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ScienceDirect
Nonlinear Analysis: Real World Applications 8 (2007) 1144 – 1170
Nonlinear Analysis
Real World Applications
www.elsevier.com/locate/na

Hopf bifurcation in a symmetric configuration of transmission lines

Zalman Balanov[a,*,1], Wieslaw Krawcewicz[b,2], Haibo Ruan[b]

[a]Department of Mathematics and Computer Sciences, Netanya Academic College, Netanya 42365, Israel
[b]Department of Mathematical and Statistical Sciences, University of Alberta, Edmonton, Alberta, Canada T6G 2G1

Received 26 April 2006; accepted 27 April 2006

Abstract

We apply the equivariant degree method to a Hopf bifurcation problem for a symmetric system of neutral functional differential equations, which reflects two symmetrically coupled configurations of the lossless transmission lines. The spectral information of the linearized system is extracted and translated into a bifurcation invariant, which carries structural information of the solution set. We calculate the values of the bifurcation invariant by following the standard computational scheme and using a specially developed Maple© package. The computational results, as well as the minimal number of bifurcating branches and their least symmetries are summarized.

MSC: primary 58E05; 58E09; secondary 35J20

1. Introduction

A tendency to create "elegant designs" very often results in the appearance of symmetries in related physical models (and, as a consequence, in mathematical models/dynamical systems). A priori it is clear that dynamical processes are typically less symmetric than the corresponding models. In addition, symmetries usually give rise to *multiple* solutions exhibiting *different* symmetry properties which, in turn, may have an enormous impact on the performance and predictability of the final project. For example, a symmetric configuration in transmission lines can increase dramatically the vulnerability of a network due to periodic fluctuations and surges. In this way we arrive at the following questions:

(a) What is a link between symmetries of a system and symmetric properties of the actual dynamics?
(b) How can one measure, predict and classify symmetric properties/minimal number of solutions to a system?

In the present paper, we illustrate how the new topological tools, such as the equivariant degree theory, provide a way to answer the above questions (a) and (b), and at the same time, help to better understand the topological nature of symmetric phenomena.

* Corresponding author.
E-mail addresses: balanov@mail.netanya.ac.il (Z. Balanov), wkrawcew@math.ualberta.ca (W. Krawcewicz), hruan@math.ualberta.ca (H. Ruan).
1 Research supported by the Alexander von Humboldt Foundation.
2 Research supported by a grant from the NSERC Canada.

1468-1218/$ - see front matter
doi:10.1016/j.nonrwa.2006.04.004

Table 1
Examples of the equivariant classification of the Hopf bifurcation with D_4 symmetries

E_j	$\varepsilon_0, \varepsilon_1, \varepsilon_3$	$\omega(\alpha_o, \beta_o)_1$	# Branches
E_0	0,1,1	$(D_4) - (\mathbb{Z}_4) - (D_1) - (\widetilde{D}_1) + (\mathbb{Z}_1)$	1
E_0	1,1,0	$-(D_4) + (D_1) + (\widetilde{D}_1) - (\mathbb{Z}_1)$	1
E_0	1,1,1	$-(D_4) + (\mathbb{Z}_4) + (D_1) + (\widetilde{D}_1) - (\mathbb{Z}_1)$	1
E_1	0,0,1	$-(\mathbb{Z}_4^t) + (D_2^d) + (\widetilde{D}_2^d) - (\mathbb{Z}_2^-)$	6
E_1	0,1,0	$(\mathbb{Z}_4^t) + (D_2^d) + (\widetilde{D}_2^d) - (\mathbb{Z}_2^-) - (D_1^z) - (\widetilde{D}_1^z) - (D_1) - (\widetilde{D}_1) + 2(\mathbb{Z}_1)$	6
E_1	0,1,1	$-(\mathbb{Z}_4^t) + (D_2^d) + (\widetilde{D}_2^d) - (\mathbb{Z}_2^-) - (D_1^z) - (\widetilde{D}_1^z) - (D_1) - (\widetilde{D}_1) + 2(\mathbb{Z}_1)$	6
E_1	1,1,0	$-(\mathbb{Z}_4^t) - (D_2^d) - (\widetilde{D}_2^d) + (\mathbb{Z}_2^-) + (D_1^z) + (\widetilde{D}_1{}^z) + (D_1) + (\widetilde{D}_1) - 2(\mathbb{Z}_1)$	6
E_1	1,1,1	$(\mathbb{Z}_4^t) - (D_2^d) - (\widetilde{D}_2^d) + (\mathbb{Z}_2^-) + (D_1^z) + (\widetilde{D}_1^z) + (D_1) + (\widetilde{D}_1) - 2(\mathbb{Z}_1)$	6
E_3	0,1,1	$(D_4^d) - (\mathbb{Z}_4^d) - (\widetilde{D}_1^z) - (D_1) + (\mathbb{Z}_1)$	2
E_3	1,0,1	$-(D_4^d) + (\mathbb{Z}_4^d)$	2
E_3	1,1,0	$-(D_4^d) + (\widetilde{D}_1^z) + (D_1) - (\mathbb{Z}_1)$	2
E_3	1,1,1	$-(D_4^d) + (\mathbb{Z}_4^d) + (\widetilde{D}_1^z) + (D_1) - (\mathbb{Z}_1)$	2

Table 2
Examples of the equivariant classification of the Hopf bifurcation with D_5 symmetries

E_j	$\varepsilon_0, \varepsilon_1, \varepsilon_2$	$\omega(\lambda_o)_1$	# Branches
E_0	1,0,1	$-(D_5) + 2(D_1) - (\mathbb{Z}_1)$	1
E_0	1,1,0	$-(D_5) + 2(D_1) - (\mathbb{Z}_1)$	1
E_0	1,1,1	$-(D_5)$	1
E_1	1,0,0	$-(\mathbb{Z}_5^{t_1}) - (D_1^z) - (D_1) + (\mathbb{Z}_1)$	8
E_1	1,0,1	$-(\mathbb{Z}_5^{t_1}) + (D_1^z) + (D_1) - (\mathbb{Z}_1)$	8
E_1	1,1,0	$-(\mathbb{Z}_5^{t_1}) + (D_1^z) + (D_1) - (\mathbb{Z}_1)$	8
E_1	1,1,1	$-(\mathbb{Z}_5^{t_1}) - (D_1^z) - (D_1) + (\mathbb{Z}_1)$	8
E_2	0,0,0	$(\mathbb{Z}_5^{t_2}) + (D_1^z) + (D_1) - (\mathbb{Z}_1)$	8
E_2	1,0,1	$-(\mathbb{Z}_5^{t_2}) + (D_1^z) + (D_1) - (\mathbb{Z}_1)$	8
E_2	1,1,0	$-(\mathbb{Z}_5^{t_2}) + (D_1^z) + (D_1) - (\mathbb{Z}_1)$	8
E_2	1,1,1	$-(\mathbb{Z}_5^{t_2}) - (D_1^z) - (D_1) + (\mathbb{Z}_1)$	8

Our effort to understand the impact of symmetries is connected to difficult topological problems underlying the foundation of the equivariant nonlinear analysis. The equivariant analysis deals with symmetric (or the so-called *equivariant*) operator equations, for which the existence, multiplicity, stability and topological structure of the solution set is analyzed by studying topological invariants associated with the corresponding operators. Such symmetric systems appear naturally (in a form of various types of differential equations) in mathematical models, related to chemical diffusion, biological oscillation and population dynamics. The equivariant degree theory, being one of the most effective methods of the equivariant analysis, as it turns out, enables us to “decipher” the mystery of symmetries. For example, the recently developed new equivariant degree techniques (cf. [3–5]) provide practical hints, which allow concrete computations for a wide range of applications. Moreover, the above techniques can be used by applied mathematicians without deep knowledge in equivariant topology and homotopy theory. The *goal* of the present paper is to study the Hopf bifurcation phenomena (i.e. the appearance of small amplitude nonconstant periodic solutions) in symmetrically coupled systems of the lossless transmission lines, by means of the equivariant degree method.

After the introduction the paper is organized as follows. Section 2 collects a minimum preliminary notions from the equivariant topology required for a proper presentation of the method. In Section 3, we establish a general framework for a treatment of neutral functional equations, especially an effective computational formula (cf. (22)), which translates the

Table 3
Examples of the equivariant classification of the Hopf bifurcation with S_4 symmetries

E_j	$\varepsilon_0, \varepsilon_1, \varepsilon_2, \varepsilon_3$	$\omega(\lambda_o)_1$	# Branches
E_0	0,0,1,1	$(S_4) - 2(D_3) - (D_2) - (\mathbb{Z}_4) + (\mathbb{Z}_3) + 2(D_1) + (\mathbb{Z}_2) - (\mathbb{Z}_1)$	1
E_0	0,1,0,0	$(S_4) - 2(D_4)$	1
E_0	0,1,0,1	$(S_4) - 2(D_4) + (\mathbb{Z}_4) - (\mathbb{Z}_3) + (D_1) + 2(\mathbb{Z}_2) - (\mathbb{Z}_1)$	1
E_0	1,0,0,0	$(S_4) - 2(D_4) - 2(D_3) + (D_2) + (\mathbb{Z}_4) + (\mathbb{Z}_3) + 2(D_1) - (\mathbb{Z}_2) - (\mathbb{Z}_1)$	1
E_0	1,0,0,1	$-(S_4) + (\mathbb{Z}_4) + (\mathbb{Z}_3) + (D_1) - (\mathbb{Z}_1)$	1
E_0	1,1,0,0	$-(S_4) + 2(D_4)$	1
E_0	1,1,0,1	$-(S_4) + 2(D_4) - (\mathbb{Z}_4) + (\mathbb{Z}_3) - (D_1) - 2(\mathbb{Z}_2) + (\mathbb{Z}_1)$	1
E_0	1,1,1,1	$-(S_4) + 2(D_4) + 2(D_3) - (D_2) - (\mathbb{Z}_4) - (\mathbb{Z}_3) - 2(D_1) + (\mathbb{Z}_2) + (\mathbb{Z}_1)$	1
E_1	0,0,0,0	(S_4^-)	1
E_1	0,1,0,0	$(S_4^-) - 2(D_4^{\hat{d}})$	4
E_1	0,1,1,0	$(S_4^-) - 2(D_4^{\hat{d}}) - 2(D_3^z) + (D_2^z) + (D_1^z) + 2(\mathbb{Z}_2) - (\mathbb{Z}_1)$	4
E_1	0,1,1,1	$(S_4^-) - 2(D_4^{\hat{d}}) - 2(D_3^z) + (D_2^z) + (\mathbb{Z}_4^-) + (\mathbb{Z}_3) + 2(D_1^z) - (\mathbb{Z}_2) - (\mathbb{Z}_1)$	4
E_1	1,0,1,0	$-(S_4^-) + 2(D_3^z) + (D_2^z) - 3(D_1^z) + (\mathbb{Z}_1)$	1
E_1	1,1,1,0	$-(S_4^-) + 2(D_4^{\hat{d}}) + 2(D_3^z) - (D_2^z) - (D_1^z) - 2(\mathbb{Z}_2) + (\mathbb{Z}_1)$	4
E_1	1,1,1,1	$-(S_4^-) + 2(D_4^{\hat{d}}) + 2(D_3^z) - (D_2^z) - (\mathbb{Z}_4^-) - (\mathbb{Z}_3) - 2(D_1^z) + (\mathbb{Z}_2) + (\mathbb{Z}_1)$	4
E_2	0,0,1,0	$(D_4^d) - (D_3) - (D_2^d) - (D_2) + (\mathbb{Z}_4^c) - (\mathbb{Z}_3^t) + (D_1^z) + 3(D_1) - (\mathbb{Z}_2^-) - (\mathbb{Z}_1)$	24
E_2	0,0,1,1	$(D_4^d) - (D_3) - (D_2^d) - (D_2) - (\mathbb{Z}_4^c) - (\mathbb{Z}_4^-) + (\mathbb{Z}_3^t) + (\mathbb{Z}_3) + (D_1) + (\mathbb{Z}_2^-) + (\mathbb{Z}_2) - (\mathbb{Z}_1)$	24
E_2	0,1,1,0	$-(D_4^d) - (D_3) + (D_2^d) + (D_2) - (\mathbb{Z}_4^c) - 2(V_4^-) - (\mathbb{Z}_3^t) - (D_1^z) - (D_1) + 5(\mathbb{Z}_2^-) + 2(\mathbb{Z}_2) - (\mathbb{Z}_1)$	24
E_2	0,1,1,1	$-(D_4^d) - (D_3) + (D_2^d) + (D_2) + (\mathbb{Z}_4^c) + (\mathbb{Z}_4^-) - 2(V_4^-) + (\mathbb{Z}_3^t) + (\mathbb{Z}_3) + (D_1) - (\mathbb{Z}_2^-) - (\mathbb{Z}_2) - (\mathbb{Z}_1)$	24
E_2	1,0,0,0	$-(D_4^d) - (D_3) - (D_2^d) - (\mathbb{Z}_4^c) - (\mathbb{Z}_3^t) + (D_1) + (\mathbb{Z}_2^-)$	24
E_2	1,0,1,0	$-(D_4^d) + (D_3) + (D_2^d) + (D_2) - (\mathbb{Z}_4^c) + (\mathbb{Z}_3^t) - (D_1^z) - 3(D_1) + (\mathbb{Z}_2^-) + (\mathbb{Z}_1)$	24
E_2	1,0,1,1	$-(D_4^d) + (D_3) + (D_2^d) + (D_2) + (\mathbb{Z}_4^c) + (\mathbb{Z}_4^-) - (\mathbb{Z}_3^t) - (\mathbb{Z}_3) - (D_1) - (\mathbb{Z}_2^-) - (\mathbb{Z}_2) + (\mathbb{Z}_1)$	24
E_2	1,1,0,0	$(D_4^d) - (D_3) + (D_2^d) + (\mathbb{Z}_4^c) + 2(V_4^-) - (\mathbb{Z}_3^t) + (D_1) - (\mathbb{Z}_2^-)$	24
E_2	1,1,1,1	$(D_4^d) + (D_3) - (D_2^d) - (D_2) - (\mathbb{Z}_4^c) - (\mathbb{Z}_4^-) + 2(V_4^-) - (\mathbb{Z}_3^t) - (\mathbb{Z}_3) - (D_1) + 2(\mathbb{Z}_2^-) + (\mathbb{Z}_2) + (\mathbb{Z}_1)$	24
E_3	0,0,1,1	$(D_4^z) - (D_3^z) - (D_2^d) - (D_2^z) - (\mathbb{Z}_4^c) - (\mathbb{Z}_4) + (\mathbb{Z}_3^t) + (\mathbb{Z}_3) + (D_1^z) + (\mathbb{Z}_2^-) + (\mathbb{Z}_2) - (\mathbb{Z}_1)$	27
E_3	0,1,0,0	$-(D_4^z) + (D_3^z) - (D_2^d) - (\mathbb{Z}_4^c) - 2(V_4^-) + (\mathbb{Z}_3^t) - (D_1^z) + (\mathbb{Z}_2^-)$	27
E_3	0,1,1,0	$-(D_4^z) - (D_3^z) + (D_2^d) + (D_2^z) - (\mathbb{Z}_4^c) - 2(V_4^-) - (\mathbb{Z}_3^t) - (D_1^z) - (D_1) + 5(\mathbb{Z}_2^-) + 2(\mathbb{Z}_2) - (\mathbb{Z}_1)$	27
E_3	0,1,1,1	$-(D_4^z) - (D_3^z) + (D_2^d) + (D_2^z) + (\mathbb{Z}_4^c) + (\mathbb{Z}_4) - 2(V_4^-) + (\mathbb{Z}_3^t) + (\mathbb{Z}_3) + (D_1^z) - (\mathbb{Z}_2^-) - (\mathbb{Z}_2) - (\mathbb{Z}_1)$	27
E_3	1,0,1,1	$-(D_4^z) + (D_3^z) + (D_2^d) + (D_2^z) + (\mathbb{Z}_4^c) + (\mathbb{Z}_4) - (\mathbb{Z}_3^t) - (\mathbb{Z}_3) - (D_1^z) - (\mathbb{Z}_2^-) - (\mathbb{Z}_2) + (\mathbb{Z}_1)$	27
E_3	1,1,0,0	$(D_4^z) - (D_3^z) + (D_2^d) + (\mathbb{Z}_4^c) + 2(V_4^-) - (\mathbb{Z}_3^t) + (D_1^z) - (\mathbb{Z}_2^-)$	27
E_3	1,1,1,0	$(D_4^z) + (D_3^z) - (D_2^d) - (D_2^z) + (\mathbb{Z}_4^c) + 2(V_4^-) + (\mathbb{Z}_3^t) + (D_1^z) + (D_1) - 5(\mathbb{Z}_2^-) - 2(\mathbb{Z}_2) + (\mathbb{Z}_1)$	27
E_3	1,1,1,1	$(D_4^z) + (D_3^z) - (D_2^d) - (D_2^z) - (\mathbb{Z}_4^c) - (\mathbb{Z}_4) + 2(V_4^-) - (\mathbb{Z}_3^t) - (\mathbb{Z}_3) - (D_1^z) + (\mathbb{Z}_2^-) + (\mathbb{Z}_2) + (\mathbb{Z}_1)$	27

spectral equivariant information provided by the characteristic equation into a bifurcation invariant expressed in terms of the equivariant degree. In Section 4, we derive and discuss two systems of symmetrically coupled (internally and externally) lossless transmission lines. Motivated by the two generic couplings, we consider in Section 5, a symmetric system of FDEs, for which a detailed analysis of the main steps preceding the computation of the associated Hopf bifurcation invariant is given. Section 6 summarizes concrete computational results (in the case of the dihedral groups D_n $(n = 4, 5)$, octahedral group S_4 and icosahedral group A_5) of the bifurcation invariants in tables (see Tables 1–4), where minimal numbers of bifurcating branches with their least symmetries are also indicated.

