Applications of Equivariant Degree for Gradient Maps to Symmetric Newtonian Systems

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Abstract

We consider $G = \Gamma \times S^1$ with Γ being a finite group, for which the complete Euler ring structure in $U(G)$ is described. The multiplication tables for $\Gamma = D_6$, S_4 and A_5 are provided in the Appendix. The equivariant degree for G-orthogonal maps is constructed using the primary equivariant degree with one free parameter. We show that the G-orthogonal degree extends the degree for G-gradiernt maps (in the case $G = \Gamma \times S^1$) introduced by K. Gęba in [19]. The obtained computational results are applied to a Γ-symmetric autonomous Newtonian system for which we study the existence of 2π-periodic solutions. For some concrete cases, we present the symmetric classification of the solution set for the considered systems.

1 Introduction

Various versions of the equivariant degree (cf. [14, 16, 18, 19, 21, 27], see also [2, 8, 15, 24, 28, 29, 30, 31, 34]), which are important tools of the equivariant analysis, provide an effective alternative to such methods as Conley index, Morse theory, minimax techniques and singularity theory. The main difficulty related to the usage of the equivariant degree seems to be its complicated construction relying on the notions from the equivariant topology, homotopy theory and algebraic topology. However, as it was shown in [2], certain equivariant degree —- the so-called *primary* equivariant degree, can be fully described by a set of *axioms*, allowing its usage outside the context of its theoretical roots. In addition, many elaborated algebraic computations are completely computerized*, making this method even more efficient.

The objective of this paper is to establish an underlying relation between the equivariant degree for gradient maps and the primary equivariant degree, and then by means of the equivariant degree theory, to study the existence problem for a system of variational ordinary differential equations in the presence of symmetries.

More precisely, suppose that V is an orthogonal Γ-representation and let $\varphi \in C^2(V;\mathbb{R})$ be a Γ-invariant function such that

(a) $(\nabla\varphi)^{-1}(0) = \{0\}$, and

(b) $\nabla\varphi(x) = Bx + o(\|x\|)$ as $\|x\| \to \infty$, where B is a Γ-equivariant linear operator.

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* The equivariant degree Maple© Library package is available at `http://krawcewicz.net/degree` or `http://www.math.ualberta.ca/∼wkrawcew/degree`.

We are interested in the existence of non-trivial periodic solutions, their multiplicity and symmetry properties, to the following system of ODEs

$$\ddot{x} = -\nabla\varphi(x). \tag{1.1}$$

It should be pointed out that in the non-symmetric case (i.e. $\Gamma = \{e\}$), the problem (1.1) was studied in [17] using the S^1-equivariant degree for gradient maps (see also [1, 10]). Here, we extend the definition for $G = \Gamma \times S^1$-equivariant degree for gradient maps using primary equivariant degree (cf. [2]-[5],[8, 24]). In this way, we can use all the computational techniques developed for the primary $\Gamma \times S^1$-equivariant degree, to fully compute the $\Gamma \times S^1$-equivariant degree for gradient maps.

2 Euler Ring $U(G)$ and Its Computations

2.1 Notations

Hereafter, G is a compact Lie group. For a closed subgroup H of G, we write $H \subset G$. Denote by (H) the conjugacy class of H in G, $N(H)$ – the *normalizer* of H in G, and $W(H) = N(H)/H$ – the *Weyl group* of H in G. Put

$$\Phi(G) := \{(H) \ : \ H \subset G\},$$

which admits a natural partial order: $(L) \leq (H)$ if L is conjugate to a subgroup of H.

Let X be a G-invariant set and $x \in X$. We adopt the following notations:

$$\begin{aligned}
G_x &:= \{g \in G \ : \ gx = x\},\\
G(x) &:= \{gx \ : \ g \in G\},\\
X^H &:= \{x \in X \ : \ H \subset G_x\},\\
X_H &:= \{x \in X \ : \ H = G_x\},\\
X^{(H)} &:= G(X^H), \quad X_{(H)} := G(X_H),\\
\mathcal{J}(X) &:= \{(H) \in \Phi(G) \ : \ \exists x \in X \ s.t. \ H = G_x\}.
\end{aligned}$$

Let V be an orthogonal G-representation. For $r > 0$, denote by

$$B_r(V) := \{v \in V \ : \ \|v\| < r\},$$

and write $B(V) := B_1(V)$ for the unit ball in V. Similar notations will also be used for an isometric Banach G-representation W.

2.2 Numbers $n(L, H)$

The following number $n(L, H)$ (cf. [2]) is needed for the computations of the Euler ring multiplication via recurrence formulas (cf. (2.3.1.1)-(2.3.2.2) in Subsection 2.3):

Definition 2.2.1. *Let $(L), (H) \in \Phi(G)$ be such that $(L) \leq (H)$. Define the set*

$$N(L, H) := \left\{g \in G : L \subset gHg^{-1}\right\},$$

and

$$n(L, H) := |N(L, H)/N(H)|,$$

where the symbol $|Y|$ stands for the cardinality of the set Y.

Remark 2.2.1. (i) Notice that the value of the number $n(L,H)$ does not depend on the choice of representatives in the conjugacy classes (L) and (H). Therefore, we assume that the number $n(L,H)$ is determined for representatives L and H such that $L \subset H$.

(ii) In the case (L) and (H) are not comparable with respect to the partial order "$\le$", we simply put $n(L,H) = 0$.

(iii) In general, it is possible that $n(L,H) = \infty$. However, in the case $\dim W(L) = \dim W(H)$, $n(L,H)$ is finite and has a very simple geometric interpretation (cf. [2]).

Lemma 2.2.1. *Let $L, H \subset G$ be such that $(L) \le (H)$ and $\dim W(H) = \dim W(L)$. Then, $n(L,H)$ represents the number of different subgroups $\widetilde{H}$ in the conjugacy class (H) such that $L \subset \widetilde{H}$. In particular, if V is an orthogonal G-representation such that (L), $(H) \in \mathcal{J}(V)$, then $V^L \cap V_{(H)}$ is a disjoint union of exactly $m = n(L,H)$ sets V_{H_j}, $j = 1, 2, \dots, m$, satisfying $(H_j) = (H)$.*

2.3 Euler Ring $U(G)$

The degree for gradient G-maps, which is defined later in Section 3, takes value in the Euler ring $U(G)$ (cf. [32]).

Definition 2.3.1. Given a compact Lie group G, the *Euler ring* $U(G)$ is the free $\mathbb{Z}$-module generated by $\Phi(G)$, i.e. $U(G) = \mathbb{Z}[\Phi(G)]$, with the multiplication $\star : U(G) \times U(G) \to U(G)$ defined on the generators by the formula

$$(H) \star (K) = \sum_{(L) \in \Phi(G)} n_L \cdot (L), \tag{2.3.1}$$

where $n_L = \chi_c\Big(\left(G/H \times G/K\right)_L / W(L)\Big)$ with χ_c standing for the Euler characteristic in Alexander-Spanier cohomology with compact support (cf. [26]).

Throughout the rest of Section 2, we assume that Γ is a finite group and $G = \Gamma \times S^1$.

Notation 2.3.1. For $k = 0, 1$, denote $\Phi_k(G) := \big\{(H) \in \Phi(G) : \dim W(H) = k\big\}$, $A_k(G) := \mathbb{Z}[\Phi_k(G)]$. Notice that $\dim G = 1$ clearly implies $\dim W(H) = \dim N(H) - \dim H \in \{0, 1\}$, so we have

$$\Phi(G) = \Phi_0(G) \cup \Phi_1(G),$$

and

$$U(G) = A_0(G) \oplus A_1(G).$$

Remark 2.3.1. Notice that $A(G) := A_0(G) \subset U(G)$ is the so-called *Burnside ring* of G (cf. [4, 32, 33]), which can be identified with the Burnside ring $A(\Gamma)$ of Γ. Indeed, the map $\Psi : \Phi_0(G) \to \Phi(\Gamma)$, defined by

$$\Psi \left[(\mathcal{H} \times S^1)\right] := (\mathcal{H}) \tag{2.3.2}$$

induces a ring isomorphism from $A(G)$ to $A(\Gamma)$.

2.3.1 Multiplication $\star|_{A_0(G)\times A_0(G)} : A_0(G) \times A_0(G) \to A_0(G)$

By Remark 2.3.1, the Euler ring multiplication $\star$, when restricted to $A_0(G) \times A_0(G)$, can be completely described by the Burnside ring multiplication on $A(\Gamma)$. Therefore, based on the description of the $A(\Gamma)$-multiplication formula obtained in [4], we have the following computational formula, defined on the generators $(\mathcal{H}), (\mathcal{K}) \in \Phi(\Gamma)$,

$$(\mathcal{H} \times S^1) \star (\mathcal{K} \times S^1) = \sum_{(\mathcal{L})\in\Phi(\Gamma)} n_{\mathcal{L}} \cdot (\mathcal{L} \times S^1), \tag{2.3.1.1}$$

where

$$n_{\mathcal{L}} = \frac{1}{|W_\Gamma(\mathcal{L})|}\left[n(\mathcal{L},\mathcal{H}) \cdot |W_\Gamma(\mathcal{H})| \cdot n(\mathcal{L},\mathcal{K}) \cdot |W_\Gamma(\mathcal{K})| - \sum_{(\widetilde{\mathcal{L}})>(\mathcal{L})} n(\mathcal{L},\widetilde{\mathcal{L}}) \cdot n_{\widetilde{\mathcal{L}}} \cdot |W_\Gamma(\widetilde{\mathcal{L}})|\right], \tag{2.3.1.2}$$

(here $W_\Gamma(\mathcal{L})$ means the Weyl group of $\mathcal{L}$ is taken in Γ).

2.3.2 Multiplication $\star|_{A_0(G)\times A_1(G)} : A_0(G) \times A_1(G) \to A_1(G)$

By using the identification $A_0(G) \simeq A(\Gamma)$ (cf. Remark 2.3.1), one can describe the $U(G)$-multiplication $\star$ restricted to $A_0(G)\times A_1(G)$, as the $A(\Gamma)$-module structure on $A_1(G)$ (cf. [2, 8, 24]).

More specifically, the elements in $\Phi_1(G)$ are the conjugacy classes of the so-called *φ-twisted l-folded* ($l \in \mathbb{N}$) subgroups in G, i.e. the subgroups of the type

$$\mathcal{H}^{\varphi,l} := \{(\gamma, z) \in \mathcal{H} \times S^1 : \varphi(\gamma) = z^l\}$$

where $\mathcal{H} \subset \Gamma$ is a subgroup and $\varphi : \mathcal{H} \to S^1$ is a group homomorphism. Then (cf. [24]),

Theorem 2.3.2.1. *Suppose that $G = \Gamma \times S^1$, where Γ is a finite group. Then $A_1(G)$ is an $A_0(G)$-module with the multiplication $\star : A_0(G) \times A_1(G) \to A_1(G)$ defined on the generators $(\mathcal{K} \times S^1) \in A_0(G)$ and $(\mathcal{H}^{\varphi,l}) \in A_1(G)$ by*

$$(\mathcal{K} \times S^1) \star (\mathcal{H}^{\varphi,l}) = \sum_{(\mathcal{L})\in\Phi(\Gamma)} n_{\mathcal{L}} \cdot \left(\mathcal{L}^{\varphi,l}\right), \tag{2.3.2.1}$$

where the coefficients $n_{\mathcal{L}} = \left|\left(G/\mathcal{K} \times S^1 \times G/\mathcal{H}^{\varphi,l}\right)_{\mathcal{L}^{\varphi,l}} / W(\mathcal{L}^{\varphi,l})\right|$ can be computed by the recurrence formula

$$\begin{aligned} n_{\mathcal{L}} = \frac{1}{|W(\mathcal{L}^{\varphi,l})/S^1|} \Big[& n(\mathcal{L},\mathcal{K}) \cdot |W_\Gamma(\mathcal{K})| \cdot n(\mathcal{L}^{\varphi,l},\mathcal{H}^{\varphi,l}) \cdot \left|W(\mathcal{H}^{\varphi,l})/S^1\right| \\ & - \sum_{(\tilde{\mathcal{L}})>(\mathcal{L})} n(\mathcal{L}^{\varphi,l},\tilde{\mathcal{L}}^{\varphi,l}) \cdot n_{\tilde{\mathcal{L}}} \cdot \left|W(\tilde{\mathcal{L}}^{\varphi,l})/S^1\right| \Big]. \end{aligned} \tag{2.3.2.2}$$

2.3.3 Multiplication $\star|_{A_1(G)\times A_1(G)} : A_1(G) \times A_1(G) \to A_1(G)$

We have the following result

Proposition 2.3.3.1. *For $G = \Gamma \times S^1$ with Γ being a finite group, the multiplication in $U(G)$, when restricted to $A_1(G) \times A_1(G)$, is trivial, i.e. for any $(\mathcal{H}^{\varphi_1,l_1}), (\mathcal{K}^{\varphi_2,l_2}) \in \Phi_1(G)$, we have*

$$(\mathcal{H}^{\varphi_1,l_1}) \star (\mathcal{K}^{\varphi_2,l_2}) = 0.$$

Proof: Put $(H) := (\mathcal{H}^{\varphi_1,l_1})$, $(K) := (\mathcal{K}^{\varphi_2,l_2})$, $X := G/H \times G/K$. According to (2.3.1), it is sufficient to show that $n_L = \chi_c\Big(X_L/W(L)\Big) = 0$ for all $(L) \in \Phi(G)$. Notice that $(g_1H, g_2K) \in X_L$ if and only if $L = g_1Hg_1^{-1} \cap g_2Kg_2^{-1}$. In particular, $X_L \neq \emptyset$ implies that $\dim W(L) = \dim W\left(g_1Hg_1^{-1} \cap g_2Kg_2^{-1}\right) \geq \min\{\dim W(H), \dim W(K)\} = 1$. On the other hand, it is clear that $\dim W(L) \leq 1$. Consequently, $X_L \neq \emptyset$ only if $\dim W(L) = 1$. Thus, without loss of generality, we assume $(L) \in \Phi_1(G)$.

Claim 1. $\chi_c\Big(X^L/W(L)\Big) = 0$ for all $(L) \in \Phi_1(G)$.

Clearly, $X^L = (G/H)^L \times (G/K)^L$ is a closed 2-dimensional submanifold of $G/H \times G/K$. We will show that each connected component of X^L has exactly *one* orbit type under the $W(L)$-action.

Take $x := (g_1H, g_2K) \in X^L$ (i.e. $L \subset g_1Hg_1^{-1} \cap g_2Kg_2^{-1}$). Write $g_1 = (\gamma_1, z_1), g_2 = (\gamma_2, z_2) \in \Gamma \times S^1$. Consider the $T^2 = S^1 \times S^1$-action on X^L given by

$$(w_1, w_2)\Big(g_1H, g_2K\Big) := \Big((\gamma_1, w_1z_1)H, (\gamma_2, w_2z_2)K\Big), \quad w := (w_1, w_2) \in T^2,\ g_1, g_2 \in G.$$

By direct verification,

$$(w_1, w_2)\Big(g_1H, g_2K\Big) = (g_1H, g_2K) \quad \Longleftrightarrow \quad w_1^{l_1} = 1, w_2^{l_2} = 1,$$

i.e. $T^2_x = \mathbb{Z}_{l_1} \times \mathbb{Z}_{l_2}$, for all $x \in X^L$. In other words, every orbit in X^L has precisely *one* orbit type $(\mathbb{Z}_{l_1} \times \mathbb{Z}_{l_2})$, thus by the existence of differentiable structure on the orbit space (cf. Theorem 4.18, [22]), X^L/T^2 is a smooth manifold of dimension $\dim X^L - \dim T^2$, which in our case, is a finite set. So we can describe X^L as a union of finitely many orbits $T^2(x) \simeq T^2$, i.e.

$$X^L = X_1^L \cup X_2^L \cup \cdots \cup X_m^L,$$

where $X_i^L \simeq T^2$ is the i-th connected component of X^L.

It is clear that two elements x, y in X^L belong to the same connected component if and only if there exists $z \in T^2$ such that $y = zx$. In addition, for $x \in X^L$, $W(L)_x = (G_x \cap N(L))/L$. Thus, to show that every connected component of X^L has exactly one orbit type, it is sufficient to show that for any $x \in X^L$, $z \in T^2$, we have $G_x = G_{zx}$. Indeed, assume $x = (g_1H, g_2K)$ with $g_1 = (\gamma_1, z_1)$, $g_2 = (\gamma_2, z_2)$. Take $(\gamma_o, w_o) \in \Gamma \times S^1$, then

$$\begin{aligned}
(\gamma_o, w_o) \in G_x \quad &\Longleftrightarrow \quad (\gamma_o, w_o)(g_1H, g_2K) = (g_1H, g_2K)\\
&\Longleftrightarrow \quad ((\gamma_o\gamma_1, w_oz_1)H, (\gamma_o\gamma_2, w_oz_2)K) = ((\gamma_1, z_1)H, (\gamma_2, z_2)K)\\
&\Longleftrightarrow \quad \gamma_1^{-1}\gamma_o\gamma_1\mathcal{H} = \mathcal{H}, \quad \gamma_2^{-1}\gamma_o\gamma_2\mathcal{K} = \mathcal{K}.
\end{aligned}$$

Since the above condition of $(\gamma_o, w_o) \in G_x$ does not depend on the choice of z_1, z_2, we have $G_x = G_{zx}$ for all $x \in X^L$ and $z \in T^2$.

Denote by (L_i) $(i = 1, 2, \ldots, k)$ the $W(L)$-orbit types of X^L. Then, we can write X^L as

$$X^L = X^L_{(L_1)} \cup X^L_{(L_2)} \cup \cdots \cup X^L_{(L_k)}. \tag{2.3.3.1}$$

By a similar argument (cf. Theorem 4.18, [22]), each $X^L_{(L_i)}/W(L)$ is a smooth manifold, and each L_i is a finite subgroup in G. Thus,

$$\dim\left(X^L_{(L_i)}/W(L)\right) = \dim\left(X^L_{(L_i)}/N(L)\right) = 2 - 1 + 0 = 1, \quad i = 1, 2, \ldots, k.$$

Therefore, combined with (2.3.3.1),

$$\dim \left(X^L/W(L)\right) = 1,$$

i.e. $X^L/W(L)$ is a 1-dimensional compact manifold, and therefore, $\chi_c\Big(X^L/N(L)\Big) = 0$.

On the other hand, it is well-known that (for example, see [32])

$$\chi_c\Big(X^L/W(L)\Big) = \sum_{(\widetilde{L})\geq(L)} \chi_c(X_{\widetilde{L}}/W(\widetilde{L})).$$

Hence, by Claim 1, we obtain

$$\begin{aligned} 0 &= \sum_{(\widetilde{L})\geq(L)} \chi_c(X_{\widetilde{L}}/W(\widetilde{L})) \\ &= \chi_c(X_L/W(L)) + \sum_{(\widetilde{L})>(L)} \chi_c(X_{\widetilde{L}}/W(\widetilde{L})) \\ &= n_L + \sum_{(\widetilde{L})>(L)} \chi_c(X_{\widetilde{L}}/W(\widetilde{L})). \qquad (2.3.3.2) \end{aligned}$$

In the case (L) is a maximal orbit type in X, we have $X_L/W(L) = X^L/W(L)$, so $n_L = \chi_c\Big(X_L/W(L)\Big) = \chi\Big(X^L/N(L)\Big) = 0$. Otherwise, by applying the induction over the orbit types in X according to the partial order, it follows from (2.3.3.2) that $n_L = \chi_c(X_L/W(L)) = 0$ for all $(L) \in \Phi_1(G)$. □

In this way we obtain:

Theorem 2.3.3.1. *Let $G = \Gamma \times S^1$ with Γ being a finite group. Then the multiplication table for the Euler ring $U(\Gamma \times S^1)$ is given by*

	$A_0(G)$	$A_1(G)$
$A_0(G)$	$A(\Gamma)$-multiplication	$A(\Gamma)$-module $A_1(G)$ multiplication
$A_1(G)$	$A(\Gamma)$-module $A_1(G)$ multiplication	0

Table 1: Multiplication Table for $U(\Gamma \times S^1)$

where we identify $A_0(G)$ with the Burnside ring $A(\Gamma)$ (see Remark 2.3.1).

As examples, we present in the Appendix the multiplication tables for $U(\Gamma \times S^1)$ in the case Γ takes value of the dihedral group D_6, the octahedral group S_4 and the icosahedral group A_5. These tables are established by using a special Maple© routines*.

* The equivariant degree Maple© Library package is available at `http://krawcewicz.net/degree` or `http://www.math.ualberta.ca/∼wkrawcew/degree`.

3 Equivariant Degree for Gradient G-Maps

In this section, we follow the construction of the G-equivariant degree for gradient G-maps introduced by K. Gęba in [18] (which we will denote by ∇_G-deg). Based on the properties of ∇_G-deg , we derive an axiomatic definition of the degree for gradient G-maps.

Let G be a compact Lie group and V be an orthogonal G-representation. Consider a C^1-differentiable G-invariant function $\varphi : V \to \mathbb{R}$. Then the gradient $\nabla\varphi : V \to V$ is a G-equivariant continuous map.

Definition 3.1. (i) A map $f : V \to V$ is called a *gradient G-map* if there exists a G-invariant function $\varphi : V \to \mathbb{R}$ of class C^1 such that $f = \nabla\varphi$. Similarly, we say a map $h : [0,1]\times V \to V$ is a *gradient G-homotopy* if there exists a G-invariant C^1-function $\psi : [0,1]\times V \to \mathbb{R}$ such that $h_t = \nabla\psi_t$, where $h_t(x) := h(t,x)$, $\psi_t(x) := \psi(t,x)$ for all $(t,x) \in [0,1]\times V$.

(ii) Let $\Omega \subset V$ be an open bounded G-invariant subset and $f : V \to V$ a continuous map. The pair (f,Ω) is called a ∇_G*-admissible* pair, if f is a gradient G-map satisfying $f(x) \neq 0$ for all $x \in \partial\Omega$. Two ∇_G-admissible pairs (f_0,Ω) and (f_1,Ω) are ∇_G*-homotopic*, if there exists a gradient G-homotopy $h : [0,1]\times V \to V$ such that $h(0,\cdot) = f_0$, $h(1,\cdot) = f_1$ with $(h(t,\cdot),\Omega)$ being ∇_G-admissible for all $t \in (0,1)$.

Take $x \in V$, put $H := G_x$, and consider the orthogonal decomposition of V

$$V = \tau_x G(x) \oplus W_x \oplus \nu_x, \tag{3.1}$$

where τM denotes the tangent bundle of M, $W_x := \tau_x V_{(H)} \ominus \tau_x G(x)$ and $\nu_x := (\tau_x V_{(H)})^{\perp}$.

Suppose $f : V \to V$ is a gradient G-map being differentiable at x and $f(x) = 0$. The derivative $Df(x)$ has a block-matrix form with respect to (3.1) (see [18] for more details)

$$Df(x) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & Kf(x) & 0 \\ 0 & 0 & Lf(x) \end{bmatrix}, \tag{3.2}$$

where $Kf(x) := Df(x)|_{W_x}$ and $Lf(x) := Df(x)|_{\nu_x}$.

