H such that $(f_0)_H^{-1}(0) \cap (D^1 \setminus D^1_o) = \emptyset$. Denote by $\xi^1 := (p_{|D^1_o})^{-1} : T^1_o \to D^1_o$ the corresponding lifting homeomorphism. Then, by continuity of Ψ, there exists $0 < \tilde{t}_1 \le 1$ such that $\bigcup_{t \in [0, \tilde{t}_1)} (\Psi_t)_H^{-1}(0) \cap (D^1 \setminus D^1_o) = \emptyset$. Since for every $t_1 \in [0, \tilde{t}_1)$, the map Ψ_t, $t \in [0, t_1]$, is a regular normal homotopy between f_0 and $f_1 := \Psi_{t_1}$, it follows from the homotopy property of the local Brouwer degree that

$$\deg((f_0)_H \circ \xi^1, T^1_o) = \deg((f_1)_H \circ \xi^1, T^1_o).$$

By the compactness of $[0,1]$, there exists a (finite) partition $0 < t_1 < \cdots < t_k = 1$ and fundamental domains $D^1, D^2, \dots, D^k$ with the corresponding lifting homeomorphisms $\xi^i := (p_{|D^i_o})^{-1} : T^i_o \to D^i_o$, such that

$$\bigcup_{t \in [t_{i-1}, t_i]} (\Psi_t)_H^{-1}(0) \cap (D^i \setminus D^i_0) = \emptyset.$$

Consequently, by induction, we obtain

$$\deg((f_0)_H \circ \xi^1, T^1_o) = \deg((f_1)_H \circ \xi^1, T^1_o) = \cdots = \deg((f_k)_H \circ \xi^k, T^k_o),$$

which implies

$$\deg((f_0)_H \circ \xi^1, T^1_o) = \deg((f_k)_H \circ \xi^k, T^k_o).$$

Proposition 7 is proved.

□

3.5. **Primary Equivariant Degree: Basic Properties.** The complete primary degree and the primary equivariant degree defined above satisfy all the reasonable properties required from any reasonable "degree theory." To see that, we need the following:

Definition 9. Let G be a compact Lie group, V an orthogonal G-representation and $f : \mathbb{R}^n \oplus V \to V$ a regular normal map such that $f(x_o) = 0$ with $G_{x_o} = H$ and $(H) \in \Phi_n(G, V)$. Let $U_{G(x_o)}$ be a G-invariant tubular neighborhood around $G(x_o)$ such that $f^{-1}(0) \cap U_{G(x_o)} = G(x_o)$. Then f is called a *tubular map* around $G(x_o)$. In addition, if $(H) \in \widetilde{\Phi}_n^+(G, V)$ and S_{x_o} is a positively oriented slice to $W(H)(x_o)$ in $\mathbb{R}^n \oplus V^H$ (cf. Remark and Definition 1), then we call $n_{x_o} = \text{sign}\det Df^H(x_o)|_{S_{x_o}}$ the *local index* of f at x_o in $U_{G(x_o)}$ (here $f^H := f|_{\Omega^H}$ and D stands for the differential). In the case $(H) \in \widetilde{\Phi}_n^-(G, V)$, we simply put $n_{x_o} = 1 \in \mathbb{Z}_2$.

Proposition 8. (cf. [13, 18]). *Let G, V, Ω and f be as in Proposition 7. Then the complete primary degree defined by (4)—(6) satisfies the following properties:*

(P1)* (Existence) *If $G\text{-Deg}^*(f, \Omega) = \sum_{(H)} n_H(H)$ is such that $n_{H_o} \neq 0$ (taken mod 2 in the case $(H_o) \in \tilde{\Phi}_n^-(G, V)$) for some $(H_o) \in \Phi_n(G, V)$, then there exists $x \in \Omega$ with $f(x) = 0$ and $G_x \supset H_o$.*

(P2)* (Additivity) *Assume that Ω_1 and Ω_2 are two G-invariant open disjoint subsets of Ω such that $f^{-1}(0) \cap \Omega \subset \Omega_1 \cup \Omega_2$. Then*

$$G\text{-Deg}^*(f, \Omega) = G\text{-Deg}^*(f, \Omega_1) + G\text{-Deg}^*(f, \Omega_2).$$

(P3)* (Homotopy) *Suppose $h : [0, 1] \times \mathbb{R}^n \oplus V \to V$ is an Ω-admissible G-equivariant homotopy. Then*

$$G\text{-Deg}^*(h_t, \Omega) = \text{const}$$

(here $h_t := h(t, \cdot, \cdot)$, $t \in [0, 1]$).

(P4)* (Suspension) *Suppose that W is another orthogonal G-representation and let U be an open, bounded G-invariant neighborhood of 0 in W. Then*

$$G\text{-Deg}^*(f \times \text{Id}, \Omega \times U) = G\text{-Deg}^*(f, \Omega).$$

(P5)* (Normalization) *Suppose f is a tubular map around $G(x_o)$, $H := G_{x_o}$, $(H) \in \Phi_n(G, V)$, with the local index n_{x_o} of f at x_o in a tubular neighborhood $U_{G(x_o)}$. Then*

$$G\text{-Deg}^*(f, U_{G(x_o)}) = n_{x_o}(H).$$

(P6)* (Elimination) *Suppose f is normal in Ω and $\Omega_H \cap f^{-1}(0) = \emptyset$ for every $(H) \in \Phi_n(G, V)$. Then*

$$G\text{-Deg}^*(f, \Omega) = 0.$$

(P7)* (Excision) *If $f^{-1}(0) \cap \Omega \subset \Omega_0$, where $\Omega_0 \subset \Omega$ is an open invariant subset, then*

$$G\text{-Deg}^*(f, \Omega) = G\text{-Deg}^*(f, \Omega_0).$$

(P8)* (Hopf property) *Suppose that $\Omega \subset \mathbb{R}^n \oplus V$ is an open invariant subset such that $\Omega_H/W(H)$ is connected for all $(H) \in \Phi_n(G, V)$ and $\Omega_K = \emptyset$ for all $(K) \in \Phi_k(G, V)$ with $k < n$. Let f, $g : \mathbb{R}^n \oplus V \to V$ be two Ω-admissible G-equivariant maps such that*

$$G\text{-Deg}^*(f, \Omega) = G\text{-Deg}^*(g, \Omega).$$

Then f and g are G-equivariantly homotopic by an Ω-admissible homotopy.

Proof: (P1)*: Assume f is regular normal and choose a regular fundamental domain D (together with the lifting homeomorphism $\xi : T_o \to D_o$ (see subsection 3.3)), for the $W(H_o)$-action on Ω_{H_o}. By assumption, $0 \neq n_{H_o} = \deg(f_{H_o} \circ \xi, T_o)$. Then, by the existence property of the (local) Brouwer degree, there exists $y_o \in T_o$ such that $f_{H_o}(\xi(y_o)) = 0$, i.e., $f_{H_o}(x_o) = 0$, where $x_o = \xi(y_o) \in D_o \subset \Omega_{H_o}$, so that $G_{x_o} = H_o$.

In the general case, take a sequence $\{f_n\}$ of G-equivariant Ω-admissible regular normal maps such that

$$\sup_{x\in\Omega} \|f_n(x) - f(x)\| < \frac{1}{n}.$$

Since for n sufficiently large f_n is G-equivariantly homotopic to f, it follows that $G\text{-Deg}^*(f,\Omega) = G\text{-Deg}^*(f_n,\Omega)$. Since f_n is normal, we obtain $f_n^{-1}(0)\cap\Omega_{H_o} \not\emptyset$, thus there is a sequence $\{x_n\} \subset \Omega_{H_o}$ such that $f_n(x_n) = 0$ for each n sufficiently large. We can assume without loss of generality that $x_n \to x$ as $n \to \infty$ and therefore $f(x) = \lim_{n\to\infty} f_n(x_n) = 0$. Since V^{H_o} is closed, $x \in V^{H_o}$ and consequently $G_x \supset H_o$.

(P2)* — (P4)*, (P7)*: To establish these properties, one can use the same idea as above: for a regular normal f (resp. h) the statements follow from (4), (5) and appropriate properties of the local Brouwer degree. In the general case it suffices to take regular normal approximations sufficiently closed to f (resp. h) and use the standard compactness argument.

(P5)*: Follows from the regular value definition of the Brouwer degree.

(P6)*: Follows from the definition of the primary equivariant degree.

(P8)*:

Step 1. Local homotopies around zeros: Denote by $\Phi_{n,0}(G,V)$ the set of all the orbit types occurring in $f^{-1}(0)\cap\Omega$. By definition of $\deg(f,\Omega)$ and Proposition 3, without loss of generality, one can assume that (i) f and g are regular normal and (ii) $\Phi_{n,0}(G,V)$ is also the set of all the orbit types occurring in $g^{-1}(0)\cap\Omega$. Further, by assumption, f and g only have zeros of primary orbit types. For each $(H) \in \Phi_{n,0}(G,V)$, choose a regular fundamental domain D on Ω_H provided by Theorem 2 with $T_o = p(D_o)$ such that $f_H^{-1}(0)\cap(D\setminus D_o) = \emptyset$ and $g_H^{-1}(0)\cap(D\setminus D_o) = \emptyset$, i.e. $p(f_H^{-1}(0)) \cup p(g_H^{-1}(0)) \subset T_o$. Notice that T_o is contractible (in particular, connected). Thus, by the Hopf Property of Brouwer degree,

$$\deg(f_H \circ \xi, T_o) = \deg(g_H \circ \xi, T_o)$$

implies that f_H is homotopic to g_H by a certain homotopy h_H on Ω_H. This homotopy can be extended, in a standard way (cf. [25, 33]), to a G-equivariant homotopy between f and g on $\Omega_{(H)}$. By Proposition 3, this homotopy can also be assumed to be regular and normal. Then, by using the normality condition, such a homotopy can be extended to an invariant neighborhood of $\Omega_{(H)}$, say $\mathcal{N}_{\Omega_{(H)}}$ (denote this homotopy by h_H). Apply the same argument to each $(H) \in \Phi_{n,0}(G,V)$ and choose for any (H) an invariant closed neighborhood $N_H \subset \mathcal{N}_{\Omega_{(H)}}$ satisfying the conditions: (i) N_H contains zeros of f and g of orbit type (H); (ii) $N_H \cap N_L = \emptyset$ as $(H) \neq (L)$. The collection of the "local" homotopies $\{h_{H_{|N_H}}\}$ for all $(H) \in \Phi_{n,0}(G,V)$, gives rise to the equivariant homotopy between f and g on the closed invariant subset $N := \bigsqcup N_H$.