The authors are grateful to Eric Woolgar for his helpful discussions and suggestions related to the derivations of the transmission line models. The Maple routines used in this work were created by Adrian Biglands.

Table 4
Examples of the equivariant classification of the Hopf bifurcation with A_5 symmetries

E_j	$\varepsilon_0, \varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4$	$\omega(\lambda_o)_1$	# Branches
E_0	1,0,1,0,1	$-(A_5)+2(D_5)+2(D_3)-(\mathbb{Z}_5)-(\mathbb{Z}_3)-4(\mathbb{Z}_2)+2(\mathbb{Z}_1)$	1
E_0	1,1,1,0,1	$-(A_5)+2(A_4)+2(D_5)-(\mathbb{Z}_5)-2(\mathbb{Z}_3)-3(\mathbb{Z}_2)+2(\mathbb{Z}_1)$	1
E_1	0,0,0,0	$(A_4)+(D_3^z)+(D_3)+(\mathbb{Z}_5^{t_1})+(\mathbb{Z}_5^{t_2})+(V_4^-)+(\mathbb{Z}_3^t)-(\mathbb{Z}_3)-(\mathbb{Z}_2^-)-(\mathbb{Z}_2)$	55
E_1	0,0,1,0,0	$(A_4)-(D_3^z)-(D_3)-(\mathbb{Z}_5^{t_1})-(\mathbb{Z}_5^{t_2})+(V_4^-)-(\mathbb{Z}_3^t)-(\mathbb{Z}_3)$ $-(\mathbb{Z}_2^-)-(\mathbb{Z}_2)+2(\mathbb{Z}_1)$	55
E_1	0,0,1,1,0	$(A_4)-(D_3^z)-(D_3)+(\mathbb{Z}_5^{t_1})+(\mathbb{Z}_5^{t_2})+(V_4^-)+(\mathbb{Z}_3^t)+(\mathbb{Z}_3)$ $+(\mathbb{Z}_2^-)+(\mathbb{Z}_2)$	55
E_1	1,0,0,0,1	$-(A_4)-(D_3^z)-(D_3)+(\mathbb{Z}_5^{t_1})+(\mathbb{Z}_5^{t_2})-(V_4^-)+(\mathbb{Z}_3^t)+3(\mathbb{Z}_3)$ $+3(\mathbb{Z}_2^-)+3(\mathbb{Z}_2)-4(\mathbb{Z}_1)$	55
E_1	1,0,1,0,1	$-(A_4)+(D_3^z)+(D_3)-(\mathbb{Z}_5^{t_1})-(\mathbb{Z}_5^{t_2})-(V_4^-)-(\mathbb{Z}_3^t)-(\mathbb{Z}_3)$ $-(\mathbb{Z}_2^-)-(\mathbb{Z}_2)+2(\mathbb{Z}_1)$	55
E_2	0,0,0,0,0	$(A_4^{t_1})+(A_4^{t_2})+(D_5)+(D_3)+(\mathbb{Z}_5^{t_1})+(\mathbb{Z}_5^{t_2})+(V_4^-)-2(\mathbb{Z}_2)$	50
E_2	0,0,1,1,0	$(A_4^{t_1})+(A_4^{t_2})-(D_5)-(D_3)+(\mathbb{Z}_5^{t_1})+(\mathbb{Z}_5^{t_2})+(\mathbb{Z}_5)+(V_4^-)$ $+(\mathbb{Z}_3)+2(\mathbb{Z}_2)-2(\mathbb{Z}_1)$	50
E_2	0,1,0,1,0	$-(A_4^{t_1})-(A_4^{t_2})+(D_5)-(D_3)-(\mathbb{Z}_5^{t_1})-(\mathbb{Z}_5^{t_2})-(\mathbb{Z}_5)-(V_4^-)+(\mathbb{Z}_1)$	50
E_2	1,0,1,0,0	$-(A_4^{t_1})-(A_4^{t_2})+(D_5)+(D_3)+(\mathbb{Z}_5^{t_1})+(\mathbb{Z}_5^{t_2})-(V_4^-)+4(\mathbb{Z}_3^t)$ $+2(\mathbb{Z}_2^-)+(\mathbb{Z}_2)-3(\mathbb{Z}_1)$	50
E_3	0,0,0,1,0	$(D_5^z)+(D_3^z)-(\mathbb{Z}_5^{t_1})-(\mathbb{Z}_5)+(V_4^-)-(\mathbb{Z}_3^t)-(\mathbb{Z}_3)-4(\mathbb{Z}_2^-)-(\mathbb{Z}_2)+3(\mathbb{Z}_1)$	44
E_3	0,0,1,0,0	$-(D_5^z)-(D_3^z)-(\mathbb{Z}_5^{t_1})+(V_4^-)-(\mathbb{Z}_3^t)-(\mathbb{Z}_2)+(\mathbb{Z}_1)$	44
E_3	0,1,0,1,0	$(D_5^z)-(D_3^z)-(\mathbb{Z}_5^{t_1})-(V_4^-)-(\mathbb{Z}_3^t)+(\mathbb{Z}_1)$	44
E_3	1,0,0,1,1	$-(D_5^z)-(D_3^z)-(\mathbb{Z}_5^{t_1})-(V_4^-)-(\mathbb{Z}_3^t)+2(\mathbb{Z}_2^-)$	44
E_3	1,0,1,0,0	$(D_5^z)+(D_3^z)+(\mathbb{Z}_5^{t_1})-(V_4^-)+(\mathbb{Z}_3^t)+(\mathbb{Z}_2)-(\mathbb{Z}_1)$	44
E_3	1,1,1,1,0	$(D_5^z)-(D_3^z)-(\mathbb{Z}_5^{t_1})-(\mathbb{Z}_5)+(V_4^-)-(\mathbb{Z}_3^t)-2(\mathbb{Z}_2^-)-(\mathbb{Z}_2)+2(\mathbb{Z}_1)$	44
E_4	0,0,0,1,0	$(D_5^z)+(D_3^z)-(\mathbb{Z}_5^{t_2})-(\mathbb{Z}_5)+(V_4^-)-(\mathbb{Z}_3^t)-(\mathbb{Z}_3)-4(\mathbb{Z}_2^-)-(\mathbb{Z}_2)+3(\mathbb{Z}_1)$	44
E_4	0,0,1,0,0	$-(D_5^z)-(D_3^z)-(\mathbb{Z}_5^{t_2})+(V_4^-)-(\mathbb{Z}_3^t)-(\mathbb{Z}_2)+(\mathbb{Z}_1)$	44
E_4	0,1,0,1,0	$(D_5^z)-(D_3^z)-(\mathbb{Z}_5^{t_2})-(V_4^-)-(\mathbb{Z}_3^t)+(\mathbb{Z}_1)$	44
E_4	1,0,0,1,1	$-(D_5^z)-(D_3^z)-(\mathbb{Z}_5^{t_2})-(V_4^-)-(\mathbb{Z}_3^t)+2(\mathbb{Z}_2^-)$	44
E_4	1,0,1,0,0	$(D_5^z)+(D_3^z)+(\mathbb{Z}_5^{t_2})-(V_4^-)+(\mathbb{Z}_3^t)+(\mathbb{Z}_2)-(\mathbb{Z}_1)$	44
E_4	1,1,1,1,0	$(D_5^z)-(D_3^z)-(\mathbb{Z}_5^{t_2})-(\mathbb{Z}_5)+(V_4^-)-(\mathbb{Z}_3^t)-2(\mathbb{Z}_2^-)-(\mathbb{Z}_2)+2(\mathbb{Z}_1)$	44

2. Preliminaries

2.1. Definitions and notations

Hereafter, $G=\Gamma\times S^1$, where Γ is a finite group and S^1 is the unit circle group. For a closed subgroup H of G, we denote by (H) the conjugacy class of H in G, $N(H)$—the normalizer of H in G, $W(H)=N(H)/H$—the Weyl group of H in G, and $\Phi(G)$—the set of all conjugacy classes in G (which admits a natural partial order: $(K)\leqslant(H)$ if K is conjugate to a subgroup of H).

Let V be an orthogonal (or isometric Banach) G-representation. For $x\in V$, denote by $G_x=\{g\in G: gx=x\}$ the *isotropy group* of x and call the conjugacy class (G_x) the *orbit type* of x in V. For a G-invariant subset $X\subset V$, put $X^H:=\{x\in X: G_x\supset H\}$ and call it the *H-fixed point subspace*.

Let Ω be an open bounded G-invariant subset of $\mathbb{R}\oplus V$, where we will always assume the trivial G-action on $\mathbb{R}$, and $f:\mathbb{R}\oplus V\to V$ a continuous equivariant map in Ω, meaning $f(gx)=gf(x)$ for all $g\in G$ and $x\in\overline{\Omega}$. f is called Ω-*admissible* if $f(x)\neq 0$ for all $x\in\partial\Omega$, and such a pair (f,Ω) will be called an *admissible pair*. Similarly, a homotopy $h : [0, 1]\times\mathbb{R}\oplus V \to V$ is called an Ω-*admissible* G-equivariant homotopy, if $h_t := h(t, \cdot)$ is an Ω-admissible G-equivariant map for all $t \in [0, 1]$.

2.2. *Primary equivariant degree with one free parameter*

Consider the set

$$\Phi_1(G) := \{(H) \in \Phi(G) : \dim W(H) = 1\}.$$

It is easy to check that the elements of $\Phi_1(G)$ are the conjugacy classes (H) of the so-called *φ-twisted l-folded* subgroups of $\Gamma \times S^1$ with $l = 1, 2, 3, \dots$, i.e.

$$H = K^{\varphi,l} := \{(\gamma, z) \in K \times S^1 : \varphi(\gamma) = z^l\},$$

where K is a subgroup of Γ and $\varphi : K \to S^1$ is a homomorphism. In the case of a 1-folded φ-twisted subgroup $K^{\varphi,1}$, we will denote it by K^φ and call it simply a *twisted* subgroup of $\Gamma \times S^1$.

Denote by

$$A_1(G) := \mathbb{Z}[\Phi_1(G)]$$

the free $\mathbb{Z}$-module generated by $\Phi_1(G)$, i.e. any element $\alpha \in A_1(G)$ can be written as a finite sum $\alpha = n_{H_1}(H_1) + n_{H_2}(H_2) + \cdots + n_{H_r}(H_r)$, $n_{H_i} \in \mathbb{Z}$.

The $\mathbb{Z}$-module $A_1(G)$ is a range of values of the so-called *primary equivariant degree* G-Deg defined on admissible pairs (f, Ω) ($\Omega \subset \mathbb{R} \oplus V$ and V is an orthogonal G-representation) and satisfying all the standard properties required from a "reasonable" degree theory. Moreover, the primary equivariant degree theory admits an axiomatic approach (see [5,3] for details). Being limited in size, we will list below only those properties which are directly referred to in the present paper.

- *Existence*: If $G\text{-Deg}(f, \Omega) = \sum_{(H)} n_H(H)$ is such that $n_{H_o} \neq 0$ for some $(H_o) \in \Phi_1(G)$, then there exists $x_o \in \Omega$ with $f(x_o) = 0$ and $G_{x_o} \supset H_o$.
- *Homotopy*: Suppose that $h : [0, 1] \times \mathbb{R} \oplus V \to V$ is an Ω-admissible G-equivariant homotopy. Then, G-Deg $(h_t, \Omega) =$ constant, where $h_t := h(t, \cdot)$.
- *Multiplicativity*: Let $A(\Gamma)$ denote the Burnside ring of Γ (see [12,16]). There exists a multiplication: $A(\Gamma) \times A_1(G) \to A_1(G)$ such that for an orthogonal Γ-representation V_o and a continuous equivariant map $f_o : V_o \to V_o$, one has

  $$G\text{-Deg}(f \times f_o, \Omega \times \mathscr{B}) = \Gamma\text{-Deg}(f_o, \mathscr{B}) \cdot G\text{-Deg}(f, \Omega),$$

  where $\mathscr{B} \subset V_o$ is the unit ball, $f_o(x) \neq 0$ for $x \in \partial B$ and Γ-Deg stands for the equivariant degree without free parameters (see [12] for details).

Remark 2.1. (i) The so-called *basic maps* $\mathfrak{b} : \mathbb{R}\oplus\mathscr{V} \to \mathscr{V}$, associated with orthogonal irreducible G-representations $\mathscr{V}$ (with nontrivial S^1-action), are the simplest homotopically nontrivial equivariant maps for which G-Deg can be easily evaluated (cf. [5,2,3]). To be more specific, define $\mathscr{O} := \{(t, v) \in \mathbb{R}\oplus\mathscr{V} : -1 < t < 1,\ \|v\| < 2\}$ and $\mathfrak{b} : \overline{\mathscr{O}} \to \mathscr{V}$ by

$$\mathfrak{b}(t, v) := (1 - \|v\| + it) \cdot v, \quad (t, v) \in \mathbb{R} \oplus \mathscr{V},$$

and call the primary degree

$$\deg_{\mathscr{V}} := G\text{-Deg}(\mathfrak{b}, \mathscr{O})$$

the *basic degree* associated with the irreducible G-representation $\mathscr{V}$. The same notion can be applied to the case without free parameter. Namely, we call $\deg_{\mathscr{V}_o} := \Gamma\text{-Deg}(-\mathrm{Id}, \mathscr{B})$ (where $\mathscr{B} \subset V_o$ is the unit ball) the *basic degree* associated with the irreducible Γ-representation $\mathscr{V}_o$.