Definition 3.2. (i) An orbit $G(x)$ is called a *regular zero orbit* of f, if $f(x) = 0$ and $Kf(x) : W_x \to W_x$ (provided by (3.2)) is an isomorphism. Let $E_-(x) \subset W_x$ denote the generalized eigenspace of $Kf(x)$ corresponding to the negative spectrum of $Kf(x)$. Then $\kappa_x := \dim E_-(x)$ is called the *Morse index* of the regular zero orbit $G(x)$. Put

$$i(G(x)) := (-1)^{\kappa(x)}, \tag{3.3}$$

or equivalently,

$$i(G(x)) := \det Kf(x) = \det Df(x)|_{W_x}.$$

(ii) For an open G-invariant subset U of $V_{(H)}$ such that $\overline{U} \subset V_{(H)}$, and a small* $\varepsilon > 0$, put

$$\mathcal{N}(U,\varepsilon) := \{y \in V \ : \ y = x + v, x \in U, v \perp \tau_x V_{(H)}, \|v\| < \varepsilon\},$$

and call it a *tubular neighborhood of type* (H). A gradient G-map $f : V \to V$, $f := \nabla\varphi$ is called (H)*-normal*, if there exists a tubular neiborhood $\mathcal{N}(U,\varepsilon)$ of type (H) such that $f^{-1}(0) \cap \Omega_{(H)} \subset \mathcal{N}(U,\varepsilon)$ and for $y \in \mathcal{N}(U,\varepsilon)$, $y = x + v$, $x \in U, v \perp \tau_x V_{(H)}$,

$$\varphi(y) = \varphi(x) + \frac{1}{2}\|v\|^2,$$

$*$ ε is assumed to be sufficiently small that the representation $y = x + v$ in $\mathcal{N}(U,\varepsilon)$ is unique.

or equivalently,

$$f(y) = f(x) + v.$$

The following notion of ∇_G-generic pair plays an essential role in the construction of the equivariant degree for G-maps presented in [18].

Definition 3.3. A ∇_G-admissible pair (f, Ω) is ∇_G*-generic* if there exists an open G-invariant subset $\Omega_o \subset \Omega$ such that

(i) $f|_{\Omega_o}$ is of class C^1;

(ii) $f^{-1}(0) \cap \Omega \subset \Omega_o$;

(iii) $f^{-1}(0) \cap \Omega_o$ is composed of regular zero orbits;

(iv) For each (H) with $f^{-1}(0) \cap \Omega_{(H)} \neq \emptyset$, there exists a tubular neighborhood $\mathcal{N}(U, \varepsilon)$ such that f is (H)-normal on $\mathcal{N}(U, \varepsilon)$.

Theorem 3.1. (Generic Approximation Theorem, cf. [18]) *For any ∇_G-admissible pair (f, Ω) there exists a ∇_G-generic pair (f_o, Ω) such that (f, Ω) and (f_o, Ω) are ∇_G-homotopic.*

Define the equivariant degree for a ∇_G-admissible pair (f, Ω) by

$$\nabla_G\text{-deg}\,(f, \Omega) := \nabla_G\text{-deg}\,(f_o, \Omega) = \sum_{(H) \in \Phi(G)} \mathfrak{n}_H \cdot (H), \tag{3.4}$$

where (f_o, Ω) is the ∇_G-generic approximation pair of (f, Ω) provided by Theorem 3.1 and

$$\mathfrak{n}_H := \sum_{(G_{x_i}) = (H)} i(G(x_i)), \tag{3.5}$$

with $G(x_i)$'s being the disjoint orbits of type (H) in $f_o^{-1}(0) \cap \Omega$.

We refer to [18] for the verification that $\nabla_G\text{-deg}\,(f, \Omega)$ is well-defined and satisfies the standard properties expected from a degree.

Now, we are in a position to formulate an alternative axiomatic definition of the degree for gradient G-maps.

Theorem 3.2. *Let G be a compact Lie group, $\Omega \subset V$ be an open bounded G-invariant subset and $f : V \to V$ be a gradient G-map. There exists a unique function $\nabla_G\text{-deg}$ associating to each ∇_G-admissible pair (f, Ω) an element $\nabla_G\text{-deg}\,(f, \Omega) \in U(G)$ such that the following properties are satisfied:*

(P1) (Existence) *If $\nabla_G\text{-deg}\,(f, \Omega) = \sum_{(H)} \mathfrak{n}_H (H)$, is such that $\mathfrak{n}_{H_o} \neq 0$ for some $(H_o) \in \Phi(G)$, then there exists $x_o \in \Omega$ with $f(x_o) = 0$ and $H_o \subset G_x$.*

(P2) (Additivity) *Suppose that Ω_1 and Ω_2 are two disjoint open G-invariant subsets of Ω such that $f^{-1}(0) \cap \Omega \subset \Omega_1 \cup \Omega_2$. Then*

$$\nabla_G\text{-deg}\,(f, \Omega) = \nabla_G\text{-deg}\,(f, \Omega_1) + \nabla_G\text{-deg}\,(f, \Omega_2).$$

(P3) (Homotopy) *If $h : [0,1] \times V \to V$ is a ∇_G-admissible homotopy, then*

$$\nabla_G\text{-deg}\,(h_t, \Omega) = \text{constant},$$

where $h_t(\cdot) := h(t, \cdot)$ for $t \in [0,1]$.

(P4) (MULTIPLICATIVITY) *Let V and W be two orthogonal G-representations, (f,Ω) and $(\widetilde{f},\widetilde{\Omega})$ two ∇_G-admissible pairs, where $\Omega\subset V$ and $\widetilde{\Omega}\subset W$. Then*

$$\nabla_G\text{-deg}\,(f\times\widetilde{f},\Omega\times\widetilde{\Omega})=\nabla_G\text{-deg}\,(f,\Omega)\star\nabla_G\text{-deg}\,(\widetilde{f},\widetilde{\Omega}),$$

where the multiplication '$\star$' is taken in the Euler ring $U(G)$.

(P5) (NORMALIZATION) *Suppose (f,Ω) is a ∇_G-generic pair such that $f^{-1}(0)\cap\Omega=G(x_o)$, for some $x_o\in\Omega$ with $H_o:=G_{x_o}$. Let $\mathcal{N}(U,\varepsilon)$ be a tubular neighborhood provided by Definition 3.3(iv) and $i(G(x_o))$ be defined by (3.3). Then*

$$\nabla_G\text{-deg}\,(f,\mathcal{N}(U,\varepsilon))=i(G(x_o))(H_o).$$

(P6) (SUSPENSION) *Suppose that W is another orthogonal G-representation and let $\mathcal{O}$ be an open bounded G-invariant neighborhood of 0 in W. Then*

$$\nabla_G\text{-deg}\,(f\times\mathrm{Id}\,,\Omega\times\mathcal{O})=\nabla_G\text{-deg}\,(f,\Omega).$$

Proof: *Existence.* The existence of ∇_G-deg satisfying (P1)-(P5) is guaranteed by its construction as shown in [18]. The suspension property (P6) is a direct consequence of (P4) and (P5). Indeed, by (P4), we have

$$\nabla_G\text{-deg}\,(f\times\mathrm{Id}\,,\Omega\times\mathcal{O})=\nabla_G\text{-deg}\,(f,\Omega)\star\nabla_G\text{-deg}\,(\mathrm{Id}\,,\mathcal{O}).$$

Since $(\mathrm{Id}\,,\mathcal{O})$ is ∇_G-generic, by (P5),

$$\nabla_G\text{-deg}\,(\mathrm{Id}\,,\mathcal{O})=i(\{0\})\,(G)=(G),$$

which is a trivial element in $U(G)$, thus (P6) follows.
Uniqueness. The uniqueness of ∇_G-deg (f,Ω) is provided by (P5), which leads to its analytic definition (see (3.4)–(3.5)). □

In what follows we will be interested in the case $G=\Gamma\times S^1$ with Γ being a finite group, for which we will show that one can pass the computations of ∇_G-deg (f,Ω) onto the computations of the primary equivariant degree for an associated map $F:\mathbb{R}\oplus V\to V$.

4 Degree for Equivariant Orthogonal Maps

In this section, using the primary degree (cf. [2]), we define the equivariant degree for G-orthogonal maps, for $G=\Gamma\times S^1$ with Γ being a finite group. This extented definition (following the result in [27] for $G=S^1$) turns out to *coincide* with the equivariant degree ∇_G-deg for gradient G-maps discussed in Section 3. In this way, one can take advantage of all the computational bases established for the primary degree (cf. [2]-[7]) to effectively compute for ∇_G-deg and study symmetric variational problems.

4.1 G-Orthogonal Maps

Definition 4.1.1. A G-equivariant map $f:V\to V$ is called *G-orthogonal* on a set $\Omega\subset V$ if f is continuous and for all $v\in\Omega$ the vector $f(v)$ is orthogonal to the orbit $G(v)$ at v. A pair (f,Ω) is called a *G-orthogonal admissible pair*, if f is G-orthogonal on Ω and $f(x)\neq 0$ for all $x\in\partial\Omega$. Two G-orthogonal admissible pairs (f_0,Ω) and (f_1,Ω) are *G-orthogonally homotopic*, if there exists a G-equivariant $[0,1]\times\Omega$-admissible map $h:[0,1]\times V\to V$ such that $h(0,\cdot)=f_0$, $h(1,\cdot)=f_1$ and $h_t:=h(t,\cdot):V\to V$ is a G-orthogonal map for all $t\in(0,1)$. Such a map h is called *G-orthogonal homotopy.*

Throughout the rest of this section, we assume that $G = \Gamma \times S^1$, for some finite group Γ.

In this case, the definition of G-orthogonality can be represented in a simple way. Consider $v \in V$ and the map $\varphi_v : G \to G(v)$ given by

$$\varphi_v(g) = gv, \quad g \in G.$$

Clearly φ_v is smooth and $D\varphi_v(1) : \tau_1(G) = \tau_1(S^1) \to \tau_v(G(v))$. Since the total space of the tangent bundle to S^1 can be written as

$$\tau(S^1) = \{(z, \gamma) \in \mathbb{C} \times S^1 : z \perp \gamma\} = \{(z, \gamma) \in \mathbb{C} \times S^1 : z = it\gamma,\ t \in \mathbb{R}\},$$

a tangent vector to the orbit $G(v)$ can be represented by

$$\tau(v) := D\varphi_v(1)(i) = \lim_{t \to 0} \frac{1}{t}\Big[e^{it}v - v\Big]. \tag{4.1.1}$$

Notice that for any $v \in V^{S^1}$, we have $\tau(v) = 0$. Thus, by using the decomposition

$$V = V^{S^1} \oplus V', \quad V' := (V^{S^1})^{\perp}, \tag{4.1.2}$$

we have that a G-equivariant map $f : V \to V$ is G-orthogonal, if and only if

$$\langle f(x, u), (0, \tau(u))\rangle = 0,$$

for every $v = (x, u) \in V = V^{S^1} \oplus V'$.

Example 4.1.1. Let $\psi : V \to \mathbb{R}$ be a continuously differentiable G-invariant function, i.e. $\psi(gv) = \psi(v)$ for all $g \in G$ and $v \in V$. Then, by the chain rule, $f := \nabla\psi : V \to V$ is G-equivariant. In addition, we have for every $v = (x, u) \in V^{S^1} \oplus V'$,

$$\frac{d}{dt}\psi(e^{it}v)\bigg|_{t=0} = \frac{d}{dt}\psi(x, e^{it}u)\bigg|_{t=0} = \langle \nabla\psi(x, u), (0, \tau(u))\rangle = 0.$$

Consequently, every G-gradient map f is G-orthogonal.

It should be pointed out that G-equivariant gradient maps do not exhaust the set of all G-orthogonal maps, as the following simple example shows.

Example 4.1.2. Let $G = S^1$ act on $V := \mathbb{R} \oplus \mathbb{C}$ by $e^{it}(x, z) = (x, e^{it} \cdot z)$, $e^{it} \in S^1$, $x \in \mathbb{R}$, $z \in \mathbb{C}$, and let $f : V \to V$ be defined by

$$f(x, z) = (x|z|^2, z).$$

By direct computation, f is G-orthogonal. However, since the derivative $Df(x, z)$ is not symmetric, in general, f is not a gradient map.

The following remark indicates that the standard linearization procedure is applicable to G-orthogonal C^1-maps.

Remark 4.1.1. Suppose that $f : V \to V$ is a G-orthogonal map of C^1 class and $v_o \in V^G$. Then, the derivative $Df(v_o) : V \to V$ is a linear G-orthogonal map. Indeed, since f is G-orthogonal, for all $v \in V$ and $h \in \mathbb{R}$, $h \neq 0$,

$$0 = \langle f(v_o + hv), \tau(v_o + hv)\rangle = \langle hf(v_o + hv), \tau(v)\rangle, \quad \text{and} \quad \langle f(v_o), \tau(v)\rangle = 0,$$

thus

$$\langle Df(v_o)v, \tau(v)\rangle = \lim_{h \to 0} \frac{1}{h}[\langle f(v_o + hv), \tau(v)\rangle - \langle f(v_o), \tau(v)\rangle] = 0.$$

4.2 S^1-Normality Condition

Given a G-orthogonal Ω-admissible map $f : V \to V$, we would like to associate to f a G-equivariant map $F : \mathbb{R} \oplus V \to V$ such that: (i) $F^{-1}(0) = \{0\} \times f^{-1}(0)$, and (ii) the equivariant homotopy properties of F are "close" to those of f. Such an association would allow us to take advantage of the developed computational techniques of the primary G-equivariant degree of F.

Observe that introducing a new dimension, in general, results in a larger set of zeros. In addition, the equivariance usually gets in conflict with the transversality. Nevertheless, in the case of G-orthogonal maps from V to V for $v \notin V^{S^1}$, the specific character of G-orthogonal maps suggests a *natural* candidate for F, namely:

$$F(t, v) = f(v) + t\tau(v), \quad t \in \mathbb{R},\ v \notin V^{S^1}. \tag{4.2.1}$$

However, in the case $v \in V^{S^1}$, F defined by (4.2.1) fails to satisfy (i) on V^{S^1}, which forces us to consider the map f on V^{S^1} and $V' = (V^{S^1})^\perp$ respectively. To this end, we need the so-called S^1-*normality* condition for orthogonal maps.

Definition 4.2.1. A G-orthogonal map $f : V \to V$ is called S^1-*normal* (on Ω) if

$$\exists_{\delta>0}\ \forall_{x\in\Omega^{S^1}}\ \forall_{u\perp V^{S^1}} \quad \|u\| < \delta \implies f(x+u) = f(x) + u. \tag{4.2.2}$$

Similarly, we say that a G-orthogonal homotopy $h : [0,1] \times V \to V$ is S^1-*normal* (on Ω) if

$$\exists_{\delta>0}\ \forall_{(t,x)\in[0,1]\times\Omega^{S^1}}\ \forall_{u\perp V^{S^1}} \quad \|u\| < \delta \implies h(t, x+u) = h(t, x) + u. \tag{4.2.3}$$

We call δ appearing in (4.2.2) and/or (4.2.3) the S^1-*normality constant.*

The following result provides us with S^1-normal approximations of G-orthogonal maps and G-orthogonal homotopies.

Theorem 4.2.1. *Suppose that (f, Ω) is a G-orthogonal admissible pair. Then, for every $\varepsilon > 0$ there exists a G-orthogonal S^1-normal Ω-admissible map $f_o : V \to V$ such that*

$$\forall_{v\in\overline{\Omega}} \quad \|f(v) - f_o(v)\| < \varepsilon. \tag{4.2.4}$$

Moreover, if $h : [0,1] \times V \to V$ is a G-orthogonal homotopy, then for every $\varepsilon > 0$ there exists a G-orthogonal and S^1-normal Ω-admissible map $h_o : [0,1] \times V \to V$ such that

$$\forall_{(t,v)\in[0,1]\times\overline{\Omega}} \quad \|h(t, v) - h_o(t, v)\| < \varepsilon. \tag{4.2.5}$$

In addition, if $h(0,\cdot) =: f_0$ and $h(1,\cdot) =: f_1$ are S^1-normal (on Ω), then the homotopy h_o can be constructed in such a way that $h_o(0,\cdot) = f_0$ and $h_o(1,\cdot) = f_1$.

Proof: Consider the decomposition (4.1.2) of V. For $v \in V$, we write $v = (x, u)$, where $x \in V^{S^1}$ and $u \in V'$. Given $\delta > 0$, define the function $\eta_\delta : \mathbb{R} \to \mathbb{R}$ by

$$\eta_\delta(\rho) := \begin{cases} 0 & \text{if } \rho \le \delta, \\ \frac{\rho-\delta}{\delta} & \text{if } \delta < \rho < 2\delta, \\ 1 & \text{if } \rho \ge 2\delta, \end{cases}$$

(see Figure 4.2).

Next, define the map $f_o : V \to V$ by

$$f_o(v) = f_o(x, u) := f(x, \eta_\delta(\|u\|)u) + (1 - \eta_\delta(\|u\|))u. \tag{4.2.6}$$

η_δ

1

ρ

δ 2δ

Figure 1: Bump function η_δ

By construction, f_o is G-orthogonal and S^1-normal on Ω (with δ as the S^1-normality constant).

Put $\varepsilon_o := \inf_{v \in \partial\Omega} \{\|f(v)\|\}$. By the Ω-admissibility of f, $\varepsilon_o > 0$. We can assume $\varepsilon \leq \frac{\varepsilon_o}{2}$. Otherwise, replace ε with $\min\{\varepsilon, \frac{\varepsilon_o}{2}\}$. We claim that for every such $0 < \varepsilon < \frac{\varepsilon_o}{2}$, there exists a proper $\delta > 0$, such that the map f_o defined by (4.2.6) satisfies (4.2.4). Since for any $v = (x, u) \in V$ with $\|u\| \geq 2\delta$, $f_o(v) = f(x, u) = f(v)$, it is sufficient to show (4.2.4) for $v = (x, u) \in \overline{\Omega}$ with $\|u\| < 2\delta$.

By the uniform continuity of f on $\overline{\Omega}$, there exists $\delta_1 > 0$ such that

$$\forall_{v,v' \in \overline{\Omega}} \ \ \|v - v'\| < \delta_1 \ \ \Rightarrow \ \ \|f(v) - f(v')\| < \frac{\varepsilon}{2}.$$

Choose $\delta := \min\{\frac{\delta_1}{2}, \frac{\varepsilon}{2}\} > 0$, thus for all $v = (x, u) \in \overline{\Omega}$ with $\|u\| < 2\delta (< \delta_1)$,

$$\begin{aligned} \|f(v) - f_o(v)\| &= \|f(x, u) - f_o(x, u)\| \\ &= \|f(x, u) - f(x, \eta_\delta(\|u\|)u) - (1 - \eta_\delta(\|u\|))u\| \\ &\leq \|f(x, u) - f(x, \eta_\delta(\|u\|)u)\| + (1 - \eta_\delta(\|u\|))\|u\| \\ &< \frac{\varepsilon}{2} + \delta < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

By the assumption $\varepsilon \leq \frac{\varepsilon_o}{2}$,

$$\forall_{v \in \overline{\Omega}} \ \ \|f(v) - f_o(v)\| < \varepsilon \leq \frac{\varepsilon_o}{2}.$$

Thus, for all $v \in \partial\Omega$,

$$\begin{aligned} \|f_o(v)\| &\geq \|f(v)\| - \|f(v) - f_o(v)\| \\ &\geq \varepsilon_o - \frac{\varepsilon_o}{2} = \frac{\varepsilon_o}{2} > 0. \end{aligned}$$

Consequently, f_o is Ω-admissible.

Define the homotopy $h : [0, 1] \times V \to V$ by

$$h(t, v) := f(x, tu + (1 - t)\eta_\delta(\|u\|)u) + (1 - t)(1 - \eta_\delta(\|u\|))u,$$

where $t \in [0, 1]$. It is clear that $h(0, \cdot) = f_o$ and $h(1, \cdot) = f$. Notice that for $v \in V$ with $\|u\| \geq 2\delta$, $h(t, v) \equiv f(x, u) = f(v)$. To check the Ω-admissibility of $h(t, \cdot)$, it is enough to show that for all $(t, v) \in [0, 1] \times \partial\Omega$ with $\|u\| < 2\delta$, we have $\|h(t, v)\| > 0$. Indeed,

$$\begin{aligned} \|h(t, v) - f(v)\| &\leq \|f(x, tu + (1 - t)\eta_\delta(\|u\|)u) - f(x, u)\| \\ &\quad + \|(1 - t)(1 - \eta_\delta(\|u\|))u\| \\ &< \frac{\varepsilon}{2} + \|u\| \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \leq \frac{\varepsilon_o}{2}, \end{aligned}$$

thus

$$\|h(t,v)\| \geq \|f(v)\| - \|h(t,v)-f(v)\| > \varepsilon_o - \frac{\varepsilon_o}{2} = \frac{\varepsilon_o}{2} > 0.$$

Consequently, h is an Ω-admissible homotopy. In order to verify that h is G-orthogonal on $\overline{\Omega}$, we notice that for $(t,v)=(t,x,u)\in[0,1]\times\overline{\Omega}$,

$$\begin{aligned}\langle h(t,x,u),(0,\tau(u))\rangle &= \langle f(x,(t+(1-t)\eta_\delta(\|u\|))u),(0,\tau(u))\rangle \\ &\quad + (1-t)(1-\eta_\delta(\|u\|))\langle u,(0,\tau(u))\rangle\rangle = 0.\end{aligned}$$

The proof of the second part of Theorem 4.2.1 (for G-orthogonal homotopies) is similar. □

4.3 Construction

Let (f,Ω) be a G-orthogonal admissible pair. By Theorem 4.2.1, there exists an S^1-normal approximation map f_o such that

$$\forall_{v\in\overline{\Omega}}\ \ \|f(v)-f_o(v)\| < \varepsilon := \frac{1}{4}\inf_{v\in\partial\Omega}\{\|f(v)\|\}. \tag{4.3.1}$$

Denote by δ the S^1-normality constant of f_o, and put

$$U_\delta := \{(t,v)\in(-1,1)\times\Omega : v=x+u,\ x\in V^{S^1},\ u\in V',\ \ \|u\|>\delta\}. \tag{4.3.2}$$

Define $F_o:\mathbb{R}\oplus V\to V$ by

$$F_o(t,v) := f_o(v)+t\tau(v), \quad (t,v)\in\mathbb{R}\oplus V, \tag{4.3.3}$$

where $\tau(v)$ is the tangent vector to the orbit $G(v)$ (cf. (4.1.1)). It is clear that F_o is G-equivariant and U_δ-admissible. Also, $f_o|_{V^{S^1}}:V^{S^1}\to V^{S^1}$ is Γ-equivariant and Ω^{S^1}-admissible.

Therefore, we can define the *orthogonal G-equivariant degree* $G\text{-Deg}^{\,o}(f,\Omega)$ of the pair (f,Ω) to be the element in $A_0(G)\oplus A_1(G)\simeq A(\Gamma)\oplus A_1(G)\simeq U(G)$ given by

$$G\text{-Deg}\,(f,\Omega) := \Big(\text{Deg}^o_\Gamma(f,\Omega),\text{Deg}^o_G(f,\Omega)\Big), \tag{4.3.4}$$

where $\text{Deg}^o_\Gamma(f,\Omega)$ is an element in $A(\Gamma)$ defined by

$$\text{Deg}^o_\Gamma(f,\Omega) := \Gamma\text{-Deg}(f_o|_{V^{S^1}},\Omega^{S^1}), \tag{4.3.5}$$

and $\text{Deg}^o_G(f,\Omega)$ is an element in $A_1(G)$ defined by

$$\text{Deg}^o_G(f,\Omega) := G\text{-Deg}\,(F_o,U_\delta), \tag{4.3.6}$$

with $G\text{-Deg}\,(F_o,U_\delta)$ standing for the primary G-degree of F_o on U_δ (see [2] for more details).