Step 2. Extension of local homotopies: Based on the local homotopies, define a map h on $A := (\{0\}\times\overline{\Omega}) \cup ([0,1]\times N) \cup (\{1\}\times\overline{\Omega})$ by letting $h(0,\cdot) =$

$f(\cdot)$, $h(1,\cdot) = g(\cdot)$ and $h(t,x) = h_H(t,x)$ for $(t,x) \in [0,1] \times N$ and x of orbit type (H). By construction, h is continuous G-equivariant. Using the equivariant Kuratowski-Dugundji Theorem (see, for instance, [25], Theorem 1.3), extend h equivariantly and continuously over $[0,1] \times \overline{\Omega}$ and denote this extension by $\hat{h}$. In general, $\hat{h}$ may have new zeros.

Step 3. Correcting $\hat{h}$ via Urysohn function: Put $\hat{A} := \hat{h}^{-1}(0) \setminus A$ (i.e. the set of the "new zeros" of $\hat{h}$). We claim that $\hat{A}$ is a closed subset in $[0,1]\times\overline{\Omega}$. Indeed, take a sequence $\{(t_n, x_n)\}$ from $\hat{A}$, and suppose $\{(t_n, x_n)\} \to (t_o, x_o)$ in $[0,1]\times\overline{\Omega}$. By continuity of $\hat{h}$, we have $\hat{h}^{-1}(0)$ is a closed subset in $[0,1]\times\Omega$, so $(t_o, x_o) \in \hat{h}^{-1}(0)$. By the normality of h, one has: $(t_o, x_o) \notin A$, i.e. $\hat{A}$ is closed. By construction, $\hat{A} \cap A = \emptyset$, thus there exists an invariant Urysohn function $\eta : [0,1] \times \overline{\Omega} \to [0,1]$ with $\eta(A) = 1$ and $\eta(\hat{A}) = 0$. Now, define a new map $\tilde{h}$ on $[0,1] \times \Omega$ by: $\tilde{h}(t,x) = \hat{h}(t \cdot \eta(t,x), x)$. It is easy to see that $\tilde{h}^{-1}(0) = h^{-1}(0)$, thus $\tilde{h}$ is a required homotopy between f and g.

□

Remark 3. One can easily reformulate Proposition 8 for the primary equivariant degree defined by (7). To this end, one should (i) replace G-Deg* by G-Deg through the whole statement; (ii) replace $\Phi_n(G,V)$ by $\Phi_n^+(G)$ in the properties $(P1)^*$, $(P5)^*$ and $(P6)^*$; (iii) require, in addition, $\Omega_K = \emptyset$ for all $(K) \in \Phi_n(G,V) \setminus \Phi_n^+(G)$ in the property $(P8)^*$. In what follows, we will refer to the corresponding properties of the primary equivariant degree as to (Pj) instead of $(Pj)^*$, $j = 1, \dots, 8$.

3.6. **Axiomatic Approach.** The following statement provides an axiomatic approach to the complete primary equivariant degree and the primary degree.

Proposition 9. *Let G be a compact Lie group.*

(i) *There exists a unique function G-Deg* assigning to each admissible pair (f,Ω) an element G-Deg$^*(f,\Omega) = \sum n_H(H)$ in $\tilde{A}_n(G,V)$, which satisfies properties (P1)*—(P6)* listed in Proposition 8;*

(ii) *There exists a unique function G-Deg assigning to each admissible pair (f,Ω) an element G-Deg$(f,\Omega) = \sum n_H(H)$ in $A_n^+(G)$, which satisfies properties (P1)—(P6) (see Proposition 8 and Remark 3).*

Proof: We only prove the statement (i), since the statement (ii) follows similarly. The *existence* part of Proposition 9 is provided by Propositions 7 and 8. To prove the *uniqueness*, take an arbitrary admissible pair (f,Ω). By the homotopy property, f can be assumed to be regular normal. By additivity (i.e. excision) and elimination properties, we can assume that $\Omega \cap f^{-1}(0)$ contains points of the orbit types $(H) \in \Phi_n(G,V)$. Since f is regular normal, the set $\Omega \cap f^{-1}(0)$ is composed of a finite number of G-orbits. Take tubular neighborhoods isolating the above orbits (this is doable, since we have finitely many zero orbits). By the additivity, the primary degree of (f,Ω) is equal to the sum of degrees of restrictions of f to the tubular neighborhoods. By the elimination axiom, the contribution of the secondary orbit types, is equal to zero. Finally, by the normalization property, the remaining orbits lead to "local indices," which determine uniquely the value of the complete primary degree G-Deg$^*(f,\Omega)$. □

4. **Axiomatic Definition of S^1-degree.** According to the general scheme outlined in the Introduction, from now on we will assume that $n = 1$.

In this and next sections, we will formulate the axioms determining the primary S^1-degree and prove that these axioms indeed uniquely define it.

Recall that any abelian compact Lie group is bi-orientable. Denote by $A_1(S^1) := A_1^+(S^1)$ the free $\mathbb{Z}$-module generated by the symbols $(\mathbb{Z}_k)$, $k = 1, 2, 3, \ldots$. Consider an orthogonal S^1-representation V, an open S^1-invariant bounded set $\Omega \subset \mathbb{R} \oplus V$, and an Ω-admissible S^1-equivariant map $f : \mathbb{R} \oplus V \to V$. Then (cf. (4)-(7)) the primary degree $S^1\text{-Deg}\,(f, \Omega)$, which we will simply call S^1*-equivariant degree*, is an element in $A_1(S^1)$ and can be written as

$$S^1\text{-Deg}\,(f, \Omega) = n_{k_1}(\mathbb{Z}_{k_1}) + n_{k_2}(\mathbb{Z}_{k_2}) + \cdots + n_{k_r}(\mathbb{Z}_{k_r}), \tag{9}$$

where $n_{k_i} \in \mathbb{Z}$.

4.1. **Basic Maps and m-Folding.** We begin our exposition with two constructions playing a substantial role in our considerations.

(i) We denote by $\mathcal{V}_k$, $k = 1, 2, 3, \ldots$, the (non-trivial) k-th real irreducible representation of the group S^1, i.e. $\mathcal{V}_k$ is the space $\mathbb{R}^2 = \mathbb{C}$ with the S^1-action given by $\gamma z := \gamma^k \cdot z$, $\gamma \in S^1$, $z \in \mathbb{C}$, and define the set

$$^k\Omega := \left\{ (t, z) \in \mathbb{R} \oplus \mathcal{V}_k : |t| < 1,\ \frac{1}{2} < |z| < 2 \right\} \tag{10}$$

and $b : \mathbb{R} \oplus \mathcal{V}_k \to \mathcal{V}_k$ by

$$b(t, z) := \big(1 - |z| + it\big) \cdot z, \quad (t, z) \in \mathbb{R} \oplus \mathcal{V}_k, \tag{11}$$

where "$\cdot$" denotes the complex multiplication in $\mathcal{V}_k = \mathbb{C}$. It is clear that the map b is S^1-equivariant and $^k\Omega$-admissible. We call the map b the S^1-*basic map* on $^k\Omega$ (or simply *basic map* if it is clear from the context what representation is involved).

(ii) Further, for every integer $m = 1, 2, 3, \ldots$, we define the homomorphism $\theta_m : S^1 \to S^1$ (called m*-folding*), by $\theta_m(\gamma) = \gamma^m$, $\gamma \in S^1$, and define the induced by θ_m homomorphism $\Theta_m : A_1(S^1) \to A_1(S^1)$, by

$$\Theta_m(\mathbb{Z}_k) := (\mathbb{Z}_{km}), \quad k = 1, 2, 3, \ldots,$$

i.e. $\Theta_m(\mathbb{Z}_k) = (\theta_m^{-1}(\mathbb{Z}_k))$, where $(\mathbb{Z}_k)$ are the free generators of $A_1(S^1)$.

Notice that if $f : \mathbb{R} \oplus V \to V$ is an Ω-admissible S^1-equivariant map for a certain open bounded S^1-invariant subset $\Omega \subset \mathbb{R} \oplus V$, then for every integer $m = 1, 2, 3, \ldots$, we can, first, define the *associated m-folded* S^1-representation $^m(V)$, which is the same vector space V with the S^1-action '$\cdot$' given by

$$\gamma \cdot v := \theta_m(\gamma) v = \gamma^m v, \quad \gamma \in S^1, \ v \in V.$$

Next, the map f considered from $\mathbb{R} \oplus {}^m(V)$ to $^m(V)$, is S^1-equivariant as well. The set Ω considered as an S^1-subset of $\mathbb{R} \oplus {}^m(V)$ will be denoted by $^m(\Omega)$. In what follows, we will say that the pair $(f, {}^m(\Omega))$ is the *m-folded* admissible pair *associated* with (f, Ω).

4.2. **Formulation of the Main Result and Consequences of Axioms.** Now, we are in a position to state the main result of this section.

Theorem 3. *There exists a unique function, denoted by* $S^1\text{-Deg}$*, assigning to each admissible pair* (f, Ω) *an element* $S^1\text{-Deg}\,(f, \Omega) \in A_1(S^1)$ *satisfying properties (P1) — (P4) (see Proposition 8 with* $G = S^1$ *and Remark 3) as well as the following ones:*

(P5)' (NORMALIZATION) *For the basic map* $b : \mathbb{R} \oplus \mathcal{V}_1 \to \mathcal{V}_1$*, we have*

$$S^1\text{-Deg}\,(b, {}^1\Omega) = (\mathbb{Z}_1).$$

(P6)' (ELIMINATION) *If* V *is a trivial* S^1*-representation, then*

$$S^1\text{-Deg}\,(f, \Omega) = 0.$$

(F) (FOLDING) *Let* ${}^m(V)$ *be the* m*-folded representation associated with* V*, and* $(f, {}^m(\Omega))$ *the* m*-folded admissible pair associated with* (f, Ω)*. Then*

$$S^1\text{-Deg}\,(f, {}^m(\Omega)) = \Theta_m\big[S^1\text{-Deg}\,(f, \Omega)\big].$$

The proof of Theorem 3 will be given in the next section. Here we present some immediate consequences of the axioms stated in Theorem 3.

Corollary 1. *Suppose* S^1-Deg *is a function provided by Theorem 3. Then:*

(P7)' (EXCISION) *Assume* Ω_o *is an* S^1*-invariant open subset of* Ω *such that* $f^{-1}(0) \cap \Omega \subset \Omega_o$*. Then*

$$S^1\text{-Deg}\,(f, \Omega) = S^1\text{-Deg}\,(f, \Omega_o).$$

(P9) (k-TH BASIC MAP) *For every* $k = 1, 2, 3, \dots,$ *and the* k*-th basic map* $b : \mathbb{R} \oplus \mathcal{V}_k \to \mathcal{V}_k$*,*

$$S^1\text{-Deg}\,(b, {}^k\Omega) = (\mathbb{Z}_k).$$

The proof of Corollary 1 is straightforward and we omit it.