(ii) By adopting the notion of *fundamental set* (cf. [1]) to the equivariant context, the concept of the primary equivariant degree can be extended to admissible pairs (f, Ω) with $\Omega \subset \mathbb{R} \oplus W$ and $f : \mathbb{R} \oplus W \to W$ being a *condensing vector field* on the Banach G-representation $\mathbb{R} \oplus W$ (cf. [9,12]). We will use for it the same symbol.

3. Symmetric Hopf bifurcation for neutral functional differential equations

We present a general framework for studying symmetric Hopf bifurcation problems for a system of neutral functional differential equations.

Let V be an orthogonal Γ-representation and $\tau \geqslant 0$ a given constant. Denote by $C_{V,\tau}$ the Banach space of continuous functions from $[-\tau, 0]$ into V equipped with the usual supremum norm $\|\varphi\|_\infty = \sup_{-\tau \leqslant \theta \leqslant 0} |\varphi(\theta)|$, $\varphi \in C_{V,\tau}$. Note that the Γ-action on V induces a natural isometric Banach Γ-representation on $C_{V,\tau}$ (as well as on $\mathbb{R} \oplus C_{V,\tau}$) given by

$$(\gamma\varphi)(\theta) := \gamma(\varphi(\theta)), \quad \gamma \in \Gamma, \quad \theta \in [-\tau, 0].$$

For a continuous function $x : \mathbb{R} \to V$ and $t \in \mathbb{R}$, define $x_t \in C_{V,\tau}$ by

$$x_t(\theta) = x(t + \theta), \quad \theta \in [-\tau, 0].$$

We consider the following one-parameter family of *neutral equations*:

$$\frac{\mathrm{d}}{\mathrm{d}t}[x(t) - b(\alpha, x_t)] = f(\alpha, x_t), \tag{1}$$

where $x : \mathbb{R} \to V$ is a continuous function,[3] and $b, f : \mathbb{R} \oplus C_{V,\tau} \to V$ satisfy the following assumptions:

(A1) b, f are continuously differentiable;
(A2) b, f are Γ-equivariant;
(A3) $b(\alpha, 0) = 0$, $f(\alpha, 0) = 0$ for all $\alpha \in \mathbb{R}$.
Also, to prevent the occurrence of the steady-state bifurcation, assume
(A4) $\det D_x f(\alpha, 0)|_V \neq 0$ for all $\alpha \in \mathbb{R}$.

In addition, assume that

(A5) b satisfies the Lipschitz condition with respect to the second variable, i.e.

$$\exists_\kappa \ 0 \leqslant \kappa < 1, \quad \text{s.t. } \|b(\alpha, \varphi) - b(\alpha, \psi)\| \leqslant \kappa \|\varphi - \psi\|_\infty \tag{2}$$

for all $\varphi, \psi \in C_{V,\tau}$, $\alpha \in \mathbb{R}$.

For $x_o \in V$, we will use the same symbol to denote the constant function $x_o(t) = x_o$. We call $(\alpha, x_o) \in \mathbb{R} \oplus V$ a *stationary* point to (1), if $f(\alpha, x_o) = 0$. By assumption (A3), $(\alpha, 0)$ is a stationary point for all $\alpha \in \mathbb{R}$. A stationary point (α, x_o) is said to be *nonsingular* if $D_x f(\alpha, x_o) I_V : V \to V$ is nonsingular.

3.1. Characteristic equation

In what follows, V^{c} denotes a complexification of the orthogonal Γ-representation V over $\mathbb{R}$. Then V^{c} has a natural structure of a complex Γ-representation given by $\gamma(z \otimes x) = z \otimes \gamma x$, for $z \in \mathbb{C}$ and $x \in V$. Also, a Γ-isotypical decomposition of the real representation V

$$V = V_0 \oplus V_1 \oplus \cdots \oplus V_r, \tag{3}$$

where $V_0 = V^\Gamma$ and V_i is modeled on the real irreducible Γ-representation $\mathcal{V}_i$, gives rise to a Γ-isotypical decomposition of the complex representation V^{c}

$$V^{\mathrm{c}} = U_0 \oplus U_1 \oplus \cdots \oplus U_s, \tag{4}$$

where $U_0 = (V^{\mathrm{c}})^\Gamma$ and U_j is modeled on the complex irreducible Γ-representation $\mathcal{U}_j$. Note that the number s of isotypical components in (4) is different in general, from the number r of isotypical components in (3), depending on the type of the irreducible representations $\mathcal{U}_j$ (cf. [6]).

[3] Formally speaking, we do not require in (1) that $x(t)$ is differentiable, but only $x(t) - b(\alpha, x_t)$ to be continuously differentiable.

Let (α, x_o) be a stationary point of (1). The linearization of (1) at (α, x_o) leads to the following *characteristic equation*,

$$\det_{\mathbb{C}} \triangle_{(\alpha,x_o)}(\lambda) = 0, \tag{5}$$

where

$$\triangle_{(\alpha,x_o)}(\lambda) := \lambda[\mathrm{Id} - D_x b(\alpha, x_o)(\mathrm{e}^{\lambda\cdot}\cdot)] - D_x f(\alpha, x_o)(\mathrm{e}^{\lambda\cdot}\cdot) \tag{6}$$

is a complex linear operator from V^{c} to V^{c}, with $(\mathrm{e}^{\lambda\cdot}\cdot)(\theta, x) = \mathrm{e}^{\lambda\theta}x$ and $D_x f(\alpha, x_o)(z \otimes x) = z \otimes D_x f(\alpha, x_o)x$ for $z \otimes x \in V^{\mathrm{c}}$ (cf. [17]).

A solution λ_o to (5) is called a *characteristic root* of system (1) at the stationary point (α, x_o). It is clear that (α, x_o) is nonsingular if and only if 0 is not a characteristic root of (1) at the stationary point (α, x_o). We say that a nonsingular stationary point (α, x_o) is a *center* if it has a purely imaginary characteristic root. A center (α, x_o) is called *isolated* if it is the only center in some neighborhood of (α, x_o) in $\mathbb{R} \oplus V$.

Put $\triangle_\alpha(\lambda) := \triangle_{(\alpha,0)}(\lambda)$. By assumption (A2), the complex linear operator $\triangle_\alpha(\lambda)$ is also Γ-equivariant. Consequently, for each isotypical component U_j of V^{c} in (4), we have $\triangle_\alpha(\lambda)(U_j) \subseteq U_j$. Denote

$$\triangle_{\alpha,j}(\lambda) := \triangle_\alpha(\lambda)|_{U_j}. \tag{7}$$

Let λ be a characteristic root of system (1) at the stationary point $(\alpha, 0)$. We will use the following notations:

$$\begin{aligned} &E_j(\lambda) := \ker \triangle_{\alpha,j}(\lambda) \subset V^{\mathrm{c}}, \\ &m_j(\lambda) := \dim_{\mathbb{C}} E_j(\lambda)/\dim_{\mathbb{C}} \mathscr{U}_j. \end{aligned} \tag{8}$$

The integer $m_j(\lambda)$ will be called the $\mathscr{U}_j$*-multiplicity* of the characteristic root λ.

In what follows, we will assume that

(A6) System (1) has an isolated center $(\alpha_o, 0)$ for some $\alpha_o \in \mathbb{R}$, with the corresponding purely imaginary characteristic root $i\beta_o$, for $\beta_o > 0$.

With the above definitions in hands, we will discuss the Hopf bifurcation problem for Eq. (1). Namely, we are interested in detecting and classifying (according to their symmetry properties) branches of small amplitude nonconstant periodic solutions to (1) bifurcating from the isolated center $(\alpha_o, 0)$.

To this end, we will associate to the problem in question the so-called *local bifurcation invariant*, being a primary equivariant degree, and follow the standard steps of the degree-theoretical treatment of the Hopf bifurcation phenomenon.

3.2. Normalization of period

By making a change of variable $x(t) = u(\beta t)$ with $\beta := 2\pi/p$, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}[u(t) - b(\alpha, u_{t,\beta})] = \frac{1}{\beta} f(\alpha, u_{t,\beta}), \tag{9}$$

where $u_{t,\beta} \in C_{V,\tau}$ is defined by

$$u_{t,\beta}(\theta) = u(t + \beta\theta), \quad \theta \in [-\tau, 0].$$

Evidently, $u(t)$ is a 2π-periodic solution of (9) if and only if $x(t)$ is a p-periodic solution of (1).

3.3. $\Gamma \times S^1$-equivariant setting in functional spaces

We use the standard identification $S^1 \simeq \mathbb{R}/2\pi\mathbb{Z}$ and denote $W := H^1(S^1; V)$ the first Sobolev space of 2π-periodic V-valued functions, $C(S^1; V)$ the space of continuous 2π-periodic V-valued functions equipped with the usual supremum norm. Note that the space W is an isometric Hilbert G-representation with the action given by

$$((\gamma, e^{i\tau})u)(t) = \gamma u(e^{i\tau}t) \quad \text{for } \gamma \in \Gamma,\ e^{i\tau} \in S^1.$$

Put $\mathbb{R}^2_+ := \mathbb{R} \times \mathbb{R}_+$.

For $u \in W$, $v \in C(S^1; V)$, $t \in \mathbb{R}$, define the following operators:

$$\begin{aligned}
&L : W \to L^2(S^1; V), \quad Lu(t) = \dot{u}(t),\\
&j : W \to C(S^1; V), \quad j(u(t)) = \widetilde{u}(t),\\
&K : W \to L^2(S^1; V), \quad Ku(t) = \frac{1}{2\pi}\int_0^{2\pi} u(s)\,ds,\\
&N_f : \mathbb{R}^2_+ \times C(S^1; V) \to L^2(S^1; V), \quad N_f(\alpha, \beta, v)(t) = f(\alpha, v_{t,\beta}),\\
&N_b : \mathbb{R}^2_+ \times C(S^1; V) \to L^2(S^1; V), \quad N_b(\alpha, \beta, v)(t) = b(\alpha, v_{t,\beta}).
\end{aligned}$$

where $\tilde{u} = u$ a.e. (cf. [14]). It can be easily shown that $(L+K)^{-1} : L^2(S^1; V) \to W$ exists and the map $\mathscr{F} : \mathbb{R}^2_+ \times W \to W$ given by

$$\mathscr{F}(\alpha, \beta, u) = (L+K)^{-1}\left[\frac{1}{\beta} N_f(\alpha, \beta, u) + K(u - N_b(\alpha, \beta, u))\right] + N_b(\alpha, \beta, u) \tag{10}$$

is a condensing map. Indeed, the map $\mathscr{F}$ is a sum of two maps, where the first map

$$(\alpha, \beta, u) \mapsto (L+K)^{-1}\left[\frac{1}{\beta} N_f(\alpha, \beta, u) + K(u - N_b(\alpha, \beta, u))\right],$$

is completely continuous, and the second map $(\alpha, \beta, u) \mapsto N_b(\alpha, \beta, u)$ is a Banach contraction with constant κ $(0 \leqslant \kappa < 1)$ (see assumptions (A1) and (A5)).

Remark 3.1. Note that, $(\alpha, \beta, u) \in \mathbb{R}^2_+ \times W$ is a 2π-periodic solution to (9) if and only if $u = \mathscr{F}(\alpha, \beta, u)$. Consequently, the occurrence of a Hopf bifurcation at $(\alpha_o, 0)$ for Eq. (1) is equivalent to a bifurcation of 2π-periodic solutions for (9) from $(\alpha_o, \beta_o, 0)$ for some $\beta_o > 0$. On the other hand, if a bifurcation at $(\alpha_o, \beta_o, 0) \in \mathbb{R}^2_+ \times W$ takes place in (9), then we *necessarily* have that the operator $\mathrm{Id} - D_u\mathscr{F}(\alpha_o, \beta_o, 0) : W \to W$ is not an isomorphism, or equivalently, $il\beta_o$, for some $l \in \mathbb{N}$, is a purely imaginary characteristic root of $(\alpha_o, 0)$, i.e. $\det_{\mathbb{C}} \triangle_{\alpha_o}(il\beta_o) = 0$ (cf. conditions (A4) and (A6)).

3.4. Dominating orbit types

In order to take advantage of the information provided by the local bifurcation invariant we need to introduce the following important concept.

Definition 3.2. An orbit type (H) in W is called *dominating*, if (H) is maximal (with respect to the usual order relation (see Section 2.1)) in the class of all φ-twisted one-folded orbit types in W (in particular, $H = K^{\varphi}$).

In what follows, the dominating orbit types will be used to estimate the *minimal number* of different periodic solutions (as well as their *symmetries*) to system (1) (see Theorem 3.4).

Remark 3.3. (i) Assume there is a solution $u_o \in W$ to (9) (for $\alpha = \alpha_o$ and some $\beta > 0$), for which one has $G_{u_o} \supset H_o$. If (H_o) is a dominating orbit type in W with $H_o = K^{\varphi}$ for some $K \subset \Gamma$ and $\varphi : K \to S^1$, then, by the maximality condition, $(G_{u_o}) = (K^{\varphi,l})$ with $l \geqslant 1$, and the corresponding orbit $G(u_o)$ is composed of exactly $|G/G_{u_o}|_{S^1}$ different periodic functions (where $|Y|_{S^1}$ denotes the number of S^1-orbits in Y). It is easy to check that the number of S^1-orbits in G/G_{u_o} is $|\Gamma/K|$ (where $|X|$ stands for the number of elements in X).

On the other hand, let x_o be a, say, p-periodic solution to (1) canonically corresponding to the above u_o. It follows from the definition of l-folding and $\Gamma \times S^1$-action on W that x_o is also a p/l-periodic solution to (1). The pair $(x_o, p/l)$

canonically determines an element $u'_o \in W$ being a solution to (9) (for $\alpha = \alpha_o$ and some β') satisfying the condition $G_{u'_o} = H_o$. In this way we obtain that (1) has at least $|\Gamma/K|$ different periodic solutions with the orbit type *exactly* (H_o) (considered in W).

(ii) Due to the maximality property of dominating orbit types and the fact that the isotropy groups increase under projections, the dominating orbit types can be easily recognized from the isotropy lattices of the irreducible representations of W.

3.5. Sufficient condition for Hopf bifurcation

For convenience, we identify $\mathbb{R}^2_+$ with a subset of $\mathbb{C}$, and given $(\alpha, \beta) \in \mathbb{R}^2_+$ put $\lambda = \alpha + i\beta$ (in particular, $\lambda_o = \alpha_o + i\beta_o$). Take an isolated center $(\alpha_o, 0)$ provided by condition (A6). Then, there exists $\delta > 0$ such that

$$a(\lambda) := \mathrm{Id} - D_u\mathscr{F}(\lambda, 0) : W \to W \tag{11}$$

is an isomorphism for $0 < |\lambda - \lambda_o| \leqslant \delta$. By implicit function theorem, there exists $\rho \in (0, \min\{1, \delta\})$ such that $u - \mathscr{F}(\lambda, u) \neq 0$ for all (λ, u) satisfying $|\lambda - \lambda_o| = \delta$ and $0 < \|u\| \leqslant \rho$.

Define the subset $\Omega \subset \mathbb{R}^2_+ \times W$ by

$$\Omega := \{(\lambda, u) \in \mathbb{R}^2_+ \times W : |\lambda - \lambda_o| < \delta,\ \ \|u\| < \rho\} \tag{12}$$

and put

$$\partial_0 := \overline{\Omega} \cap (\mathbb{R}^2_+ \times \{0\}) \quad \text{and} \quad \partial_\rho := \{(\lambda, u) \in \overline{\Omega} : \|u\| = \rho\}.$$

Next, take a G-invariant function $\varsigma : \overline{\Omega} \to \mathbb{R}$ satisfying the conditions:

$$\begin{cases} \varsigma(\lambda, u) > 0 & \text{for } (\lambda, u) \in \partial_\rho, \\ \varsigma(\lambda, u) < 0 & \text{for } (\lambda, u) \in \partial_0. \end{cases}$$

(Following Ize, such a function is usually called *auxiliary*). An auxiliary function can be easily constructed, for example,

$$\varsigma(\lambda, u) = |\lambda - \lambda_o|(\|u\| - \rho) + \|u\| - \frac{\rho}{2}; \quad (\lambda, u) \in \overline{\Omega}. \tag{13}$$

Define the map $\mathfrak{F}_\varsigma : \overline{\Omega} \to \mathbb{R} \oplus W$ by

$$\mathfrak{F}_\varsigma(\lambda, u) = (\varsigma(\lambda, u), u - \mathscr{F}(\lambda, u)), \quad (\lambda, u) \in \overline{\Omega} \tag{14}$$

(see formula (10) and Fig. 1).