We claim that the definition (4.3.4)-(4.3.6) is independent of the choice of a G-orthogonal S^1-normal approximation f_o. Indeed, assume that $f'_o:V\to V$ is another S^1-normal approximation of f such that

$$\forall_{v\in\overline{\Omega}}\ \ \|f(v)-f'_o(v)\| < \varepsilon. \tag{4.3.7}$$

Let δ' be the S^1-normality constant of f'_o, and $U_{\delta'}$ be given by (4.3.2). Define $F'_o:\mathbb{R}\oplus V\to V$ by

$$F'_o(t,v) := f'_o(v)+t\tau(v), \quad (t,v)\in\mathbb{R}\oplus V.$$

Put $\bar{\delta} := \min\{\delta, \delta'\}$, and define $U_{\bar{\delta}}$ by (4.3.2). By the excision property of the primary degree, we have

$$G\text{-Deg}\,(F_o, U_{\bar{\delta}}) = G\text{-Deg}\,(F_o, U_\delta),$$

and

$$G\text{-Deg}\,(F_o', U_{\bar{\delta}}) = G\text{-Deg}\,(F_o', U_{\delta'}).$$

Also, by (4.3.1) and (4.3.7), we have that f_o and f_o' are G-orthogonally homotopic on Ω. In particular, $f_o|_{V^{S^1}}$ and $f_o'|_{V^{S^1}}$ are Γ-homotopic on Ω^{S^1}, thus, by the homotopy property of the primary degree,

$$\Gamma\text{-Deg}(\bar{f}_o, \Omega^{S^1}) = \Gamma\text{-Deg}(\bar{f}_o', \Omega^{S^1}).$$

Moreover, F_o and F_o' are G-orthogonally homotopic on $U_{\bar{\delta}}$, so by the homotopy property of the primary degree, we have

$$G\text{-Deg}\,(F_o, U_{\bar{\delta}}) = G\text{-Deg}\,(F_o', U_{\bar{\delta}}).$$

Therefore,

$$G\text{-Deg}\,(F_o, U_\delta) = G\text{-Deg}\,(F_o', U_{\delta'}).$$

Definition 4.3.1. (cf. [2]). Let V be an orthogonal G-representation, $k \in \{0,1\}$, $\Omega \subset \mathbb{R}^k \oplus V$ an open bounded invariant set and $f : \mathbb{R}^k \oplus V \to V$ an Ω-admissible G-equivariant map. We say that f is *regular normal* in Ω, if

(i) f is of class C^1;

(ii) for every $(H) \in \Phi_k(G, \Omega)$ and $x \in f^{-1}(0) \cap \Omega_H$, the following (H)-*normality* condition at x is satisfied: *There exists $\delta_x > 0$ such that for all $w \in \nu_x(\Omega_\alpha)$ with $\|w\| < \delta_x$,*

$$f(x + w) = f(x) + w = w;$$

(iii) for every $(H) \in \Phi_k(G, \Omega)$, zero is a regular value of $f_H := f|_{\Omega_H} : \Omega_H \to V^H$.

Remark 4.3.1. In the case of G-equivariant maps, with or without free parameters, the so-called *regular normal approximation* theorem was established in [26], which states that every G-equivariant map can be approximated (on a compact set) by a regular normal G-map. If (f_o, Ω) is an admissible pair such that f_o is G-gradient generic map in Ω then $f_o|_{\Omega^{S^1}}$ is regular normal (without free parameterrs) and the map F_o corresponding to f_o (defined by (26)) is also regular normal (in this case it is a G-map with one free parameter).

Definition 4.3.2. Let V be an orthogonal G-representation, $k \in \{0,1\}$ and $f : \mathbb{R}^k \oplus V \to V$ a regular normal map such that $f(x_o) = 0$ with $G_{x_o} = H$ and $(H) \in \Phi_k(G)$. Let $U_{G(x_o)}$ be a G-invariant tubular neighborhood around $G(x_o)$ such that $f^{-1}(0) \cap U_{G(x_o)} = G(x_o)$. Then f is called a *tubular map* around $G(x_o)$. In addition, if S_{x_o} is a positively oriented slice to $W(H)(x_o)$ in $\mathbb{R}^k \oplus V^H$ (cf. [2]), then we call $n_{x_o} = \text{sign}\det Df^H(x_o)|_{S_{x_o}}$ the *local index* of f at x_o in $U_{G(x_o)}$ (here $f^H := f|_{\Omega^H}$).

In this way, we obtain the following

Theorem 4.3.1. *To each G-orthogonal Ω-admissible map $f : V \to V$, one can associate the orthogonal G-equivariant degree $G\text{-Deg}^{\,o}(f, \Omega) \in U(G)$ (see formula (4.3.4)) satisfying the properties:*

(P1) (Existence) *If $G\text{-Deg}^{\,o}(f, \Omega) = (\text{Deg}^o_\Gamma(f, \Omega), \text{Deg}^o_G(f, \Omega))$ is such that*

$$\text{Deg}^o_\Gamma(f, \Omega) = \sum_{(\mathcal{H}) \in A(\Gamma)} n_{\mathcal{H}} \cdot (\mathcal{H}) \neq 0$$

or

$$\mathrm{Deg}^o_G(f,\Omega) = \sum_{(H)\in A_1(G)} n_H \cdot (H) \neq 0$$

i.e. $n_{\mathcal{H}_o} \neq 0$ *for some* $(\mathcal{H}_o) \in \Phi(\Gamma)$ *or* $n_{H_o} \neq 0$ *for some* $(H_o) \in \Phi_1(G)$*, then there exists* $x_o \in \Omega$ *such that* $f(x_o) = 0$ *and*

(a) $G_{x_o} \supset \mathcal{H}_o \times S^1$ *if* $(\mathcal{H}_o) \in \Phi(\Gamma)$*;*

(b) $G_{x_o} \supset H_o$ *if* $(H_o) \in \Phi_1(G)$*.*

(P2) (Additivity) *Suppose that* Ω_1 *and* Ω_2 *are two disjoint open* G*-invariant subsets of* Ω *such that* $f^{-1}(0) \cap \Omega \subset \Omega_1 \cup \Omega_2$*. Then*

$$G\text{-}\mathrm{Deg}^o(f,\Omega) = G\text{-}\mathrm{Deg}^o(f,\Omega_1) + G\text{-}\mathrm{Deg}^o(f,\Omega_2).$$

(P3) (Homotopy) *If* $h : [0,1] \times V \to V$ *is a* G*-orthogonal* Ω*-admissible homotopy, then*

$$G\text{-}\mathrm{Deg}^o(h_t,\Omega) = \text{constant}, \qquad \textit{for all } t \in [0,1],$$

where $h_t(\cdot) := h(t,\cdot)$ *for* $t \in [0,1]$*.*

(P4) (Multiplicativity) *Let* V *and* W *be two orthogonal* G*-representations,* (f,Ω) *and* $(\widetilde{f},\widetilde{\Omega})$ *two* ∇_G*-admissible pairs, where* $\Omega \subset V$ *and* $\widetilde{\Omega} \subset W$*. Then*

$$G\text{-}\mathrm{Deg}^o(f \times \widetilde{f}, \Omega \times \widetilde{\Omega}) = G\text{-}\mathrm{Deg}^o(f,\Omega) \star G\text{-}\mathrm{Deg}^o(\widetilde{f},\widetilde{\Omega}),$$

where the multiplication '$\star$*' is taken in the Euler ring* $U(G)$*.*

(P5) (Normalization) *Suppose* f *is a tubular map around* $G(x_o)$*, where* $(H) := (G_{x_o}) \in \Phi_k(G)$*, for some* $k \in \{0,1\}$*. In particular, write* $H = \mathcal{H} \times S^1$ *if* $k = 0$*. Denote by* n_{x_o} *the local index of* f *at* x_o *in a tubular neighborhood* $U_{G(x_o)}$*. Then*

$$G\text{-}\mathrm{Deg}^o(f,U_{G(x_o)}) = \begin{cases} (n_{x_o}(\mathcal{H}),0) & \textit{if } k=0, \\ (0,n_{x_o}(H)) & \textit{if } k=1. \end{cases}$$

(P6) (Suspension) *Let* W *be another orthogonal* G*-representation and* $\mathcal{O} \subset W$ *an open bounded* G*-invariant neighborhood of* 0 *in* W*. Then*

$$G\text{-}\mathrm{Deg}^o(f \times \mathrm{Id}, \Omega \times \mathcal{O}) = G\text{-}\mathrm{Deg}^o(f,\Omega).$$

Proof: All the above properties are direct consequences of the corresponding properties of the primary degree with one free parameter and primary degree without free parameter (cf. [2, 24]). □

We have the following result

Theorem 4.3.2. *Let* $G = \Gamma \times S^1$ *(with* Γ *being a finite group) and* (f,Ω) *be a* ∇_G*-admissible pair. Then*

$$G\text{-}\mathrm{Deg}^o(f,\Omega) = \nabla_G\text{-}\mathrm{deg}(f,\Omega),$$

under the identification $A_0(G) \simeq A(\Gamma)$ *(cf. (2.3.2)).*

Proof: By Theorem 3.1, we can assume that f is a ∇_G-generic map on Ω. Then, by definition, $f^{-1}(0)\cap\Omega$ is composed of several regular zero orbits, say $G(x_p)$ of type (H_p), $p=0,1,\dots,r$, and for each $G(x_p)$, there is a tubular neighborhood $\mathcal{N}(U_p,\varepsilon_p)$ such that $f(x)=f(u)+v$ for all $x=u+v\in\mathcal{N}(U_p,\varepsilon)$, $u\in V_{(H_p)}$, $v\perp\tau_x V_{(H_p)}$. Put $\varepsilon=\min_p\{\varepsilon_p\}$ and $\Omega_o:=\bigcup_p\mathcal{N}(U_p,\varepsilon)$. Then, $Lf(x)=\mathrm{Id}$ for all $x\in\Omega_o$. In particular, f is S^1-normal on Ω_o.

Since $f^{-1}(0)\cap\Omega\subset\Omega_o$, by the excision property,

$$\nabla_G\text{-deg}\,(f,\Omega)=\nabla_G\text{-deg}\,(f,\Omega_o)=\sum_p\nabla_G\text{-deg}\,(f,\mathcal{N}(U_p,\varepsilon)),$$

and

$$\nabla_G\text{-deg}\,(f,\mathcal{N}(U_p,\varepsilon))=i(G(x_p))(H_p).$$

In the case $(H_p)\in\Phi_0(G)$, i.e. $H_p=\mathcal{H}\times S^1$ for some $\mathcal{H}\subset S^1$, $x_p\in\mathcal{N}(U_p,\varepsilon)\subset V^{S^1}$. Since f is generic on Ω^{S^1}, f is a regular normal map on $\mathcal{N}(U_p,\varepsilon)$. Thus,

$$\Gamma\text{-Deg}(f|_{V^{S^1}},\mathcal{N}(U_p,\varepsilon))=n_{x_p}(\mathcal{H}).$$

We need to show that $n_{x_p}=i(G(x_p))$. Indeed, since

$$G(x_p)\simeq G/G_{x_p}=\frac{\Gamma\times S^1}{\mathcal{H}\times S^1}\simeq\Gamma/\mathcal{H}$$

is a finite set, we have $\tau_{x_p}(G(x_p))=\{0\}$. Hence, the decomposition (3.1) reduces to

$$V=\tau_{x_p}V_{(H_p)}\oplus\nu_x,$$

and the block-matrix (3.2) reduces to

$$Df(x_p)=\begin{bmatrix}Kf(x_p)&0\\0&Lf(x_p)\end{bmatrix},$$

and $Lf(x_p)=\mathrm{Id}$. Also, notice that $\tau_{x_p}V_{(H_p)}=\tau_{x_p}V_{(\mathcal{H}\times S^1)}\subset V^{S^1}$, so

$$n_{x_p}=\text{sign}\,\det(Df(x_p)|_{V^{S^1}})=\text{sign}\,\det(Df(x_p)|_{\tau_{x_p}V_{(H_p)}})=\text{sign}\,\det Kf(x_p)=i(G(x_p)).$$

In the case $(H_p)\in\Phi_1(G)$, i.e. $H=\mathcal{H}^{\varphi,l}$ for some $\mathcal{H}\subset\Gamma$, $\varphi:\mathcal{H}\to S^1$, $l\in\mathbb{N}$. Since f is S^1-normal on Ω_o and generic on Ω, by the construction of $F:\mathbb{R}\oplus V\to V$(cf. (4.3.3)), we have F is regular normal on U_δ (where δ is the S^1-normality constant for f). In particular, F is regular normal on $\{0\}\times\mathcal{N}(U_p,\varepsilon)$, i.e. F is a tubular map around $G(x_p)$. Thus,

$$G\text{-Deg}\,(F,\{0\}\times\mathcal{N}(U_p,\varepsilon))=n_{x_p}(H_p).$$

We need to show that $n_{x_p}=i(G(x_p))$. Since $\nu_{x_p}\cap V^{H_p}=\{0\}$, we have $DF(x_p)|_{\mathbb{R}\oplus V^{H_p}}:\mathbb{R}\oplus V^{H_p}\to V^{H_p}$ has the following form

$$DF(x_p)=\begin{bmatrix}1&*&*\\0&*&KF(x_p)\end{bmatrix},$$

with respect to $V^{H_p}=(\tau_{x_p}(G(x_p))\cap V^{H_p})\oplus(W_{x_p}\cap V^{H_p})$. The slice S_{x_p} is orthogonal to $\tau_{x_p}(G(x_p))$, thus $DF(x_p)|_{S_{x_p}}:S_{x_p}\to V^{H_p}$ is given by

$$DF(x_p)=\begin{bmatrix}1&*\\0&Kf(x_p)\end{bmatrix}.$$

Therefore,

$$n_{x_p}=\text{sign}\,\det(DF(x_p)|_{S_{x_p}})=\text{sign}\,\det(Kf(x_p))=i(G(x_p)).$$

Consequently, $G\text{-Deg}^{\,o}(f,\Omega)=\nabla_G\text{-deg}\,(f,\Omega)$. □

5 Degree of Equivariant Gradient Linear Maps

In this section, we derive the computational formulas for the degree of G-equivariant gradient *linear* maps. The standard linearization procedure allows us to accomplish computations of the equivariant degrees for much more complicated gradient maps.

5.1 Computational Formula for Degree of G-Gradient Linear Maps

Throughout this section, $G = \Gamma \times S^1$ for a finite group Γ and V stands for an orthogonal G-representation. Consider a symmetric G-equivariant linear isomorphism $A : V \to V$. Clearly, $(A, B(V))$ is a ∇_G-admissible pair and $\nabla_G\text{-deg}\,(A, B(V))$ is well defined. One can compute $\nabla_G\text{-deg}\,(A, B(V))$ using the so-called *basic degrees* and the multiplicativity property of degree.

Let $\mathcal{V}_i$, $i = 0, 1, \dots, r$ be the complete list of all real irreducible Γ-representations, where $\mathcal{V}_0$ denotes the trivial irreducible Γ-representation. Then, we have the following Γ-*isotypical decomposition* of V

$$V = V_0 \oplus V_1 \oplus \cdots \oplus V_r, \tag{5.1.1}$$

where V_i is modeled on the $\mathcal{V}_i$ for $i = 0, 1, \dots, r$. In particular, $V_0 = V^\Gamma$. Let V^c denote a complexification of V over $\mathbb{R}$. Then V^c has a natural structure of a complex Γ-representation given by $\gamma(z \otimes x) = z \otimes \gamma x$, for $z \in \mathbb{C}$ and $x \in V$. Simialry, we consider a Γ-isotypical decomposition of the complex representation V^c

$$V^c = U_0 \oplus U_1 \oplus \cdots \oplus U_s, \tag{5.1.2}$$

where $U_0 = (V^c)^\Gamma$ and U_j is modeled on the complex irreducible Γ-representation $\mathcal{U}_j$ (cf. [12]).

For each complex Γ-representation $\mathcal{U}_j$, $j = 0, 1, \dots, s$, and $l = 1, 2, \dots$, we can define a $\Gamma \times S^1$-action on $\mathcal{U}_j$ by

$$(\gamma, z)w = z^l \cdot (\gamma w), \quad (\gamma, z) \in \Gamma \times S^1, \quad w \in \mathcal{U}_j,$$

where '$\cdot$' is the complex multiplication. This real $\Gamma \times S^1$-representation remains irreducible and is denoted by $\mathcal{V}_{j,l}$.

Consider the decomposition (4.1.2) of V. Since A is a G-gradient linear isomorphism, we have that $\bar{A} := A|_{V^{S^1}} : V^{S^1} \to V^{S^1}$ and $A' := A|_{V'} : V' \to V'$ are two G-gradient linear isomorphisms. By the multiplicativity property of the degree for G-gradient maps,

$$\nabla_G\text{-deg}\,(A, B(V)) = \nabla_G\text{-deg}\,(\bar{A}, B(V^{S^1})) \star \nabla_G\text{-deg}\,(A', B(V')).$$

To compute $\nabla_G\text{-deg}\,(\bar{A}, B(V^{S^1}))$, denote by $\sigma_-(\bar{A})$ the negative spectrum of $\bar{A} : V^{S^1} \to V^{S^1}$. For every $\mu \in \sigma_-(\bar{A})$, denote the corresponding eigenspace by $E(\mu)$. Since $E(\mu) \subset V^{S^1}$ is Γ-invariant, it has the following Γ-isotypical decomposition

$$E(\mu) = E_0(\mu) \oplus E_1(\mu) \oplus \cdots \oplus E_r(\mu),$$

where the component $E_i(\mu)$ is modeled on the irreducible Γ-representation $\mathcal{V}_i$, $i = 0, 1, \dots, r$. Define

$$m_i(\mu) = \dim E_i(\mu) / \dim \mathcal{V}_i, \quad i = 0, 1, \dots, r,$$

which will be called the $\mathcal{V}_i$-*multiplicity* of the eigenvalue μ.

Define the *basic degree* for the irreducible Γ-representation $\mathcal{V}_i$ by

$$\deg_{\mathcal{V}_i} := \nabla_G\text{-deg}\,(-\mathrm{Id}\,, B(\mathcal{V}_i)) \in A(G). \tag{5.1.3}$$

Then, again by the multiplicativity property, we obtain

$$\nabla_G\text{-deg}\,(\bar{A}, B(V^{S^1})) = \prod_{\mu \in \sigma_-(\bar{A})} \prod_{i=0}^{r} (\deg_{\mathcal{V}_i})^{m_i(\mu)}. \tag{5.1.4}$$

Since $A' : V' \to V'$ is symmetric, thus it is diagonalizable. Denote by $\sigma_-(A') = \{\xi_1, \xi_2, \dots, \xi_l\}$ (resp. $\sigma_+(A') = \{\xi_{l+1}, \xi_{l+2}, \dots, \xi_m\}$) the negative (resp. positive) spectrum of A', and for each eigenvalue ξ of A' denote by $E(\xi)$ the corresponding eigenspace in V'. Put

$$V'_- := \bigoplus_{\xi \in \sigma_-(A')} E(\xi), \quad \text{and} \quad V'_+ := \bigoplus_{\xi \in \sigma_+(A')} E(\xi),$$

i.e. $V' = V'_- \oplus V'_+$, and define $A'_- := A'|_{V'_-}$ and $A'_+ := A'|_{V'_+}$. Then, for $s \in [0,1]$, define the operators

$$B'_s := \Big(-s\text{Id} + (1-s)A'_-, s\text{Id} + (1-s)A'_+ \Big) : V'_- \oplus V'_+ \to V'_- \oplus V'_+.$$

Obviously, for any $s \in [0,1]$, the linear isomorphism $B'_s : V' \to V'$ is a symmetric G-equivariant operator, thus it is a G-gradient linear map. Put $B' = B'_1$.

For each eigenspace $E(\xi)$, $\xi \in \sigma_-(A')$, we consider the G-isotypical decomposition

$$E(\xi) = \bigoplus_{j,l} E_{j,l}(\xi),$$

where the component $E_{j,l}(\xi)$ is modelled on the irreducible G-representation $\mathcal{V}_{j,l}$, and define the $\mathcal{V}_{j,l}$-*multiplicity* of ξ by

$$m_{j,l}(\xi) := \dim E_{j,l}(\xi) / \dim \mathcal{V}_{j,l}. \tag{5.1.5}$$

Put $\text{Deg}_{\mathcal{V}_{j,l}} := \nabla_G\text{-deg}\,(-\text{Id}\,, B(\mathcal{V}_{j,l})) \in U(G)$ and define $\text{deg}_{\mathcal{V}_{j,l}}$ by the identity

$$\text{Deg}_{\mathcal{V}_{j,l}} = (G) + \text{deg}_{\mathcal{V}_{j,l}}.$$

One can directly verify that $\text{deg}_{\mathcal{V}_{j,l}} \in A_1(G)$. We will call $\text{deg}_{\mathcal{V}_{j,l}}$ the *basic degree* for the irreducible G-representation $\mathcal{V}_{j,l}$.

By the multiplicativity property of degree,

$$\nabla_G\text{-deg}\,(A', B(V')) = \prod_{\xi \in \sigma_-(A')} \prod_{j,l} (\text{Deg}_{\mathcal{V}_{j,l}})^{m_{j,l}(\xi)}.$$

Since (G) is the neutral element with respect to $\star$ in $U(G)$ and the multiplication $\star : A_1(G) \times A_1(G) \to U(G)$ is trivial (see Proposition 2.3.3.1),

$$\nabla_G\text{-deg}\,(A', B(V')) = (G) + \sum_{\xi \in \sigma_-(A')} \sum_{j,l} m_{j,l}(\xi) \text{deg}_{\mathcal{V}_{j,l}}.$$

In this way, we have the following

Proposition 5.1.1. *Let $A : V \to V$ be a linear symmetric G-equivariant isomorphism. Then*

$$\nabla_G\text{-deg}\,(A, B(V)) = \nabla_G\text{-deg}\,(\bar{A}, B(V^{S^1})) + \nabla_G\text{-deg}\,(\bar{A}, B(V^{S^1})) \star \sum_{\xi \in \sigma_-(A')} \sum_{j,l} m_{j,l}(\xi) deg_{\mathcal{V}_{j,l}},$$

where $\nabla_G\text{-deg}\,(\bar{A}, B(V^{S^1}))$ is given by (5.1.4).

By Theorem 4.3.2, we have

Corollary 5.1.1. *Let $A : V \to V$ be a linear symmetric G-equivariant isomorphism and $B(V) \subset V$ be the unit ball. Then*

$$G\text{-Deg}^{\,o}(A, B(V)) = \left(Deg^o_\Gamma(A, B(V)), Deg^o_G(A, B(V))\right),$$

where $Deg^o_\Gamma(A, B(V))$ is given by (5.1.4) and $Deg^o_G(A, B(V)) = Deg^o_\Gamma(A, B(V)) \star \sum_{\xi \in \sigma_-(A')} \sum_{j,l} m_{j,l}(\xi) deg_{\mathcal{V}_{j,l}}$.

For convenience, we list the values of the basic degrees $\deg_{\mathcal{V}_i}$ and $\deg_{\mathcal{V}_{j,l}}$ in the case $\Gamma = D_6, S_4, A_5$, which can be obtained from a special Maple© routines* (see the Appendix for the explanation of the used notations).

Example 5.1.1. $\Gamma = D_6$

There are six real irreducible representations of D_6: the trivial representation $\mathcal{V}_0$, two 2-dimensional representations $\mathcal{V}_1$ and $\mathcal{V}_2$, a 1-dimensional representation $\mathcal{V}_3$ induced by $\varphi : D_6 \to D_2$ with $\ker\varphi = \mathbb{Z}_6$, and another two 1-dimensional representations $\mathcal{V}_4$ and $\mathcal{V}_4$ induced by $\varphi : D_6 \to \mathbb{Z}_2$ with $\ker\varphi = D_3$ and $\ker\varphi = \widetilde{D_3}$ respectively.