Corollary 2. *Let* $b^- : \mathbb{R} \oplus \mathcal{V}_k \to \mathcal{V}_k$*,* $k = 1, 2, 3, \dots,$ *be defined by*

$$b^-(t, z) = \big(1 - |z| - it\big) \cdot z, \quad t \in \mathbb{R}, \;\; z \in \mathcal{V}_k. \tag{12}$$

Assume S^1-Deg *is a function provided by Theorem 4.1. Then*

$$S^1\text{-Deg}\,(b^-, {}^k\Omega) = -(\mathbb{Z}_k). \tag{13}$$

Proof: We consider the set

$$\Omega := \left\{(t, z) \in \mathbb{R} \oplus \mathcal{V}_k : |t| < 2, \;\; \frac{1}{2} < |z| < 2\right\}$$

and the function $\alpha : \mathbb{R} \to \mathbb{R}$ defined by

$$\alpha(t) = \begin{cases} 1 & \text{if} \;\; t < -1 \;\text{ or }\; t > \frac{3}{2}, \\ -t & \text{if} \;\; -1 \le t < \frac{1}{4}, \\ t - \frac{1}{2} & \text{if} \;\; \frac{1}{4} \le t \le \frac{3}{2}. \end{cases}$$

Define the homotopy $h : [0,1] \times \mathbb{R} \oplus \mathcal{V}_k \to \mathcal{V}_k$ by

$$h_\lambda(t, z) = \Big(\lambda\big(1 - |z|\big) + i\big((1 - \lambda) + \lambda\alpha(t)\big)\Big) \cdot z, \quad z \in \mathcal{V}_k, \; t \in \mathbb{R}, \; \lambda \in [0,1].$$

It is clear that h_λ is an Ω-admissible homotopy such that $h_0(t, z) = i \cdot z$, which implies (by (P1)) that $S^1\text{-Deg}\,(h_0, \Omega) = 0$ and, therefore (by (P3)),

$$S^1\text{-Deg}\,(h_1, \Omega) = 0. \tag{14}$$

Obviously, $h_1^{-1}(0) \cap \Omega = \big\{(t, z) \in \mathbb{R} \oplus \mathcal{V}_k : |z| = 1, \, t = 0, \, \frac{1}{2}\big\}$. Put

$$\Omega_1 := \left\{(t, z) \in \mathbb{R} \oplus \mathcal{V}_k : |t| < \frac{1}{4}, \;\; \frac{1}{2} < |z| < 2\right\},$$

$$\Omega_2 := \left\{(t, z) : \left|t - \frac{1}{2}\right| < \frac{1}{4}, \;\; \frac{1}{2} < |z| < 2\right\}.$$

Then (by (P2) and (14))

$$S^1\text{-Deg}\,(h_1,\Omega_1) + S^1\text{-Deg}\,(h_1,\Omega_2) = 0. \tag{15}$$

By (P7)' (resp. (P3)), we have

$$S^1\text{-Deg}\,(h_1,\Omega_1) = S^1\text{-Deg}\,(b^-, {}^k\Omega) \quad \Big(\text{resp.} \quad S^1\text{-Deg}\,(h_1,\Omega_2) = S^1\text{-Deg}\,(b, {}^k\Omega)\Big).$$

Therefore, by (P9) and (15), $S^1\text{-Deg}\,(b^-, {}^k\Omega) = -(\mathbb{Z}_k)$. □

5. **Proof of Theorem 3.**

5.1. **Positive Orientation in a Slice and Central Lemma.** The proof of Theorem 3 is essentially based on a regular value argument. To formulate and prove the corresponding statement (see Lemma 3), we will analyze the general notion of positive orientation on a slice (see Remark and Definition 1) in a relevant setting.

Take the standard orientation on $\mathbb{C}$ and consider $S^1 \subset \mathbb{C}$ as an oriented submanifold. Let W be a non-trivial $(n+1)$-dimensional S^1-representation. Take a non-zero $x \in W$ and assume the orbit $G(x)$ does not intersect W^G. Using the above orientation on S^1, we assign a tangent vector v_{n+1} to the orbit $G(x)$ at the point x, which indicates the natural orientation of $G(x)$.

We consider the slice S_x to the orbit $G(x)$ at x:

$$S_x := \Big\{ w \in W : w \bullet v_{n+1} = 0 \Big\},$$

where "•" denotes the standard inner product in W.

In S_x we define the *positive orientation*: choose a basis $\{v_1, v_2, \ldots, v_n\} \subset S_x$ such that the *change-of-basis* matrix from the basis $\{v_1, v_2, \ldots, v_n, v_{n+1}\} \subset W$ to the standard basis $\{e_1, e_2, \ldots, e_n, e_{n+1}\} \subset W \simeq \mathbb{R}^{n+1}$, has a positive determinant. Then the basis $\{v_1, v_2, \ldots, v_n\}$ defines the positive orientation of S_x (cf. Remark and Definition 1).

We are now in a position to state:

Lemma 3. (Central Lemma) *Let $f : \mathbb{R} \oplus V \to V$ be a regular normal Ω-admissible map such that $f^{-1}(0) \cap \Omega$ consists of one S^1-orbit $G(x_o)$. Suppose that $G_{x_o} = \mathbb{Z}_{k_o}$ and denote by S_{x_o} the positively oriented slice at x_o to the orbit $G(x_o)$. Assume that S^1-Deg is a function provided by Theorem 3. Then*

$$S^1\text{-Deg}\,(f,\Omega) = n_o\,(\mathbb{Z}_{k_o}),$$

where n_o is the local index of f at x_o (cf. Definition 9).

5.2. **Proof of Lemma 3. Step 1: Simplification of the S^1-Action ("Unfolding").** We consider the S^1-isotypical decomposition of the space V, i.e.

$$V = V^G \oplus V_{k_1} \oplus V_{k_2} \oplus \cdots \oplus V_{k_r}, \tag{16}$$

where V_{k_j} is modeled on the S^1-irreducible representation $\mathcal{V}_{k_j}$ (which means that any irreducible subrepresentation of V_{k_j} is equivalent to $\mathcal{V}_{k_j}$). Assume that $x_o = y_0 + y_1 + \cdots + y_r$, where $y_0 \in \mathbb{R} \oplus V^G$, $y_j \in V_{k_j}$. If $y_j \neq 0$, then $G_{y_j} = \mathbb{Z}_{k_j}$, which implies (since $G_{x_o} = \mathbb{Z}_{k_o}$) that k_j is a multiple of k_o. Indeed, notice that $G_{x+y} = G_x \cap G_y$, thus $G_{x_o} = \mathbb{Z}_{k_o} \subset \mathbb{Z}_{k_j} = G_{y_j}$.

In addition, since V is an orthogonal S^1-representation, the isotypical components V_{k_j} and V_{k_i}, for $k_j \neq k_i$, are orthogonal one to another. Consequently, if k_j is not a multiple of k_o, then the isotypical component V_{k_j} is orthogonal to $\mathbb{R} \oplus V^G$ and to every component V_{k_i}, for which k_i is a multiple of k_o. In particular, this implies that V_{k_j} is orthogonal to the subspace $\mathbb{R}\oplus V^H$, where $H := G_{x_o} = \mathbb{Z}_{k_o}$. Since, by assumption, f is normal in Ω, it maps the small vectors $v \in (\mathbb{R}\oplus V^H)^\perp$ near the orbit $G(x_o)$ identically on themselves, i.e. $f(x_o+v)=v$. In other words, this property implies that the map f, on a small neighborhood of $G(x_o)$ can be considered (up to a certain admissible homotopy) as the product map $f_o\times \mathrm{Id}$, with $f_o := f_{|\mathbb{R}\oplus V^H}$. By the suspension property (P4), we have

$$S^1\text{-Deg}\,(f,\Omega) = S^1\text{-Deg}\,(f_o\times \mathrm{Id}, \Omega_o\times B) S^1\text{-Deg}\,(f_o,\Omega_o),$$

where $\Omega_o = \Omega\cap(\mathbb{R}\oplus V^H)$ and B denotes the unit ball in $(\mathbb{R}\oplus V^H)^\perp$. Thus,

$$\text{sign det}\, Df(x_o)|_{S_{x_o}} = \text{sign det}\, Df_o(x_o)|_{S'_{x_o}},$$

where $S'_{x_o} := S_{x_o}\cap(\mathbb{R}\oplus V^H)$. In this way, we can assume without loss of generality that in the decomposition (16)

$$k_1 = k_o\cdot n_1,\ k_2 = k_o\cdot n_2,\ \dots,\ k_r = k_o\cdot n_r,$$

and $k_o = \gcd(k_1,k_2,\dots,k_r)$. Since in this case, the subgroup $H=\mathbb{Z}_{k_o}$ acts trivially on V, we can define the action of $S^1 \simeq S^1/H$ on the space V, which is also an orthogonal S^1-representation, denoted by $\tilde V$ (for the purpose of distinguishing it from V). Moreover, the map f is also S^1-equivariant with respect to this new action. Denote by $\tilde\Omega$ the set Ω considered as an S^1-subspace of $\tilde V$. Then (f,Ω) is the k_o-folded admissible pair associated with the admissible pair $(f,\tilde\Omega)$. Therefore, by the folding property (F), we have

$$S^1\text{-Deg}\,(f,\Omega) = \Theta_{k_o}\Big[S^1\text{-Deg}\,(f,\tilde\Omega)\Big].$$

Consequently, it is sufficient to show that

$$S^1\text{-Deg}\,(f,\tilde\Omega) = n_o\,(\mathbb{Z}_1).$$

In the remaining part of the proof, we will simply assume that $G_{x_o}=\mathbb{Z}_1$.

Step 2: Reduction to a tubular neighborhood. Take a tubular neighborhood

$$\Omega' = G\big(x_o + B(0,\varepsilon)\big),\quad B(0,\varepsilon) := \big\{v\in S_{x_o} : \|v\|<\varepsilon\big\},$$

where $0<\varepsilon<\|x_o\|$, around the orbit $G(x_o)$. Then every point $x\in\Omega'$ has a unique representation as $\gamma x_o+\gamma v$, for some $v\in B(0,\varepsilon)$ and $\gamma\in S^1$.

Define the linear operator

$$A := Df(x_o)_{|S_{x_o}} : S_{x_o}\to V,$$

and the map $f_0 := \overline{\Omega'}\to V$ by

$$f_0\big(\gamma(x_o+v)\big) = \gamma(Av),\quad \gamma\in S^1,\ \ v\in B(0,\varepsilon),$$

which is clearly S^1-equivariant. Notice that

$$S^1\text{-Deg}\,(f_0,\Omega') = S^1\text{-Deg}\,(f,\Omega).$$

Indeed, we can always assume that $\varepsilon>0$ was chosen to be sufficiently small, so the homotopy

$$h(\lambda,\gamma(x_o+v)) = \gamma\big[\lambda Av + (1-\lambda)f(x_o+v)\big],\quad \lambda\in[0,1],\ \ \gamma\in S^1,\ \ v\in S_{x_o},$$

is Ω'-admissible.