By construction, $\mathfrak{F}_\varsigma$ is G-equivariant and Ω-admissible condensing field. Put (cf. Remark 2.1(ii))

$$\omega(\lambda_o) := G\text{-Deg}(\mathfrak{F}_\varsigma, \Omega) \in A_1(G), \tag{15}$$

and call $\omega(\lambda_o)$ the *local G-invariant* for the Γ-symmetric Hopf bifurcation at the point $(\lambda_o, 0)$.

Following the same ideas as in the proof of the Theorem 3.2 in [7] (see also [10,5]), one can easily establish:

Theorem 3.4. *Given system* (1), *assume conditions* (A1)–(A6) *to be satisfied. Take* $\mathscr{F}$ *defined by* (10) *and construct* Ω *according to* (12). *Let* $\varsigma : \overline{\Omega} \to \mathbb{R}$ *be a G-invariant auxiliary function* (*see* (13)) *and let* $\mathfrak{F}_\varsigma$ *be defined by* (14).

(i) *Assume* (*cf.* (15)) $\omega(\lambda_o) \neq 0$, *i.e.*

$$\omega(\lambda_o) = \sum_{(H)} n_H(H) \quad \text{and} \quad n_{H_o} \neq 0 \tag{16}$$

for some $(H_o) \in \Phi_1(G)$. *Then, there exists a branch of nontrivial solutions to* (1) *bifurcating from the point* $(\alpha_o, 0)$ (*with the limit frequency* $l\beta_o$ *for some* $l \in \mathbb{N}$). *More precisely, the closure of the set composed of all*

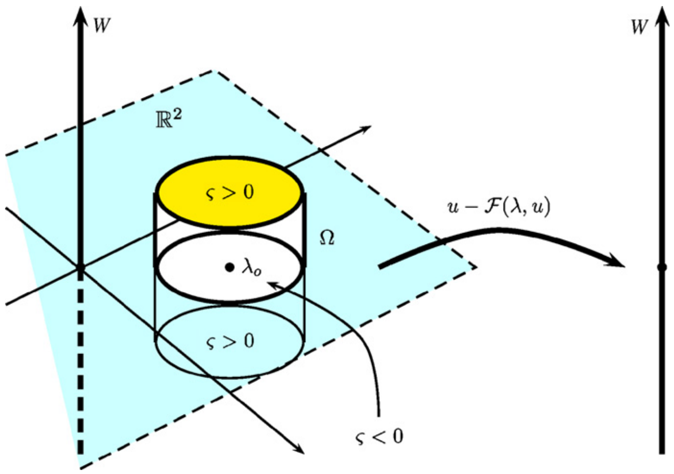

Fig. 1. Auxiliary function for Hopf bifurcation.

nontrivial solutions $(\lambda, u) \in \Omega$ *to* (9), *i.e.*

$$\overline{\{(\lambda, u) \in \Omega\colon\ \mathfrak{F}(\lambda, u) = 0,\ u \neq 0\}}$$

contains a compact connected subset C *such that*

$$(\lambda_o, 0) \in C \quad \text{and} \quad C \cap \partial_r \neq \emptyset, \quad C \subset \mathbb{R}^2_+ \times W^{H_o},$$

$(\lambda_o = \alpha_o + i\beta_o)$ *which, in particular, implies that for every* $(\alpha, \beta, u) \in C$ *we have* $G_u \supset H_o$.

(ii) *If, in addition,* (H_o) *is a dominating orbit type in* W, *then there exist at least* $|G/H_o|_{S^1}$ *different branches of periodic solutions to Eq.* (1) *bifurcating from* $(\alpha_o, 0)$ (*with the limit frequency* $l\beta_o$ *for some* $l \in \mathbb{N}$). *Moreover, for each* (α, β, u) *belonging to these branches of* (*nontrivial*) *solutions one has* $(G_u) = (H_o)$ (*considered in the space* W).

Remark 3.5. (i) It is usually the case that there is more than one dominating orbit type in W contributing to the lower estimate of all bifurcating branches of solutions.

(ii) In addition, if there is also a coefficient $n_K \neq 0$ such that (K) is not a dominating orbit type, but $n_H = 0$ for all dominating orbit types (H) such that $(K) < (H)$, then we can also predict the existence of multiple branches by analyzing all the dominating orbit types (H) larger than (K). However, the exact orbit type of these branches (as well as the corresponding estimate) cannot be determined precisely.

To apply Theorem 3.4 to classify symmetries of periodic solutions to concrete symmetric FDEs, we use a sequence of reductions that allows us to establish an effective formula for computing/estimating $\omega(\lambda_o)$.

3.6. Computation of the local $\Gamma \times S^1$-invariant and its first coefficients

We will present a computational formula of the local $\Gamma \times S^1$-invariant $\omega(\lambda_o)$, based on the homotopy and multiplicativity properties of the primary equivariant degree (cf. Section 2.2). For details and justification of the derivation, we refer to [3].

Using the standard linearization argument and homotopy property of the primary equivariant degree, it can be verified that

$$\omega(\lambda_o) = G\text{-Deg}(\mathfrak{F}_o, \Omega), \tag{17}$$

where $\mathfrak{F}_o(\lambda, u_o) := (\varsigma(\lambda, u_o), u_o - D_u\mathscr{F}(\lambda, 0)u_o)$ and $\mathscr{F}$ is defined by (10).

On the other hand, the Hilbert $\Gamma \times S^1$-representation $W = H^1(S^1; V)$ permits an S^1-isotypical decomposition

$$W = V \oplus \overline{\bigoplus_{l=1}^{\infty} W_l}, \tag{18}$$

where $V := W^{S^1}$ stands for the subspace of the constant functions in W, which is naturally a real Γ-representation, and the subspace W_l is defined by

$$W_l := \{\mathrm{e}^{ilt}(x_n + iy_n) : x_n, y_n \in V\},$$

which is isomorphic to V^{c} as a complex Γ-representation. Taking into account (3), (4) and (18), we have the following G-isotypical decomposition of W:

$$W = \bigoplus_{i=0}^{r} V_i \oplus \overline{\bigoplus_{j,l} V_{j,l}}, \tag{19}$$

where $V_{j,l}$ stands for a G-isotypical component of W_l modeled on a G-irreducible representation $\mathscr{V}_{j,l}$ $(j=0, 1, \ldots, s)$.

Put

$$a_i(\lambda) := \mathrm{Id} - D_u\mathscr{F}(\lambda, 0)|_{V_i}, \quad a_{j,l}(\lambda) := \mathrm{Id} - D_u\mathscr{F}(\lambda, 0)|_{V_{j,l}} \text{ for } |\lambda - \lambda_o| \leqslant \delta.$$

It can be shown that

$$a_i(\lambda) = -\frac{1}{\beta} D_x f(\alpha, 0)|_{V_i}, \quad a_{j,l}(\lambda) = \frac{1}{il\beta} \triangle_{\alpha,j}(il\beta), \tag{20}$$

where $\lambda = \alpha + i\beta \in \mathbb{C}$.

Combining the multiplicativity property of the primary degree with the so-called Splitting Lemma (see [5,3]), one obtains:

$$\begin{aligned} \omega(\lambda_o) &= G\text{-Deg}(\mathfrak{F}_o, \Omega) \\ &= \prod_{i=0}^{r} \Gamma\text{-Deg}(a_i(\lambda_o), \mathscr{B}_i) \cdot \sum_{j,l} G\text{-Deg}(a_{j,l}(\lambda_o), \Omega \cap (\mathbb{R}^2_+ \times V_{j,l})), \end{aligned} \tag{21}$$

where $\mathscr{B}_i$ denotes the unit ball in V_i. Moreover, based on the values of the basic degrees $\deg_{\mathscr{V}_i}$ and $\deg_{\mathscr{V}_{j,l}}$ associated with $\mathscr{V}_i$ and $\mathscr{V}_{j,l}$ (cf. Remark 2.1), and the so-called $\mathscr{V}_{j,l}$-isotypical *crossing numbers* $\mathrm{t}_{j,l}(\alpha_o, \beta_o)$ (cf. [3]), formula (21) can be reduced to the following one:

$$\omega(\lambda_o) = \prod_{\mu \in \sigma_-} \prod_{i=0}^{r} (\deg_{\mathscr{V}_i})^{m_i(\mu)} \cdot \sum_{j,l} \mathrm{t}_{j,l}(\alpha_o, \beta_o) \deg_{\mathscr{V}_{j,l}}, \tag{22}$$

where σ_- denotes the set of all negative eigenvalues of $\bigoplus_{i=0}^{r} a_i(\lambda_o)$, $m_i(\mu)$ is the $\mathscr{V}_i$-*multiplicity* of μ given by

$$m_i(\mu) := \dim_{\mathbb{R}} (E_i(\mu)) / \dim_{\mathbb{R}} \mathscr{V}_i, \tag{23}$$

where $E_i(\mu)$ is the eigenspace of μ when restricted to V_i, and $\mathrm{t}_{j,l}(\alpha_o, \beta_o)$ can be obtained via the $\mathscr{U}_j$-multiplicity of $i\beta_o$ (cf. (8)) by

$$\mathrm{t}_{j,l}(\alpha_o, \beta_o) = -\mathrm{sign} \left.\frac{\mathrm{d}u}{\mathrm{d}\alpha}\right|_{\alpha=\alpha_o} m_j(il\beta_o), \tag{24}$$

where $u(\alpha)$ stands for the real part of solution to (5) for $x_o = 0$.

Remark 3.6. Although the entire value of the invariant $\omega(\lambda_o)$ should be considered to fully classify the symmetric Hopf bifurcation branches for system (1), in order to simplify our exposition (by reducing the number of additional

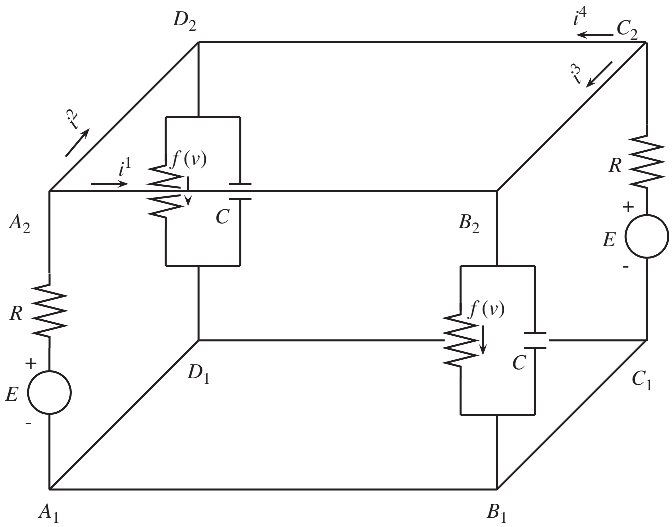

Fig. 2. Symmetric model of transmission lines: internal coupling.

cases), we will restrict our computations to the coefficients $n_{H_o} = n_{K_o^{\varphi,1}}$, which will be called *first coefficients*, and we will denote the corresponding part of the invariant $\omega(\lambda_o)$ by $\omega(\lambda_o)_1$. Thus, by (22)

$$\omega(\lambda_o)_1 = \prod_{\mu\in\sigma_-}\prod_{i=0}^{r}(\deg_{\mathscr{V}_i})^{m_i(\mu)} \cdot \sum_j \mathfrak{t}_{j,1}(\alpha_o, \beta_o)\deg_{\mathscr{V}_{j,1}}. \tag{25}$$

In fact, the first coefficients turn out to be sufficient to detect the solutions corresponding to the dominating orbit types.

4. Symmetric configurations of lossless transmission line models

In this section, we consider two simple (but generic) types of symmetric configurations for the lossless transmission line models, and derive the corresponding symmetric neutral functional differential equations, which give insight of reasonable symmetries one could expect in suchmodels.

4.1. Configuration 1: internal coupling

Consider first a cube of symmetrically coupled lossless transmission line networks between two recipients and two power stations. Assume all coupled networks are identical, each of which is a uniformly distributed lossless transmission line with the inductance L_s and parallel capacitance C_s per unit length. To derive the network equations, we place the x-axis in the direction of the line, with two ends of the normalized line at $x = 0$ and 1 (see Fig. 2).[4]

Denote by $i^j(x,t)$ the current flowing in the jth line at time t and distance x down the line and $v^j(x,t)$ the voltage across the line at t and x, for $j = 1, 2, 3, 4$. It is well-known that (see, for instance, [13]) the functions $i^j := i^j(x,t)$ and $v^j := v^j(x,t)$ obey the following partial differential equations (*Telegrapher's equation*)

$$\begin{cases} \dfrac{\partial v^j}{\partial x} = -L_s\dfrac{\partial i^j}{\partial t}, \\ C_s\dfrac{\partial v^j}{\partial t} = -\dfrac{\partial i^j}{\partial x}. \end{cases} \tag{26}$$

[4] This example of internal coupling can be easily generalized to a coupling of N recipients and N power stations with an $N > 2$.

When these networks are coupled symmetrically in the way shown in Fig. 2, the vertical lines have coupling terms from the preceding and succeeding lines at each end $x=0$ and $x=1$, thus it gives rise to the boundary conditions

$$\begin{cases} E = v_0^1 + (i_0^1 + i_0^2)R, \\ i_1^1 + i_1^3 = f(v_1^1) + C\,\dfrac{\mathrm{d}v_1^1}{\mathrm{d}t}, \\ E = v_0^3 + (i_0^3 + i_0^4)R, \\ i_1^2 + i_1^4 = f(v_1^2) + C\,\dfrac{\mathrm{d}v_1^2}{\mathrm{d}t}, \\ v_0^1 = v_0^2, \quad v_0^3 = v_0^4, \\ v_1^1 = v_1^3, \quad v_1^2 = v_1^4, \end{cases} \tag{27}$$

where $i_\delta^j = i_\delta^j(t) := i^j(\delta, t)$, $v_\delta^j = v_\delta^j(t) := v^j(\delta, t)$ for $\delta \in \{0, 1\}$, E is the constant direct current voltage and $f(v_1^j)$ is the current through the nonlinear resistor in the direction shown in Fig. 2.

For mathematical simplicity, we assume that:

(E1) the boundary value problem (26)–(27) admits a unique solution $(v_*^j, i_*^j) := (v_*^j(x,t), i_*^j(x,t))$, for $j = 1, 2, 3, 4$ such that $\partial i_*^j/\partial x = \partial v_*^j/\partial x = 0$ (the so-called *equilibrium point*).