The values of $\deg_{\mathcal{V}_i}$ are

$$\begin{aligned}
\deg_{\mathcal{V}_0} &= -(D_6 \times S^1),\\
\deg_{\mathcal{V}_1} &= (D_6 \times S^1) - (\widetilde{D}_1 \times S^1) - (D_1 \times S^1) + (\mathbb{Z}_1 \times S^1),\\
\deg_{\mathcal{V}_2} &= (D_6 \times S^1) - 2(D_2 \times S^1) + (\mathbb{Z}_2 \times S^1),\\
\deg_{\mathcal{V}_3} &= (D_6 \times S^1) - (\mathbb{Z}_6 \times S^1),\\
\deg_{\mathcal{V}_4} &= (D_6 \times S^1) - (D_3 \times S^1),\\
\deg_{\mathcal{V}_5} &= (D_6 \times S^1) - (\widetilde{D}_3 \times S^1).
\end{aligned}$$

Example 5.1.2. $\Gamma = S_4$:

There are five real irreducible representations of S_4: the trivial representation $\mathcal{V}_0$, the one-dimensional representation $\mathcal{V}_1$ corresponding to the homomorphism $\varphi : S_4 \to \mathbb{Z}_2 \subset O(1)$, where $\ker\varphi = A_4$, the two-dimensional representation $\mathcal{V}_2$ corresponding to the homomorphism $\psi : S_4 \to S_4/V_4 = S_3 \simeq D_3 \subset O(2)$, and two different three-dimensional representations of S_4, one of them being the natural representation $\mathcal{V}_3$ of S_4, while the other $\mathcal{V}_4$ being the tensor product $\mathcal{V}_1 \otimes \mathcal{V}_3$ of the natural three-dimensional representation with the non-trivial one-dimensional representation.

The values of $\deg_{\mathcal{V}_i}$ are

$$\begin{aligned}
\deg_{\mathcal{V}_0} &= -(S_4 \times S^1),\\
\deg_{\mathcal{V}_1} &= (S_4 \times S^1) - (A_4 \times S^1),\\
\deg_{\mathcal{V}_2} &= (S_4 \times S^1) - 2(D_4 \times S^1) + (V_4 \times S^1),\\
\deg_{\mathcal{V}_3} &= (S_4 \times S^1) - 2(D_3 \times S^1) - (D_2 \times S^1) + 3(D_1 \times S^1) - (\mathbb{Z}_1 \times S^1),\\
\deg_{\mathcal{V}_4} &= (S_4 \times S^1) - (\mathbb{Z}_4 \times S^1) - (D_1 \times S^1) - (\mathbb{Z}_3 \times S^1) + (\mathbb{Z}_1 \times S^1).
\end{aligned}$$

Example 5.1.3. $G = A_5 \times S^1$

There are five irreducible representations of A_5: $\mathcal{V}_0$ — the trivial representation, $\mathcal{V}_1$ — the natural 4-dimensional representation of A_5, $\mathcal{V}_2$ — the 5-dimensional representation of A_5, and two 3-dimensional representations $\mathcal{V}_3$ and $\mathcal{V}_4$.

The values of $\deg_{\mathcal{V}_i}$ are

$$\begin{aligned}
\deg_{\mathcal{V}_0} &= -(A_5 \times S^1)\\
\deg_{\mathcal{V}_1} &= (A_5 \times S^1) - 2(A_4 \times S^1) - 2(D_3 \times S^1) + 3(\mathbb{Z}_2 \times S^1) + 3(\mathbb{Z}_3 \times S^1) - 2(\mathbb{Z}_1 \times S^1)\\
\deg_{\mathcal{V}_2} &= (A_5 \times S^1) - 2(D_5 \times S^1) - 2(D_3 \times S^1) + 3(\mathbb{Z}_2 \times S^1) - (\mathbb{Z}_1 \times S^1)\\
\deg_{\mathcal{V}_3} &= \deg_{\mathcal{V}_3} = (A_5 \times S^1) - (\mathbb{Z}_5 \times S^1) - (\mathbb{Z}_3 \times S^1) - (\mathbb{Z}_2 \times S^1) + (\mathbb{Z}_1 \times S^1)
\end{aligned}$$

* The equivariant degree Maple© Library package is available at `http://krawcewicz.net/degree` or `http://www.math.ualberta.ca/~wkrawcew/degree`.

Example 5.1.4. $G = D_6 \times S^1$

There are six irreducible G-representations, $\mathcal{V}_{j,1}$, $j = 0,1,2,3,4,5$, obtained by taking complexifications of each real irreducible D_6-representation.

The values of $\deg_{\mathcal{V}_{j,1}}$ are

$$\begin{aligned}
\deg_{\mathcal{V}_{0,1}} &= (D_6^d)\\
\deg_{\mathcal{V}_{1,1}} &= (\mathbb{Z}_6^{t_1}) + (D_2^d) + (D_2^{\hat{d}}) - (\mathbb{Z}_2^-)\\
\deg_{\mathcal{V}_{2,1}} &= (\mathbb{Z}_6^{t_2}) + (D_2^z) + (D_2) - (\mathbb{Z}_2)\\
\deg_{\mathcal{V}_{3,1}} &= (D_6^z)\\
\deg_{\mathcal{V}_{4,1}} &= (D_6^d)\\
\deg_{\mathcal{V}_{5,1}} &= (D_6^{\hat{d}}).
\end{aligned}$$

For $l > 1$, $\deg_{\mathcal{V}_{j,l}} = \Theta_l[\deg_{\mathcal{V}_{j,1}}]$, where $\Theta_l : A_1(G) \to A_1(G)$ is defined on generators by

$$\Theta[(\mathcal{H}^{\varphi,1})] := (\mathcal{H}^{\varphi,1}). \tag{5.1.6}$$

Example 5.1.5. $G = S_4 \times S^1$

The irreducible representations $\mathcal{V}_{j,1}$ are obtained from the complexifications of $\mathcal{V}_j$, $j = 0,1,2,3,4$.

The values of $\deg_{\mathcal{V}_{j,1}}$ are

$$\begin{aligned}
\deg_{\mathcal{V}_{0,1}} &= (S_4)\\
\deg_{\mathcal{V}_{1,1}} &= (S_4^-)\\
\deg_{\mathcal{V}_{2,1}} &= (A_4^t) + (D_4) + (D_4^{\hat{d}}) - (V_4)\\
\deg_{\mathcal{V}_{3,1}} &= (\mathbb{Z}_4^c) + (D_4^d) + (D_2^d) + (D_3) + (\mathbb{Z}_3^t) - (\mathbb{Z}_2^-) - (D_1)\\
\deg_{\mathcal{V}_{4,1}} &= (\mathbb{Z}_4^c) + (D_4^z) + (D_2^d) + (D_3^z) + (\mathbb{Z}_3^t) - (\mathbb{Z}_2^-) - (D_1^z).
\end{aligned}$$

Example 5.1.6. $G = A_5 \times S^1$

Again, by taking complexifications of $\mathcal{V}_j$, we obtain the G-representations, $\mathcal{V}_{j,1}$, $j = 0,1,2,3,4$.

The values of $\deg_{\mathcal{V}_{j,1}}$ are

$$\begin{aligned}
\deg_{\mathcal{V}_{0,1}} &= (A_5)\\
\deg_{\mathcal{V}_{1,1}} &= (A_4) + (D_3) + (D_3^z) + (V_4^-) + (\mathbb{Z}_3^t) + (\mathbb{Z}_5^{t_1}) + (\mathbb{Z}_5^{t_2}) - (\mathbb{Z}_2) - (\mathbb{Z}_3) - (\mathbb{Z}_2^-)\\
\deg_{\mathcal{V}_{2,1}} &= (D_5) + (D_3) + (A_4^{t_1}) + (A_4^{t_2}) + (V_4^-) + (\mathbb{Z}_5^{t_1}) + (\mathbb{Z}_5^{t_2}) - 2(\mathbb{Z}_2)\\
\deg_{\mathcal{V}_{3,1}} &= (D_5^z) + (V_4^-) + (D_3^z) + (\mathbb{Z}_5^{t_1}) + (\mathbb{Z}_3^t) - 2(\mathbb{Z}_2^-)\\
\deg_{\mathcal{V}_{4,1}} &= (D_5^z) + (V_4^-) + (D_3^z) + (\mathbb{Z}_5^{t_2}) + (\mathbb{Z}_3^t) - 2(\mathbb{Z}_2^-)
\end{aligned}$$

6 Symmetric Autonomous Newtonian System

Let V be an orthogonal Γ-representation, consider a C^2-differentiable Γ-invariant function $\varphi : V \to \mathbb{R}$. Then the gradient $\nabla\varphi : V \to V$ is a Γ-equivariant C^1-differentiable map. We will assume that

(A1) $\quad \nabla\varphi(x) = 0 \iff x = 0.$

We are interested in finding non-zero solutions to the following system of ODEs:

$$\begin{cases} \ddot{x} = -\nabla\varphi(x), & x(t) \in V, \\ x(0) = x(2\pi), & \dot{x}(0) = \dot{x}(2\pi), \end{cases} \tag{6.1}$$

where x is twice weakly differentiable with respect to t.

Suppose that A, $B : V \to V$ are two symmetric Γ-equivariant linear isomorphisms such that

(A2) $\nabla^2\varphi(0) = A$.

(A3) $\nabla\varphi(x) = Bx + o(\|x\|)$ as $\|x\| \to \infty$, i.e.

$$\lim_{\|x\|\to\infty} \frac{\|\nabla\varphi(x) - Bx\|}{\|x\|} = 0.$$

Remark 6.1. Notice that the conditions (A1)–(A3) imply that

$$\Gamma\text{-Deg}(-A, B(V)) = \Gamma\text{-Deg}(-B, B(V)). \tag{6.2}$$

Indeed, one can use the standard linearization argument to show that, by (A2), there exists $\varepsilon > 0$ such that

$$\Gamma\text{-Deg}(-A, B(V)) = \Gamma\text{-Deg}(-A, B_\varepsilon(V)) = \Gamma\text{-Deg}(-\nabla\varphi, B_\varepsilon(V)).$$

On the other hand, for $R > 0$ being sufficiently large number, we have, by (A3),

$$\Gamma\text{-Deg}(-B, B(V)) = \Gamma\text{-Deg}(-B, B_R(V)) = \Gamma\text{-Deg}(-\nabla\varphi, B_R(V)).$$

Since (A1) implies $-\nabla\varphi^{-1}(0) = \{0\}$, by the excision property of the Γ-equivariant degree,

$$\Gamma\text{-Deg}(-\nabla\varphi, B_\varepsilon(V)) = \Gamma\text{-Deg}(-\nabla\varphi, B_R(V)),$$

so (6.2) follows.

Denote by $\sigma(A)$ (resp. $\sigma(B)$) the spectrum of A (resp. the spectrum of B). We assume

(A4) $(\sigma(A) \cup \sigma(B)) \cap \{k^2 \;:\; k = 0, 1, 2, \dots\} = \emptyset$.

Remark 6.2. Suppose that $C : V \to V$ is a symmetric linear operator such that $\sigma(C) \cap \{k^2 \;:\; k = 0, 1, 2, \dots\} = \emptyset$, then the system

$$\begin{cases} -\ddot{x} = Cx, & x(t) \in V, \\ x(0) = x(2\pi), & \dot{x}(0) = \dot{x}(2\pi) \end{cases}$$

has no non-zero solutions. Therefore, the condition (A4) can be translated as a requirement that the linearization of (6.1) at $x = 0$ and $x = \infty$ have no non-zero solutions.

Example 6.1. One can easily construct an example of a Γ-invariant function $\varphi : V \to \mathbb{R}$ satisfying the assumptions (A1)–(A4). For instance, let $\eta : \mathbb{R} \to \mathbb{R}$ be a C^2-differentiable function such that $\eta'(t) > 0$ for all $t \in \mathbb{R}$ and $\lim_{t\to\infty} \eta'(t) = b > 0$. Also, assume that $2\eta'(0), 2b \notin \{k^2 \;:\; k = 0, 1, 2, \dots\}$. Then, $\varphi(x) := \eta(\|x\|^2)$ is Γ-invariant and the gradient $\nabla\varphi(x) = 2\eta'(\|x\|^2)x$, satisfies (A1) and clearly $\nabla\varphi(0)h = 2\eta'(0)h$.

On the other hand,

$$\begin{aligned} \lim_{\|x\|\to\infty} \frac{\|\nabla\varphi(x) - 2bx\|}{\|x\|} &= \lim_{\|x\|\to\infty} \frac{\|(2\eta'(\|x\|^2) - 2b)x\|}{\|x\|} \\ &= \lim_{\|x\|\to\infty} |2\eta'(\|x\|^2) - 2b| = 0, \end{aligned}$$

so (A2) and (A3) are clearly satisfied with $A = 2\eta'(0)\,\mathrm{Id}$, $B = 2b\,\mathrm{Id}$.

6.1 Functional Setting

In order to reformulate the problem (6.1) as a variational problem, consider the Sobolev space $W := H^1(S^1;V)$. It is a natural G-representation for $G = \Gamma \times S^1$, where the G-action is defined by

$$\big((\gamma, e^{i\tau})u\big)(t) := \gamma u(t+\tau), \quad \gamma \in \Gamma, \;\; \tau \in \mathbb{R}, \;\; u \in W.$$

Moreover, W is a Hilbert G-representation with the inner product

$$\langle u, v\rangle_{H^1} := \int_0^{2\pi} \langle \dot u(t), \dot v(t)\rangle + \langle u(t), v(t)\rangle dt, \quad u, v \in W.$$

We will denote by $\|\cdot\|_{H^1}$ the induced norm by $\langle\cdot,\cdot\rangle_{H^1}$ on W.

Define $\Psi : W \to \mathbb{R}$ by

$$\Psi(u) := \int_0^{2\pi} \left(\frac{1}{2}\|\dot u(t)\|^2 - \varphi(u(t))\right) dt,$$

(where $\|\cdot\|$ stands for the L^2-norm). Clearly, the functional Ψ is G-invariant and C^2-differentiable. Indeed, one can easily verify that

$$D\Psi(u)(v) = \int_0^{2\pi} \langle \dot u(t), \dot v(t)\rangle - \langle \nabla\varphi(u(t)), v(t)\rangle\, dt.$$

Notice that if $D\Psi(u) \equiv 0$ for some $u \in W$, then $u \in H^2(S^1;V)$ and u is a solution to (6.1). Consequently, the problem (6.1) can be reformulated as

$$\nabla\Psi(u) = 0. \tag{6.1.1}$$

To determine an explicit formula for $\nabla\Psi$, we represent Ψ as

$$\Psi(u) = \frac{1}{2}\|u\|^2_{H^1} - \widetilde\Phi(u), \quad u \in W,$$

where

$$\widetilde\Phi(u) = \int_0^{2\pi} \widetilde\varphi(u(t))dt, \quad \widetilde\varphi(h) = \varphi(h) + \frac{1}{2}\|h\|^2, \;\; h \in V.$$

Clearly, $\nabla\Psi(u) = u - \nabla\widetilde\Phi(u)$.

Introduce the following maps:

$$\begin{aligned} L &: H^2(S^1;V) \to L^2(S^1;V), & Lu &= -\ddot u + u, \\ j &: H^2(S^1;V) \to H^1(S^1;V), & ju &= u, \\ N_{\nabla\widetilde\varphi} &: C(S^1;V) \to L^2(S^1;V), & N_{\nabla\widetilde\varphi}(u) &= \nabla\widetilde\varphi(u) = \nabla\varphi(u) + \mathrm{Id}\,. \end{aligned} \tag{6.1.2}$$

Since the equation

$$\langle \nabla\widetilde\Phi(u), v\rangle_{H^1} = D\widetilde\Phi(u)(v),$$

translates to

$$\int_0^{2\pi} \left(\left\langle \frac{d}{dt}\nabla\widetilde\Phi(u)(t), \dot v(t)\right\rangle + \langle \nabla\widetilde\Phi(u)(t), v(t)\rangle\right) dt = \int_0^{2\pi} \langle \nabla\widetilde\varphi(u(t)), v(t)\rangle\, dt,$$

for all $v \in H^1(S^1;V)$, we obtain that $\nabla\widetilde{\Phi}(u)$ is a weak solution y to the system

$$\begin{cases} -\ddot{y} + y = \nabla\widetilde{\varphi}(u), \\ y(0) = y(2\pi),\ \dot{y}(0) = \dot{y}(2\pi). \end{cases}$$

Therefore, one obtains

$$\nabla\widetilde{\Phi}(u) = j \circ L^{-1} \circ N_{\nabla\widetilde{\varphi}}(u), \quad u \in W,$$

which leads to

$$\nabla\Psi(u) = u - j \circ L^{-1} \circ N_{\nabla\widetilde{\varphi}}(u), \quad u \in W.$$

Put $\mathfrak{F} := \nabla\Psi : W \to W$. Then, (cf. (6.1.1))

$$x \quad \text{is a solution to (6.1)} \quad \Longleftrightarrow \quad \mathfrak{F}(x) = 0, \quad x \in W.$$

Notice that since j is a compact inclusion, $\mathfrak{F}$ is a completely continuous G-equivariant field on W. In particular, $\mathfrak{F}$ is a G-orthogonal map, since $\mathfrak{F}$ is a gradient G-map.

By (A2)–(A4), for sufficiently small $\varepsilon > 0$ (resp. sufficiently large $R > 0$) the map $\mathfrak{F}$ is $B_\varepsilon(W)$-admissible (resp. $B_R(W)$-admissible). Thus, one can define the orthogonal G-equivariant degrees of $\mathfrak{F}$ on $B_\varepsilon(W)$ (resp. on $B_R(W)$), which is denoted by $G\text{-Deg}^o(\mathfrak{F}, B_\varepsilon(W))$ (resp. $G\text{-Deg}^o(\mathfrak{F}, B_R(W))$). Observe that, by the excision property of the orthogonal degree, if $G\text{-Deg}^o(\mathfrak{F}, B_R(W)) - G\text{-Deg}^o(\mathfrak{F}, B_ve(W)) \neq 0$, then there is a solution (6.1.1) (or equivalently, to the system (6.1)) in $B_R(W) \setminus B_\varepsilon(W)$ (cf. [17]).

6.2 Existence Result

Define the G-orthogonal isomorphisms $\mathcal{A}$, $\mathcal{B} : W \to W$ by

$$\mathcal{A} := \text{Id} - j \circ L^{-1} \circ (A + \text{Id}), \quad \mathcal{B} := \text{Id} - j \circ L^{-1} \circ (B + \text{Id}). \tag{6.2.1}$$

By (A2)–(A3) and using the standard linearization argument, we have

$$G\text{-Deg}^o(\mathfrak{F}, B_\varepsilon(W)) = G\text{-Deg}^o(\mathcal{A}, B_\varepsilon(W)) = G\text{-Deg}^o(\mathcal{A}, B(W)), \tag{6.2.2}$$

$$G\text{-Deg}^o(\mathfrak{F}, B_R(W)) = G\text{-Deg}^o(\mathcal{B}, B_R(W)) = G\text{-Deg}^o(\mathcal{B}, B(W)), \tag{6.2.3}$$

which leads to the following existence result for the system (6.1):

Theorem 6.2.1. *Let $\varphi : V \to \mathbb{R}$ be a Γ-equivariant C^2-differentiable function satisfying the assumptions (A1)–(A4), and suppose the maps $\mathcal{A}$ and $\mathcal{B}$ are given by (6.2.1) with*

$$G\text{-Deg}^o(\mathcal{A}, B(W)) - G\text{-Deg}^o(\mathcal{B}, B(W)) = (\deg_0, \deg_1) \in A(\Gamma) \times A_1^t(G). \tag{6.2.4}$$

Then $\deg_0 = 0$ *and if*

$$\deg_1 = \sum_{(H)} n_H \cdot (H) \neq 0$$

i.e. $n_{H_o} \neq 0$, for some orbit type (H_o) in W', then there exists a non-constant periodic solution x_o to (6.1) satisfying $G_{x_o} \supset H_o$. In addition, if $H_o = K_o^{\psi,k}$ (for some subgroup $K_o \subset \Gamma$ and homomorphism $\psi : K_o \to S^1$) is such that $(K_o^{\psi,1})$ is a dominating orbit type in W, then there exist at least $|\Gamma/K_o|$ different non-constant periodic solutions with the orbit type at least $(K_o^{\psi,k})$.*

* We call $(H^{\varphi,1})$ a *dominating orbit type* in W if it is a maximal orbit type among all the 1-folded orbit types in W (cf. [4]).

Proof: Recall that by the definition of the orthogonal degree (cf. (4.3.4)-(4.3.6)),

$$\deg_0 = \mathrm{Deg}^o_\Gamma(\mathcal{A}, B(W)) - \mathrm{Deg}^o_\Gamma(\mathcal{B}, B(W)),$$

where

$$\mathrm{Deg}^o_\Gamma(\mathcal{A}, B(W)) = \Gamma\text{-Deg}(\mathcal{A}|_{W^{S^1}}, O_1),$$
$$\mathrm{Deg}^o_\Gamma(\mathcal{B}, B(W)) = \Gamma\text{-Deg}(\mathcal{B}|_{W^{S^1}}, O_1),$$

(O_1 stands for the unit ball in W^{S^1}). Observe that $W^{S^1} \simeq V$, thus by (6.1.2), we have $L|_{W^{S^1}} = \mathrm{Id}$, which implies $\mathcal{A}|_{W^{S^1}} = -A$ and $\mathcal{B}|_{W^{S^1}} = -B$ (cf. (6.2.1)), i.e.

$$\mathrm{Deg}^o_\Gamma(\mathcal{A}, B(W)) = \Gamma\text{-Deg}(-A, O_1)$$
$$\mathrm{Deg}^o_\Gamma(\mathcal{B}, B(W)) = \Gamma\text{-Deg}(-B, O_1).$$

Combined with (6.2), we conclude $\deg_0 = 0$.

By (6.2.2)–(6.2.3) and the excision property of the orthogonal degree, if

$$G\text{-Deg}^{\,o}(\mathcal{A}, B(W)) - G\text{-Deg}^{\,o}(\mathcal{B}, B(W)) \neq 0,$$

then there exists a solution to the system (6.1) in $B_R(W) \setminus B_\varepsilon(W)$. Moreover, by (A1), $x = 0$ is the only constant solution to (6.1). Therefore, there exists a non-constant solution to the system (6.1) in $B_R(W) \setminus B_\varepsilon(W)$.

Suppose that $n_{H_o} \neq 0$, where $(H_o) = (K_o^{\psi,k})$ and $(K_o^{\psi,1})$ is a dominating orbit type in W. Then, by the existence property of the orthogonal degree, there exists a solution $u \in B_R(W) \setminus B_\varepsilon(W)$ to the system (6.1) such that $G_u \supset H_o$. Due to (A1), we have that $(G_u) = (K^{\widetilde{\psi},\widetilde{k}})$ for some K with $K_o \subset K \subsetneq \Gamma$ and a homomorphism $\widetilde{\psi} : K \to S^1$ with $\widetilde{\psi}|_{K_o} = \psi$, $\widetilde{k} \geq k$. Since $(K_o^{\psi,1})$ is a maximal orbit type in the set of all 1-folded twisted orbit types in W, thus $(K_o^{\psi,\widetilde{k}})$ is a maximal orbit type in the set of all $\widetilde{k}$-folded twisted orbit types in W. Consequently, $(K^{\widetilde{\psi},\widetilde{k}}) = (K_o^{\psi,\widetilde{k}})$. Therefore, there exist at least $|\Gamma/K_o|$ different non-constant periodic solutions with the *exact* orbit type $(K_o^{\psi,\widetilde{k}})$. In other words, there exist at least $|\Gamma/K_o|$ different non-constant periodic solutions with the orbit type *at least* $(K_o^{\psi,k})$. □

6.3 Computation of $\deg_1$

For simplicity, assume that*

(A5) the operators A and B have only positive eigenvalues.