Step 3: Reduction to One Isotypical Component. We consider the path $x_\lambda = \lambda e + (1-\lambda)x_o$, $\lambda\in[0,1]$, where e is a unit vector belonging to the isotypical component V_1. Let S_{x_λ} be the slice to the orbit $G(x_\lambda)$ at the point x_λ, and $B_\lambda =$

$\{v \in S_{x_\lambda} : \|v\| < \varepsilon\}$ for $\min\{\|x_o\|, 1\} > \varepsilon > 0$. We put $\Omega_\lambda : G(x_\lambda + B_\lambda)$, $A_\lambda := Df(x_\lambda)_{|S_{x_\lambda}}$ and define $f_\lambda : \overline{\Omega_\lambda} \to V$, $\lambda \in [0,1]$, by

$$f_\lambda(\gamma(x_\lambda + v)) = \gamma(A_\lambda v), \quad v \in S_{x_\lambda}, \ \gamma \in S^1.$$

By the excision property (P7)' and the homotopy property (P3), we have

$$S^1\text{-Deg}\,(f_1, \Omega_1) = S^1\text{-Deg}\,(f_\lambda, \Omega_\lambda) S^1\text{-Deg}\,(f_0, \Omega') = S^1\text{-Deg}\,(f, \Omega).$$

Notice that, using a path in the space of linear isomorphisms from S_e to V, the matrix A can be deformed to a block matrix $\tilde{A}$, which is Id on the isotypical components $V_{k_2}, \ldots, V_{k_r}$. Since $S_e = \{v \in \mathbb{R} \oplus V : v \bullet e = 0\}$, by applying the suspension property (P4), we can assume that $V = V^G \oplus V_1$, $e \in V_1$.

Step 4: Reduction to Basic Maps. Suppose that $V_1 = \mathbb{C}^k = \mathbb{R}^{2k}$ and $e = (0, 0, \ldots, 0, 1, 0)$. Since the orbit $G(e)$ consists of the points $(0,\ 0,\ \ldots,\ 0,\ \cos\tau,\ \sin\tau) \in \mathbb{R}^{2k}$, the tangent vector to $G(e)$ at e is the vector $v_{2k+1} = (0, 0, \ldots, 0, 1)$, and consequently the slice S_e consists of all vectors of the form $(\alpha_1, \alpha_2, \ldots, \alpha_{2k-1}, 0)$, $\alpha_j \in \mathbb{R}$. By taking the standard basis in S_e, which in this case defines the positive orientation of S_e, we can use the fact that there exists a path A_λ ($\lambda \in [0,1]$), in $GL(2k, \mathbb{R})$ connecting the matrix $\tilde{A}$ to the matrix:

$$A_1 : \begin{bmatrix} 0 & 0 & \ldots & 0 & 0 & -1 \\ 0 & 1 & \ldots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \ldots & 0 & 1 & 0 \\ 1 & 0 & \ldots & 0 & 0 & 0 \end{bmatrix}$$

if sign $\det Df(x_o)|_{S_{x_o}} > 0$, and

$$A_1 := \begin{bmatrix} 0 & 0 & \ldots & 0 & 0 & -1 \\ 0 & 1 & \ldots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \ldots & 0 & 1 & 0 \\ -1 & 0 & \ldots & 0 & 0 & 0 \end{bmatrix}$$

if sign $\det Df(x_o)|_{S_{x_o}} < 0$. The path A_λ defines an Ω_1-admissible homotopy

$$f_{1+\lambda}\big(\gamma(e + v)\big) = \gamma(A_\lambda v), \quad v \in S_e, \ \gamma \in S^1.$$

Let us consider an element $(t, v) \in \mathbb{R} \oplus V$, which is represented as

$$(t, v) = v_0 + \tilde{v}_1 + \gamma s e, \quad v_0 \in V^G, \ \tilde{v}_1 \in \mathbb{C}^{k-1} \times \{0\} \subset \mathbb{C}^k = V_1, \ \gamma \in S^1, \ s \in \mathbb{R}_+.$$

Then we have

$$\begin{aligned} f_2(t, v) &= f_2(t, v_0 + \tilde{v}_1 + \gamma s e) = f_2(\gamma(t, v_0 + \gamma^{-1}\tilde{v}_1 + s e) \\ &= \gamma\big(A_1(t, v_0 + \gamma^{-1}\tilde{v}_1 + s e\big) = \gamma(v_0 + \gamma^{-1}\tilde{v}_1) + \gamma \tilde{A}_1(t, s) \\ &= v_0 + \tilde{v}_1 + \gamma \tilde{A}_1(t, s), \end{aligned}$$

where $\tilde{A}_1 := \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ if sign $\det Df(x_o)|_{S_{x_o}} > 0$ and $\tilde{A}_1 := \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$ if sign $\det Df(x_o)|_{S_{x_o}} < 0$. The above identities show that the map f_2 is "normal" with respect to the vectors $v_0 + \tilde{v}_1$, i.e. $f_2 = \tilde{f}_2 \times \mathrm{Id}$, where $\tilde{f}_2 : \mathbb{R} \oplus \mathbb{C} \to \mathbb{C}$ is given by:

$$\tilde{f}_2(t, \gamma s e) = \gamma\big(\tilde{A}_1(t, s)\big), \quad \gamma \in S^1, \ s \in \mathbb{R}_+, \ t \in \mathbb{R}.$$

Therefore, by the suspension property (P4), we have

$$S^1\text{-Deg}\,(f_2,\Omega_1) = S^1\text{-Deg}\,(\tilde{f}_2,\tilde{\Omega}_1),$$

where $\tilde{\Omega}_1 := \big\{(t,z)\in\mathbb{R}\oplus\mathbb{C} : |t|<1,\ \frac{1}{2}<|z|<2\big\}$ is equivariantly homotopically equivalent to Ω_1, and the S^1-action on $\mathbb{C}$ is the standard complex multiplication.

Let us consider the maps $b(t,z)=\big(1-|z|+it\big)\cdot z$ and $b^-(t,z)=\big(1-|z|-it\big)\cdot z$, defined on $\tilde{\Omega}_1$, to which we can apply the linearization procedure along the orbit $G(z_o)$, $z_o=(0,1,0)\in\mathbb{R}\oplus\mathbb{C}$. More precisely, we consider the derivatives $Db(0,1,0)$ and $Db^-(0,1,0)$ restricted to S_e, which can be easily evaluated:

$$B_+ := Db(0,1,0)_{|S_e}\begin{bmatrix} 0 & -1\\ 1 & 0\end{bmatrix};\quad B_- := Db^-(0,1,0)_{|S_e} = \begin{bmatrix} 0 & -1\\ -1 & 0\end{bmatrix} \tag{17}$$

Then, by applying the formula $f_\pm(t,\gamma s) := \gamma(B_\pm(t,s))$, $\gamma\in S^1$, $s\in\mathbb{R}_+$ and $t\in\mathbb{R}$, we observe that f_+ (resp. f_-) is equivariantly homotopic to the basic map b (resp. b^-). Therefore, if sign det $Df(x_o)|_{S_{x_o}} = 1$, then there exists an $\tilde{\Omega}_1$-admissible homotopy between b and $\tilde{f}_2$, and if sign det $D_{S_{x_o}}f(x_o) = -1$, then there exists an $\tilde{\Omega}_1$-admissible homotopy between b^- and $\tilde{f}_2$. Consequently, by the normalization property (P5) and Corollary 2, we obtain that

$$S^1\text{-Deg}\,(f,\Omega) = n_o\,(\mathbb{Z}_1),$$

which completes the proof. □

5.3. **Proof of Theorem 3.** *Existence.* We claim that the primary degree defined by the formulae (4)—(7) (with $n=1$ and $G=S^1$) satisfies the properties listed in Theorem 3. Indeed, Properties (P1)—(P4), (P6)' are provided by Proposition 8. Property (P5)' follows from (17). To show (F), consider an admissible pair (f,Ω) and the associated m-folded pair $(f,{}^m(\Omega))$. By the homotopy and excision properties, we can assume that f is regular normal on Ω (and, consequently, on ${}^m(\Omega)$). Take some orbit type $(\mathbb{Z}_k)$ occurring in Ω and let D be a regular fundamental domain for $\Omega_{\mathbb{Z}_k}$. Then D is a regular fundamental domain for ${}^m(\Omega)_{\mathbb{Z}_{km}}$. Since f is the same for both cases, the result follows from (4).

Uniqueness. Let $S^1\text{-}\widetilde{\text{Deg}}$ be a function satisfying Properties (P1)—(P4), (P5)', (P6)' and (F). Let V be an orthogonal S^1-representation, $\Omega\subset\mathbb{R}\oplus V$ an S^1-invariant open bounded region, and $f:\mathbb{R}\oplus V\to V$ an equivariant Ω-admissible map. We will show that

$$S^1\text{-}\widetilde{\text{Deg}}\,(f,\Omega) = S^1\text{-Deg}\,(f,\Omega).$$

By Proposition 3 and homotopy property (P3), without lost of generality one can assume that f is regular normal. By the normality, there exists an open S^1-invariant subset $\Omega_o\subset\Omega$ such that $Z := f^{-1}(0)\cap\Omega^{S^1} = f^{-1}(0)\cap\Omega_o$, i.e. Ω_o is an *isolating invariant neighborhood* of Z. In addition, we can assume that $f_{|\Omega_o}$ (up to an Ω_o-admissible homotopy) is a product map $f^{S^1}\times\text{Id}$, where $f^{S^1} := f_{|\mathbb{R}\oplus V^{S^1}}$, and Id is the identity operator on the space $(\mathbb{R}\oplus V^{S^1})^\perp$. Then, by the suspension property (P4) and the elimination property (P6)', we have

$$S^1\text{-}\widetilde{\text{Deg}}\,(f,\Omega_o) = S^1\text{-}\widetilde{\text{Deg}}\,(f^{S^1}\times\text{Id},\Omega_o^{S^1}\times B) = S^1\text{-}\widetilde{\text{Deg}}\,(f^{S^1},\Omega_o^{S^1}) = 0,$$

where B denotes the unit ball in $(\mathbb{R}\oplus V^{S^1})^\perp$.

Since f is assumed to be regular, we have that

$$f^{-1}(0)\cap\Omega = Z\cup S^1(x_1)\cup\cdots\cup S^1(x_m),$$

where $S^1(x_j)$, $j = 1, 2, \dots, m$, are isolated orbits. We can choose open invariant sets $\Omega_j \subset \Omega$ such that $\Omega_j \supset S^1(x_j)$, $\Omega_j \cap \Omega_i = \emptyset$, $i \neq j$, $i, j = 0, 1, 2, \dots, m$. Then, by applying the additivity property (P2), we obtain that

$$\begin{aligned} S^1\text{-}\widetilde{\mathrm{Deg}}\,(f,\Omega) &= S^1\text{-}\widetilde{\mathrm{Deg}}\,(f,\Omega_0) + S^1\text{-}\widetilde{\mathrm{Deg}}\,(f,\Omega_1) + \dots + S^1\text{-}\widetilde{\mathrm{Deg}}\,(f,\Omega_m) \\ &= S^1\text{-}\widetilde{\mathrm{Deg}}\,(f,\Omega_1) + \dots + S^1\text{-}\widetilde{\mathrm{Deg}}\,(f,\Omega_m). \end{aligned}$$

For each of the orbits $S^1(x_j)$, $j = 1, \dots, m$, we consider the positively oriented slice S_j at the point x_j, and we denote by $D_j f(x_j)$ the matrix of the derivative $Df(x_j)_{|S_j}$, with respect to a basis in S_j defining the positive orientation on it.