Thus, the equilibrium point (v_*^j, i_*^j), $j = 1, 2, 3, 4$ satisfies the following *equilibrium equations*:

$$\begin{cases} E = v_*^1 + (i_*^1 + i_*^2)R, \\ i_*^1 + i_*^3 = f(v_*^1) + C\,\dfrac{\mathrm{d}v_*^1}{\mathrm{d}t}, \\ E = v_*^3 + (i_*^3 + i_*^4)R, \\ i_*^2 + i_*^4 = f(v_*^2) + C\,\dfrac{\mathrm{d}v_*^2}{\mathrm{d}t}. \end{cases} \tag{28}$$

Now, subtract the first four equations in (27) by (28), we obtain

$$\begin{cases} 0 = v_0^1 - v_*^1 + (i_0^1 - i_*^1 + i_0^2 - i_*^2)R, \\ i_1^1 - i_*^1 + i_1^3 - i_*^3 = f(v_1^1) - f(v_*^1) + C\,\dfrac{\mathrm{d}}{\mathrm{d}t}(v_1^1 - v_*^1), \\ 0 = v_0^3 - v_*^3 + (i_0^3 - i_*^3 + i_0^4 - i_*^4)R, \\ i_1^2 - i_*^2 + i_1^4 - i_*^4 = f(v_1^2) - f(v_*^2) + C\,\dfrac{\mathrm{d}}{\mathrm{d}t}(v_1^2 - v_*^2). \end{cases} \tag{29}$$

By changing variables, $\mathscr{X}_\delta^j = v_\delta^j - v_*^j$, $\mathscr{Y}_\delta^j = i_\delta^j - i_*^j$ (for $\delta = 0, 1$), and putting

$$g(\mathscr{X}_1^j) := f(\mathscr{X}_1^j + v_*^j) - f(v_*^j) = f(v_1^j) - f(v_*^j), \tag{30}$$

the boundary conditions (27) reduce to

$$\begin{cases} 0 = \mathscr{X}_0^1 + (\mathscr{Y}_0^1 + \mathscr{Y}_0^2)R, \\ \mathscr{Y}_1^1 + \mathscr{Y}_1^3 = g(\mathscr{X}_1^1) + C\,\dfrac{\mathrm{d}\mathscr{X}_1^1}{\mathrm{d}t}, \\ 0 = \mathscr{X}_0^3 + (\mathscr{Y}_0^3 + \mathscr{Y}_0^4)R, \\ \mathscr{Y}_1^2 + \mathscr{Y}_1^4 = g(\mathscr{X}_1^2) + C\,\dfrac{\mathrm{d}\mathscr{X}_1^2}{\mathrm{d}t}, \\ \mathscr{X}_0^1 = \mathscr{X}_0^2, \quad \mathscr{X}_0^3 = \mathscr{X}_0^4, \\ \mathscr{X}_1^1 = \mathscr{X}_1^3, \quad \mathscr{X}_1^2 = \mathscr{X}_1^4. \end{cases}$$

For simplicity, we replace the symbols $\mathscr{X}^j_\delta$ and $\mathscr{Y}^j_\delta$ with v^j_δ and i^j_δ, respectively (for $\delta = 0, 1$),

$$\begin{cases} 0 = v^1_0 + (i^1_0 + i^2_0)R, \\ i^1_1 + i^3_1 = g(v^1_1) + C\,\dfrac{\mathrm{d}v^1_1}{\mathrm{d}t}, \\ 0 = v^3_0 + (i^3_0 + i^4_0)R, \\ i^2_1 + i^4_1 = g(v^2_1) + C\,\dfrac{\mathrm{d}v^2_1}{\mathrm{d}t}, \\ v^1_0 = v^2_0, \quad v^3_0 = v^4_0, \\ v^1_1 = v^3_1, \quad v^2_1 = v^4_1. \end{cases} \tag{31}$$

Our goal is to reduce the boundary value problem (26) and (31) to a system of symmetric FDEs. To this end, recall that the general solution to (26) (the so-called *d'Alembert solution*) takes the form:

$$\begin{cases} v^j(x,t) = \dfrac{1}{2}\,[\phi^j(x-at) + \psi^j(x+at)], \\ i^j(x,t) = \dfrac{1}{2b}\,[\phi^j(x-at) - \psi^j(x+at)], \end{cases} \tag{32}$$

where

$$a = \frac{1}{\sqrt{L_s C_s}}, \quad b = \sqrt{\frac{L_s}{C_s}} \tag{33}$$

are, respectively, the propagation velocity of waves and the characteristic impedance of the line, and $\phi^j \in C^1((-\infty, 1]; \mathbb{R})$, $\psi^j \in C^1([0,\infty); \mathbb{R})$ (see, for instance, [15]).

Next, we will essentially use the identity

$$i^j(x,t) + i^j\left(x, t - \frac{2}{a}\right) = i^j\left(x-1, t-\frac{1}{a}\right) + i^j\left(x+1, t-\frac{1}{a}\right), \tag{34}$$

supported by the following simple computation:

$$\begin{aligned} i^j(x,t) &= \frac{1}{2b}\,[\phi^j(x-at) - \psi^j(x+at)] \\ &= \frac{1}{2b}\left[\phi^j\left(x-1-a\left(t-\frac{1}{a}\right)\right) - \psi^j\left(x+1+a\left(t-\frac{1}{a}\right)\right)\right] \\ &= \frac{1}{2b}\left[\phi^j\left(x-1-a\left(t-\frac{1}{a}\right)\right) - \psi^j\left(x-1+a\left(t-\frac{1}{a}\right)\right)\right] + \frac{1}{2b}\left[\psi^j\left(x-1+a\left(t-\frac{1}{a}\right)\right)\right. \\ &\quad \left. - \phi^j\left(x+1-a\left(t-\frac{1}{a}\right)\right)\right] + \frac{1}{2b}\left[\phi^j\left(x+1-a\left(t-\frac{1}{a}\right)\right) - \psi^j\left(x+1+a\left(t-\frac{1}{a}\right)\right)\right] \\ &= \frac{1}{2b}\left[\phi^j\left(x-1-a\left(t-\frac{1}{a}\right)\right) - \psi^j\left(x-1+a\left(t-\frac{1}{a}\right)\right)\right] - \frac{1}{2b}\left[\phi^j\left(x-a\left(t-\frac{2}{a}\right)\right)\right. \\ &\quad \left. - \psi^j\left(x+a\left(t-\frac{2}{a}\right)\right)\right] + \frac{1}{2b}\left[\phi^j\left(x+1-a\left(t-\frac{1}{a}\right)\right) - \psi^j\left(x+1+a\left(t-\frac{1}{a}\right)\right)\right] \\ &= i^j\left(x-1, t-\frac{1}{a}\right) - i^j\left(x, t-\frac{2}{a}\right) + i^j\left(x+1, t-\frac{1}{a}\right). \end{aligned}$$

In particular, by substituting $x = 1$ in (34), we have

$$i^j\left(2, t-\frac{1}{a}\right) = i^j_1(t) + i^j_1\left(t-\frac{2}{a}\right) - i^j_0\left(t-\frac{1}{a}\right). \tag{35}$$

Return to the boundary conditions (31). Using (32), we obtain:

$$\begin{cases} \phi^1(-at) = \dfrac{R-b}{R+b}\,\psi^1(at) - \dfrac{2bR}{R+b}\,i^2_0(t), \\ \phi^3(-at) = \dfrac{R-b}{R+b}\,\psi^3(at) - \dfrac{2bR}{R+b}\,i^4_0(t). \end{cases}$$

Consequently,

$$\begin{aligned}
C\,\frac{\mathrm{d}v_1^1}{\mathrm{d}t} &= i_1^1 + i_1^3 - g(v_1^1)\\
&= \frac{\phi^1(1-at) - v_1^1}{b} + \frac{\phi^3(1-at) - v_1^3}{b} - g(v_1^1)\\
&= \frac{\phi^1(-a(t-\frac{1}{a})) - v_1^1}{b} + \frac{\phi^3(-a(t-\frac{1}{a})) - v_1^3}{b} - g(v_1^1)\\
&= \frac{\frac{R-b}{R+b}\psi^1(at-1) - \frac{2bR}{R+b} i_0^2(t-\frac{1}{a}) - v_1^1}{b} + \frac{\frac{R-b}{R+b}\psi^3(at-1) - \frac{2bR}{R+b} i_0^4(t-\frac{1}{a}) - v_1^3}{b} - g(v_1^1).
\end{aligned} \tag{36}$$

Similarly, we also have

$$\begin{aligned}
&C\,\frac{R-b}{R+b}\,\frac{\mathrm{d}v_1^1}{\mathrm{d}t}\,(t-\frac{2}{a})\\
&= \frac{R-b}{R+b}\left[i_1^1\left(t-\frac{2}{a}\right) + i_1^3\left(t-\frac{2}{a}\right) - g\left(v_1^1\left(t-\frac{2}{a}\right)\right)\right]\\
&= \frac{R-b}{R+b}\left[\frac{v_1^1(t-\frac{2}{a}) - \psi^1(1+a(t-\frac{2}{a}))}{b} + \frac{v_1^3(t-\frac{2}{a}) - \psi^3(1+a(t-\frac{2}{a}))}{b} - g\left(v_1^1\left(t-\frac{2}{a}\right)\right)\right]\\
&= \frac{1}{b}\,\frac{R-b}{R+b}\left[v_1^1\left(t-\frac{2}{a}\right) + v_1^3\left(t-\frac{2}{a}\right)\right] - \frac{1}{b}\,\frac{R-b}{R+b}[\psi^1(at-1) + \psi^3(at-1)]\\
&\quad - \frac{R-b}{R+b}\,g\left(v_1^1\left(t-\frac{2}{a}\right)\right).
\end{aligned} \tag{37}$$

Combining (36) and (37) results in

$$\begin{aligned}
C\left[\frac{\mathrm{d}v_1^1}{\mathrm{d}t} + \frac{R-b}{R+b}\,\frac{\mathrm{d}v_1^1}{\mathrm{d}t}\left(t-\frac{2}{a}\right)\right] &= -\frac{2R}{R+b}\left[i_0^2\left(t-\frac{1}{a}\right) + i_0^4\left(t-\frac{1}{a}\right)\right] - \frac{1}{b}\,(v_1^1 + v_1^3)\\
&\quad + \frac{1}{b}\,\frac{R-b}{R+b}\left[v_1^1\left(t-\frac{2}{a}\right) + v_1^3\left(t-\frac{2}{a}\right)\right]\\
&\quad - g(v_1^1) - \frac{R-b}{R+b}\,g\left(v_1^1\left(t-\frac{2}{a}\right)\right).
\end{aligned} \tag{38}$$

On the other hand, since by (32),

$$\begin{aligned}
i_0^2\left(t-\frac{1}{a}\right) &= \frac{1}{2b}[\phi^2(1-at) - \psi^2(at-1)]\\
&= \frac{1}{2b}[2v_1^2 - \psi^2(1+at) - \psi^2(at-1)]\\
&= \frac{1}{b}\,v_1^2 - \frac{1}{2b}[\psi^2(1+at) + \psi^2(at-1)]\\
&= \frac{1}{b}\,v_1^2 - \frac{1}{2b}\left[\psi^2(1+at) + 2v_1^2\left(t-\frac{2}{a}\right) - \phi^2(3-at)\right]\\
&= \frac{1}{b}\,v_1^2 - \frac{1}{b}\,v_1^2\left(t-\frac{2}{a}\right) + \frac{1}{2b}[\phi^2(3-at) - \psi^2(1+at)]\\
&= \frac{1}{b}\,v_1^2 - \frac{1}{b}\,v_1^2\left(t-\frac{2}{a}\right) + i^2\left(2, t-\frac{1}{a}\right),
\end{aligned}$$

it follows from (35) that

$$i_0^2\left(t-\frac{1}{a}\right) = \frac{1}{2b}\,v_1^2 - \frac{1}{2b}\,v_1^2\left(t-\frac{2}{a}\right) + \frac{1}{2}\left[i_1^2(t) + i_1^2\left(t-\frac{2}{a}\right)\right]. \tag{39}$$

Symmetrically, a similar statement is valid for i_0^4, i.e.

$$i_0^4\left(t-\frac{1}{a}\right)=\frac{1}{2b}\,v_1^4-\frac{1}{2b}\,v_1^4\left(t-\frac{2}{a}\right)+\frac{1}{2}\left[i_1^4(t)+i_1^4\left(t-\frac{2}{a}\right)\right]. \tag{40}$$

Using the boundary conditions (31) and (39)–(40), we have

$$\begin{aligned}i_0^2\left(t-\frac{1}{a}\right)+i_0^4\left(t-\frac{1}{a}\right)&=\frac{1}{b}\,v_1^2-\frac{1}{b}\,v_1^2\left(t-\frac{2}{a}\right)+\frac{1}{2}\left[g(v_1^2)+C\,\frac{\mathrm{d}v_1^2}{\mathrm{d}t}\right.\\&\left.\quad+g\left(v_1^2\left(t-\frac{2}{a}\right)\right)+C\,\frac{\mathrm{d}v_1^2}{\mathrm{d}t}\left(t-\frac{2}{a}\right)\right].\end{aligned} \tag{41}$$

Therefore, by substituting (41) into (38) and using the last equality from (31), we obtain:

$$\begin{aligned}&C\left[\frac{\mathrm{d}v_1^1}{\mathrm{d}t}+\frac{R-b}{R+b}\,\frac{\mathrm{d}v_1^1}{\mathrm{d}t}\left(t-\frac{2}{a}\right)\right]\\&=-\frac{2R}{R+b}\left[\frac{1}{b}\,v_1^2-\frac{1}{b}\,v_1^2\left(t-\frac{2}{a}\right)\right]-\frac{R}{R+b}\left[g(v_1^2)+C\,\frac{\mathrm{d}v_1^2}{\mathrm{d}t}+g\left(v_1^2\left(t-\frac{2}{a}\right)\right)+C\,\frac{\mathrm{d}v_1^2}{\mathrm{d}t}\left(t-\frac{2}{a}\right)\right]\\&\quad-\frac{2}{b}\,v_1^1+\frac{2}{b}\,\frac{R-b}{R+b}\,v_1^1\left(t-\frac{2}{a}\right)-g(v_1^1)-\frac{R-b}{R+b}\,g\left(v_1^1\left(t-\frac{2}{a}\right)\right)\\&=-C\,\frac{R}{R+b}\left[\frac{\mathrm{d}v_1^2}{\mathrm{d}t}+\frac{\mathrm{d}v_1^2}{\mathrm{d}t}\left(t-\frac{2}{a}\right)\right]\\&\quad-\frac{2}{b}\,v_1^1+\frac{2}{b}\,\frac{R-b}{R+b}\,v_1^1\left(t-\frac{2}{a}\right)-g(v_1^1)-\frac{R-b}{R+b}\,g\left(v_1^1\left(t-\frac{2}{a}\right)\right)\\&\quad-\frac{2R}{R+b}\left[\frac{1}{b}\,v_1^2-\frac{1}{b}\,v_1^2\left(t-\frac{2}{a}\right)\right]-\frac{R}{R+b}\left[g(v_1^2)+g\left(v_1^2\left(t-\frac{2}{a}\right)\right)\right],\end{aligned}$$

which, after rearrangement, yields:

$$\begin{aligned}&C\left[\frac{\mathrm{d}v_1^1}{\mathrm{d}t}+\frac{R}{R+b}\,\frac{\mathrm{d}v_1^2}{\mathrm{d}t}+\frac{R-b}{R+b}\,\frac{\mathrm{d}v_1^1}{\mathrm{d}t}\left(t-\frac{2}{a}\right)+\frac{R}{R+b}\,\frac{\mathrm{d}v_1^2}{\mathrm{d}t}\left(t-\frac{2}{a}\right)\right]\\&=-\frac{2}{b}\,v_1^1+\frac{2}{b}\,\frac{R-b}{R+b}\,v_1^1\left(t-\frac{2}{a}\right)-\frac{2}{b}\,\frac{R}{R+b}\left[v_1^2-v_1^2\left(t-\frac{2}{a}\right)\right]\\&\quad-g(v_1^1)-\frac{R-b}{R+b}\,g\left(v_1^1\left(t-\frac{2}{a}\right)\right)-\frac{R}{R+b}\left[g(v_1^2)+g\left(v_1^2\left(t-\frac{2}{a}\right)\right)\right].\end{aligned} \tag{42}$$