For the Γ-isotypical decomposition of V^c given by (5.1.2), put

$$\widetilde{m}_j := \dim U_j / \dim \mathcal{U}_j. \tag{6.3.1}$$

Consider the "complexified" operator $A : V^c \to V^c$ given by $A(z \otimes v) := z \otimes Av$ (for which the same notation is used). For each $\mu \in \sigma(A)$, denote by $\widetilde{E}(\mu)$ the eigenspace of μ considered in V^c and call

$$\widetilde{m}_j(\mu) := \frac{\dim \widetilde{E}_j(\mu)}{\dim \mathcal{U}_j} := \frac{\dim \left(\widetilde{E}(\mu) \cap U_j\right)}{\dim \mathcal{U}_j}, \tag{6.3.2}$$

* In the case A and B have negative eigenvalues, the argument remains valid for the "positive" parts of $\sigma(A)$ and $\sigma(B)$.

the $\mathcal{U}_j$*-multiplicity* of μ.

Put $A^j := A|_{U_j}$ and

$$\sigma_j^k(A) := \{\mu \in \sigma(A^j) \ : \ k^2 < \mu < (k+1)^2\},$$

thus by the assumption (A4),

$$\sigma(A^j) = \bigcup_{k=0}^{\infty} \sigma_j^k(A).$$

Recall $\mathcal{A}' := \mathcal{A}|_{W'}$, $W' := (W^{S^1})^\perp$. The definition of $\mathcal{A}$ (cf. (6.2.1)) clearly implies that

$$\begin{aligned}\sigma(\mathcal{A}') &= \left\{1 - \frac{\mu+1}{l^2+1} \ : \ \mu \in \sigma(A), \ l = 1, 2, \dots \right\} \\ &= \left\{1 - \frac{\mu+1}{l^2+1} \ : \ \mu \in \sigma_j^k(A), \ j = 0, 1, \dots, s, \ k = 0, 1, \dots, l = 1, 2, \dots \right\}.\end{aligned}$$

Consequently, the negative spectrum $\sigma_-(\mathcal{A}')$ of $\mathcal{A}'$ can be described by

$$\sigma_-(\mathcal{A}') = \left\{1 - \frac{\mu+1}{l^2+1} \ : \ \mu \in \sigma_j^k(A), \ j = 0, 1, \dots, s, \ k = 0, 1, \dots, l = 1, \dots, k \right\}. \tag{6.3.3}$$

Moreover, for an eigenvalue $1 - \frac{\mu+1}{l^2+1}$ of $\mathcal{A}'|_{W_l} : W_l \to W_l$, we have (cf. (5.1.5))

$$m_{j,l}\left(1 - \frac{\mu+1}{l^2+1}\right) = \widetilde{m}_j(\mu), \quad l = 1, 2, \dots \tag{6.3.4}$$

Therefore, by (6.3.3)–(6.3.4), the second component of G-$\mathrm{Deg}^{\,o}(\mathcal{A}, B(W))$ equals to (cf. Corollary 5.1.1)

$$\begin{aligned}\mathrm{Deg}^o_G(\mathcal{A}, B(W)) &= \mathrm{Deg}^o_\Gamma(\mathcal{A}, B(W)) \star \sum_{\xi \in \sigma_-(\mathcal{A}')} m_{j,l}(\xi) \deg_{\mathcal{V}_{j,l}} \\ &= \mathrm{Deg}^o_\Gamma(\mathcal{A}, B(W)) \star \sum_{j=0}^{s}\sum_{k=0}^{\infty}\sum_{l=1}^{k} \sum_{\mu \in \sigma_j^k(A)} m_{j,l}(1 - \frac{\mu+1}{l^2+1}) \deg_{\mathcal{V}_{j,l}} \\ &= \mathrm{Deg}^o_\Gamma(\mathcal{A}, B(W)) \star \sum_{j=0}^{s}\sum_{k=0}^{\infty} \sum_{\mu \in \sigma_j^k(A)} \widetilde{m}_j(\mu) \sum_{l=1}^{k} \deg_{\mathcal{V}_{j,l}}.\end{aligned} \tag{6.3.5}$$

On the other hand, $A^j : U_j \to U_j$ is completely diagonalizable, thus (cf. (6.3.1))

$$\widetilde{m}_j = \sum_{\mu \in \sigma(A^j)} \widetilde{m}_j(\mu) = \sum_{k=0}^{\infty} \sum_{\mu \in \sigma_j^k(A)} \widetilde{m}_j(\mu). \tag{6.3.6}$$

Now, by putting

$$\widetilde{m}_j^k(A) := \sum_{\mu \in \sigma_j^k(A)} \widetilde{m}_j(\mu),$$

we can simplify (6.3.5) to the following form:

$$\mathrm{Deg}^o_G(\mathcal{A}, B(W)) = \mathrm{Deg}^o_\Gamma(\mathcal{A}, B(W)) \star \sum_{j=0}^{s}\sum_{k=0}^{\infty} \widetilde{m}_j^k(A) \sum_{l=1}^{k} \deg_{\mathcal{V}_{j,l}}.$$

Notice that (cf. (6.3.6))

$$\widetilde{m}_j = \sum_{k=0}^{\infty} \widetilde{m}_j^k(A). \tag{6.3.7}$$

Following the same lines for the operator $\mathcal{B}$, by assumption (A5), one obtains

$$\mathrm{Deg}_G^o(\mathcal{B}, B(W)) = \mathrm{Deg}_\Gamma^o(\mathcal{B}, B(W)) \star \sum_{j=0}^{s}\sum_{k=0}^{\infty} \widetilde{m}_j^k(B) \sum_{l=1}^{k} \deg_{\mathcal{V}_{j,l}},$$

and

$$\widetilde{m}_j = \sum_{k=0}^{\infty} \widetilde{m}_j^k(B), \tag{6.3.8}$$

where

$$\widetilde{m}_j^k(B) := \sum_{\eta \in \sigma_j^k(B)} \widetilde{m}_j(\eta),$$

with $\widetilde{m}_j(\eta)$ being the $\mathcal{U}_j$-isotypical multiplicity of η (cf. (6.3.2)).

By Theorem 6.2.1, $\deg_0 = 0$, thus $\mathrm{Deg}_\Gamma^o(\mathcal{A}, B(W)) = \mathrm{Deg}_\Gamma^o(\mathcal{B}, B(W))$. Put

$$\mathrm{Deg}_\Gamma^o := \mathrm{Deg}_\Gamma^o(\mathcal{A}, B(W)) = \mathrm{Deg}_\Gamma^o(\mathcal{B}, B(W)). \tag{6.3.9}$$

Therefore, by (6.2.4),

$$\begin{aligned}
\deg_1 &= \mathrm{Deg}_G^o(\mathcal{A}, B(W)) - \mathrm{Deg}_G^o(\mathcal{B}, B(W)) \\
&= \mathrm{Deg}_\Gamma^o \star \sum_{j=0}^{s}\sum_{k=0}^{\infty} \left(\left(\widetilde{m}_j^k(A) - \widetilde{m}_j^k(B)\right) \sum_{l=1}^{k} \deg_{\mathcal{V}_{j,l}} \right) \\
&= \prod_{\mu \in \sigma_-(\bar{A})} \prod_{i=0}^{r} (\deg_{\mathcal{V}_i})^{m_i(\mu)} \star \sum_{j=0}^{s}\sum_{k=0}^{\infty} \left(\mathfrak{m}_j^k \sum_{l=1}^{k} \deg_{\mathcal{V}_{j,l}} \right),
\end{aligned} \tag{6.3.10}$$

where

$$\mathfrak{m}_j^k := \widetilde{m}_j^k(A) - \widetilde{m}_j^k(B). \tag{6.3.11}$$

Definition 6.3.1. We call the number $\mathfrak{m}_j^k$ given by (6.3.11) the *k-th $\mathcal{U}_j$-isotypical compartmental defect number* for the map $\mathfrak{F}$, for $j = 0, 1, \dots, s$ and $k = 0, 1, \dots$.

The following lemma describes the possible combinations of the $\mathcal{U}_j$-isotypical compartmental defect numbers $\mathfrak{m}_j^k$, $k = 0, 2, \dots$, subject to conditions (6.3.7)–(6.3.8):

Lemma 6.3.1. *Let a, N be positive integers, $(n_k)_{k=1}^N$ and $(m_k)_{k=1}^N$ be two N-part partitions of a, i.e.*

$$a = n_1 + n_2 + \dots + n_N = m_1 + m_2 + \dots + m_N,$$

where n_k's and m_k's are non-negative integers. Put

$$b_k := n_k - m_k, \quad k = 1, 2, \dots, N,$$
$$b^+ := \sum_{b_k > 0} b_k, \quad b^- := \sum_{b_k < 0} b_k,$$

where a sum over the empty set is assumed to be 0.

Then $(b_k)_{k=1}^N$ is a partition of 0 *with $0 \le b^+ \le a$ and $-a \le b^- \le 0$.*

Proof: Assume that $(n_k)_{k=1}^N$ and $(m_k)_{k=1}^N$ are partitions of a, i.e.

$$a = n_1 + n_2 + \cdots + n_N = m_1 + m_2 + \cdots + m_N.$$

Then, clearly, $(b_k)_{k=1}^N = (n_k - m_k)_{k=1}^N$ is a partition of 0 and, by definition, $b^+ \geq 0$, $b^- \leq 0$. Moreover, since $n_k \geq 0$ and $m_k \geq 0$ for all k,

$$b^+ = \sum_{b_k>0} b_k = \sum_{n_k>m_k} (n_k - m_k) \leq \sum_{n_k>m_k} n_k \leq \sum_{k=1}^N n_k = a$$

$$b^- = \sum_{b_k<0} b_k = \sum_{n_k<m_k} (n_k - m_k) \geq \sum_{n_k<m_k} (-m_k) \geq -\sum_{k=1}^N m_k = -a,$$

which concludes the proof. □

6.4 Concrete Existence Results for Selected Symmetries

We present here the computational results for several Γ-representations, which were described previously in Section 5.1. We use the same notations as in Section 5.1.

We introduce the following condition which is specific to all examples considered later:

Condition 6.4.1. *(i) Decomposition (5.1.1) contains isotypical components modelled only on irreducible representations of real type (in particular, $r = s$).*

(ii) For each $\mu \in \sigma(A)$ there exists a single isotypical component $V_i = V_{i_\mu}$ in (5.1.1) which (completely) contains the eigenspace $E(\mu)$.

By Condition 6.4.1(ii),

$$m_i(\mu) = \begin{cases} \dim_{\mathbb{R}} E(\mu)/\dim_{\mathbb{R}} \mathcal{V}_i & i = i_\mu, \\ 0 \quad i \neq i_\mu. \end{cases} \tag{6.4.1}$$

Also notice that $(\deg_{\mathcal{V}_i})^2 = (G)$ for all i (cf. (5.1.3)). Put

$$\varepsilon_i = \sum_{\mu \in \sigma(A)} m_{i_\mu}(\mu) \mod 2.$$

Thus,

$$\mathrm{Deg}_\Gamma^o = \prod_{i=0}^r \left(\deg_{\mathcal{V}_i}\right)^{\varepsilon_i}.$$

Consequently, the computational formula (6.3.10) reduces to

$$\deg_1 = \prod_{i=0}^r \left(\deg_{\mathcal{V}_i}\right)^{\varepsilon_i} \star \sum_{j=0}^s \sum_{k=0}^\infty \left(\mathfrak{m}_j^k \sum_{l=1}^k \deg_{\mathcal{V}_{j,l}}\right). \tag{6.4.2}$$

Consider the system (6.1) assuming that (A1)–(A5) and Condition 6.4.1 are satisfied. As the symmetry group Γ, take the dihedral groups D_4, $D_5{}^*$, D_6, the octahedral group S_4 and the icosahedral group A_5. More specifically, assume that $V := \mathbb{R}^n$ is an orthogonal Γ-representation,

In a similar way, the multiplication tables and the lists of basic degrees can be obtained in the case $\Gamma = D_4$, D_5 by using the special Maple© routines, where the notations used are also explained.

where Γ acts on $u = (u_1, u_2, \dots, u_n) \in V$ by permuting its coordinates. Moreover, for $\Gamma = D_n$, assume that C is of the type

$$C = \begin{bmatrix} c & d & 0 & \dots & 0 & d \\ d & c & d & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ d & 0 & 0 & \dots & d & c \end{bmatrix}.$$

For $\Gamma = S_4$, C is of the type

$$C = \begin{bmatrix} c & d & 0 & d & 0 & d & 0 & 0 \\ d & c & d & 0 & 0 & 0 & d & 0 \\ 0 & d & c & d & 0 & 0 & 0 & d \\ d & 0 & d & c & d & 0 & 0 & 0 \\ 0 & 0 & 0 & d & c & d & 0 & d \\ d & 0 & 0 & 0 & d & c & d & 0 \\ 0 & d & 0 & 0 & 0 & d & c & d \\ 0 & 0 & d & 0 & d & 0 & d & c \end{bmatrix}.$$

For $\Gamma = A_5$, C is of the type

$$\mathrm{C}=\begin{bmatrix}
c & d & 0 & 0 & d & 0 & 0 & d & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
d & c & d & 0 & 0 & 0 & 0 & 0 & 0 & d & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & d & c & d & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & d & c & d & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d & 0 & 0 & 0 & 0 & 0 & 0 \\
d & 0 & 0 & d & c & d & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & d & c & d & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & d & c & d & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d & 0 & 0 & 0 \\
d & 0 & 0 & 0 & 0 & 0 & d & c & d & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & d & c & d & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d & 0 & 0 \\
0 & d & 0 & 0 & 0 & 0 & 0 & 0 & d & c & d & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d & c & d & 0 & 0 & 0 & 0 & 0 & 0 & d & 0 \\
0 & 0 & d & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d & c & d & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d & c & d & 0 & 0 & 0 & 0 & 0 & d \\
0 & 0 & 0 & d & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d & c & d & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & d & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d & c & d & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d & c & d & 0 & 0 & d \\
0 & 0 & 0 & 0 & 0 & 0 & d & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d & c & d & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d & c & d & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d & 0 & 0 & 0 & 0 & 0 & 0 & d & c & d \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d & 0 & 0 & d & 0 & 0 & d & c
\end{bmatrix}.$$

For definiteness, let $c = 4.5$, $d = 1$ for the matrix A, and $c = 9.5$, $d = 1$ for the matrix B.

Dihedral Symmetries D_4. In the case $\Gamma = D_4$, we have $V = \mathcal{V}_0 \oplus \mathcal{V}_1 \oplus \mathcal{V}_3$, to which we associate the sequence $(\varepsilon_0, \varepsilon_1, \varepsilon_3) = (1, 1, 1)$ and $\sigma(A) = \{\xi_0^0 = 6.5, \xi_1^1 = 4.5, \xi_3^3 = 2.5\}$, $\sigma(B) = \{\xi_0^0 = 11.5, \xi_1^1 = 9.5, \xi_3^3 = 7.5\}$. Thus, we have the following non-zero $\widetilde{m}_j^k$'s for A and B:

$$\begin{aligned} \widetilde{m}_0^2(A) &= 1, \quad \widetilde{m}_0^3(B) = 1, \\ \widetilde{m}_1^2(A) &= 1, \quad \widetilde{m}_1^3(B) = 1, \\ \widetilde{m}_3^1(A) &= 1, \quad \widetilde{m}_3^2(B) = 1. \end{aligned}$$

Consequently, we have the non-zero isotypical defect numbers

$$\mathfrak{m}_0^2 = 1, \quad \mathfrak{m}_0^3 = -1, \quad \mathfrak{m}_1^2 = 1, \quad \mathfrak{m}_1^3 = -1, \quad \mathfrak{m}_3^1 = 1, \quad \mathfrak{m}_3^2 = -1.$$

Hence,

$$\begin{aligned}
&\sum_{j=0}^{s}\sum_{k=0}^{\infty}\left(\mathfrak{m}_j^k\sum_{l=1}^{k}\deg_{\mathcal{V}_{j,l}}\right)\\
&= 1\cdot\Big(\deg_{\mathcal{V}_{0,1}}+\deg_{\mathcal{V}_{0,2}}\Big)+(-1)\cdot\Big(\deg_{\mathcal{V}_{0,1}}+\deg_{\mathcal{V}_{0,2}}+\deg_{\mathcal{V}_{0,3}}\Big)\\
&+1\cdot\Big(\deg_{\mathcal{V}_{1,1}}+\deg_{\mathcal{V}_{1,2}}\Big)+(-1)\cdot\Big(\deg_{\mathcal{V}_{1,1}}+\deg_{\mathcal{V}_{1,2}}+\deg_{\mathcal{V}_{1,3}}\Big)\\
&+1\cdot\deg_{\mathcal{V}_{3,1}}+(-1)\cdot\Big(\deg_{\mathcal{V}_{3,1}}+\deg_{\mathcal{V}_{3,2}}\Big)\\
&=-\deg_{\mathcal{V}_{0,3}}-\deg_{\mathcal{V}_{1,3}}-\deg_{\mathcal{V}_{3,2}}.
\end{aligned}$$

Finally,

$$\begin{aligned}
\deg_1 &= \Theta_3\left[\texttt{showdegree[D4]}\,(1,1,0,1,0,-1,-1,0,0,0)\right]\\
&\quad+\Theta_2\left[\texttt{showdegree[D4]}\,(1,1,0,1,0,0,0,0,-1,0)\right],
\end{aligned}$$

where Θ is as described by (5.1.6) and showdegree[Γ] is a special Maple© procedure available at http://krawcewicz.net/degree.

The dominating orbit types in W are (D_4), $(\mathbb{Z}_4^t):=(\mathbb{Z}_4^{t_1})$, (D_2^d), $(\widetilde{D}_2^d)$ and (D_4^d). The value of $\deg_1$ is listed in Table 2.

Dihedral Symmetries D_5. In the case $\Gamma=D_5$, we have $V=\mathcal{V}_0\oplus\mathcal{V}_1\oplus\mathcal{V}_2$, to which we associate the sequence $(\varepsilon_0,\varepsilon_1,\varepsilon_2)=(1,1,1)$ and $\sigma(A)=\{\xi_0^0=6.5,\xi_1^1=4.5+\frac{\sqrt{5}-1}{2},\xi_2^2=4.5-\frac{\sqrt{5}+1}{2}\}$, $\sigma(B)=\{\xi_0^0=11.5,\xi_1^1=9.5+\frac{\sqrt{5}-1}{2},\xi_2^2=9.5-\frac{\sqrt{5}+1}{2}\}$. Thus, we have the following non-zero $\widetilde{m}_j^k$'s for A and B:

$$\begin{aligned}
\widetilde{m}_0^2(A)&=1, &\widetilde{m}_0^3(B)&=1,\\
\widetilde{m}_1^2(A)&=1, &\widetilde{m}_1^3(B)&=1,\\
\widetilde{m}_2^1(A)&=1, &\widetilde{m}_2^2(B)&=1.
\end{aligned}$$

Consequently, we have the non-zero isotypical defect numbers

$$\mathfrak{m}_0^2=1,\quad \mathfrak{m}_0^3=-1,\quad \mathfrak{m}_1^2=1,\quad \mathfrak{m}_1^3=-1,\quad \mathfrak{m}_2^1=1,\quad \mathfrak{m}_2^2=-1.$$

Hence,

$$\begin{aligned}
&\sum_{j=0}^{s}\sum_{k=0}^{\infty}\left(\mathfrak{m}_j^k\sum_{l=1}^{k}\deg_{\mathcal{V}_{j,l}}\right)\\
&= 1\cdot\Big(\deg_{\mathcal{V}_{0,1}}+\deg_{\mathcal{V}_{0,2}}\Big)+(-1)\cdot\Big(\deg_{\mathcal{V}_{0,1}}+\deg_{\mathcal{V}_{0,2}}+\deg_{\mathcal{V}_{0,3}}\Big)\\
&+1\cdot\Big(\deg_{\mathcal{V}_{1,1}}+\deg_{\mathcal{V}_{1,2}}\Big)+(-1)\cdot\Big(\deg_{\mathcal{V}_{1,1}}+\deg_{\mathcal{V}_{1,2}}+\deg_{\mathcal{V}_{1,3}}\Big)\\
&+1\cdot\deg_{\mathcal{V}_{2,1}}+(-1)\cdot\Big(\deg_{\mathcal{V}_{2,1}}+\deg_{\mathcal{V}_{2,2}}\Big)\\
&=-\deg_{\mathcal{V}_{0,3}}-\deg_{\mathcal{V}_{1,3}}-\deg_{\mathcal{V}_{2,2}}.
\end{aligned}$$

Finally,

$$\begin{aligned}
\deg_1 &= \Theta_3\left[\texttt{showdegree[D5]}\,(1,1,1,0,-1,-1,0,0)\right]\\
&\quad+\Theta_2\left[\texttt{showdegree[D5]}\,(1,1,1,0,0,0,-1,0)\right].
\end{aligned}$$

The dominating orbit types in W are (D_5), $(\mathbb{Z}_5^{t_1})$, $(\mathbb{Z}_5^{t_2})$ and (D_1^z). The value of $\deg_1$ is listed in Table 2.

Γ	$\deg_1$	# Sols
D_4	$(D_4^3) + (\mathbb{Z}_4^{t,3}) - (D_2^{d,3}) + (\widetilde{D}_2^{d,3}) - (D_2^3)$ $-(\mathbb{Z}_2^{-,3}) + (D_1^{z,3}) - (\widetilde{D}_1^{z,3}) + 2(D_1^3) - 2(\widetilde{D}_1^3)$ $+(D_4^{d,2}) - (D_2^2) - (\widetilde{D}_1^{z,2}) + (D_1^2)$	8
D_5	$(D_5^3) + (\mathbb{Z}_5^{t_1,3}) + (D_1^{z,3}) + (D_1^3) - (\mathbb{Z}_1^3)$ $+(\mathbb{Z}_5^{t_2,2}) + (D_1^{z,2}) + (D_1^2) - (\mathbb{Z}_1^2)$	10
D_6	$(D_6^{d,3}) + (\mathbb{Z}_6^{t_1,3}) - (D_3^3) - 3(D_2^{d,3}) - (D_2^{\hat{d},3})$ $-(\mathbb{Z}_3^{t,3}) + 2(\widetilde{D}_1^{z,3}) + (\widetilde{D}_1^3) + (D_1^3) + 2(\mathbb{Z}_2^{-,3})$ $-2(\mathbb{Z}_1^3) + (D_6^{d,2}) + (\mathbb{Z}_6^{t_2,2}) - (D_3^2) - (D_2^{z,2})$ $-2(D_2^{d,2}) - (D_2^2) - (\mathbb{Z}_3^{t,2}) + 2(\widetilde{D}_1^{z,2}) + (\widetilde{D}_1^2)$ $+(D_1^2) + (\mathbb{Z}_2^{-,2}) + (\mathbb{Z}_2^2) - 2(\mathbb{Z}_1^2)$	11
S_4	$(S_4^3) - (A_4^3) + (D_4^{d,3}) - 3(D_3^3) - (D_2^{d,3})$ $-2(D_2^3) - (\mathbb{Z}_4^{c,3}) - (\mathbb{Z}_4^{-,3}) - (\mathbb{Z}_4^3) - (V_4^{-,3})$ $-(\mathbb{Z}_3^{t,3}) + (\mathbb{Z}_3^3) + 3(D_1^3) + (\mathbb{Z}_2^{-,3}) + 2(\mathbb{Z}_2^3)$ $-(\mathbb{Z}_1^3) + (S_4^{-,2}) - (A_4^2) + (D_4^{z,2}) - 3(D_3^{z,2})$ $-(D_2^{d,2}) - 2(D_2^{z,2}) - (\mathbb{Z}_4^{c,2}) - (\mathbb{Z}_4^{-,2}) - (\mathbb{Z}_4^2)$ $-(V_4^{-,2}) - (\mathbb{Z}_3^{t,2}) + (\mathbb{Z}_3^2) + 3(D_1^{z,2}) + (\mathbb{Z}_2^{-,2})$ $+2(\mathbb{Z}_2^2) - (\mathbb{Z}_1^2)$	32
A_5	$(A_5^3) + (A_4^{t_1,3}) + (A_4^{t_2,3}) + (A_4^3) - (D_5^{z,3})$ $-3(D_5^3) - 2(D_3^{z,3}) - 4(D_3^3) - 3(\mathbb{Z}_5^{t_1,3}) - 2(\mathbb{Z}_5^{t_2,3})$ $+3(V_4^{-,3}) - 6(\mathbb{Z}_3^{t,3}) - (\mathbb{Z}_3^3) - 3(\mathbb{Z}_2^{-,3}) + 5(\mathbb{Z}_1^3)$ $(A_4^2) - (D_5^{z,2}) - 2(D_3^{z,2}) - (D_3^2) - (\mathbb{Z}_5^{t_1,2})$ $-2(\mathbb{Z}_5^{t_2,2}) + 2(V_4^{-,2}) - 2(\mathbb{Z}_3^{t,2}) - (\mathbb{Z}_3^2) - (\mathbb{Z}_2^{-,2})$ $-2(\mathbb{Z}_2^2) + 3(\mathbb{Z}_1^2)$	66

Table 2: Existence results for the system (6.1) with symmetry group Γ.