Applying the Central Lemma and Properties (P2), (P7)', one obtains

$$\begin{aligned} S^1\text{-}\widetilde{\mathrm{Deg}}\,(f,\Omega) &= \sum_{j=1}^{m} S^1\text{-}\widetilde{\mathrm{Deg}}\,(f,\Omega_j) \sum_{j=1}^{m} \mathrm{sign}\,\det Df(x_j)|_{S_j} \cdot (\mathbb{Z}_{k_j}) \\ &= \sum_{j=1}^{m} S^1\text{-}\mathrm{Deg}\,(f,\Omega_j) = S^1\text{-}\mathrm{Deg}\,(f,\Omega). \end{aligned}$$

6. **Computation of S^1-Degree Via Reduction to Basic Maps.**

6.1. **Statement of the Problem.** The goal of this section is to show how the axiomatic approach described in the previous two sections allows us to calculate the S^1-degree for an important class of S^1-equivariant maps which naturally appear in symmetric Hopf bifurcation problems.

We start with a simple observation that every S^1-representation admits a so-called *natural complex structure*, which turns out to be a convenient setting for the discussion of Hopf bifurcation problems and a natural way of describing the S^1-action to carry out certain computations. To be more specific, let V be an S^1-representation with $V^{S^1} = \{0\}$. Then one can define on V a complex structure sensitive to the S^1-action as follows. Assume, for a moment, that $V = \mathcal{V}_k$. Then, for $z \in \mathbb{C}$ we put $z = |z|e^{i\theta}$, for some $\theta \in [0, 2\pi)$. The complex multiplication of $v \in \mathcal{V}_k$ by the number z is defined by

$$z \cdot v := |z| e^{\frac{i\theta}{k}} v. \tag{18}$$

Suppose, further, that V is (in general) reducible, and we have the following S^1-isotypical decomposition:

$$V = V_{k_1} \oplus V_{k_2} \oplus \dots \oplus V_{k_s}, \tag{19}$$

where V_{k_j} is modeled on the irreducible S^1-representation $\mathcal{V}_{k_j}$, $j = 1, 2, \dots, s$. Since for every j, $\mathcal{V}_{k_j}$ can be equipped with the complex structure according to (18), every isotypical component from (19) also admits such a structure. In this way, we obtain on V a complex structure which we will call *natural complex structure*.

Let Γ be a compact Lie group. The problem of studying Γ-symmetric Hopf bifurcations in many cases can be reduced to the following one (cf. [4]):

Let $G = \Gamma \times S^1$ and let V be an orthogonal G-representation with $V^{S^1} = \{0\}$ (here S^1 is identified with $\{1\} \times S^1$). Suppose V (considered as the S^1-representation) is equipped with the natural complex structure and put

$$\mathcal{O} := \Big\{(\lambda, v) \in \mathbb{C} \oplus V : \|v\| < 2,\ \frac{1}{2} < |\lambda| < 4\Big\}. \tag{20}$$

Assume G acts trivially on $\mathbb{C}$ and $\mathbb{R}$ and take a continuous map $a : S^1 \to GL^G(V)$, where $GL^G(V)$ stands for the set of all G-equivariant linear invertible maps in V. Define a G-equivariant map $f_a : \overline{\mathcal{O}} \to \mathbb{R} \oplus V$ by

$$f_a(\lambda, v) \left(|\lambda|(\|v\| - 1) + \|v\| + 1, a\left(\frac{\lambda}{|\lambda|}\right) v \right), \quad (\lambda, v) \in \overline{\mathcal{O}}. \tag{21}$$

How can one compute the primary degree G-Deg$\,(f_a, \mathcal{O})$?

Our approach to the above problem involves the following four components:

(i) *Recurrence Formula* (see Proposition 13) allowing a reduction of the general problem to the computation of the corresponding S^1-degree;
(ii) *Splitting Lemma* (cf. Lemma 4) allowing a reduction to subrepresentations;
(iii) *Homotopy factorization* (cf. Corollaries 3 and 4) allowing a factorization of a given map through canonical representatives of the elements of $\pi_1(GL^G(k, \mathbb{C}))$ and next deformations to the so-called $\mathbb{C}$-complementing maps being natural "complex counterparts" for the k-th basic maps (cf. Definition 10);
(iv) *Suspension procedure* allowing a reduction of the computation of the S^1-degree of $\mathbb{C}$-complementing maps to the one of k-th basic maps (cf. Proposition 10).

The last three techniques come together at the end of this section (see Theorem 4 where the S^1-degree for (21) is given). Observe that the Splitting Lemma is presented in a form much more general than what is needed to establish Theorem 4.

6.2. **$\mathbb{C}$-Complementing Maps and Suspension Procedure.** We start with the following:

Definition 10. Let $b : \mathbb{R} \oplus \mathcal{V}_k \to \mathcal{V}_k$ (resp. $b^- : \mathbb{R} \oplus \mathcal{V}_k \to \mathcal{V}_k$) be the k-th basic map (resp. a map defined by (12)) and let ${}^k\Omega$ be defined by (10). Assume that $\mathcal{V}_k$ is equipped with the natural complex structure and $\mathcal{O}$ is given by (20). Suppose that $f : \mathbb{C} \oplus \mathcal{V}_k \to \mathbb{R} \oplus \mathcal{V}_k$ (resp. $f^- : \mathbb{C} \oplus \mathcal{V}_k \to \mathbb{R} \oplus \mathcal{V}_k$) is defined by $f(\lambda, v) = \Big(|\lambda|(\|v\| - 1) + \|v\| + 1, \lambda \cdot v\Big)$ (resp. $f^-(\lambda, v) = (|\lambda|(\|v\| - 1) + \|v\| + 1, \overline{\lambda} \cdot v)$), where $\lambda \in \mathbb{C}$, $v \in \mathcal{V}_k$. Then the pair $(f, \mathcal{O})$ (resp. $(f^-, \mathcal{O})$) is called a $\mathbb{C}$-*complementing pair* to $(b, {}^k\Omega)$ (resp. to $(b^-, {}^k\Omega)$).

It is clear that $(f, \mathcal{O})$, $(f^-, \mathcal{O})$, $(b, {}^k\Omega)$, and $(b^-, {}^k\Omega)$ are admissible pairs. The following statement justifies the above definition.

Proposition 10. *Let $(f, \mathcal{O})$ (resp. $(f^-, \mathcal{O})$) be a $\mathbb{C}$-complementing pair to $(b, {}^k\Omega)$ (resp. $(b^-, {}^k\Omega)$). Then f (resp. f^-) is S^1-homotopic (by an $\mathcal{O}$-admissible homotopy) to a map $\overline{f_1}$ (resp. $\overline{f_1^-}$), which is a suspension of b (resp. b^-) on an open subset containing zeros of $\overline{f_1}$ (resp. $\overline{f_1^-}$). In particular,*

$$S^1\text{-Deg}\,(f, \mathcal{O}) = S^1\text{-Deg}\,(b, {}^k\Omega) = (\mathbb{Z}_k), \tag{22}$$

$$S^1\text{-Deg}\,(f^-, \mathcal{O}) = S^1\text{-Deg}\,(b^-, {}^k\Omega) = -(\mathbb{Z}_k). \tag{23}$$

Proof: We will consider only the case of the map f (the proof for the map f^- is similar). To begin with, observe that the map

$$f_1(\lambda, v) = \left(|\lambda|(\|v\| - 1) + \|v\| + 1, \frac{\lambda}{|\lambda|} \cdot v \right),$$

defined for $(\lambda,v)\in\overline{\mathcal{O}}$, is S^1-homotopic (by an $\mathcal{O}$-admissible homotopy) to the map f (since for any $(\lambda,v)\in\partial\mathcal{O}$, the vectors $f(\lambda,v))$ and $f_1(\lambda,v)$ do not point the opposite directions). Thus, we have

$$S^1\text{-Deg}\,(f,\mathcal{O})=S^1\text{-Deg}\,(f_1,\mathcal{O}).$$

Let us define the function $\eta:\mathbb{R}\to\mathbb{R}$ by

$$\eta(t)=\begin{cases}0 & \text{if } t<\frac{1}{2}\\ t-\frac{1}{2} & \text{if } \frac{1}{2}\le t\le\frac{3}{2}\\ 1 & \text{if } t>\frac{3}{2},\end{cases}$$

and put $\theta(v)=\eta(\|v\|)$, for $v\in\mathcal{V}_k$. Set

$$f_\theta(\lambda,v)=(1-\theta(v))\big(f_1(\lambda,0)+v\big)+\theta(v)f_1(\lambda,v), \tag{24}$$

where $(\lambda,v)\in\overline{\mathcal{O}}$. Obviously, f_1 is S^1-homotopic to f_θ by an $\mathcal{O}$-admissible homotopy, i.e. we have

$$S^1\text{-Deg}\,(f_1,\mathcal{O})=S^1\text{-Deg}\,(f_\theta,\mathcal{O})$$

for any $\theta\in[0,1]$. By direct computation, $f_\theta^{-1}(0)=Z_0\cup Z_1\subset\Omega$ where $Z_0:=\Big\{(\lambda,0)\in\mathbb{C}\oplus\mathcal{V}_k:|\lambda|=1\Big\}$ and $Z_1:=\Big\{(-3,v)\in\mathbb{C}\oplus\mathcal{V}_k:\|v\|=1\Big\}$.