By the same argument, we obtain:

$$\begin{aligned}&C\left[\frac{\mathrm{d}v_1^2}{\mathrm{d}t}+\frac{R}{R+b}\,\frac{\mathrm{d}v_1^1}{\mathrm{d}t}+\frac{R-b}{R+b}\,\frac{\mathrm{d}v_1^2}{\mathrm{d}t}\left(t-\frac{2}{a}\right)+\frac{R}{R+b}\,\frac{\mathrm{d}v_1^1}{\mathrm{d}t}\left(t-\frac{2}{a}\right)\right]\\&=-\frac{2}{b}\,v_1^2+\frac{2}{b}\,\frac{R-b}{R+b}\,v_1^2\left(t-\frac{2}{a}\right)-\frac{2}{b}\,\frac{R}{R+b}\left[v_1^1-v_1^1\left(t-\frac{2}{a}\right)\right]\\&\quad-g(v_1^2)-\frac{R-b}{R+b}\,g\left(v_1^2\left(t-\frac{2}{a}\right)\right)-\frac{R}{R+b}\left[g(v_1^1)+g\left(v_1^1\left(t-\frac{2}{a}\right)\right)\right].\end{aligned} \tag{43}$$

In terms of matrices, system (42)–(43) can be rewritten as

$$C\left[S_1\frac{\mathrm{d}}{\mathrm{d}t}x(t)-S_2\frac{\mathrm{d}}{\mathrm{d}t}x(t-r)\right]=-S_3x(t)-S_4x(t-r)-S_5G(x(t))+S_6G(x(t-r)),\tag{44}$$

where

$$r=\frac{2}{a},\quad x(t)=\begin{bmatrix}v_1^1(t)\\ v_1^2(t)\end{bmatrix},\quad G(x(t))=\begin{bmatrix}g(v_1^1(t))\\ g(v_1^2(t))\end{bmatrix},$$

$$S_1=\begin{bmatrix}1 & \dfrac{R}{R+b}\\ \dfrac{R}{R+b} & 1\end{bmatrix},\quad S_2=\begin{bmatrix}\dfrac{b-R}{R+b} & -\dfrac{R}{R+b}\\ -\dfrac{R}{R+b} & \dfrac{b-R}{R+b}\end{bmatrix},$$

$$S_3=\begin{bmatrix}\dfrac{2}{b} & \dfrac{2}{b}\dfrac{R}{R+b}\\ \dfrac{2}{b}\dfrac{R}{R+b} & \dfrac{2}{b}\end{bmatrix}=\frac{2}{b}S_1,\quad S_4=\begin{bmatrix}-\dfrac{2}{b}\dfrac{R-b}{R+b} & -\dfrac{2}{b}\dfrac{R}{R+b}\\ -\dfrac{2}{b}\dfrac{R}{R+b} & -\dfrac{2}{b}\dfrac{R-b}{R+b}\end{bmatrix}=\frac{2}{b}S_2,$$

$$S_5=\begin{bmatrix}1 & \dfrac{R}{R+b}\\ \dfrac{R}{R+b} & 1\end{bmatrix}=S_1,\quad S_6=\begin{bmatrix}\dfrac{b-R}{R+b} & -\dfrac{R}{R+b}\\ -\dfrac{R}{R+b} & \dfrac{b-R}{R+b}\end{bmatrix}=S_2.$$

Multiplying (44) by S_1^{-1} (recall that $b\neq 0$ (see (33))), we arrive at

$$\frac{\mathrm{d}}{\mathrm{d}t}[x(t)-Qx(t-r)]=-\frac{2}{bC}x(t)-\frac{2}{bC}Qx(t-r)-\frac{1}{C}G(x(t))+\frac{1}{C}QG(x(t-r)),\tag{45}$$

where $Q=S_1^{-1}S_2$.

Note that system (44) embodies the symmetric situation, namely the internal coupling, in the following way: let $\Gamma:=D_2$ act on $V:=\mathbb{R}^2$ by permuting the coordinates of vectors $x=[\begin{smallmatrix}v^1\\ v^2\end{smallmatrix}]\in V$, then system (44) is symmetric with respect to the Γ-action on V.

4.2. Configuration 2: external coupling

A second example of symmetric coupling was considered in [17] , where N recipients are mutually coupled via lossless transmission line network which are interconnected by a common resistor R_o between neighboring recipients, and extensively connected with N power stations (Fig. 3).

Denote by $i^j(x,t)$ the current flowing in the jth line at time t and distance x down the line and $v^j(x,t)$ the voltage across the line at t and x, for $j=1,\dots,N$. The same Telegrapher's equation (26) holds for $i^j(x,t)$ and $v^j(x,t)$. However, the boundary conditions need to be modified for this external coupling. For $j=1,\dots,N$, we have

$$\begin{cases}E=v_0^j+i_0^jR,\\ i_1^j=f(v_1^j)+C\dfrac{\mathrm{d}v_1^j}{\mathrm{d}t}-(I^{j-1}(t)-I^j(t)),\\ v_1^j-v_1^{j+1}=I^j(t)R_o,\end{cases}\tag{46}$$

where $I^0(t):=I^N(t)$, $v^{N+1}:=v^1$, I^j's are the so-called coupling terms (see [17]).

For mathematical simplicity, we assume that (cf. (E1)):

(E2) the boundary value problem (26) and (46) admits a unique equilibrium point $(v_*^j,i_*^j):=(v_*^j(x,t),i_*^j(x,t))$, for $j=1,\dots,N$.

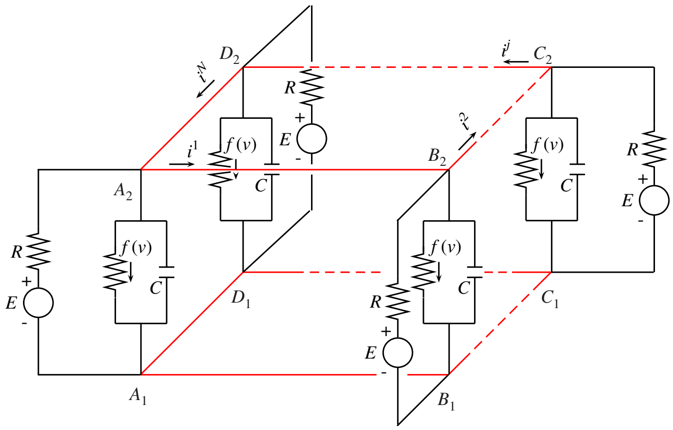

Fig. 3. Symmetric model of transmission lines: external coupling.

By a change of variables provided by (30), the boundary conditions (46) can be translated to

$$\begin{cases} 0 = v_0^j + i_0^j R, \\ i_1^j = g(v_1^j) + C\,\dfrac{\mathrm{d}v_1^j}{\mathrm{d}t} - \dfrac{1}{R_o}(v_1^{j+1} - 2v_1^j + v_1^{j-1}). \end{cases} \tag{47}$$

We are now in a position to reduce the boundary value problem (26) and (47) to a symmetric system of FDEs. By (47) and (32), we have

$$\phi^j(-at) = \frac{R-b}{R+b}\,\psi^j(at),$$

and

$$\begin{aligned} C\,\frac{\mathrm{d}v_1^j}{\mathrm{d}t} &= i_1^j - g(v_1^j) + \frac{1}{R_o}\left(v_1^{j+1} - 2v_1^j + v_1^{j-1}\right) \\ &= \frac{\phi^j(1-at) - v_1^j}{b} - g(v_1^j) + \frac{1}{R_o}\left(v_1^{j+1} - 2v_1^j + v_1^{j-1}\right) \\ &= \frac{\phi^j(-a(t-\frac{1}{a})) - v_1^j}{b} - g(v_1^j) + \frac{1}{R_o}\left(v_1^{j+1} - 2v_1^j + v_1^{j-1}\right) \\ &= \frac{(R-b/R+b)\psi^j(at-1) - v_1^j}{b} - g(v_1^j) + \frac{1}{R_o}\left(v_1^{j+1} - 2v_1^j + v_1^{j-1}\right). \end{aligned}$$

Similarly,

$$\begin{aligned} &C\,\frac{R-b}{R+b}\,\frac{\mathrm{d}v_1^j}{\mathrm{d}t}\left(t - \frac{2}{a}\right) \\ &= \frac{R-b}{R+b}\,i_1^j\left(t-\frac{2}{a}\right) - \frac{R-b}{R+b}\,g\left(v_1^j\left(t-\frac{2}{a}\right)\right) + \frac{1}{R_o}\,\frac{R-b}{R+b}\left(v_1^{j+1} - 2v_1^j + v_1^{j-1}\right) \\ &= \frac{R-b}{R+b}\,\frac{v_1^j(t-\frac{2}{a}) - \psi^j(at-1))}{b} - \frac{R-b}{R+b}\,g\left(v_1^j\left(t-\frac{2}{a}\right)\right) + \frac{1}{R_o}\,\frac{R-b}{R+b}\left(v_1^{j+1} - 2v_1^j + v_1^{j-1}\right). \end{aligned}$$

Therefore,

$$C\left[\frac{\mathrm{d}v_1^j}{\mathrm{d}t}+\frac{R-b}{R+b}\frac{\mathrm{d}v_1^j}{\mathrm{d}t}\left(t-\frac{2}{a}\right)\right]=-\frac{1}{b}v_1^j+\frac{1}{b}\frac{R-b}{R+b}v_1^j\left(t-\frac{2}{a}\right)-g(v_1^j)-\frac{R-b}{R+b}g\left(v_1^j\left(t-\frac{2}{a}\right)\right)$$
$$+\frac{1}{R_o}\left(v_1^{j+1}-2v_1^j+v_1^{j-1}\right)+\frac{1}{R_o}\frac{R-b}{R+b}\left(v_1^{j+1}-2v_1^j+v_1^{j-1}\right). \quad (48)$$

In terms of matrices, we rewrite (48) as

$$\frac{\mathrm{d}}{\mathrm{d}t}[x(t)-\alpha x(t-r)]=-\frac{1}{bC}Px(t)-\frac{1}{bC}\alpha Px(t-r)-\frac{1}{C}G(x(t))+\frac{1}{C}\alpha G(x(t-r)), \quad (49)$$

where

$$r=\frac{2}{a},\quad x(t)=\begin{bmatrix} v_1^1(t)\\ \vdots \\ v_1^N(t)\end{bmatrix},\quad G(x(t))=\begin{bmatrix} g(v_1^1(t))\\ \vdots \\ g(v_1^N(t))\end{bmatrix},$$

$$\alpha=-\frac{R-b}{R+b},\quad P=\begin{bmatrix} 1+\frac{2b}{R_o} & -\frac{b}{R_o} & 0 & \cdots & 0 & -\frac{b}{R_o}\\ -\frac{b}{R_o} & 1+\frac{2b}{R_o} & -\frac{b}{R_o} & \cdots & 0 & 0\\ \vdots & & \ddots & & \vdots & \vdots\\ -\frac{b}{R_o} & 0 & 0 & \cdots & -\frac{b}{R_o} & 1+\frac{2b}{R_o}\end{bmatrix}.$$

Note that system (49) is a $\Gamma := D_N$-symmetric system in the following sense: consider Γ acting on $V := \mathbb{R}^N$ by permuting the coordinates of vectors $x=\begin{bmatrix} v^1\\ \vdots \\ v^N\end{bmatrix}\in V$, then system (49) is symmetric with respect to the Γ-action on V.

5. Hopf bifurcation results for symmetric configurations of transmission line models

Being motivated by the two generic models of symmetric couplings (cf. (45) and (49)), we will present below a general symmetric system of functional differential equations and discuss several crucial elements in computations of its associated bifurcation invariant, which are the prerequisite for the usage of our Maple© package.

5.1. Statement of the problem

We combine the two different coupling models discussed in Section 4 in the following symmetric functional differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t}[x(t)-\alpha Qx(t-r)]=-P_1x(t)-\alpha QP_2x(t-r)-aG(x(t))+a\alpha QG(x(t-r)), \quad (50)$$

where a and r are positive constants, α is the bifurcation parameter and

$$x(t)=\begin{bmatrix} x^1(t)\\ x^2(t)\\ \vdots \\ x^n(t)\end{bmatrix}\in\mathbb{R}^n,\quad G(x(t))=\begin{bmatrix} g(x^1(t))\\ g(x^2(t))\\ \vdots \\ g(x^n(t))\end{bmatrix}\in\mathbb{R}^n.$$

In addition, we assume

(H1) $g:\mathbb{R}\to\mathbb{R}$ is continuously differentiable, $g(0)=g'(0)=0$;

(H2) A finite group Γ acts on $\mathbb{R}^n$ by permuting the coordinates of vectors $x \in V := \mathbb{R}^n$, meaning geometrically it permutes the vertices of a certain related polygon or polyhedron;
(H3) (i) Q, P_1, P_2 are $n \times n$-matrices, which commute pairwisely;
(ii) Q, P_1, P_2 commute with the Γ-action on V;
(H4) $|\alpha| \cdot \|Q\| < 1$.

Remark 5.1. (i) Since Q, P_1, P_2 are pairwisely commuting matrices, they can be diagonalized simultaneously. In other words, Q, P_1, P_2 share the same eigenspaces with respect to a certain choice of a basis of V. We will use the symbols ξ, ζ and η to denote the eigenvalues of Q, P_1, and P_2, respectively, corresponding to the same eigenvector $v \in V$. Further, assume that ζ and η satisfy the following condition:

(H5) In the case $\zeta\eta > 0$, $\sqrt{\zeta\eta} \neq ((2k+1)/2r)\pi$ for any $k \in \mathbb{Z}$.
(ii) By assumption (H4), system (50) satisfies (A5).
(iii) It is clear that system (50) is symmetric with respect to the Γ-action on V and $(\alpha, 0)$ is a stationary point for all α.

In this way, we are dealing here with a Γ-symmetric system of functional differential equations, and we are interested in studying the nontrivial periodic solutions that bifurcate from the stationary point.

5.2. Characateristic equation and isolated centers

By linearizing system (50) at $x = 0$, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}[x(t) - \alpha Q x(t-r)] = -P_1 x(t) - \alpha Q P_2 x(t-r).$$

Substituting $x = \mathrm{e}^{\lambda t} v$ for $\lambda \in \mathbb{C}$, $0 \neq v \in V$, we have

$$\lambda \mathrm{e}^{\lambda t} v - \alpha Q \lambda \mathrm{e}^{\lambda(t-r)} v = -P_1 \mathrm{e}^{\lambda t} v - \alpha Q P_2 \mathrm{e}^{\lambda(t-r)} v,$$

i.e.

$$[\lambda \mathrm{Id} - \alpha Q \lambda \mathrm{e}^{-\lambda r} + P_1 + \alpha Q P_2 \mathrm{e}^{-\lambda r}] v = 0.$$

Therefore, we have the following characteristic equation for system (50)

$$\det\nolimits_{\mathbb{C}} \triangle_{(\alpha,0)}(\lambda) = 0, \tag{51}$$

where

$$\triangle_{(\alpha,0)}(\lambda) := (\lambda \mathrm{Id} - \alpha Q \lambda \mathrm{e}^{-\lambda r}) + P_1 + \alpha Q P_2 \mathrm{e}^{-\lambda r}.$$

Next, we need to find possible values of α such that (51) has a purely imaginary root $i\beta$ for some $\beta > 0$, so we could detect the potential bifurcation points $(\alpha, \beta, 0) \in \mathbb{R}^2_+ \times W$.