Dihedral Symmetries D_6**.** In the case $\Gamma = D_6$, we have $V = \mathcal{V}_0 \oplus \mathcal{V}_1 \oplus \mathcal{V}_2 \oplus \mathcal{V}_4$, to which we associate the sequence $(\varepsilon_0, \varepsilon_1, \varepsilon_2, \varepsilon_4) = (1,1,1,1)$ and $\sigma(A) = \{\xi_0^0 = 6.5, \xi_1^1 = 5.5, \xi_2^2 = 3.5, \xi_4^4 = 2.5\}$, $\sigma(B) = \{\xi_0^0 = 11.5, \xi_1^1 = 10.5, \xi_2^2 = 8.5, \xi_4^4 = 7.5\}$. Thus, we have the following non-zero $\widetilde{m}_j^k$'s for A and B:

$$\begin{aligned} \widetilde{m}_0^2(A) &= 1, \quad \widetilde{m}_0^3(B) = 1, \\ \widetilde{m}_1^2(A) &= 1, \quad \widetilde{m}_1^3(B) = 1, \\ \widetilde{m}_2^1(A) &= 1, \quad \widetilde{m}_2^2(B) = 1, \\ \widetilde{m}_4^1(A) &= 1, \quad \widetilde{m}_4^2(B) = 1. \end{aligned}$$

Consequently, we have the non-zero isotypical defect numbers

$$\begin{aligned} &\mathfrak{m}_0^2 = 1, \quad \mathfrak{m}_0^3 = -1, \quad \mathfrak{m}_1^2 = 1, \quad \mathfrak{m}_1^3 = -1, \\ &\mathfrak{m}_2^1 = 1, \quad \mathfrak{m}_2^2 = -1, \quad \mathfrak{m}_4^1 = 1, \quad \mathfrak{m}_4^2 = -1. \end{aligned}$$

Hence,

$$\begin{aligned} &\sum_{j=0}^{s}\sum_{k=0}^{\infty}\left(\mathfrak{m}_j^k \sum_{l=1}^{k} \deg_{\mathcal{V}_{j,l}}\right) \\ &= 1 \cdot \left(\deg_{\mathcal{V}_{0,1}} + \deg_{\mathcal{V}_{0,2}}\right) + (-1) \cdot \left(\deg_{\mathcal{V}_{0,1}} + \deg_{\mathcal{V}_{0,2}} + \deg_{\mathcal{V}_{0,3}}\right) \\ &+ 1 \cdot \left(\deg_{\mathcal{V}_{1,1}} + \deg_{\mathcal{V}_{1,2}}\right) + (-1) \cdot \left(\deg_{\mathcal{V}_{1,1}} + \deg_{\mathcal{V}_{1,2}} + \deg_{\mathcal{V}_{1,3}}\right) \\ &+ 1 \cdot \deg_{\mathcal{V}_{2,1}} + (-1) \cdot \left(\deg_{\mathcal{V}_{2,1}} + \deg_{\mathcal{V}_{2,2}}\right) \\ &+ 1 \cdot \deg_{\mathcal{V}_{4,1}} + (-1) \cdot \left(\deg_{\mathcal{V}_{4,1}} + \deg_{\mathcal{V}_{4,2}}\right) \\ &= -\deg_{\mathcal{V}_{0,3}} - \deg_{\mathcal{V}_{1,3}} - \deg_{\mathcal{V}_{2,2}} - \deg_{\mathcal{V}_{4,2}}. \end{aligned}$$

Finally,

$$\begin{aligned} \deg_1 = &\,\Theta_3 \left[\texttt{showdegree[D6]}\,(1,1,1,0,1,0,-1,-1,0,0,0,0)\right] \\ &+ \Theta_2 \left[\texttt{showdegree[D6]}\,(1,1,1,0,1,0,0,0,-1,0,-1,0)\right]. \end{aligned}$$

The dominating orbit types in W are (D_6), (D_6^d), $(\mathbb{Z}_6^{t_1})$, $(\mathbb{Z}_6^{t_2})$, $(D_2^{\hat{d}})$ and (D_2^{z}). The value of $\deg_1$ is listed in Table 2.

Octahedral Symmetries S_4**.** For the octahedral group S_4 we consider the representation $V = \mathbb{R}^8$, which has the isotypical decomposition $V = \mathcal{V}_0 \oplus \mathcal{V}_1 \oplus \mathcal{V}_3 \oplus \mathcal{V}_4$, to which we associate the sequence $(\varepsilon_0, \varepsilon_1, \varepsilon_3, \varepsilon_4) = (1,1,1,1)$, and $\sigma(A) = \{\xi_0^0 = 7.5, \xi_1^1 = 1.5, \xi_2^3 = 5.5, \xi_3^4 = 3.5\}$, $\sigma(B) = \{\xi_0^0 = 12.5, \xi_1^1 = 6.5, \xi_2^3 = 10.5, \xi_3^4 = 8.5\}$. Thus, we have the following non-zero $\widetilde{m}_j^k$'s for A and B:

$$\begin{aligned} \widetilde{m}_0^2(A) &= 1, \quad \widetilde{m}_0^3(B) = 1, \\ \widetilde{m}_1^1(A) &= 1, \quad \widetilde{m}_1^2(B) = 1, \\ \widetilde{m}_3^2(A) &= 1, \quad \widetilde{m}_3^3(B) = 1, \\ \widetilde{m}_4^1(A) &= 1, \quad \widetilde{m}_4^2(B) = 1. \end{aligned}$$

Consequently, we have the non-zero isotypical defect numbers

$$\begin{aligned} &\mathfrak{m}_0^2 = 1, \quad \mathfrak{m}_0^3 = -1, \quad \mathfrak{m}_1^1 = 1, \quad \mathfrak{m}_1^2 = -1, \\ &\mathfrak{m}_3^2 = 1, \quad \mathfrak{m}_3^3 = -1, \quad \mathfrak{m}_4^1 = 1, \quad \mathfrak{m}_4^2 = -1. \end{aligned}$$

Hence,

$$
\begin{aligned}
&\sum_{j=0}^{s}\sum_{k=0}^{\infty}\left(\mathfrak{m}_j^k\sum_{l=1}^{k}\deg_{\mathcal{V}_{j,l}}\right)\\
&= 1\cdot\left(\deg_{\mathcal{V}_{0,1}}+\deg_{\mathcal{V}_{0,2}}\right)+(-1)\cdot\left(\deg_{\mathcal{V}_{0,1}}+\deg_{\mathcal{V}_{0,2}}+\deg_{\mathcal{V}_{0,3}}\right)\\
&+1\cdot\deg_{\mathcal{V}_{1,1}}+(-1)\cdot\left(\deg_{\mathcal{V}_{1,1}}+\deg_{\mathcal{V}_{1,2}}\right)\\
&+1\cdot\left(\deg_{\mathcal{V}_{3,1}}+\deg_{\mathcal{V}_{3,2}}\right)+(-1)\cdot\left(\deg_{\mathcal{V}_{3,1}}+\deg_{\mathcal{V}_{3,2}}+\deg_{\mathcal{V}_{3,3}}\right)\\
&+1\cdot\deg_{\mathcal{V}_{4,1}}+(-1)\cdot\left(\deg_{\mathcal{V}_{4,1}}+\deg_{\mathcal{V}_{4,2}}\right)\\
&= -\deg_{\mathcal{V}_{0,3}}-\deg_{\mathcal{V}_{1,2}}-\deg_{\mathcal{V}_{3,3}}-\deg_{\mathcal{V}_{4,2}}.
\end{aligned}
$$

Finally,

$$
\begin{aligned}
\deg_1 &= \Theta_3\left[\texttt{showdegree[S4]}\,(1,1,0,1,1,-1,0,0,-1,0)\right]\\
&\quad+\Theta_2\left[\texttt{showdegree[S4]}\,(1,1,0,1,1,0,-1,0,0,-1)\right].
\end{aligned}
$$

The dominating orbit types in W are (S_4), (S_4^-), (D_4^d), (D_2^d), $(\mathbb{Z}_4^c):=(\mathbb{Z}_4^{t_1})$, $(\mathbb{Z}_3^t):=(\mathbb{Z}_3^{t_1})$ and (D_4^z). The value of $\deg_1$ is listed in Table 2.

Icosahedral Symmetries A_5. Finally, we consider the system (6.1) with the group of symmetries $G=A_5\times S^1$, where A_5 denotes the icosahedral group. The A_5-representation $V=\mathbb{R}^{20}$ has the following isotypical decomposition

$$
V=\mathcal{V}_0\oplus(\mathcal{V}_1\oplus\mathcal{V}_1)\oplus\mathcal{V}_2\oplus\mathcal{V}_3\oplus\mathcal{V}_4,
$$

to which we associate the sequence $(\varepsilon_0,\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4)=(1,0,1,1,1)$, and $\sigma(A)=\{\xi_0^0=7.5,\xi_1^1=4.5,\xi_2^1=2.5,\xi_3^2=5.5,\xi_4^3=4.5+\sqrt{5},\xi_5^4=4.5-\sqrt{5}\}$, $\sigma(B)=\{\{\xi_0^0=12.5,\xi_1^1=9.5,\xi_2^1=7.5,\xi_3^2=10.5,\xi_4^3=9.5+\sqrt{5},\xi_5^4=9.5-\sqrt{5}\}\}$. Thus, we have the following non-zero $\widetilde{m}_j^k$'s for A and B:

$$
\begin{aligned}
\widetilde{m}_0^2(A)&=1, &\widetilde{m}_0^3(B)&=1,\\
\widetilde{m}_1^1(A)&=1, &\widetilde{m}_1^3(B)&=1,\\
\widetilde{m}_1^2(A)&=1, &\widetilde{m}_1^2(B)&=1,\\
\widetilde{m}_2^2(A)&=1, &\widetilde{m}_2^3(B)&=1,\\
\widetilde{m}_3^2(A)&=1, &\widetilde{m}_3^3(B)&=1,\\
\widetilde{m}_4^1(A)&=1, &\widetilde{m}_4^2(B)&=1.
\end{aligned}
$$

Consequently, we have the non-zero isotypical defect numbers

$$
\begin{aligned}
&\mathfrak{m}_0^2=1,\quad \mathfrak{m}_0^3=-1,\quad \mathfrak{m}_1^1=1,\quad \mathfrak{m}_1^3=-1,\quad \mathfrak{m}_2^2=1,\\
&\mathfrak{m}_2^3=-1,\quad \mathfrak{m}_3^2=1,\quad \mathfrak{m}_3^3=-1,\quad \mathfrak{m}_4^1=1,\quad \mathfrak{m}_4^2=-1.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
&\sum_{j=0}^{s}\sum_{k=0}^{\infty}\left(\mathfrak{m}_j^k\sum_{l=1}^{k}\deg_{\mathcal{V}_{j,l}}\right)\\
&= 1\cdot\left(\deg_{\mathcal{V}_{0,1}}+\deg_{\mathcal{V}_{0,2}}\right)+(-1)\cdot\left(\deg_{\mathcal{V}_{0,1}}+\deg_{\mathcal{V}_{0,2}}+\deg_{\mathcal{V}_{0,3}}\right)\\
&+1\cdot\deg_{\mathcal{V}_{1,1}}+(-1)\cdot\left(\deg_{\mathcal{V}_{1,1}}+\deg_{\mathcal{V}_{1,2}}+\deg_{\mathcal{V}_{1,3}}\right)\\
&+1\cdot\left(\deg_{\mathcal{V}_{2,1}}+\deg_{\mathcal{V}_{2,2}}\right)+(-1)\cdot\left(\deg_{\mathcal{V}_{2,1}}+\deg_{\mathcal{V}_{2,2}}+\deg_{\mathcal{V}_{2,3}}\right)\\
&+1\cdot\left(\deg_{\mathcal{V}_{3,1}}+\deg_{\mathcal{V}_{3,2}}\right)+(-1)\cdot\left(\deg_{\mathcal{V}_{3,1}}+\deg_{\mathcal{V}_{3,2}}+\deg_{\mathcal{V}_{3,3}}\right)\\
&+1\cdot\deg_{\mathcal{V}_{4,1}}+(-1)\cdot\left(\deg_{\mathcal{V}_{4,1}}+\deg_{\mathcal{V}_{4,2}}\right)\\
&=-\deg_{\mathcal{V}_{0,3}}-\deg_{\mathcal{V}_{1,2}}-\deg_{\mathcal{V}_{1,3}}-\deg_{\mathcal{V}_{2,3}}-\deg_{\mathcal{V}_{3,3}}-\deg_{\mathcal{V}_{4,2}}\,.
\end{aligned}
$$

Finally,

$$
\begin{aligned}
\deg_1 &= \Theta_3\left[\texttt{showdegree[A5]}\,(1,0,1,1,1,-1,-1,-1,-1,0)\right]\\
&\quad+\Theta_2\left[\texttt{showdegree[A5]}\,(1,0,1,1,1,0,-1,0,0,-1)\right].
\end{aligned}
$$

The dominating orbit types: (A_5), (D_3^z), (V_4^-), $(\mathbb{Z}_5^{t_1})$, $(\mathbb{Z}_5^{t_2})$, $(A_4^{t_1})$, $(A_4^{t_2})$ and (D_5^z). The value of $\deg_1$ is listed in Table 2.

Appendix

Example 1. Let Γ be the dihedral group D_6 of order 12, as the group of rotations 1, μ, μ^2, μ^3, μ^4, μ^5 of the complex plane (where μ is the multiplication by $e^{\frac{2\pi i}{6}}$) plus the reflections κ, $\kappa\mu$, $\kappa\mu^2$, $\kappa\mu^3$, $\kappa\mu^4$, $\kappa\mu^5$ with κ being the operator of complex conjugation. The subgroups in D_6, up to the conjugacy class, are listed below:

$$
\begin{aligned}
&D_6=\{1,\mu,\mu^2,\mu^3,\mu^4,\mu^5,\kappa,\kappa\mu,\kappa\mu^2,\kappa\mu^3,\kappa\mu^4,\kappa\mu^5\},\\
&\widetilde{D}_3=\{1,\mu^2,\mu^4,\kappa\mu,\kappa\mu^3,\kappa\mu^5\},\quad D_3=\{1,\mu^2,\mu^4,\kappa,\kappa\mu^2,\kappa\mu^4\},\\
&\mathbb{Z}_6=\{1,\mu,\mu^2,\mu^3,\mu^4,\mu^5\}\\
&D_2=\{1,-1,\kappa,-\kappa\},\\
&\mathbb{Z}_3=\{1,\mu^2,\mu^4\},\\
&\widetilde{D}_1=\{1,\kappa\mu\},\quad D_1=\{1,\kappa\},\\
&\mathbb{Z}_2=\{1,-1\},\quad \mathbb{Z}_1=\{1\}.
\end{aligned}
$$

The non-trivial* twisted one-folded subgroups in $D_6 \times S^1$, up to the conjugacy class are:

$$
\begin{aligned}
D_6^z &= \{(1,1),(\mu,1),(\mu^2,1),(\mu^3,1),(\mu^4,1),(\mu^5,1),(\kappa,-1),\\
&\quad (\kappa\mu,-1),(\kappa\mu^2,-1),(\kappa\mu^3,-1),(\kappa\mu^4,-1),(\kappa\mu^5,-1)\},\\
D_6^{\hat{d}} &= \{(1,1),(\mu,-1),(\mu^2,1),(\mu^3,-1),(\mu^4,1),(\mu^5,-1),(\kappa,-1),\\
&\quad (\kappa\mu,1),(\kappa\mu^2,-1),(\kappa\mu^3,1),(\kappa\mu^4,-1),(\kappa\mu^5,1)\},\\
D_6^d &= \{(1,1),(\mu,-1),(\mu^2,1),(\mu^3,-1),(\mu^4,1),(\mu^5,-1),(\kappa,1),\\
&\quad (\kappa\mu,-1),(\kappa\mu^2,1),(\kappa\mu^3,-1),(\kappa\mu^4,1),(\kappa\mu^5,-1)\},\\
\mathbb{Z}_6^d &= \{(1,1),(\mu,-1),(\mu^2,1),(\mu^3,-1)(\mu^4,1),(\mu^5,-1)\},\\
\mathbb{Z}_6^{t_1} &= \{(1,1),(\mu,\mu),(\mu^2,\mu^2),(\mu^3,\mu^3)(\mu^4,\mu^4),(\mu^5,\mu^5)\},\\
\mathbb{Z}_6^{t_2} &= \{(1,1),(\mu,\mu^2),(\mu^2,\mu^4),(\mu^3,1)(\mu^4,\mu^2),(\mu^5,\mu^4)\},\\
\widetilde{D}_3^z &= \{(1,1),(\mu^2,1),(\mu^4,1),(\kappa\mu,-1),(\kappa\mu^3,-1),(\kappa\mu^5,-1)\},\\
D_3^z &= \{(1,1),(\mu^2,1),(\mu^4,1),(\kappa,-1),(\kappa\mu^2,-1),(\kappa\mu^4,-1)\},\\
D_2^z &= \{(1,1),(-1,1),(\kappa,-1),(-\kappa,-1)\},\\
D_2^d &= \{(1,1),(-1,-1),(\kappa,1),(-\kappa,-1)\},\\
D_2^{\hat{d}} &= \{(1,1),(-1,-1),(\kappa,-1),(-\kappa,1)\},\\
\mathbb{Z}_3^t &= \{(1,1),(\mu^2,\mu^2),(\mu^4,\mu^4)\},\\
\widetilde{D}_1^z &= \{(1,1),(\kappa\mu,-1)\}, \quad D_1^z = \{(1,1),(\kappa,-1)\},\\
\mathbb{Z}_2^- &= \{(1,1),(-1,-1)\}.
\end{aligned}
$$

We present the $U(D_6 \times S^1)$ multiplication table in Table 3, where only the *one*-folded twisted subgroups are included**

Example 2. Consider the permutation group $\Gamma = S_4$ of four symbols $\{1,2,3,4\}$, which is isomorphic to the octahedral group of symmetries, preserving orientation of a regular cube. The subgoups of S_4, up to the conjugacy classes, are

$$
\begin{aligned}
S_4 &= \{(1),(12),(13),(14),(23),(24),(34),(123),(132),(124),\\
&\quad (142),(134),(143),(234),(243),(12)(34),(13)(24),(14)(23),\\
&\quad (1234),(1243),(1324),(1342),(1423),(1432)\},\\
A_4 &= \{(1),(12)(34),(123),(132),(13)(24),(142),\\
&\quad (124),(14)(23),(134),(143),(243),(234)\},\\
D_4 &= \{(1),(1324),(12)(34),(1423),(34),(14)(23),(12),(13)(24)\},\\
D_3 &= \{(1),(123),(132),(12),(23),(13)\},\\
D_2 &= \{(1),(12)(34),(12),(34)\},\\
V_4 &= \{(1),(12)(34),(13)(24),(14)(23)\},\\
\mathbb{Z}_4 &= \{(1),(1324),(12)(34),(1423)\},\\
\mathbb{Z}_3 &= \{(1),(123),(132)\}, \quad \mathbb{Z}_2 = \{(1),(12)(34)\},\\
D_1 &= \{(1),(12)\}, \quad \mathbb{Z}_1 = \{(1)\}.
\end{aligned}
$$

* Every subgroup $\mathcal{H} \subset \Gamma$, which can be naturally identified with $\mathcal{H} \times \{1\} \subset \Gamma \times S^1$, is a (trivailly) twisted one-folded subgroup given by the homomorphism $\varphi : \mathcal{H} \to \{1\}$.

** For l-folded twisted subgroups, where $l > 1$, the multiplication table can be extended systematically.

The following lists the non-trivial twisted one-folded subgroups in $S_4 \times S^1$ (up to the conjugacy):

$$\begin{aligned}
S_4^- = \{&((1),1),((12),-1),((12)(34),1),((123),1),((1234),-1),\\
&((13),-1),((13)(24),1),((132),1),((1342),-1),((14),-1),\\
&((14)(23),1),((142),1),((1324),-1),((23),-1),((124),1),\\
&((1243),-1),((24),-1),((134),1),((1423),-1),((34),-1),\\
&((143),1),((1432),-1),((243),1),((234),1)\},\\
A_4^t = \{&((1),1),((12)(34),1),((13)(24),1),((14)(23),1),((123),\gamma),\\
&((132),\gamma^2),((142),\gamma),((124),\gamma^2),((134),\gamma),\\
&((143),\gamma^2),((243),\gamma),((234),\gamma^2)\},\\
D_4^z = \{&((1),1),((1324),1),((12)(34),1),((1423),1),((34),-1),\\
&((14)(23),-1),((12),-1),((13)(24),-1)\},\\
D_4^d = \{&((1),1),((1324),-1),((12)(34),1),((1423),-1),((34),1),\\
&((14)(23),-1),((12),1),((13)(24),-1)\},\\
D_4^{\hat{d}} = \{&((1),1),((1324),-1),((12)(34),1),((1423),-1),((34),-1),\\
&((14)(23),1),((12),-1),((13)(24),1)\},\\
D_3^z = \{&((1),1),((123),1),((132),1),((12),-1),((23),-1),((13),-1)\},\\
D_2^z = \{&((1),1),((12)(34),1),((12),-1),((34),-1)\},\\
D_2^d = \{&((1),1),((12)(34),-1),((12),1),((34),-1)\},\\
V_4^- = \{&((1),1),((12)(34),1),((13)(24),-1),((14)(23),-1)\},\\
\mathbb{Z}_4^- = \{&((1),1),((1324),-1),((12)(34),1),((1423),-1)\},\\
\mathbb{Z}_4^c = \{&((1),1),((1324),i),((12)(34),-1),((1423),-i)\},\\
\mathbb{Z}_3^t = \{&((1),1),((123),\gamma),((132),\gamma^2)\},\\
\mathbb{Z}_2^- = \{&((1),1),((12)(34),-1)\},\\
D_1^z = \{&((1),1),((12),-1)\},
\end{aligned}$$

where $\gamma = e^{\frac{2\pi}{3}i}$. We present the multiplication table for $U(S_4 \times S^1)$ in Table 4.