Put

$$\Omega_0:=\left\{(\lambda,v):\frac{1}{2}<|\lambda|<\frac{3}{2},\ \|v\|<\frac{1}{2}\right\}$$

and

$$\Omega_1:=\left\{(\lambda,v):|\lambda+3|<\frac{1}{2},\ \frac{1}{2}<\|v\|<\frac{3}{2}\right\}.$$

Then, by the additivity property of the S^1-degree, we have

$$S^1\text{-Deg}\,(f_\theta,\mathcal{O})=S^1\text{-Deg}\,(f_\theta,\Omega_0)+S^1\text{-Deg}\,(f_\theta,\Omega_1).$$

Since for $(\lambda,v)\in\Omega_0$, we have $f_\theta(\lambda,v)=(1-|\lambda|,v)$, it follows from the suspension property that

$$S^1\text{-Deg}\,(f_\theta,\Omega_0)=S^1\text{-Deg}\,(\varphi_o,B_o),$$

where $B_o=\{\lambda\in\mathbb{C}:\frac{1}{2}<|\lambda|<3\}$ and $\varphi_o:\overline{B}_o\to\mathbb{R}$ is defined by $\varphi_o(\lambda)=1-|\lambda|$. Clearly, φ_o is homotopic by a B_o-admissible homotopy to a constant map $\varphi_1\equiv 5$, thus $S^1\text{-Deg}\,(\varphi_o,B_o)=0$, so we have

$$S^1\text{-Deg}\,(f,\mathcal{O})=S^1\text{-Deg}\,(f_\theta,\Omega_1).$$

Replacing the $\mathbb{R}$-component of (24) $\theta(v)$ (resp. $\|v\|$) by $\|v\|-\frac{1}{2}$ (resp. 1), one obtains the map

$$\begin{aligned}\tilde f_\theta(\lambda,v)&=\left(\frac{1}{2}(3-|\lambda|),\Big(1-\theta(v)+\theta(v)\cdot\frac{\lambda}{|\lambda|}\Big)\cdot v\right)\\ &=\left(\frac{1}{2}(3-|\lambda|),\frac{3(1+|\lambda|)-(2|\lambda|+6)\|v\|+(2\|v\|-1)(\lambda+3)}{2|\lambda|}\cdot v\right)\end{aligned}$$

where $(\lambda,v)\in\Omega_1$ (recall, $\theta(v)=\|v\|-\frac{1}{2}$ on Ω_1).

Obviously, $\tilde f_\theta$ has no zeros on $\partial\Omega_1$. Moreover, for any $(\lambda,v)\in\partial\Omega_1$ the vectors $f_\theta(\lambda,v)$ and $\tilde f_\theta(\lambda,v)$ do not point in opposite directions. Therefore, f_θ and $\tilde f_\theta$ are S^1-homotopic by Ω_1-admissible homotopy and

$$S^1\text{-Deg}\,(f_\theta,\Omega_1)=S^1\text{-Deg}\,(\tilde f_\theta,\Omega_1).$$

Next, replacing in the V-component of $\tilde{f}_\theta$ the value $|\lambda|$ (resp. $2\|v\|-1$) by 3 (resp. 1), one obtains the map

$$\hat{f}_1(\lambda, v) = \left(\frac{1}{2}(3-|\lambda|), \frac{12(1-\|v\|)+(\lambda+3)}{6}\cdot v\right),$$

where $(\lambda, v) \in \Omega_1$.

At this moment, we can apply the change of variables $\lambda' = \lambda + 3$, leading to the set $\Omega_2 : \left\{(\lambda', v) : |\lambda'| < \frac{1}{2},\ \frac{1}{2} < \|v\| < \frac{3}{2}\right\}$ and (after an appropriate S^1-homotopy) the map $\tilde{f}_1 : \overline{\Omega_2} \to \mathbb{R} \oplus V$, given by

$$\tilde{f}_1(\alpha + i\beta, v)\left(\frac{1}{2}\alpha, \frac{12(1-\|v\|)+(\alpha+i\beta)}{6}\cdot v\right), \quad \lambda' = \alpha + i\beta,$$

(here, we used the fact that $3 - |\lambda| = 3 - \sqrt{(\alpha-3)^2 + (\beta)^2}$ is S^1-homotopic to α, since $|\beta| \le |\lambda'| < \frac{1}{2}$, which guarantees no zeros of such a homotopy crossing $\partial\Omega_2$), which is clearly Ω_2-admissibly S^1-homotopic to the map

$$\overline{f_1}(\alpha + i\beta, v) = (\alpha, (1 - \|v\| + i\beta)\cdot v).$$

Obviously, $\overline{f}_1$ is a suspension of the basic map b, therefore

$$S^1\text{-Deg}\,(\overline{f_1}, \Omega_2) = S^1\text{-Deg}\,(b, {}^k\Omega),$$

and since

$$S^1\text{-Deg}\,(\overline{f_1}, \Omega_2) = S^1\text{-Deg}\,(\tilde{f}_\theta, \Omega_1) = S^1\text{-Deg}\,(f, \mathcal{O}),$$

the equality (22) follows. $\square$

6.3. **Homotopy Factorization: Properties of $GL^G(V)$.** Let V be an orthogonal G-representation and let $GL^G(V)$ be the group of all equivariant linear invertible operators on V. We first recall some standard algebraic facts about a decomposition of $GL^G(V)$.

Proposition 11. (cf. [20]) *Let*

$$V = U_{k_1} \oplus \cdots \oplus U_{k_r}, \tag{25}$$

be the G-isotypical decomposition, where a component U_{k_j} is modeled on an irreducible representation $\mathcal{U}_{k_j}$. Then:

(i) *$GL^G(V) = \bigoplus_{j=1}^r GL^G(U_{k_j})$;*

(ii) *for any isotypical component U_{k_j} from (25) we have $GL^G(U_{k_j}) \simeq GL(m, \mathbb{F})$, where $m = \dim U_{k_j}/\dim \mathcal{U}_{k_j}$ and $\mathbb{F} \simeq GL^G(\mathcal{U}_{k_j})$, i.e. $\mathbb{F} = \mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$, depending on the type of the irreducible representation $\mathcal{U}_{k_j}$.*

Next, we will discuss homotopy properties of the group $GL(m, \mathbb{C})$. We keep the following notations: for a continuous map $\varphi : S^1 \to \mathbb{C} \setminus \{0\}$, $S^1 \subset \mathbb{C}$, the symbol $\deg(\varphi, S^1)$ stands for its Brouwer degree; for $A \in GL(m, \mathbb{C})$ the symbol $\det_{\mathbb{C}} A$ stands for the complex determinant. We have

Proposition 12. (see, for instance, [16, 23]).

(i) *Two continuous maps Φ, $\Psi : S^1 \to GL(m, \mathbb{C})$, $m \ge 1$, are homotopic if and only if the maps $\varphi := \det_{\mathbb{C}} \circ \Phi$ and $\psi := \det_{\mathbb{C}} \circ \Psi$ are homotopic, i.e.*

$$\deg(\varphi, S^1) = \deg(\psi, S^1).$$

(ii) *For every map $\Phi : S^1 \to GL(m,\mathbb{C})$, there exists $l \in \mathbb{Z}$ such that Φ is homotopic to Φ_l given by*

$$\Phi_l(\gamma) := \begin{bmatrix} \gamma^l & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix}, \quad \gamma \in S^1.$$

In particular, for $\varphi_l := \det_{\mathbb{C}} \circ \Phi_l$, one has $\deg(\varphi_l, S^1) = l$.

Combining Propositions 11 and 12 we obtain the following statement related to the homotopy factorization procedure.

Corollary 3. *Let G be a compact Lie group, V an orthogonal G-representation, and U_{k_o} an isotypical component of V modeled on an irreducible G-representation $\mathcal{U}_{k_o}$ of the complex type. Assume $m = \dim U_{k_o}/\dim \mathcal{U}_{k_o}$. Then*

(i) *$GL^G(U_{k_o}) \simeq GL(m,\mathbb{C})$;*

(ii) *for each $a \in \pi_1(GL^G(U_{k_o}))$ there exists a representative $\varphi_a : S^1 \to GL(m,\mathbb{C})$, such that*

$$\varphi_a(\lambda) = \begin{bmatrix} \lambda^l & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix}, \quad \lambda \in S^1,$$

for some $l \in \mathbb{Z}$. In particular, we have an isomorphism $\mu_{k_o} : \pi_1(GL^G(V_{k_o})) \to \mathbb{Z}$, where $\mu_{k_o}(a) = l$.

6.4. **Splitting Lemma.**

Lemma 4. (Splitting Lemma) *Let G be a compact Lie group, V_1 and V_2 orthogonal G-representations, $V = V_1 \oplus V_2$. Assume that the G-isotypical decomposition of V contains only components modeled on irreducible G-representations of complex type. Suppose that $a_j : S^1 \to GL^G(V_j)$, $j = 1,2$, are two continuous maps and $a : S^1 \to GL^G(V)$ is given by*

$$a(\lambda) = a_1(\lambda) \oplus a_2(\lambda), \quad \lambda \in S^1.$$

Assume $\mathcal{O}$ and f_a are defined by (20) and (21), respectively. Put

$$\mathcal{O}_j := \Big\{(\lambda, v_j) \in \mathbb{C} \oplus V_j : \|v_j\| < 2,\ \frac{1}{2} < |\lambda| < 4\Big\},$$

$$f_{a_j}(\lambda, v_j) := \left(|\lambda|(\|v_j\| - 1) + \|v_j\| + 1, a_j\left(\frac{\lambda}{|\lambda|}\right) v_j\right),$$

where $j = 1,2$, $v_j \in V_j$. Then

$$G\text{-Deg}\,(f_a, \mathcal{O}) = G\text{-Deg}\,(f_{a_1}, \mathcal{O}_1) + G\text{-Deg}\,(f_{a_2}, \mathcal{O}_2).$$

Proof: We can assume without loss of generality that $a_j : S^1 \to GL^G(V_j) \cap O(V_j)$ is analytic, i.e. there exists an analytic extension of a_j to a neighborhood of S^1 in $\mathbb{C}$ (here $O(V_j)$ stands for the group of orthogonal operators on V_j, $j = 1,2$). Introduce the functions $q_j : \mathbb{R} \to \mathbb{R}$, $j = 1,2$,

$$q_j(t) = \begin{cases} 1 & \text{if } 0 \le t < s_j; \\ -\frac{1}{\epsilon_j}(t - t_j) & \text{if } s_j \le t < t_j; \\ 0 & \text{if } t \ge t_j, \end{cases} \quad \text{where} \quad \begin{cases} s_j = \frac{j}{j+4} - \frac{1}{2(j+4)^2}; \\ t_j = \frac{j}{j+4} + \frac{1}{2(j+4)^2}; \\ \epsilon_j = t_j - s_j = \frac{1}{(j+4)^2}. \end{cases}$$

Then define for $(\lambda, v_1, v_2) \in \overline{\mathcal{O}} \subset \mathbb{C} \oplus V_1 \oplus V_2$ the map

$$\tilde{f}_a(\lambda, v_1, v_2) := \Big(\theta(\lambda, v_1, v_2), \beta_1(\lambda, v_1), \beta_2(\lambda, v_1, v_2)\Big),$$

with

$$\begin{aligned}
\theta(\lambda, v_1, v_2) &= |\lambda|(\|v_1 + v_2\| - 1) + \|v_1 + v_2\| + 1,\\
\beta_1(\lambda, v_1) &= q_2(\|v_1\|)v_1 + (1 - q_2(\|v_1\|)a_1\left(\frac{\lambda}{|\lambda|}\right)v_1,\\
\beta_2(\lambda, v_1, v_2) &= q_1(\|v_1 + v_2\|)v_2 + (1 - q_1(\|v_1 + v_2\|)a_2\left(\frac{\lambda}{|\lambda|}\right)v_2.
\end{aligned}$$

The maps f_a and $\tilde{f}_a$ are G-homotopic by an $\mathcal{O}$-admissible homotopy.