By Remark 5.1(i), when restricted to the same eigenspace of Q, P_1 and P_2, the characteristic equation (51) reduces to the following algebraic equation

$$(\lambda + \zeta)\mathrm{e}^{\lambda r} - \alpha\xi(\lambda - \eta) = 0. \tag{52}$$

By replacing in (52) λ with $i\beta$ for some $\beta \neq 0$, and separating the real and imaginary parts, we obtain

$$\begin{cases} \zeta\cos(\beta r) - \beta\sin(\beta r) = -\eta\alpha\xi, \\ \zeta\sin(\beta r) + \beta\cos(\beta r) = \beta\alpha\xi. \end{cases} \tag{53}$$

which leads to

$$\tan(\beta r) = \begin{cases} \dfrac{\beta(\zeta + \eta)}{\beta^2 - \zeta\eta} & \text{if } \beta^2 \neq \zeta\eta, \\ \infty & \text{if } \beta^2 = \zeta\eta. \end{cases} \tag{54}$$

However, it can be verified that under the assumption (H5), the second case in (54) cannot occur.

Hence, we have the following (note that $\beta \neq 0$)

$$\sin(\beta r) = \delta \frac{\beta(\zeta+\eta)}{\zeta^2+\beta^2}\sqrt{\frac{\zeta^2+\beta^2}{\eta^2+\beta^2}}, \quad \cos(\beta r) = \delta \frac{\beta^2-\zeta\eta}{\zeta^2+\beta^2}\sqrt{\frac{\zeta^2+\beta^2}{\eta^2+\beta^2}}, \tag{55}$$

where $\delta \in \{\pm 1\}$ depending on the range of βr. Also, observe that in the case $\xi = 0$, (53) does not permit any nonzero solution of β. So we suppose $\xi \neq 0$, then (53) yields:

$$\alpha = \frac{\delta}{\xi}\sqrt{\frac{\zeta^2+\beta^2}{\eta^2+\beta^2}}. \tag{56}$$

Using (56), we simplify (55) to

$$\sin(\beta r) = \frac{\alpha\xi\beta(\zeta+\eta)}{\zeta^2+\beta^2}, \quad \cos(\beta r) = \frac{\alpha\xi(\beta^2-\zeta\eta)}{\zeta^2+\beta^2}. \tag{57}$$

Clearly, assumption (A6) is satisfied for system (50). We summarize the corresponding information in the statement following below (the needed arguments can be easily deducted from graphing (54)).

Lemma 5.2. *Given system* (50) *satisfying* (H1) *and* (H3), *fix a triple of reals* ζ, η *and* ξ *as in Remark* 5.1(i) *satisfying* (H5). *Then the equation*

$$\tan(\beta r) = \frac{\beta(\zeta+\eta)}{\beta^2-\zeta\eta}$$

has infinitely many positive solutions β_k*'s* ($k \in \mathbb{N}$), *such that*

(a) $0 < \beta_k < \beta_l$ *for* $k < l$;
(b) $\lim_{k\to\infty} \beta_k = \infty$;
(c) *for each* β_k, *the point* $(\alpha_k, 0)$ *is an isolated center for system* (50), *where*

$$\alpha_k = \frac{\delta}{\xi}\sqrt{\frac{\zeta^2+\beta_k^2}{\eta^2+\beta_k^2}}, \quad \delta = \pm 1.$$

Moreover,

(1) *In the case* $\zeta\eta > 0$, *by putting* $k_o := \lfloor (r\sqrt{\zeta\eta}/\pi) + \frac{1}{2} \rfloor$, *where the symbol* $\lfloor\cdot\rfloor$ *stands for the greatest integer function, we have*
  (1d) *If* $k_o = r\sqrt{\zeta\eta}/\pi$, *then*

$$\beta_k \in \begin{cases} \left(\frac{2k-1}{2r}\pi, \frac{k}{r}\pi\right) & \text{for } k < k_o, \\ \left(\frac{k}{r}\pi, \frac{2k+1}{2r}\pi\right) & \text{for } k \geqslant k_o. \end{cases}$$

  (1d′) *Otherwise*,

$$\beta_k \in \begin{cases} \left(\frac{2k-1}{2r}\pi, \frac{k}{r}\pi\right) & \text{for } k \leqslant k_o, \\ \left(\frac{k-1}{r}\pi, \frac{2k-1}{2r}\pi\right) & \text{for } k > k_o. \end{cases}$$

(2) *In the case* $\zeta\eta < 0$ *and* $\zeta + \eta < 0$, *we have*
  (2d) $\beta_k \in (\frac{2k-1}{2r}\pi, \frac{k}{r}\pi)$ *for* $k \in \mathbb{N}$.
(3) *In the case* $\zeta\eta < 0$ *and* $\zeta + \eta > 0$, *we have*
  (3d) *if* $\zeta + \eta \leqslant -\zeta\eta$, *then* $\beta_k \in ((k/r)\pi, ((2k+1)/2r)\pi)$ *for* $k \in \mathbb{N}$;
  (3d′) *otherwise*, $\beta_k \in ((k-1)/r)\pi, ((2k-1)/2r)\pi)$ *for* $k \in \mathbb{N}$.

5.3. Negative spectrum σ_- and $\mathcal{V}_i$-multiplicity

To use the computational formula (25), we need the information on the negative spectrum σ_- of the linear operator $\bigoplus_{i=0}^r a_i(\lambda_o)$ and the $\mathcal{V}_i$-multiplicity $m_i(\mu)$ for each $\mu \in \sigma_-$.

Under assumption (H1), system (50) gives rise to (cf. (20))

$$\bigoplus_{i=0}^{r} a_i(\lambda_o) = -\frac{1}{\beta_o} D_x f(\alpha_o, 0) = \frac{1}{\beta_o}(P_1 + \alpha_o Q P_2) : V \to V,$$

for each isolated center $(\alpha_o, 0)$ $(\lambda_o = \alpha_o + i\beta_o)$.

Under the notations of Remark 5.1(i), we will assume for a fixed triple of ξ, ζ, η that

(H6) $\zeta + \alpha_o \xi \eta \neq 0$.

Then, system (50) satisfies the assumption (A4).

Recall $\alpha_o = \frac{\delta}{\xi}\sqrt{(\zeta^2 + \beta_o^2)/(\eta^2 + \beta_o^2)}$ (cf. (56)), thus the negative spectrum σ_- of $\bigoplus_{i=0}^r a_i(\lambda_o)$ is determined by

$$\begin{aligned}\sigma_- &= \left\{\mu = \frac{1}{\beta_o}(\zeta + \alpha_o\xi\eta) : \frac{1}{\beta_o}(\zeta + \alpha_o\xi\eta) < 0\right\} \\ &= \left\{\mu = \frac{1}{\beta_o}(\zeta + \alpha_o\xi\eta) : \frac{1}{\beta_o}\left(\zeta + \delta\sqrt{\frac{\zeta^2 + \beta_o^2}{\eta^2 + \beta_o^2}}\,\eta\right) < 0\right\} \\ &= \left\{\mu = \frac{1}{\beta_o}(\zeta + \alpha_o\xi\eta) : \zeta\sqrt{\eta^2 + \beta_o^2} + \delta\eta\sqrt{\zeta^2 + \beta_o^2} < 0\right\} \\ &= \left\{\mu = \frac{1}{\beta_o}(\zeta + \alpha_o\xi\eta) : \zeta + \delta\eta < 0\right\},\end{aligned} \tag{58}$$

where we used the fact that $\operatorname{sign}(\zeta\sqrt{\eta^2 + \beta_o^2} + \delta\eta\sqrt{\zeta^2 + \beta_o^2}) = \operatorname{sign}(\zeta + \delta\eta)$.

In all the examples considered in the sequel, the following condition is satisfied

Condition (R). (i) Decomposition (3) contains isotypical components modeled only on irreducible representations of *real* type (in particular (cf. (3) and (4)), $r = s$).

(ii) For each $\mu \in \sigma_-$ there exists a *single* isotypical component $V_i := V_{i_\mu}$ in (3) which (completely) contains the eigenspace $E(\mu)$.

Therefore, formula (23) of the $\mathcal{V}_i$-multiplicity $m_i(\mu)$ reduces to

$$m_i(\mu) = \begin{cases} \dim_{\mathbb{R}} E(\mu)/\dim_{\mathbb{R}} \mathcal{V}_i, & i = i_\mu, \\ 0, & i \neq i_\mu. \end{cases} \tag{59}$$

5.4. Crossing numbers $t_{j,1}$ and $\mathcal{V}_{j,1}$-multiplicity

To proceed with the computational formula (25), we also need to evaluate the crossing numbers $t_{j,1}(\alpha_o, \beta_o)$, in particular (cf. (24)), to determine $(\operatorname{sign}(\mathrm{d}/\mathrm{d}\alpha)\, u(\alpha_o))$, and the $\mathcal{V}_{j,1}$-multiplicity $m_j(i\beta_o)$.

By substituting $\lambda = u + iv$ in (52) and separating the real and imaginary parts, we obtain

$$\begin{cases} \mathrm{e}^{ur}(u + \zeta)\cos(vr) - \mathrm{e}^{ur} v \sin(vr) = \xi\alpha(u - \eta), \\ \mathrm{e}^{ur}(u + \zeta)\sin(vr) + \mathrm{e}^{ur} v \cos(vr) = \xi\alpha v. \end{cases} \tag{60}$$

By implicit differentiation of (60) with respect to α at α_o, $u = 0$, $v = \beta_o$, we obtain

$$\begin{cases} A\dfrac{\mathrm{d}u}{\mathrm{d}\alpha}(\alpha_o) - B\dfrac{\mathrm{d}v}{\mathrm{d}\alpha}(\alpha_o) = -\eta\xi, \\ B\dfrac{\mathrm{d}u}{\mathrm{d}\alpha}(\alpha_o) + A\dfrac{\mathrm{d}v}{\mathrm{d}\alpha}(\alpha_o) = \beta_o\xi, \end{cases} \tag{61}$$

where

$$\begin{cases} A = r(\zeta\cos(\beta_o r) - \beta_o\sin(\beta_o r)) + (\cos(\beta_o r) - \alpha_o\xi), \\ B = r(\zeta\sin(\beta_o r) + \beta_o\cos(\beta_o r)) + \sin(\beta_o r). \end{cases} \tag{62}$$

Substituting (57) into (62) leads to

$$\begin{cases} A = -\dfrac{\alpha_o\xi}{\zeta^2+\beta_o^2}[\eta r(\zeta^2+\beta_o^2) + \zeta(\zeta+\eta)], \\ B = \dfrac{\alpha_o\xi}{\zeta^2+\beta_o^2}[\beta_o r(\zeta^2+\beta_o^2) + \beta_o(\zeta+\eta)]. \end{cases} \tag{63}$$

Thus, it follows from (63) and (61) that

$$\begin{aligned} \frac{\mathrm{d}u}{\mathrm{d}\alpha}(\alpha_o) &= \frac{1}{A^2+B^2}(-\eta\xi A + \beta_o\xi B) \\ &= \frac{\alpha_o\xi^2}{A^2+B^2}\left[r(\eta^2+\beta_o^2) + \frac{1}{\zeta^2+\beta_o^2}(\eta\zeta+\beta_o^2)(\zeta+\eta)\right]. \end{aligned} \tag{64}$$

Lemma 5.3. *Let* $(\alpha_o, 0)$ *be an isolated center for system* (50) *and* $i\beta$ *the corresponding characteristic root. Assume that for* α *close to* α_o, *the characteristic roots have the form* $u(\alpha) + iv(\alpha)$. *Assume, finally,*

(i) $r \geqslant 1$;
(ii) $\beta > 1$.

Then,

$$\operatorname{sign}\left(\frac{\mathrm{d}u}{\mathrm{d}\alpha}(\alpha_o)\right) = \operatorname{sign}(\alpha_o).$$

Proof. Directly from (64), it suffices to show

$$\Upsilon(\eta,\zeta) := r(\eta^2+\beta_o^2) + \frac{1}{\zeta^2+\beta_o^2}(\eta\zeta+\beta_o^2)(\zeta+\eta) > 0.$$

Put

$$\Phi(\eta,\zeta) := \eta^2+\beta_o^2 + \frac{1}{\zeta^2+\beta_o^2}(\eta\zeta+\beta_o^2)(\zeta+\eta).$$

By assumption (i), $\Upsilon(\eta,\zeta) \geqslant \Phi(\eta,\zeta)$ for all η, ζ, thus we only need to show

$$\Phi(\eta,\zeta) > 0.$$

Case 1: If $\eta = 0$, $\Phi(0,\zeta) = \beta_o^2 + (1/(\zeta^2+\beta_o^2))\,\beta_o^2\zeta = \beta_o^2((\zeta^2+\zeta+\beta_o^2)/(\zeta^2+\beta_o^2)) \overset{(ii)}{>} 0$.
Case 2: If $\eta \neq 0$, then we can write $(\eta,\zeta) = (\eta, t\eta)$ for a unique $t \in \mathbb{R}$. Thus,

$$\Phi(\eta, t\eta) = \eta^2+\beta_o^2 + \frac{1}{t^2\eta^2+\beta_o^2}(t\eta^2+\beta_o^2)(t+1)\eta.$$

Seeking a contradiction, assume

$$\Phi(\eta_o, t_o\eta_o) \leqslant 0 \tag{65}$$

at some $(\eta_o, t_o\eta_o)$ and put

$$\Psi(t) := \Phi(\eta_o, t\eta_o).$$

Since $\lim_{t\to\pm\infty}\Psi(t)=\eta_o^2+\beta_o^2+\eta_o \overset{(ii)}{>} 0$, it follows from (65) that $\Psi(t)$ has a nonpositive minimum value at some $t_{\min}$. An elementary calculus argument implies:

$$t_{\min}=\begin{cases}\dfrac{\beta_o}{\eta_o} & \text{if } \eta_o<0,\\ -\dfrac{\beta_o}{\eta_o} & \text{if } \eta_o>0.\end{cases}$$

Thus,

$$\Psi(t_{\min})=\begin{cases}\eta_o^2+\beta_o^2+\dfrac{(\eta_o+\beta_o)^2}{2\beta_o} & \text{if } \eta_o<0,\\ \eta_o^2+\beta_o^2-\dfrac{(\eta_o-\beta_o)^2}{2\beta_o} & \text{if } \eta_o>0.\end{cases}$$

Clearly, in the case $\eta_o<0$, $\Psi(t_{\min})>0$, and for $\eta_o>0$

$$\Psi(t_{\min})=\eta_o^2+\beta_o^2-\frac{(\eta_o-\beta_o)^2}{2\beta_o}\overset{(ii)}{>}\eta_o^2+\beta_o^2-\frac{(\eta_o-\beta_o)^2}{2}=\frac{(\eta_o+\beta_o)^2}{2}\geqslant 0,$$

and a contradiction arises, which asserts the conclusion. □

To evaluate the $\mathscr{V}_{j,1}$-multiplicity $m_j(i\beta_o)$, recall that $\lambda_o=i\beta_o$ is a solution to (5), i.e. $\det_{\mathbb{C}}\triangle_{\alpha_o}(i\beta_o)=0$. In the case of system (50), it is equivalent to (cf. (52))

$$(i\beta_o+\zeta)e^{i\beta_o r}-\alpha_o\xi(i\beta_o-\eta)=0.$$

Under Condition (R), we have each $E(i\beta_o)$ is completely contained in a *single* isotypical component U_j for some $j=j_{\beta_o}$ in (4). Thus,

$$m_j(i\beta_o)=\begin{cases}\dim_{\mathbb{C}}E(i\beta_o)/\dim_{\mathbb{C}}\mathscr{U}_j, & j=j_{\beta_o},\\ 0, & j\neq j_{\beta_o}.\end{cases}\tag{66}$$

Without loss of generality,[5] we can assume $\alpha_o<0$, so by Lemma 5.3, sign $(\mathrm{d}u/\mathrm{d}\alpha)(\alpha_o)=-1$. Therefore, by (24), we have

$$\mathrm{t}_{j,1}(\alpha_o,\beta_o)=\begin{cases}\dim_{\mathbb{C}}E(i\beta_o)/\dim_{\mathbb{C}}\mathscr{U}_j, & j=j_{\beta_o},\\ 0, & j\neq j_{\beta_o}.\end{cases}\tag{67}$$

6. Concrete results for selected symmetry groups and usage of Maple© package

In this section, assuming conditions (H1)–(H6) and $\alpha<0$ to be satisfied by system (50), we will present quantative results for some specific symmetry group Γ, where Γ takes values from the dihedral groups D_4, D_5, the octahedral group S_4 and the icosahedral group A_5.