Example 3. Let $\Gamma = A_5$ be the alternating group of order 60, i.e. A_5 is the group of even permutations of five symbols $\{1,2,3,4,5\}$, which is isomorphic to the icosahedral group of symmetries of a regular dodecahedron. Up to the conjugacy, the subgroups in A_5 are listed below (besides A_5

and $\mathbb{Z}_1$):

$$
\begin{aligned}
A_4 &= \{(1),\,(12)(34),\,(13)(24),\,(14)(23),\,(123),\,(132),\\
&\quad (124),\,(142),\,(134),(143),\,(234),\,(243)\},\\
D_5 &= \{(1),\,(12345),\,(13524),\,(14253),\,(15432),\,(12)(35),\\
&\quad (13)(54),\,(14)(23),\,(15)(24),\,(25)(34)\},\\
D_3 &= \{(1),\,(123),\,(132),\,(12)(45),\,(13)(45),\,(23)(45)\},\\
\mathbb{Z}_5 &= \{(1),\,(12345),\,(13524),\,(14253),\,(15432)\},\\
V_4 &= \{(1),\,(12)(34),\,(13)(24),\,(23)(14)\},\\
\mathbb{Z}_3 &= \{(1)\,,(123),\,(132)\},\\
\mathbb{Z}_2 &= \{(1),\,(12)(34)\},
\end{aligned}
$$

The non-trivial twisted one-folded subgroups in $A_5 \times S^1$ are (up to the conjugacy):

$$
\begin{aligned}
A_4^{t_1} &= \Big\{\big((1),1\big),\,\big((12)(34),1\big),\,\big((13)(24),1\big),\,\big((14)(23),1\big),\,\big((123),\gamma\big),\,\big((132),\gamma^2\big),\\
&\quad \big((124),\gamma^2\big),\,\big((142),\gamma\big),\,\big((134),\gamma\big),\,\big((143),\gamma^2\big),\,\big((234),\gamma^2\big),\,\big((243),\gamma\big)\Big\}\\
A_4^{t_2} &= \Big\{\big((1),1\big),\,\big((12)(34),1\big),\,\big((13)(24),1\big),\,\big((14)(23),1\big),\,\big((123),\gamma^2\big),\,\big((132),\gamma\big),\\
&\quad \big((124),\gamma\big),\,\big((142),\gamma^2\big),\,\big((134),\gamma^2\big),\,\big((143),\gamma\big),\,\big((234),\gamma\big),\,\big((243),\gamma^2\big)\Big\}\\
D_5^z &= \Big\{\big((1),1\big),\,\big((12345),1\big),\,\big((13524),1\big),\,\big((15432),1\big),\,\big(14253),1\big),\,\big((12)(35),-1\big),\\
&\quad \big((13)(45),-1\big),\,\big((14)(23),-1\big),\,\big((15)(24),-1\big),\,\big((25)(34),-1\big)\Big\},\\
D_3^z &= \Big\{\big((1),1\big),\,\big((123),1\big),\,\big((132),1\big),\,\big((12)(45),-1\big),\,\big((13)(45),-1\big),\big((23)(45),-1\big)\Big\}\\
\mathbb{Z}_5^{t_1} &= \Big\{\big((1),1\big),\,\big((12345),\xi\big),\,\big((13524),\xi^2\big),\,\big((14253),\xi^3\big),\,\big((15432),\xi^4\big)\Big\}\\
\mathbb{Z}_5^{t_2} &= \Big\{\big((1),1\big),\,\big((12345),\xi^2\big),\,\big((13524),\xi^4\big),\,\big((14253),\xi\big),\,\big((15432),\xi^3\big)\Big\}\\
V_4^- &= \Big\{\big((1),1\big),\,\big((12)(34),-1\big),\,\big((13)(24),-1\big),\,\big((23)(14),1\big)\Big\}\\
\mathbb{Z}_3^t &= \Big\{\big((1),1\big),\,\big((123),\gamma\big),\,\big((132),\gamma^2\big)\Big\}\\
\mathbb{Z}_2^- &= \Big\{\big((1),1\big),\,\big((12)(34),-1\big)\Big\},
\end{aligned}
$$

where $\xi = e^{\frac{2\pi}{5}i}$, $\gamma = e^{\frac{2\pi}{3}i}$. We present the $U(A_5 \times S^1)$ multiplication table in Table 5.

	$(D_6\times S^1)$	$(\widetilde{D}_3\times S^1)$	$(D_3\times S^1)$	$(\mathbb{Z}_6\times S^1)$	$(D_2\times S^1)$	$(\mathbb{Z}_3\times S^1)$	$(\widetilde{D}_1\times S^1)$	$(D_1\times S^1)$	$(\mathbb{Z}_2\times S^1)$	$(\mathbb{Z}_1\times S^1)$
$(D_6\times S^1)$	$(D_6\times S^1)$	$(\widetilde{D}_3\times S^1)$	$(D_3\times S^1)$	$(\mathbb{Z}_6\times S^1)$	$(D_2\times S^1)$	$(\mathbb{Z}_3\times S^1)$	$(\widetilde{D}_1\times S^1)$	$(D_1\times S^1)$	$(\mathbb{Z}_2\times S^1)$	$(\mathbb{Z}_1\times S^1)$
$(\widetilde{D}_3\times S^1)$	$(\widetilde{D}_3\times S^1)$	$2(\widetilde{D}_3\times S^1)$	$(\mathbb{Z}_3\times S^1)$	$(\mathbb{Z}_3\times S^1)$	$(\widetilde{D}_1\times S^1)$	$3(\mathbb{Z}_3\times S^1)$	$2(\widetilde{D}_1\times S^1)$	$(\mathbb{Z}_1\times S^1)$	$(\mathbb{Z}_1\times S^1)$	$2(\mathbb{Z}_1\times S^1)$
$(D_3\times S^1)$	$(D_3\times S^1)$	$(\mathbb{Z}_3\times S^1)$	$2(D_3\times S^1)$	$(\mathbb{Z}_3\times S^1)$	$(D_1\times S^1)$	$2(\mathbb{Z}_3\times S^1)$	$(\mathbb{Z}_1\times S^1)$	$2(D_1\times S^1)$	$(\mathbb{Z}_1\times S^1)$	$2(\mathbb{Z}_1\times S^1)$
$(\mathbb{Z}_6\times S^1)$	$(\mathbb{Z}_6\times S^1)$	$(\mathbb{Z}_3\times S^1)$	$(\mathbb{Z}_3\times S^1)$	$2(\mathbb{Z}_6\times S^1)$	$(\mathbb{Z}_2\times S^1)$	$2(\mathbb{Z}_3\times S^1)$	$(\mathbb{Z}_1\times S^1)$	$(\mathbb{Z}_1\times S^1)$	$2(\mathbb{Z}_2\times S^1)$	$2(\mathbb{Z}_1\times S^1)$
$(D_2\times S^1)$	$(D_2\times S^1)$	$(\widetilde{D}_1\times S^1)$	$(D_1\times S^1)$	$(\mathbb{Z}_2\times S^1)$	$(D_2\times S^1)+(\mathbb{Z}_2\times S^1)$	$(\mathbb{Z}_1\times S^1)$	$(\widetilde{D}_1\times S^1)+(\mathbb{Z}_1\times S^1)$	$(D_1\times S^1)+(\mathbb{Z}_1\times S^1)$	$3(\mathbb{Z}_2\times S^1)$	$3(\mathbb{Z}_1\times S^1)$
$(\mathbb{Z}_3\times S^1)$	$(\mathbb{Z}_3\times S^1)$	$2(\mathbb{Z}_3\times S^1)$	$2(\mathbb{Z}_3\times S^1)$	$2(\mathbb{Z}_3\times S^1)$	$(\mathbb{Z}_1\times S^1)$	$4(\mathbb{Z}_3\times S^1)$	$2(\mathbb{Z}_1\times S^1)$	$2(\mathbb{Z}_1\times S^1)$	$2(\mathbb{Z}_1\times S^1)$	$4(\mathbb{Z}_1\times S^1)$
$(\widetilde{D}_1\times S^1)$	$(\widetilde{D}_1\times S^1)$	$2(\widetilde{D}_1\times S^1)$	$(\mathbb{Z}_1\times S^1)$	$(\mathbb{Z}_1\times S^1)$	$(\widetilde{D}_1\times S^1)+(\mathbb{Z}_1\times S^1)$	$2(\mathbb{Z}_1\times S^1)$	$2(\widetilde{D}_1\times S^1)+(\mathbb{Z}_1\times S^1)$	$3(\mathbb{Z}_1\times S^1)$	$3(\mathbb{Z}_1\times S^1)$	$6(\mathbb{Z}_1\times S^1)$
$(D_1\times S^1)$	$(D_1\times S^1)$	$(\mathbb{Z}_1\times S^1)$	$2(D_1\times S^1)$	$(\mathbb{Z}_1\times S^1)$	$(D_1\times S^1)+(\mathbb{Z}_1\times S^1)$	$2(\mathbb{Z}_1\times S^1)$	$3(\mathbb{Z}_1\times S^1)$	$2(D_1\times S^1)+(\mathbb{Z}_1\times S^1)$	$3(\mathbb{Z}_1\times S^1)$	$6(\mathbb{Z}_1\times S^1)$
$(\mathbb{Z}_2\times S^1)$	$(\mathbb{Z}_2\times S^1)$	$(\mathbb{Z}_1\times S^1)$	$(\mathbb{Z}_1\times S^1)$	$2(\mathbb{Z}_2\times S^1)$	$3(\mathbb{Z}_2\times S^1)$	$2(\mathbb{Z}_1\times S^1)$	$3(\mathbb{Z}_1\times S^1)$	$3(\mathbb{Z}_1\times S^1)$	$6(\mathbb{Z}_2\times S^1)$	$6(\mathbb{Z}_1\times S^1)$
$(\mathbb{Z}_1\times S^1)$	$(\mathbb{Z}_1\times S^1)$	$2(\mathbb{Z}_1\times S^1)$	$2(\mathbb{Z}_1\times S^1)$	$2(\mathbb{Z}_1\times S^1)$	$3(\mathbb{Z}_1\times S^1)$	$4(\mathbb{Z}_1\times S^1)$	$6(\mathbb{Z}_1\times S^1)$	$6(\mathbb{Z}_1\times S^1)$	$6(\mathbb{Z}_1\times S^1)$	$12(\mathbb{Z}_1\times S^1)$
(D_6)	(D_6)	$(\widetilde{D}_3)$	(D_3)	$(\mathbb{Z}_6)$	(D_2)	$(\mathbb{Z}_3)$	$(\widetilde{D}_1)$	(D_1)	$(\mathbb{Z}_2)$	$(\mathbb{Z}_1)$
$(\widetilde{D}_3)$	$(\widetilde{D}_3)$	$2(\widetilde{D}_3)$	$(\mathbb{Z}_3)$	$(\mathbb{Z}_3)$	$(\widetilde{D}_1)$	$3(\mathbb{Z}_3)$	$2(\widetilde{D}_1)$	$(\mathbb{Z}_1)$	$(\mathbb{Z}_1)$	$2(\mathbb{Z}_1)$
(D_3)	(D_3)	$(\mathbb{Z}_3)$	$2(D_3)$	$(\mathbb{Z}_3)$	(D_1)	$2(\mathbb{Z}_3)$	$(\mathbb{Z}_1)$	$2(D_1)$	$(\mathbb{Z}_1)$	$2(\mathbb{Z}_1)$
$(\mathbb{Z}_6)$	$(\mathbb{Z}_6)$	$(\mathbb{Z}_3)$	$(\mathbb{Z}_3)$	$2(\mathbb{Z}_6)$	$(\mathbb{Z}_2)$	$2(\mathbb{Z}_3)$	$(\mathbb{Z}_1)$	$(\mathbb{Z}_1)$	$2(\mathbb{Z}_2)$	$2(\mathbb{Z}_1)$
(D_2)	(D_2)	$(\widetilde{D}_1)$	(D_1)	$(\mathbb{Z}_2)$	$(D_2)+(\mathbb{Z}_2)$	$(\mathbb{Z}_1)$	$(\widetilde{D}_1)+(\mathbb{Z}_1)$	$(D_1)+(\mathbb{Z}_1)$	$3(\mathbb{Z}_2)$	$3(\mathbb{Z}_1)$
$(\mathbb{Z}_3)$	$(\mathbb{Z}_3)$	$2(\mathbb{Z}_3)$	$2(\mathbb{Z}_3)$	$2(\mathbb{Z}_3)$	$(\mathbb{Z}_1)$	$4(\mathbb{Z}_3)$	$2(\mathbb{Z}_1)$	$2(\mathbb{Z}_1)$	$2(\mathbb{Z}_1)$	$4(\mathbb{Z}_1)$
$(\widetilde{D}_1)$	$(\widetilde{D}_1)$	$2(\widetilde{D}_1)$	$(\mathbb{Z}_1)$	$(\mathbb{Z}_1)$	$(\widetilde{D}_1)+(\mathbb{Z}_1)$	$2(\mathbb{Z}_1)$	$2(\widetilde{D}_1)+(\mathbb{Z}_1)$	$3(\mathbb{Z}_1)$	$3(\mathbb{Z}_1)$	$6(\mathbb{Z}_1)$
(D_1)	(D_1)	$(\mathbb{Z}_1)$	$2(D_1)$	$(\mathbb{Z}_1)$	$(D_1)+(\mathbb{Z}_1)$	$2(\mathbb{Z}_1)$	$3(\mathbb{Z}_1)$	$2(D_1)+(\mathbb{Z}_1)$	$3(\mathbb{Z}_1)$	$6(\mathbb{Z}_1)$
$(\mathbb{Z}_2)$	$(\mathbb{Z}_2)$	$(\mathbb{Z}_1)$	$(\mathbb{Z}_1)$	$2(\mathbb{Z}_2)$	$3(\mathbb{Z}_2)$	$2(\mathbb{Z}_1)$	$3(\mathbb{Z}_1)$	$3(\mathbb{Z}_1)$	$6(\mathbb{Z}_2)$	$6(\mathbb{Z}_1)$
$(\mathbb{Z}_1)$	$(\mathbb{Z}_1)$	$2(\mathbb{Z}_1)$	$2(\mathbb{Z}_1)$	$2(\mathbb{Z}_1)$	$3(\mathbb{Z}_1)$	$4(\mathbb{Z}_1)$	$6(\mathbb{Z}_1)$	$6(\mathbb{Z}_1)$	$6(\mathbb{Z}_1)$	$12(\mathbb{Z}_1)$
(D_6^z)	(D_6^z)	$(\widetilde{D}_3^z)$	(D_3^z)	$(\mathbb{Z}_6)$	(D_2^z)	$(\mathbb{Z}_3)$	$(\widetilde{D}_1^z)$	(D_1^z)	$(\mathbb{Z}_2)$	$(\mathbb{Z}_1)$
$(D_6^{\hat{d}})$	$(D_6^{\hat{d}})$	$(\widetilde{D}_3)$	(D_3^z)	$(\mathbb{Z}_6^d)$	$(D_2^{\hat{d}})$	$(\mathbb{Z}_3)$	$(\widetilde{D}_1)$	(D_1^z)	$(\mathbb{Z}_2^-)$	$(\mathbb{Z}_1)$
(D_6^d)	(D_6^d)	$(\widetilde{D}_3^z)$	(D_3)	$(\mathbb{Z}_6^d)$	(D_2^d)	$(\mathbb{Z}_3)$	$(\widetilde{D}_1^z)$	(D_1)	$(\mathbb{Z}_2^-)$	$(\mathbb{Z}_1)$
$(\mathbb{Z}_6^d)$	$(\mathbb{Z}_6^d)$	$(\mathbb{Z}_3)$	$(\mathbb{Z}_3)$	$2(\mathbb{Z}_6^d)$	$(\mathbb{Z}_2^-)$	$3(\mathbb{Z}_3)$	$(\mathbb{Z}_1)$	$(\mathbb{Z}_1)$	$2(\mathbb{Z}_2^-)$	$2(\mathbb{Z}_1)$
$(\mathbb{Z}_6^{t_1})$	$(\mathbb{Z}_6^{t_1})$	$(\mathbb{Z}_3^t)$	$(\mathbb{Z}_3^t)$	$2(\mathbb{Z}_6^{t_1})$	$(\mathbb{Z}_2^-)$	$2(\mathbb{Z}_3^t)$	$(\mathbb{Z}_1)$	$(\mathbb{Z}_1)$	$2(\mathbb{Z}_2^-)$	$2(\mathbb{Z}_1)$
$(\mathbb{Z}_6^{t_2})$	$(\mathbb{Z}_6^{t_2})$	$(\mathbb{Z}_3^t)$	$(\mathbb{Z}_3^t)$	$2(\mathbb{Z}_6^{t_2})$	$(\mathbb{Z}_2)$	$2(\mathbb{Z}_3^t)$	$(\mathbb{Z}_1)$	$(\mathbb{Z}_1)$	$2(\mathbb{Z}_2)$	$2(\mathbb{Z}_1)$
$(\widetilde{D}_3^z)$	$(\widetilde{D}_3^z)$	$2(\widetilde{D}_3^z)$	$(\mathbb{Z}_3)$	$(\mathbb{Z}_3)$	$(\widetilde{D}_1^z)$	$2(\mathbb{Z}_3)$	$2(\widetilde{D}_1^z)$	$(\mathbb{Z}_1)$	$(\mathbb{Z}_1)$	$2(\mathbb{Z}_1)$
(D_3^z)	(D_3^z)	$(\mathbb{Z}_3)$	$2(D_3^z)$	$(\mathbb{Z}_3)$	(D_1^z)	$2(\mathbb{Z}_3)$	$(\mathbb{Z}_1)$	$2(D_1^z)$	$(\mathbb{Z}_1)$	$2(\mathbb{Z}_1)$
(D_2^z)	(D_2^z)	$(\widetilde{D}_1^z)$	(D_1^z)	$(\mathbb{Z}_2)$	$(D_2^z)+(\mathbb{Z}_2)$	$(\mathbb{Z}_1)$	$(\widetilde{D}_1^z)+(\mathbb{Z}_1)$	$(D_1^z)+(\mathbb{Z}_1)$	$3(\mathbb{Z}_2)$	$3(\mathbb{Z}_1)$
(D_2^d)	(D_2^d)	$(\widetilde{D}_1^z)$	(D_1)	$(\mathbb{Z}_2^-)$	$(D_2^d)+(\mathbb{Z}_2^-)$	$(\mathbb{Z}_1)$	$(\widetilde{D}_1^z)+(\mathbb{Z}_1)$	$(D_1)+(\mathbb{Z}_1)$	$3(\mathbb{Z}_2^-)$	$3(\mathbb{Z}_1)$
$(D_2^{\hat{d}})$	$(D_2^{\hat{d}})$	$(\widetilde{D}_1)$	(D_1^z)	$(\mathbb{Z}_2^-)$	$(D_2^{\hat{d}})+(\mathbb{Z}_2^-)$	$(\mathbb{Z}_1)$	$(\widetilde{D}_1)+(\mathbb{Z}_1)$	$(D_1^z)+(\mathbb{Z}_1)$	$3(\mathbb{Z}_2^-)$	$3(\mathbb{Z}_1)$
$(\mathbb{Z}_3^t)$	$(\mathbb{Z}_3^t)$	$2(\mathbb{Z}_3^t)$	$2(\mathbb{Z}_3^t)$	$2(\mathbb{Z}_3^t)$	$(\mathbb{Z}_1)$	$4(\mathbb{Z}_3^t)$	$2(\mathbb{Z}_1)$	$2(\mathbb{Z}_1)$	$2(\mathbb{Z}_1)$	$4(\mathbb{Z}_1)$
$(\widetilde{D}_1^z)$	$(\widetilde{D}_1^z)$	$2(\widetilde{D}_1^z)$	$(\mathbb{Z}_1)$	$(\mathbb{Z}_1)$	$(\widetilde{D}_1^z)+(\mathbb{Z}_1)$	$2(\mathbb{Z}_1)$	$2(\widetilde{D}_1^z)+2(\mathbb{Z}_1)$	$3(\mathbb{Z}_1)$	$3(\mathbb{Z}_1)$	$6(\mathbb{Z}_1)$
(D_1^z)	(D_1^z)	$(\mathbb{Z}_1)$	$2(D_1^z)$	$(\mathbb{Z}_1)$	$(D_1^z)+(\mathbb{Z}_1)$	$2(\mathbb{Z}_1)$	$3(\mathbb{Z}_1)$	$2(D_1^z)+2(\mathbb{Z}_1)$	$3(\mathbb{Z}_1)$	$6(\mathbb{Z}_1)$
$(\mathbb{Z}_2^-)$	$(\mathbb{Z}_2^-)$	$(\mathbb{Z}_1)$	$(\mathbb{Z}_1)$	$2(\mathbb{Z}_2^-)$	$3(\mathbb{Z}_2^-)$	$2(\mathbb{Z}_1)$	$3(\mathbb{Z}_1)$	$3(\mathbb{Z}_1)$	$6(\mathbb{Z}_2^-)$	$6(\mathbb{Z}_1)$