Let us examine zeros of the map $\tilde{f}_a$. It is clear that

$$Z_0 := \Big\{(\lambda, 0, 0) : |\lambda| = 1\Big\} \subset \tilde{f}_a^{-1}(0).$$

Observe that if $(\lambda, v_1, v_2) \in \tilde{f}_a^{-1}(0)$ is such that $v_1 \neq 0$ (resp. $v_2 \neq 0$) then $v_2 = 0$ (resp. $v_1 = 0$). Indeed, suppose that $(\lambda, v_1, v_2) \in \tilde{f}_a^{-1}(0)$ is such that $v_1 \neq 0 \neq v_2$. Then, by comparing the norms of the both sides in the following equalities: $q_2(\|v_1\|)v_1 = -(1 - q_2(\|v_1\|)a_1\left(\frac{\lambda}{|\lambda|}\right)v_1$ and $q_1(\|v_1 + v_2\|)v_2 = -(1 - q_1(\|v_1 + v_2\|)a_2\left(\frac{\lambda}{|\lambda|}\right)v_2$, we obtain

$$q_2(\|v_1\|) = 1 - q_2(\|v_1\|) \quad \text{and} \quad q_1(\|v_1 + v_2\|) = 1 - q_1(\|v_1 + v_2\|),$$

which implies

$$q_2(\|v_1\|) = q_1(\|v_1 + v_2\|) = \frac{1}{2},$$

so $\|v_1\| = \frac{1}{3}$ and $\|v_1 + v_2\| = \frac{1}{5}$, but this is a contradiction because v_1 is orthogonal to v_2 and thus $\|v_1 + v_2\| \geq \|v_1\|$.

Therefore, we can first suppose that $(\lambda, v_1, 0) \in \tilde{f}_a^{-1}(0)$, $v_1 \neq 0$, so $\|v_1\| = \frac{1}{3}$. Then $\theta(\lambda, v_1, 0) = 0$ and $\beta_1(\lambda, v_1) = 0$ imply $|\lambda|\left(\frac{1}{3} - 1\right) + \frac{1}{3} + 1 = 0$, i.e. $|\lambda| = 2$. On the other hand, since $q_2(\frac{1}{3}) = \frac{1}{2}$,

$$\beta_1(\lambda, v_1) = \frac{1}{2}\left[v_1 - a_1\left(\frac{\lambda}{|\lambda|}\right)v_1\right] = 0, \quad v_1 \neq 0,$$

λ satisfies the equation

$$\det{}_{\mathbb{C}}\left[\mathrm{Id} - a_1\left(\frac{\lambda}{|\lambda|}\right)\mathrm{Id}\right] = 0, \quad |\lambda| = 2. \tag{26}$$

Since the map $\omega \to \det_{\mathbb{C}}[\mathrm{Id} - a_1(\omega)\mathrm{Id}]$ is analytic in a neighborhood of S^1 in $\mathbb{C}$, the equation

$$\det{}_{\mathbb{C}}[\mathrm{Id} - a_1(\omega)\mathrm{Id}] = 0, \quad \omega \in S^1,$$

has only a finite number of solutions, and consequently the equation (26) also has finitely many solutions, say $\lambda_1, \dots, \lambda_n$. Put

$$Z_k := \Big\{(\lambda_k, v_1, 0) : \|v_1\| = \frac{1}{3}\Big\}, \quad k = 1, \dots, n.$$

If $(\lambda, v_1, 0) \in \tilde{f}_a^{-1}(0)$, $v_1 \neq 0$, then $(\lambda, v_1, 0) \in \mathbb{Z}_1 \cup \cdots \cup Z_n$. Similarly, if $(\lambda, 0, v_2) \in \tilde{f}_a^{-1}(0)$, $v_2 \neq 0$, then $\|v_2\| = \frac{1}{5}$ and $|\lambda| = \frac{3}{2}$, and there exists a finite number of solutions $\lambda'_1, \dots, \lambda'_m$ to the equation

$$\det_{\mathbb{C}}\left[\mathrm{Id} - a_2\left(\frac{\lambda}{|\lambda|}\right)\mathrm{Id}\right] = 0, \quad |\lambda| = \frac{3}{2}.$$

Put $Z'_l := \left\{(\lambda'_l, 0, v_2) : \|v_2\| = \frac{1}{5}\right\}$, $\quad l = 1, \dots, m$. In this way, we have proved that $\tilde{f}_a^{-1}(0) \subset Z_0 \cup Z_1 \cup \dots \cup Z_n \cup Z'_1 \cup \dots \cup Z'_m$. By applying the excision property to G-invariant separating neighborhoods of Z_k, Z'_l, $k = 0, 1, \dots, n$, $l = 1, \dots, m$, and using appropriate deformations of $\tilde{f}_a$ on these sets, we obtain the map $\hat{f}_a$ such that $\hat{f}_a(\lambda, v_1, v_2) = (\theta(\lambda, v_1, v_2), \beta_1(\lambda, v_1), v_2)$ for (λ, v_1, v_2) in a neighborhood of Z_k, $k = 1, \dots, n$, and $\hat{f}_a(\lambda, v_1, v_2) = (\theta(\lambda, v_1, v_2), v_1, \beta_2(\lambda, 0, v_2))$ for (λ, v_1, v_2) in a neighborhood of Z'_l, $l = 1, \dots, m$. Notice that $\tilde{f}_a$ in a neighborhood of Z_0 is homotopic to a map without zeros.

The conclusion then follows from the suspension and excision properties. □

6.5. **S^1-Degree Formulae.** Here we combine the above results to compute the S^1-degree of (21). We start with the following:

Corollary 4. *Let $V = \mathcal{V}_k$ be the k-th irreducible S^1-representation ($k > 0$) equipped with the natural complex structure, $l \in \mathbb{Z}$ and*

$$\tilde{f}(\lambda, v) = \left(|\lambda|(\|v\| - 1) + \|v\| + 1, \left(\frac{\lambda}{|\lambda|}\right)^l v\right), \quad (\lambda, v) \in \overline{\mathcal{O}},$$

where $\mathcal{O}$ is given by (20). Then $S^1\text{-Deg}\,(\tilde{f}, \mathcal{O}) = l(\mathbb{Z}_k)$.

Proof: For the sake of definiteness, assume that $l > 0$ (the case $l \le 0$ can be treated using a similar argument), and consider the map

$$\tilde{f} \times \mathrm{Id} : \overline{\mathcal{O}} \times \overline{B_{l-1}} \to \mathbb{R} \oplus \mathcal{V}_k \oplus \Big[\underbrace{\mathcal{V}_k \oplus \dots \oplus \mathcal{V}_k}_{l-1}\Big],$$

where $B_{l-1} = \underbrace{B(\mathcal{V}_k) \times \dots \times B(\mathcal{V}_k)}_{l-1}$ and $B(\mathcal{V}_k)$ denotes the unit ball in $\mathcal{V}_k$. Then, by the suspension property,

$$S^1\text{-Deg}\,(\tilde{f}, \mathcal{O}) = S^1\text{-Deg}\left(\tilde{f} \times \mathrm{Id}, \mathcal{O} \times B_{l-1}\right).$$

Obviously, $\tilde{f} \times \mathrm{Id}$ is equivariantly homotopic, by an $\mathcal{O} \times B_{l-1}$-admissible homotopy, to f_a given by (21), where $v \in V = \underbrace{\mathcal{V}_k \oplus \dots \oplus \mathcal{V}_k}_{l}$ and $a : S^1 \to GL^{S^1}(V)$ is defined by

$$a(\gamma)\begin{bmatrix} \gamma^l & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix}, \quad \gamma \in S^1.$$

From Proposition 12 it follows that f_a is equivariantly homotopic (by an $\mathcal{O} \times B_{l-1}$-admissible homotopy) to f_b given by

$$f_b(\lambda, v) = \left(|\lambda|\|(\|v\| - 1) + \|v\| + 1, b\left(\frac{\lambda}{|\lambda|}\right)v\right),$$

with $b : S^1 \to GL^{S^1}(V)$ defined by

$$b(\gamma) = \begin{bmatrix} \gamma & 0 & \dots & 0 \\ 0 & \gamma & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \gamma \end{bmatrix}, \quad \gamma \in S^1.$$

Since $S^1\text{-Deg}\,(\tilde{f}, \mathcal{O}) = S^1\text{-Deg}\,(f_b, \mathcal{O} \times B_{l-1})$, by the Splitting Lemma and Proposition 10, we have

$$S^1\text{-Deg}\,(\tilde{f}, \mathcal{O}) = \underbrace{(\mathbb{Z}_k) + \dots + (\mathbb{Z}_k)}_{l}\, l(\mathbb{Z}_k).$$

The proof of Corollary 4 is complete. □

By combining Proposition 12, Corollary 3, Splitting Lemma and Corollary 4, we immediately obtain:

Theorem 4. *Let V be an orthogonal S^1-representation with $V^{S^1} = \{0\}$, admitting the isotypical decomposition (19) and equipped with the natural complex structure. Let $\mathcal{O}$ (resp. f_a) be defined by (20) (resp. (21)). Then*

$$S^1\text{-Deg}\,(f_a, \mathcal{O}) = \sum_{j=1}^{s} l_j (\mathbb{Z}_{k_j}),$$

where $l_j := \deg\left(\det_{\mathbb{C}} \circ a_j, S^1\right), \quad a_j(\lambda) := a(\lambda)|_{V_{k_j}} : V_{k_j} \to V_{k_j}$, *for* $j = 1, \dots, s$.

As an immediate consequence of Theorem 4, we obtain

Corollary 5. *Let V and $\mathcal{O}$ be as in Theorem 4. Let $l_j \in \mathbb{Z}$, $j = 1, \dots, s$, be given integers and assume that $\dim_{\mathbb{C}} V_{k_j} = m_j$. Define $f : \overline{\mathcal{O}} \to \mathbb{R} \oplus V$ by $f(\lambda, v_1, \dots, v_s) = \Big(|\lambda|\big(\|v\| - 1\big) + \|v\| + 1, \lambda^{l_1} v_1, \dots, \lambda^{l_s} v_s\Big)$, where $\lambda \in \mathbb{C} \setminus \{0\}$, $v_j \in V_{k_j}$. Then*

$$S^1\text{-Deg}\,(f, \mathcal{O}) = \sum_{j=1}^{s} m_j l_j (\mathbb{Z}_{k_j}).$$

7. **Recurrence Formula.**

7.1. **Preliminaries.** In this section we present the so-called Recurrence Formula to provide the computation of the primary G-degree via appropriate S^1-degree and coefficients depending on G only. To formulate and prove the corresponding result (see Proposition 13), we will introduce/recall several notions and notations.