For the details and notations related to the lists of irreducible representations, basic degrees, multiplication tables and identification of the dominating orbit types, we refer to [2].

Below we will briefly summarize our discussions presented in Sections 5.2–5.4, and describe the *input data* to the Maple package used to compute the equivariant bifurcation invariant.

Recall that, by (25),

$$\omega(\lambda_o)_1=\omega_\Gamma\cdot\omega_G,$$

[5] In the case $\alpha_o>0$, the value of $\mathrm{t}_{j,1}(\alpha_o,\beta_o)$ can be obtained simply by reversing its sign (cf. (24)).

where

$$\omega_\Gamma = \prod_{\mu\in\sigma_-}\prod_{i=0}^{r}(\deg_{\mathscr{V}_i})^{m_i(\mu)},$$

and

$$\omega_G = \sum_j \mathrm{t}_{j,1}(\alpha_o,\beta_o)\deg_{\mathscr{V}_{j,1}} \in A_1(G).$$

By formula (59),

$$\omega_\Gamma = \prod_{i=0}^{r}(\deg_{\mathscr{V}_i})^{\sum_{\mu\in\sigma_-} m_{i_\mu}(\mu)}.$$

Note that $(\deg_{\mathscr{V}_i})^2 = (\Gamma)$, which is a trivial element in the Burnside ring $A(\Gamma)$. Therefore, by putting

$$\varepsilon_i := \sum_{\mu\in\sigma_-} m_{i_\mu}(\mu) \ (\mathrm{mod}\, 2), \quad i=0,1,\ldots,r,$$

one obtains

$$\omega_\Gamma = \prod_{i=0}^{r}(\deg_{\mathscr{V}_i})^{\varepsilon_i}.$$

Clearly, the sequence $\{\varepsilon_0,\varepsilon_1,\ldots,\varepsilon_r\}$ permits only possibly finitely many different values.

By formula (67),

$$\omega_G = \dim_{\mathbb{C}} E(i\beta_o)/\dim_{\mathbb{C}} \mathscr{U}_{j_{\beta_o}} \deg_{\mathscr{V}_{j_{\beta_o},1}}.$$

We will use the notation $\mathrm{t}_{j_{\beta_o}} := \dim_{\mathbb{C}} E(i\beta_o)/\dim_{\mathbb{C}} \mathscr{U}_{j_{\beta_o}}$, which stands for the $\mathscr{U}_j$-multiplicity of $i\beta_o$. Thus $\mathrm{t}_{j_{\beta_o}}$ also permits only possibly finitely many different values.

Therefore, we have the following formula for the local bifurcation invariant

$$\omega(\lambda_o)_1 = \prod_{i=0}^{r}(\deg_{\mathscr{V}_i})^{\varepsilon_i}\cdot \mathrm{t}_{j_{\beta_o}}\deg_{\mathscr{V}_{j_{\beta_o},1}}. \tag{68}$$

The input data for the computation of the local invariant thus consists of two finite sequences:

$$\{\varepsilon_0,\varepsilon_1,\ldots,\varepsilon_r\}, \quad \{\mathrm{t}_0,\mathrm{t}_1,\ldots,\mathrm{t}_r\},$$

which are forwarded to the following command from the Maple© package[6]:

$$\omega(\lambda_o)_1 := \mathtt{showdegree}(\varepsilon_0,\varepsilon_1,\ldots,\varepsilon_r,\mathrm{t}_0,\mathrm{t}_1,\ldots,\mathrm{t}_r).$$

Since the value of $\mathrm{t}_{j_{\beta_o}}$ depends only on the $E_{j_{\beta_o}}(i\beta_o)$, we put $E_{j_{\beta_o}} := E_{j_{\beta_o}}(i\beta_o)$ and present our quantative results in a form of a matrix

$E_{j_{\beta_o}}$	$\varepsilon_{i_1},\varepsilon_{i_2},\ldots,\varepsilon_{i_m}$	$\omega(\lambda_o)_1$	# Branches

,

where we only list $\{\varepsilon_{i_1},\varepsilon_{i_2},\ldots,\varepsilon_{i_m}\}\subset\{\varepsilon_0,\varepsilon_1,\ldots,\varepsilon_r\}$ for those ε_{i_k}, which can realize the value 1.

Remark 6.1. Although by Lemma 5.2(c), we are potentially dealing with *infinitely* many bifurcation points, only *finitely* many different values of $\omega(\lambda_o)_1$ can occur, which is related to the fact that the value of $\omega(\lambda_o)_1$ is determined by only possibly *finitely* many different choices of the values of the two sequences $\{\varepsilon_0,\varepsilon_1,\ldots,\varepsilon_r\}$ and $\{\mathrm{t}_0,\mathrm{t}_1,\ldots,\mathrm{t}_r\}$.

[6] The package is available at ⟨http://krawcewicz.net/degree⟩.

The case $\Gamma = D_4$. We have the D_4-isotypical decompositions

$$V = \mathscr{V}_0 \oplus \mathscr{V}_1 \oplus \mathscr{V}_3, \quad V^c = \mathscr{U}_0 \oplus \mathscr{U}_1 \oplus \mathscr{U}_3,$$

thus $\{\varepsilon_{i_1}, \varepsilon_{i_2}, \dots, \varepsilon_{i_m}\} = \{\varepsilon_0, \varepsilon_1, \varepsilon_3\}$, and there are three types of bifurcation points (α_o, β_o) correspondingly. Since getting the complete list of the bifurcation invariants $\omega(\lambda_o)_1$ for system (50) is a simple task of applying the Maple© package for the group $\Gamma = D_4$ by

$$\omega(\lambda_o)_1 = \texttt{showdegree}[D_4](\varepsilon_0, \varepsilon_1, 0, \varepsilon_3, 0, \mathrm{t}_0, \mathrm{t}_1, 0, \mathrm{t}_3, 0),$$

we present in Table 1 only some selected results for the group D_4.

The case $\Gamma = D_5$. We have the following D_5-isotypical decompositions

$$V = \mathscr{V}_0 \oplus \mathscr{V}_1 \oplus \mathscr{V}_2, \quad V^c = \mathscr{U}_0 \oplus \mathscr{U}_1 \oplus \mathscr{U}_2,$$

thus $\{\varepsilon_{i_1}, \varepsilon_{i_2}, \dots, \varepsilon_{i_m}\}=\{\varepsilon_0, \varepsilon_1, \varepsilon_2\}$, and there are three types of bifurcation points (α_o, β_o) correspondingly. We present the list of selected bifurcation invariants $\omega(\lambda_o)_1$ for system (50) in Table 2. To treat the remaining cases, one can use the Maple© package for D_5,

$$\omega(\lambda_o)_1 = \texttt{showdegree}[D_5](\varepsilon_0, \varepsilon_1, \varepsilon_3, 0, \mathrm{t}_0, \mathrm{t}_1, \mathrm{t}_2, 0).$$

The case $\Gamma = S_4$. We have the following S_4-isotypical decompositions

$$V = \mathscr{V}_0 \oplus \mathscr{V}_1 \oplus \mathscr{V}_3 \oplus \mathscr{V}_4, \quad V^c = \mathscr{U}_0 \oplus \mathscr{U}_1 \oplus \mathscr{U}_3 \oplus \mathscr{U}_4,$$

thus $\{\varepsilon_{i_1}, \varepsilon_{i_2}, \dots, \varepsilon_{i_m}\} = \{\varepsilon_0, \varepsilon_1, \varepsilon_3, \varepsilon_4\}$, and there are four types of bifurcation points (α_o, β_o) correspondingly. The list of selected bifurcation invariants $\omega(\lambda_o)_1$ for system (50) is presented in Table 3. One can use the Maple© package for the group $\Gamma = S_4$ to obtain the remaining invariants,

$$\omega(\lambda_o)_1 = \texttt{showdegree}[S_4](\varepsilon_0, \varepsilon_1, 0, \varepsilon_3, \varepsilon_4, \mathrm{t}_0, \mathrm{t}_1, 0, \mathrm{t}_3, \mathrm{t}_4).$$

The case $\Gamma = A_5$. We have the following A_5-isotypical decompositions

$$V = \mathscr{V}_0 \oplus [\mathscr{V}_1 \oplus \mathscr{V}_1] \oplus \mathscr{V}_2 \oplus \mathscr{V}_3 \oplus \mathscr{V}_4,$$
$$V^c = \mathscr{U}_0 \oplus [\mathscr{U}_1 \oplus \mathscr{U}_1] \oplus \mathscr{U}_2 \oplus \mathscr{U}_3 \oplus \mathscr{U}_4,$$

thus $\{\varepsilon_{i_1}, \varepsilon_{i_2}, \dots, \varepsilon_{i_m}\} = \{\varepsilon_0, \varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4\}$, and there are five types of bifurcation points (α_o, β_o) correspondingly. A partial list of the bifurcation invariants $\omega(\lambda_o)_1$ for system (50) is presented in Table 4, which was established by using the Maple© package for the group $\Gamma = A_5$,

$$\omega(\lambda_o)_1 = \texttt{showdegree}[A_5](\varepsilon_0, \varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4, \mathrm{t}_0, \mathrm{t}_1, \mathrm{t}_2, \mathrm{t}_3, \mathrm{t}_4).$$

Let us explain quickly how to estimate, based on the value of $\omega(\lambda_o)_1$, the minimal number of branches bifurcating from $(\alpha_o, 0)$ as well as their symmetries. In Fig. 4, we present the isotropy lattice (for twisted one-folded orbit types) for W. The dominating orbit types (H) in W, which are (A_5), (D_3^z), (V_4^-), $(\mathbb{Z}_5^{t_1})$, $(\mathbb{Z}_5^{t_2})$, $(A_4^{t_1})$, $(A_4^{t_2})$ and (D_5^z), are indicated by bold characters. They are maximal elements in this lattice under the usual partial order relation. We use the notation $(H)^{[n]}$ to indicate that there are n S^1-orbits in an orbit $G(x)$ of the type (H). Consider, for example,

$$\omega(\lambda_o)_1 = -(A_4) - (\mathbf{D_3^z}) - (D_3) + (\mathbb{Z}_\mathbf{5}^{\mathbf{t_1}}) + (\mathbb{Z}_\mathbf{5}^{\mathbf{t_2}}) - (\mathbf{V_4^-}) + (\mathbb{Z}_3^t) + 3(\mathbb{Z}_3).$$

Then, we will definitely have 10 branches of periodic solutions with the symmetries (D_3^z), 12 with the symmetries $(\mathbb{Z}_5^{t_1})$, 12 with the symmetries $(\mathbb{Z}_5^{t_2})$, and 15 with the symmetries (V_4^-). Since (A_5) is the dominating orbit type larger than the orbit types (A_4), (D_3), it follows that there must be at least one additional branch of periodic solutions. On the other hand, the only dominating orbit types larger than $(\mathbb{Z}_3^t)$ are $(A_4^{t_1})$ and $(A_4^{t_2})$, therefore, the nontriviality of the $(\mathbb{Z}_3^t)$-coefficient implies the existence of 5 additional branches of periodic solutions. All together, we predict 55 branches of nontrivial periodic solutions.

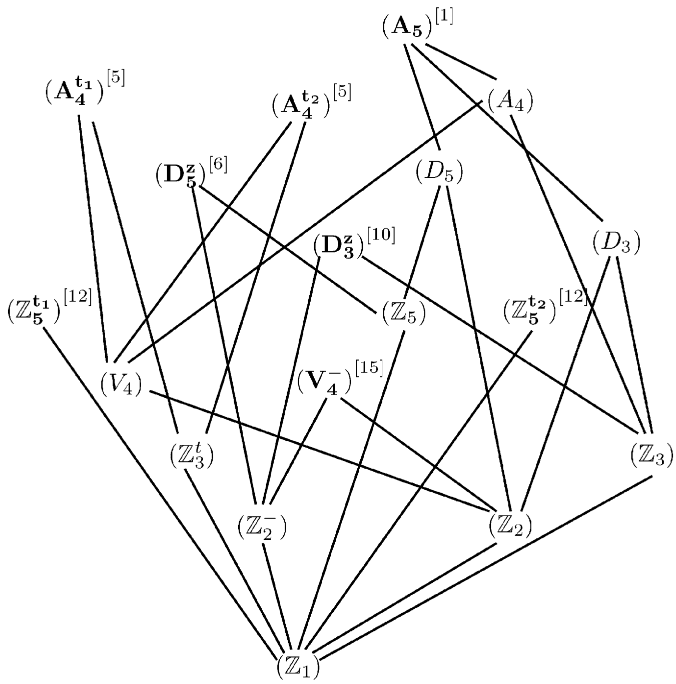

Fig. 4. Twisted one-folded orbit types in W for $\Gamma = A_5$.

References

[1] R.R. Akhmerov, M.I. Kamenskii, A.S. Potapov, A.E. Rodkina, B.N. Sadovskii, Measures of Noncompactness and Condensing Operators, Birkhäuser, Boston, Basel, Berlin, 1992.

[2] Z. Balanov, M. Farzamirad, W. Krawcewicz, Symmetric systems of van der Pol equations, Topol. Meth. Nonlin. Anal. 27 (1) (2006) 29–90.

[3] Z. Balanov, M. Farzamirad, W. Krawcewicz, H. Ruan, Applied equivariant degree. Part II: symmetric Hopf bifurcation for functional differential equations, Discrete Continuous Dyn. Systems Series A 16 (2006).

[4] Z. Balanov, W. Krawcewicz, H. Ruan, Applied equivariant degree. Part III: global symmetric Hopf bifurcation for functional differential equations, Proceedings of the Latvian Academy of Sciences, Section B: Natural, Exact and Applied Sciences 59 (6) (2005) 20–30.

[5] Z. Balanov, W. Krawcewicz, H. Ruan, Applied equivariant degree. Part I: an axiomatic approach to primary degree, Discrete Continuous Dyn. Systems Series A 15 (2006) 983–1016.

[6] T. Bröcker, T. Tom Dieck, Representations of Compact Lie Groups, Springer, Berlin, 1985.

[7] L.H. Erbe, K. Geba, W. Krawcewicz, J. Wu, S^1-degree and global Hopf bifurcation theory of functional differential equations, J. Differential Equations 97 (1992) 227–239.

[9] J. Ize, A. Vignoli, Equivariant Degree Theory, De Gruyter Series in Nonlinear Analysis and Applications, 2003.

[10] W. Krawcewicz, P. Vivi, J. Wu, Computational formulae of an equivariant degree with applications to symmetric bifurcations, Nonlinear Stud. 4 (1997) 89–119.

[12] W. Krawcewicz, J. Wu, Theory of Degrees with Applications to Bifurcations and Differential Equations, CMS Series of Monographs, Wiley, New York, 1997.

[13] A. Magnusson, V.K. Tripathi, Transmission Lines and Wave Propagation, third ed., CRC Press, Boca Raton, 1992.

[14] S.L. Sobolev, Some Applications of Functional Analysis in Mathematical Physics, American Mathematical Society, Providence, RI, 1991.

[15] A.N. Tikhonov, A.A. Samarskii, Equations of Mathematical Physics, The Macmillan Company, Pergamon Press, 1963.

[16] T. Tom Dieck, Transformation Groups, de Gruyter, Berlin, 1987.

[17] J. Wu, Theory and Applications of Partial Functional Differential Equations, Springer, New York, 1996.