Table 3: Multiplication Table for $U(D_6\times S^1)$

	$(A_4\times S^1)$	$(D_4\times S^1)$	$(D_3\times S^1)$	$(D_2\times S^1)$	$(V_4\times S^1)$	$(\mathbb{Z}_4\times S^1)$	$(\mathbb{Z}_3\times S^1)$	$(\mathbb{Z}_2\times S^1)$
$(A_4\times S^1)$	$2(A_4\times S^1)$	$(V_4\times S^1)$	$(\mathbb{Z}_3\times S^1)$	$(\mathbb{Z}_2\times S^1)$	$2(V_4\times S^1)$	$(\mathbb{Z}_2\times S^1)$	$2(\mathbb{Z}_3\times S^1)$	$2(\mathbb{Z}_2\times S^1)$
$(D_4\times S^1)$	$(V_4\times S^1)$	$(D_4\times S^1)+(V_4\times S^1)$	$(D_1\times S^1)$	$(D_2\times S^1)+(\mathbb{Z}_2\times S^1)$	$3(V_4\times S^1)$	$(\mathbb{Z}_4\times S^1)+(\mathbb{Z}_2\times S^1)$	$(\mathbb{Z}_1\times S^1)$	$3(\mathbb{Z}_2\times S^1)$
$(D_3\times S^1)$	$(\mathbb{Z}_3\times S^1)$	$(D_1\times S^1)$	$(D_3\times S^1)+(D_1\times S^1)$	$2(D_1\times S^1)$	$(\mathbb{Z}_1\times S^1)$	$(\mathbb{Z}_1\times S^1)$	$(\mathbb{Z}_3\times S^1)+(\mathbb{Z}_1\times S^1)$	$2(\mathbb{Z}_1\times S^1)$
$(D_2\times S^1)$	$(\mathbb{Z}_2\times S^1)$	$(D_2\times S^1)+(\mathbb{Z}_2\times S^1)$	$2(D_1\times S^1)$	$2(D_2\times S^1)+(\mathbb{Z}_1\times S^1)$	$3(\mathbb{Z}_2\times S^1)$	$(\mathbb{Z}_2\times S^1)+(\mathbb{Z}_1\times S^1)$	$2(\mathbb{Z}_1\times S^1)$	$2(\mathbb{Z}_2\times S^1)+2(\mathbb{Z}_1\times S^1)$
$(V_4\times S^1)$	$2(V_4\times S^1)$	$3(V_4\times S^1)$	$(\mathbb{Z}_1\times S^1)$	$3(\mathbb{Z}_2\times S^1)$	$6(V_4\times S^1)$	$3(\mathbb{Z}_2\times S^1)$	$2(\mathbb{Z}_1\times S^1)$	$6(\mathbb{Z}_2\times S^1)$
$(\mathbb{Z}_4\times S^1)$	$2(\mathbb{Z}_2\times S^1)$	$(\mathbb{Z}_4\times S^1)+(\mathbb{Z}_2\times S^1)$	$(\mathbb{Z}_1\times S^1)$	$(\mathbb{Z}_2\times S^1)+(\mathbb{Z}_1\times S^1)$	$3(\mathbb{Z}_2\times S^1)$	$2(\mathbb{Z}_4\times S^1)+(\mathbb{Z}_1\times S^1)$	$2(\mathbb{Z}_1\times S^1)$	$2(\mathbb{Z}_2\times S^1)+2(\mathbb{Z}_1\times S^1)$
$(\mathbb{Z}_3\times S^1)$	$2(\mathbb{Z}_3\times S^1)$	$(\mathbb{Z}_1\times S^1)$	$(\mathbb{Z}_3\times S^1)+(\mathbb{Z}_1\times S^1)$	$2(\mathbb{Z}_1\times S^1)$	$2(\mathbb{Z}_1\times S^1)$	$2(\mathbb{Z}_1\times S^1)$	$2(\mathbb{Z}_3\times S^1)+2(\mathbb{Z}_1\times S^1)$	$4(\mathbb{Z}_1\times S^1)$
$(\mathbb{Z}_2\times S^1)$	$2(\mathbb{Z}_2\times S^1)$	$3(\mathbb{Z}_2\times S^1)$	$2(\mathbb{Z}_1\times S^1)$	$2(\mathbb{Z}_2\times S^1)+2(\mathbb{Z}_1\times S^1)$	$6(\mathbb{Z}_2\times S^1)$	$2(\mathbb{Z}_2\times S^1)+2(\mathbb{Z}_1\times S^1)$	$4(\mathbb{Z}_1\times S^1)$	$4(\mathbb{Z}_2\times S^1)+4(\mathbb{Z}_1\times S^1)$
$(D_1\times S^1)$	$(\mathbb{Z}_1\times S^1)$	$(D_1\times S^1)+(\mathbb{Z}_1\times S^1)$	$2(D_1\times S^1)+(\mathbb{Z}_1\times S^1)$	$2(D_1\times S^1)+2(\mathbb{Z}_1\times S^1)$	$3(\mathbb{Z}_1\times S^1)$	$3(\mathbb{Z}_1\times S^1)$	$4(\mathbb{Z}_1\times S^1)$	$6(\mathbb{Z}_1\times S^1)$
$(\mathbb{Z}_1\times S^1)$	$2(\mathbb{Z}_1\times S^1)$	$3(\mathbb{Z}_1\times S^1)$	$4(\mathbb{Z}_1\times S^1)$	$6(\mathbb{Z}_1\times S^1)$	$6(\mathbb{Z}_1\times S^1)$	$6(\mathbb{Z}_1\times S^1)$	$8(\mathbb{Z}_1\times S^1)$	$12(\mathbb{Z}_1\times S^1)$
(A_4)	$2(A_4)$	(V_4)	$(\mathbb{Z}_3)$	$(\mathbb{Z}_2)$	$2(V_4)$	$(\mathbb{Z}_2)$	$2(\mathbb{Z}_3)$	$2(\mathbb{Z}_2)$
(D_4)	(V_4)	$(D_4)+(V_4)$	(D_1)	$(D_2)+(\mathbb{Z}_2)$	$3(V_4)$	$(\mathbb{Z}_4)+(\mathbb{Z}_2)$	$(\mathbb{Z}_1)$	$3(\mathbb{Z}_2)$
(D_3)	$(\mathbb{Z}_3)$	(D_1)	$(D_3)+(D_1)$	$2(D_1)$	$(\mathbb{Z}_1)$	$(\mathbb{Z}_1)$	$(\mathbb{Z}_3)+(\mathbb{Z}_1)$	$2(\mathbb{Z}_1)$
(D_2)	$(\mathbb{Z}_2)$	$(D_2)+(\mathbb{Z}_2)$	$2(D_1)$	$2(D_2)+(\mathbb{Z}_1)$	$3(\mathbb{Z}_2)$	$(\mathbb{Z}_2)+(\mathbb{Z}_1)$	$2(\mathbb{Z}_1)$	$2(\mathbb{Z}_2)+2(\mathbb{Z}_1)$
(V_4)	$2(V_4)$	$3(V_4)$	$(\mathbb{Z}_1)$	$3(\mathbb{Z}_2)$	$6(V_4)$	$3(\mathbb{Z}_2)$	$2(\mathbb{Z}_1)$	$6(\mathbb{Z}_2)$
$(\mathbb{Z}_4)$	$(\mathbb{Z}_2)$	$(\mathbb{Z}_4)+(\mathbb{Z}_2)$	$(\mathbb{Z}_1)$	$(\mathbb{Z}_2)+(\mathbb{Z}_1)$	$3(\mathbb{Z}_2)$	$2(\mathbb{Z}_4)+(\mathbb{Z}_1)$	$2(\mathbb{Z}_1)$	$2(\mathbb{Z}_2)+2(\mathbb{Z}_1)$
$(\mathbb{Z}_3)$	$2(\mathbb{Z}_3)$	$(\mathbb{Z}_1)$	$(\mathbb{Z}_3)+(\mathbb{Z}_1)$	$2(\mathbb{Z}_1)$	$2(\mathbb{Z}_1)$	$2(\mathbb{Z}_1)$	$2(\mathbb{Z}_3)+2(\mathbb{Z}_1)$	$4(\mathbb{Z}_1)$
$(\mathbb{Z}_2)$	$2(\mathbb{Z}_2)$	$3(\mathbb{Z}_2)$	$2(\mathbb{Z}_1)$	$2(\mathbb{Z}_2)+2(\mathbb{Z}_1)$	$6(\mathbb{Z}_2)$	$2(\mathbb{Z}_2)+2(\mathbb{Z}_1)$	$4(\mathbb{Z}_1)$	$4(\mathbb{Z}_2)+4(\mathbb{Z}_1)$
(D_1)	$(\mathbb{Z}_1)$	$(D_1)+(\mathbb{Z}_1)$	$2(D_1)+(\mathbb{Z}_1)$	$2(D_1)+2(\mathbb{Z}_1)$	$3(\mathbb{Z}_1)$	$3(\mathbb{Z}_1)$	$4(\mathbb{Z}_1)$	$6(\mathbb{Z}_1)$
$(\mathbb{Z}_1)$	$2(\mathbb{Z}_1)$	$3(\mathbb{Z}_1)$	$4(\mathbb{Z}_1)$	$6(\mathbb{Z}_1)$	$6(\mathbb{Z}_1)$	$6(\mathbb{Z}_1)$	$8(\mathbb{Z}_1)$	$12(\mathbb{Z}_1)$
(S_4^-)	(A_4)	$(D_4^{\hat{d}})$	(D_3^z)	(D_2^z)	(V_4)	$(\mathbb{Z}_4^-)$	$(\mathbb{Z}_3)$	$(\mathbb{Z}_2)$
(A_4^t)	$2(A_4^t)$	(V_4)	$(\mathbb{Z}_3^t)$	$(\mathbb{Z}_2)$	$2(V_4)$	$(\mathbb{Z}_2)$	$2(\mathbb{Z}_3^t)$	$2(\mathbb{Z}_2)$
(D_4^z)	(V_4^-)	$(D_4^z)+(V_4^-)$	(D_1^z)	$(D_2^z)+(\mathbb{Z}_2^-)$	$3(V_4^-)$	$(\mathbb{Z}_4)+(\mathbb{Z}_2^-)$	$(\mathbb{Z}_1)$	$(\mathbb{Z}_2)+2(\mathbb{Z}_2^-)$
(D_4^d)	(V_4^-)	$(D_4^d)+(V_4^-)$	(D_1)	$(D_2)+(\mathbb{Z}_2^-)$	$3(V_4^-)$	$(\mathbb{Z}_4^-)+(\mathbb{Z}_2^-)$	$(\mathbb{Z}_1)$	$(\mathbb{Z}_2)+2(\mathbb{Z}_2^-)$
$(D_4^{\hat{d}})$	(V_4)	$(D_4^{\hat{d}})+(V_4)$	(D_1^z)	$(D_2^z)+(\mathbb{Z}_2)$	$3(V_4)$	$(\mathbb{Z}_4^-)+(\mathbb{Z}_2)$	$(\mathbb{Z}_1)$	$3(\mathbb{Z}_2)$
(D_3^z)	$(\mathbb{Z}_3)$	(D_1^z)	$(D_3^z)+(D_1^z)$	$2(D_1^z)$	$(\mathbb{Z}_1)$	$(\mathbb{Z}_1)$	$(\mathbb{Z}_3)+(\mathbb{Z}_1)$	$2(\mathbb{Z}_1)$
(D_2^z)	$(\mathbb{Z}_2)$	$(D_2^z)+(\mathbb{Z}_2)$	$2(D_1^z)$	$2(D_2^z)+(\mathbb{Z}_1)$	$3(\mathbb{Z}_2)$	$(\mathbb{Z}_2)+(\mathbb{Z}_1)$	$2(\mathbb{Z}_1)$	$2(\mathbb{Z}_2)+2(\mathbb{Z}_1)$
(D_2^d)	$(\mathbb{Z}_2^-)$	$(D_2^d)+(\mathbb{Z}_2^-)$	$(D_1)+(D_1^z)$	$2(D_2^d)+(\mathbb{Z}_1)$	$3(\mathbb{Z}_2^-)$	$(\mathbb{Z}_2^-)+(\mathbb{Z}_1)$	$2(\mathbb{Z}_1)$	$2(\mathbb{Z}_2^-)+2(\mathbb{Z}_1)$
(V_4^-)	$2(V_4^-)$	$3(V_4^-)$	$(\mathbb{Z}_1)$	$(\mathbb{Z}_2)+2(\mathbb{Z}_2^-)$	$6(V_4^-)$	$(\mathbb{Z}_2)+2(\mathbb{Z}_2^-)$	$2(\mathbb{Z}_1)$	$2(\mathbb{Z}_2)+4(\mathbb{Z}_2^-)$
$(\mathbb{Z}_4^-)$	$(\mathbb{Z}_2)$	$(\mathbb{Z}_4^-)+(\mathbb{Z}_2)$	$(\mathbb{Z}_1)$	$(\mathbb{Z}_2)+(\mathbb{Z}_1)$	$3(\mathbb{Z}_2)$	$2(\mathbb{Z}_4^-)+(\mathbb{Z}_1)$	$2(\mathbb{Z}_1)$	$2(\mathbb{Z}_2)+2(\mathbb{Z}_1)$
$(\mathbb{Z}_4^c)$	$2(\mathbb{Z}_2^-)$	$(\mathbb{Z}_4^c)+(\mathbb{Z}_2^-)$	$(\mathbb{Z}_1)$	$(\mathbb{Z}_2^-)+(\mathbb{Z}_1)$	$3(\mathbb{Z}_2^-)$	$2(\mathbb{Z}_4^c)+(\mathbb{Z}_1)$	$2(\mathbb{Z}_1)$	$2(\mathbb{Z}_2^-)+2(\mathbb{Z}_1)$
$(\mathbb{Z}_3^t)$	$2(\mathbb{Z}_3^t)$	$(\mathbb{Z}_1)$	$(\mathbb{Z}_3^t)+(\mathbb{Z}_1)$	$2(\mathbb{Z}_1)$	$2(\mathbb{Z}_1)$	$2(\mathbb{Z}_1)$	$2(\mathbb{Z}_3^t)+2(\mathbb{Z}_1)$	$4(\mathbb{Z}_1)$
$(\mathbb{Z}_2^-)$	$2(\mathbb{Z}_2^-)$	$3(\mathbb{Z}_2^-)$	$2(\mathbb{Z}_1)$	$2(\mathbb{Z}_2^-)+2(\mathbb{Z}_1)$	$6(\mathbb{Z}_2^-)$	$2(\mathbb{Z}_2^-)+2(\mathbb{Z}_1)$	$4(\mathbb{Z}_1)$	$4(\mathbb{Z}_2^-)+4(\mathbb{Z}_1)$
(D_1^z)	$(\mathbb{Z}_1)$	$(D_1^z)+(\mathbb{Z}_1)$	$2(D_1^z)+(\mathbb{Z}_1)$	$2(D_1^z)+2(\mathbb{Z}_1)$	$3(\mathbb{Z}_1)$	$3(\mathbb{Z}_1)$	$4(\mathbb{Z}_1)$	$6(\mathbb{Z}_1)$

Table 4: Multiplication Table for $U(S_4\times S^1)$

	$(A_5 \times S^1)$	$(A_4 \times S^1)$	$(D_5 \times S^1)$	$(D_3 \times S^1)$	$(\mathbb{Z}_5 \times S^1)$	$(V_4 \times S^1)$	$(\mathbb{Z}_3 \times S^1$
$(A_5 \times S^1)$	$(A_5 \times S^1)$	$(A_4 \times S^1)$	$(D_5 \times S^1)$	$(D_3 \times S^1)$	$(\mathbb{Z}_5 \times S^1)$	$(V_4 \times S^1)$	$(\mathbb{Z}_3 \times S^1$
$(A_4 \times S^1)$	$(A_4 \times S^1)$	$(A_4 \times S^1) + (\mathbb{Z}_3 \times S^1)$	$(\mathbb{Z}_2 \times S^1)$	$(\mathbb{Z}_3 \times S^1) + (\mathbb{Z}_2 \times S^1)$	$(\mathbb{Z}_1 \times S^1)$	$(V_4 \times S^1) + (\mathbb{Z}_1 \times S^1)$	$2(\mathbb{Z}_3 \times S^1) + ($
$(D_5 \times S^1)$	$(D_5 \times S^1)$	$(\mathbb{Z}_2 \times S^1)$	$(D_5 \times S^1) + (\mathbb{Z}_2 \times S^1)$	$2(\mathbb{Z}_2 \times S^1)$	$(\mathbb{Z}_5 \times S^1) + (\mathbb{Z}_1 \times S^1)$	$3(\mathbb{Z}_2 \times S^1)$	$2(\mathbb{Z}_1 \times S$
$(D_3 \times S^1)$	$(D_3 \times S^1)$	$(\mathbb{Z}_3 \times S^1) + (\mathbb{Z}_2 \times S^1)$	$2(\mathbb{Z}_2 \times S^1)$	$(D_3 \times S^1) + (\mathbb{Z}_2 \times S^1) + (\mathbb{Z}_1 \times S^1)$	$2(\mathbb{Z}_1 \times S^1)$	$3(\mathbb{Z}_2 \times S^1) + (\mathbb{Z}_1 \times S^1)$	$(\mathbb{Z}_3 \times S^1) + 3($
$(\mathbb{Z}_5 \times S^1)$	$(\mathbb{Z}_5 \times S^1)$	$(\mathbb{Z}_1 \times S^1)$	$(\mathbb{Z}_5 \times S^1) + (\mathbb{Z}_1 \times S^1)$	$2(\mathbb{Z}_1 \times S^1)$	$2(\mathbb{Z}_5 \times S^1) + 2(\mathbb{Z}_1 \times S^1)$	$3(\mathbb{Z}_1 \times S^1)$	$4(\mathbb{Z}_1 \times S$
$(V_4 \times S^1)$	$(V_4 \times S^1)$	$(V_4 \times S^1) + (\mathbb{Z}_1 \times S^1)$	$3(\mathbb{Z}_2 \times S^1)$	$3(\mathbb{Z}_2 \times S^1) + (\mathbb{Z}_1 \times S^1)$	$3(\mathbb{Z}_1 \times S^1)$	$3(V_4 \times S^1) + 3(\mathbb{Z}_1 \times S^1)$	$5(\mathbb{Z}_1 \times S$
$(\mathbb{Z}_3 \times S^1)$	$(\mathbb{Z}_3 \times S^1)$	$2(\mathbb{Z}_3 \times S^1) + (\mathbb{Z}_1 \times S^1)$	$2(\mathbb{Z}_1 \times S^1)$	$(\mathbb{Z}_3 \times S^1) + 3(\mathbb{Z}_1 \times S^1)$	$4(\mathbb{Z}_1 \times S^1)$	$5(\mathbb{Z}_1 \times S^1)$	$2(\mathbb{Z}_3 \times S^1) + 6($
$(\mathbb{Z}_2 \times S^1)$	$(\mathbb{Z}_2 \times S^1)$	$(\mathbb{Z}_2 \times S^1) + 2(\mathbb{Z}_1 \times S^1)$	$2(\mathbb{Z}_2 \times S^1) + 2(\mathbb{Z}_1 \times S^1)$	$2(\mathbb{Z}_2 \times S^1) + 4(\mathbb{Z}_1 \times S^1)$	$6(\mathbb{Z}_1 \times S^1)$	$3(\mathbb{Z}_2 \times S^1) + 6(\mathbb{Z}_1 \times S^1)$	$10(\mathbb{Z}_1 \times S$
$(\mathbb{Z}_1 \times S^1)$	$(\mathbb{Z}_1 \times S^1)$	$5(\mathbb{Z}_1 \times S^1)$	$6(\mathbb{Z}_1 \times S^1)$	$10(\mathbb{Z}_1 \times S^1)$	$12(\mathbb{Z}_1 \times S^1)$	$15(\mathbb{Z}_1 \times S^1)$	$20(\mathbb{Z}_1 \times S$
(A_5)	(A_5)	(A_4)	(D_5)	(D_3)	$(\mathbb{Z}_5)$	(V_4)	$(\mathbb{Z}_3)$
(A_4)	(A_4)	$(A_4) + (\mathbb{Z}_3)$	$(\mathbb{Z}_2)$	$(\mathbb{Z}_3) + (\mathbb{Z}_2)$	$(\mathbb{Z}_1)$	$(V_4) + (\mathbb{Z}_1)$	$2(\mathbb{Z}_3) + ($
(D_5)	(D_5)	$(\mathbb{Z}_2)$	$(D_5) + (\mathbb{Z}_2)$	$2(\mathbb{Z}_2)$	$(\mathbb{Z}_5) + (\mathbb{Z}_1)$	$3(\mathbb{Z}_2)$	$2(\mathbb{Z}_1)$
(D_3)	(D_3)	$(\mathbb{Z}_3) + (\mathbb{Z}_2)$	$2(\mathbb{Z}_2)$	$(D_3) + (\mathbb{Z}_2) + (\mathbb{Z}_1)$	$2(\mathbb{Z}_1)$	$3(\mathbb{Z}_2) + (\mathbb{Z}_1)$	$(\mathbb{Z}_3) + 3($
$(\mathbb{Z}_5)$	$(\mathbb{Z}_5)$	$(\mathbb{Z}_1)$	$(\mathbb{Z}_5) + (\mathbb{Z}_1)$	$2(\mathbb{Z}_1)$	$2(\mathbb{Z}_5) + 2(\mathbb{Z}_1)$	$3(\mathbb{Z}_1)$	$4(\mathbb{Z}_1)$
(V_4)	(V_4)	$(V_4) + (\mathbb{Z}_1)$	$3(\mathbb{Z}_2)$	$3(\mathbb{Z}_2) + (\mathbb{Z}_1)$	$3(\mathbb{Z}_1)$	$3(V_4) + 3(\mathbb{Z}_1)$	$5(\mathbb{Z}_1)$
$(\mathbb{Z}_3)$	$(\mathbb{Z}_3)$	$2(\mathbb{Z}_3) + (\mathbb{Z}_1)$	$2(\mathbb{Z}_1)$	$(\mathbb{Z}_3) + 3(\mathbb{Z}_1)$	$4(\mathbb{Z}_1)$	$5(\mathbb{Z}_1)$	$2(\mathbb{Z}_3) + 6($
$(\mathbb{Z}_2)$	$(\mathbb{Z}_2)$	$(\mathbb{Z}_2) + 2(\mathbb{Z}_1)$	$2(\mathbb{Z}_2) + 2(\mathbb{Z}_1)$	$2(\mathbb{Z}_2) + 4(\mathbb{Z}_1)$	$6(\mathbb{Z}_1)$	$3(\mathbb{Z}_2) + 6(\mathbb{Z}_1)$	$10(\mathbb{Z}_1)$
$(\mathbb{Z}_1)$	$(\mathbb{Z}_1)$	$5(\mathbb{Z}_1)$	$6(\mathbb{Z}_1)$	$10(\mathbb{Z}_1)$	$12(\mathbb{Z}_1)$	$15(\mathbb{Z}_1)$	$20(\mathbb{Z}_1)$
$(A_4^{t_1})$	$(A_4^{t_1})$	$(A_4^{t_1}) + (\mathbb{Z}_3^t)$	$(\mathbb{Z}_2)$	$(\mathbb{Z}_2) + (\mathbb{Z}_3^t)$	$(\mathbb{Z}_1)$	$(V_4) + (\mathbb{Z}_1)$	$2(\mathbb{Z}_3^t) + ($
$(A_4^{t_2})$	$(A_4^{t_2})$	$(A_4^{t_2}) + (\mathbb{Z}_3^t)$	$(\mathbb{Z}_2)$	$(\mathbb{Z}_2) + (\mathbb{Z}_3^t)$	$(\mathbb{Z}_1)$	$(V_4) + (\mathbb{Z}_1)$	$2(\mathbb{Z}_3^t) + ($
(D_5^z)	(D_5^z)	$(\mathbb{Z}_2^-)$	$(D_5^z) + (\mathbb{Z}_2^-)$	$2(\mathbb{Z}_2^-)$	$(\mathbb{Z}_5) + (\mathbb{Z}_1)$	$3(\mathbb{Z}_2^-)$	$2(\mathbb{Z}_1)$
(D_3^z)	(D_3^z)	$(\mathbb{Z}_3) + (\mathbb{Z}_2^-)$	$2(\mathbb{Z}_2^-)$	$(D_3^z) + (\mathbb{Z}_2^-) + (\mathbb{Z}_1)$	$2(\mathbb{Z}_1)$	$3(\mathbb{Z}_2^-) + (\mathbb{Z}_1)$	$(\mathbb{Z}_3) + 3($
$(\mathbb{Z}_5^{t_1})$	$(\mathbb{Z}_5^{t_1})$	$(\mathbb{Z}_1)$	$(\mathbb{Z}_5^{t_1}) + (\mathbb{Z}_1)$	$2(\mathbb{Z}_1)$	$2(\mathbb{Z}_5^{t_1}) + 2(\mathbb{Z}_1)$	$3(\mathbb{Z}_1)$	$4(\mathbb{Z}_1)$
$(\mathbb{Z}_5^{t_2})$	$(\mathbb{Z}_5^{t_2})$	$(\mathbb{Z}_1)$	$(\mathbb{Z}_5^{t_2}) + (\mathbb{Z}_1)$	$2(\mathbb{Z}_1)$	$2(\mathbb{Z}_5^{t_2}) + 2(\mathbb{Z}_1)$	$3(\mathbb{Z}_1)$	$4(\mathbb{Z}_1)$
(V_4^-)	(V_4^-)	$(V_4^-) + (\mathbb{Z}_1)$	$2(\mathbb{Z}_2^-) + (\mathbb{Z}_2)$	$2(\mathbb{Z}_2^-) + (\mathbb{Z}_2) + (\mathbb{Z}_1)$	$3(\mathbb{Z}_1)$	$3(V_4^-) + 3(\mathbb{Z}_1)$	$5(\mathbb{Z}_1)$
$(\mathbb{Z}_3^t)$	$(\mathbb{Z}_3^t)$	$2(\mathbb{Z}_3^t) + (\mathbb{Z}_1)$	$2(\mathbb{Z}_1)$	$(\mathbb{Z}_3^t) + 3(\mathbb{Z}_1)$	$6(\mathbb{Z}_1)$	$5(\mathbb{Z}_1)$	$2(\mathbb{Z}_3^t) + 6($
$(\mathbb{Z}_2^-)$	$(\mathbb{Z}_2^-)$	$(\mathbb{Z}_2^-) + 2(\mathbb{Z}_1)$	$2(\mathbb{Z}_2^-) + 2(\mathbb{Z}_1)$	$2(\mathbb{Z}_2^-) + 4(\mathbb{Z}_1)$	$6(\mathbb{Z}_1)$	$3(\mathbb{Z}_2^-) + 6(\mathbb{Z}_1)$	$10(\mathbb{Z}_1)$

Table 5: Multiplication Table for $U(A_5 \times S^1)$.

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