(a): The Number $\deg_1(f, \Omega)$. Let V be an orthogonal S^1-representation, $\Omega \subset \mathbb{R} \oplus V$ an open bounded S^1-invariant set, and $f : \mathbb{R} \oplus V \to V$ an Ω-admissible S^1-equivariant map. Consider the S^1-degree defined by (9) and put

$$\deg_{k_i}(f, \Omega) := n_{k_i}, \quad i = 1, 2, \dots, r.$$

This notation is motivated by the fact that each of the integer coefficients in (9) satisfies the usual additivity, homotopy, excision, and suspension properties.

(b): Primary Degree and Relative Bi-Orientable Orbit Types. In Subsection 3.3, we have indicated one reason for excluding relative bi-orientable orbit types from the construction of the primary G-degree. Another one rests on the fact that the inclusion of these types would lead, in general, to unnecessary complications related to the validity of other important properties of the primary degree (for instance, the so-called multiplicativity property (cf. [2, 3])). At the same time, for many important classes of groups appearing in applications (for instance, the so-called twisted groups (see [2, 3])), the appearance of relatively bi-orientable orbit types does not affect the computational formulae.

Nevertheless, potential applications of the equivariant degree, which are based on the Recurrence Formula, probably are not exhausted by the twisted groups. Therefore, for the sake of completeness we present the Recurrence Formula with the relatively bi-orientable orbit types being taken into account. To be more specific, assume G is a compact Lie group, V is an orthogonal G-representation, $\Omega \subset \mathbb{R}\oplus V$ an open bounded G-invariant subset and $f : \mathbb{R}\oplus V \to V$ an Ω-admissible G-equivariant map. We will assume that the primary degree $G\text{-Deg}\,(f,\Omega)$ is extended to the orbit types $(H) \in \tilde{\Phi}_1^+(G,V) \setminus \Phi_1^+(G,V)$ (to this end one should: (a) use the fact that $\Omega_H/W(H) = \Omega_{H_o}/W(H_o)$ (see Proposition 1), i.e. any regular fundamental domain for the $W(H_o)$-action on Ω_{H_o} is automatically a regular fundamental domain for the $W(H)$-action on Ω_H; (b) apply formula (4) to f_{H_o}). Thus,

$$G\text{-Deg}\,(f,\Omega) = \sum_{(H)\in\tilde{\Phi}_1^+(G,V)} n_H\ (H). \tag{27}$$

Further, since we assumed that there is chosen a fixed invariant orientation on $W(H)_o$ for every $(H) \in \tilde{\Phi}_1^+(G,V)$, S^1 can be canonically identified with the connected component of $1 \in W(H)_o$. Thus we have $S^1 \subset W(H)$ (in the case $G = \Gamma\times S^1$, the inclusion $S^1 \subset W(H)$ is in fact uniquely defined for twisted groups H) and also S^1 acts freely on Ω_H as a result of the free $W(H)$-action on Ω_H. Therefore, the restriction $f^H := f_{|\mathbb{R}\oplus V^H}$ is S^1-equivariant and has $(\mathbb{Z}_1)$ as its "smallest" orbit type with respect to the partial order defined on the set of all conjugacy classes of closed subgroups of $W(H)$.

(c): Result. Below we formulate the main result of this section which, to some extent, may be counted as the Borsuk-Ulam type Theorem in the case of one free parameter.

Proposition 13. (Recurrence Formula) *Let V be an orthogonal G-representation, $\Omega \subset \mathbb{R}\oplus V$ an open bounded invariant subset and $f : \mathbb{R}\oplus V \to V$ a G-equivariant Ω-admissible map. Then*

$$G\text{-Deg}\,(f,\Omega) = \sum_{(H)\in\tilde{\Phi}_1^+(G)} n_H \cdot (H),$$

where

$$n_H = \left[\deg_1(f^H,\Omega^H) - \sum_{(K)>(H)} n_K\, n(H,K)\,|W(K)/S^1|\right] \Big/ \left|W(H)/S^1\right|$$

and $f^H = f_{|\Omega^H}$.

Observe that a particular case of Proposition 13 was established in [22], where an argument based on the S^1-fixed point index was utilized.

7.2. **Proof of Proposition 13.** The proof of Proposition 13 is based on two lemmas below:

Lemma 5. *Let V, Ω and f be as in Proposition 13 and assume that f is regular normal and G-Deg(f,Ω) is given by (27). Then for $(H_o)\in\tilde{\Phi}_1^+(G,V)$*

$$n_{H_o}=\deg_1(f^{H_o},\Omega_{H_o})\Big/|W(H_o)/S^1|. \tag{28}$$

In other words, Lemma 5 states that the algebraic count of the $W(H_o)$-orbits of solutions for the equation $f^{H_o}(x)=0$ can be achieved by using the S^1-degree $\deg_1(f^{H_o},\Omega_{H_o})$ and purely algebraic characteristics depending on the group G only.

Proof: Let us consider an (H_o) in $\tilde{\Phi}_1^+(G,V)$. By the regular normality of f, the set of solution of $f^{H_o}(x)=0,\quad x\in\Omega_{H_o}$, is composed of a finite number of $W(H_o)$-orbits $W(H_o)(x_1)\cup\cdots\cup W(H_o)(x_k)$, where each of the orbits $W(H_o)(x_j)$, in turn, can be represented as a union of m copies of S^1-orbits, where $m=|W(H_o)/S^1|$, i.e. $W(H_o)(x_j)=S^1(x_{j,1})\cup\cdots\cup S^1(x_{j,m})$.

For each orbit $W(H_o)(x_j)$ we define the positive orientation on the slice S_{x_j} (cf. Remark and Definition 1). By formula (4),

$$n_{H_o}=\sum_{j=1}^{k}\text{sign det}\,Df^{H_o}(x_j)|_{S_{x_j}}. \tag{29}$$

Similarly,

$$\deg_1(f^{H_o},\Omega_{H_o})\sum_{j=1}^{n}\sum_{l=1}^{m}\text{sign det}\,Df^{H_o}(x_{j,l})|_{S_{x_{j,l}}}=m\sum_{j=1}^{k}\text{sign det}\,Df^{H_o}(x_j)|_{S_{x_j}}, \tag{30}$$

where $S_{x_{j,l}}$ denotes the slice to the S^1-orbit $S^1(x_{j,l})$. Comparing (29) and (30) yields (28). □

Lemma 6. *Let V, Ω and f be as in Lemma 5 and $(L)\in\tilde{\Phi}_1^+(V,G)$. Then*

$$\deg_1(f^L,\Omega^L)=\sum_{(H)\geq(L)}n(L,H)\,\deg_1(f^H,\Omega_H).$$

where $(H)\in\tilde{\Phi}_1^+(G,V)$.

Proof: Since f is regular normal and

$$V^L=\bigcup_{H\supset L}V_H,$$

it is clear that the set $Z:=(f^L)^{-1}(0)$ of zeros of f^L is such that $Z_H=Z\cap V_H$ is compact for every $(H)\in\Phi_1(G,V)$, $H\supset L$ (recall that $\Phi_1(G,V)$ stands for the set of all orbit types (H) in V such that $\dim W(H)=1$ with no additional bi-orientability requirement). Let $\mathcal{U}(Z_H)$ be an isolating neighborhood of Z_H in V^L and put $\mathcal{W}(Z_H):=\mathcal{U}(Z_H)\cap V_H$. Then, by normality of f, suspension and excision properties of the S^1-degree, it follows

$$\deg_1(f^L,\mathcal{U}(Z_H))=\deg_1(f^H,\mathcal{W}(Z_H))=\deg_1(f^H,\Omega_H).$$

Consequently, using the additivity of the S^1-degree and the geometric meaning of the numbers $n(L,H)$ (see Lemma 1), combined with Proposition 4, we obtain

$$\begin{aligned}\deg_1(f^L,\Omega^L) &= \sum_{H\supset L}\deg_1(f^L,\mathcal{U}(Z_H)) + \sum_{\tilde H\supset L}\deg_1(f^L,\mathcal{U}(Z_{\tilde H}))\\ &= \sum_{H\supset L}\deg_1(f^H,\Omega_H) + \sum_{\tilde H\supset L}\deg_1(f^{\tilde H},\Omega_{\tilde H})\\ &= \sum_{(H)\geq(L)} n(L,H)\,\deg_1(f^H,\Omega_H) + \sum_{(\tilde H)\geq(L)} n(L,\tilde H)\,\deg_1(f^{\tilde H},\Omega_{\tilde H}),\end{aligned}$$

where $\dim W(H)=1=\dim W(\tilde H)$, $W(H)_o$ is bi-orientable and $W(\tilde H)_o$ is not bi-orientable. However, the value of $\deg_1(f^{\tilde H},\Omega_{\tilde H})$ depends on an orientation of the $W(\tilde H)$-orbits in $\Omega_{\tilde H}$, which in this case is not uniquely determined. Therefore, by changing the orientation, instead of $\deg_1(f^{\tilde H},\Omega_{\tilde H})$, the value $-\deg_1(f^{\tilde H},\Omega_{\tilde H})$ can also be obtained. Consequently, we obtain the equality

$$\begin{aligned}&\sum_{(H)\geq(L)} n(L,H)\,\deg_1(f^H,\Omega_H) + \sum_{(\tilde H)\geq(L)} n(L,\tilde H)\,\deg_1(f^{\tilde H},\Omega_{\tilde H})\\ &= \sum_{(H)\geq(L)} n(L,H)\,\deg_1(f^H,\Omega_H) - \sum_{(\tilde H)\geq(L)} n(L,\tilde H)\,\deg_1(f^{\tilde H},\Omega_{\tilde H}),\end{aligned}$$

which implies

$$\sum_{(\tilde H)\geq(L)} n(L,\tilde H)\,\deg_1(f^{\tilde H},\Omega_{\tilde H}) = 0.$$

In this way, we obtain

$$\deg_1(f^L,\Omega^L) = \sum_{(H)\geq(L)} n(L,H)\,\deg_1(f^H,\Omega_H),$$

where $(H)\in\tilde\Phi_1^+(G,V)$. □

Proof of Proposition 13. By the homotopy property of the primary degree, we can assume without loss of generality, that f is a regular normal map in Ω.

Consider the fixed point space $\mathbb{R}\oplus V^{H_o}$ and the $W(H_o)$-equivariant restriction $f^{H_o}:\mathbb{R}\oplus V^{H_o}\to V^{H_o}$ of f. By Lemma 5, the number $n_{H_o}\cdot|W(H_o)/S^1|$ represents the S^1-degree $\deg_1(f^{H_o},\Omega_{H_o})$. On the other hand, by Lemma 6, we obtain

$$n_{H_o}\cdot|W(H_o)/S^1| = \deg_1(f^{H_o},\Omega^{H_o}) - \sum_{(K)>(H_o)} n(H_o,K)\,n_K\cdot|W(K)/S^1|,$$

where $(K)\in\tilde\Phi_1^+(G,V)$, and the result follows. □

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