Periodic Solutions to $O(2)$-Symmetric Variational Problems: $O(2) \times S^1$-Equivariant Gradient Degree Approach

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This paper is dedicated to Alex Ioffe and Simeon Reich on the occasion of their birthdays.

Abstract. To study symmetric properties of solutions to equivariant variational problems, Kazimierz Gęba introduced the so-called G-equivariant gradient degree taking its values in the Euler ring $U(G)$. In this paper, we develop several techniques to evaluate the multiplication structure of $U(\Gamma \times S^1)$, where Γ is a compact Lie group. In addition, some methods for the evaluation of the $\Gamma \times S^1$-equivariant degree, based on its connections with other equivariant degrees, are proposed. Finally, the obtained results are applied to a periodic-Dirichlet mixed boundary value problem for an elliptic asymptotically linear variational equation with $O(2)$-symmetries.

1. Introduction

Subject and Goal. Let W be a Hilbert representation of compact Lie group G and $f : W \to \mathbb{R}$ a smooth invariant function. The problem of classifying symmetric properties of solutions to the equation

$$\nabla f(x) = 0, \quad x \in \Omega, \tag{1}$$

(Ω – an open bounded invariant set) has been attacked by many authors using various methods: Lusternik-Schnirelman theory (cf. [**27, 28**]), equivariant Conley index theory (cf. [**3**]), Morse-Floer techniques (cf. [**2, 14**]), to mention a few (see also [**6, 13, 20**]). The degree-theoretic treatment of problem (1) (for $G = S^1$) was initiated in [**8**], where a rational-valued gradient S^1-homotopy invariant was introduced (see also [**10**], where a similar invariant was considered in the context of systems with first integral). Recently, K. Gęba (cf. [**17**]) suggested a method to study the above problem using the so-called equivariant gradient degree (for more information on the equivariant gradient degree, we refer to [**33**]). Under reasonable conditions, the equivariant gradient degree turns out to be the full equivariant

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gradient homotopy invariant (cf. [**9**]). In particular, it contains the essential equivariant topological information on the solution set for (1), which may have a very complicated structure related to a large number of different orbit types. Due to this complexity, one cannot expect that the above invariant is a simple integer (or even a rational). In fact, it takes its values in an algebraic object known as the Euler ring of the group G (denoted by $U(G)$) introduced by T. tom Dieck (cf. [**11, 12**].

To be more specific, it is well-known that the Brouwer degree of a (non-equivariant) gradient map, is an algebraic "count" of solutions satisfying the additivity and multiplicativity (with respect to the product map) properties. Moreover, it can be expressed as the Euler characteristic of the cohomological Conley index (or Morse-Floer complex), which takes its values in the ring $\mathbb{Z} = U(\mathbb{Z}_1)$. A passage to the equivariant setting requires: (a) an algebraic "count" of orbits of solutions rather than individual solutions, (b) a separate treatment of orbits of different types, which should be (c) compatible with the "count" of orbits in products. For a continuous compact Lie group, such a "count" can be achieved, in a parallel way to the Brouwer degree, by using the Euler characteristic of the appropriate orbit spaces. The Euler ring (see Definition 3.1), as well as the equivariant gradient degree, constitute the formalization of the above stream of ideas. The *main goal* of this paper is to present some ideas and methods allowing (i) effective computations of $U(G)$ in the case $G = \Gamma \times S^1$, where Γ is a compact Lie group (see [**18, 12**] for the computations of $U(SO(3))$, and (ii) to establish computational techniques for the $\Gamma \times S^1$-equivariant gradient degree by providing connections with the other equivariant degrees (cf. [**1, 22**]).

As an example, we completely evaluate the ring structure of $U(O(2) \times S^1)$ and establish computational formulae for $O(2) \times S^1$-linear gradient maps. The obtained results can be applied to various variational problems with symmetries (existence, bifurcation (both, local and global), continuation, etc.) to classify symmetric properties of solutions (cf. [**31, 32, 29, 22**]).

Overview. After the Introduction, the paper is organized as follows. In Section 2, we present several facts from equivariant topology and list the properties of the Euler characteristic relevant to our discussion. In Section 3, after giving the definition of the Euler ring, we explore its connections with Burnside ring and (in the case $G = \Gamma \times S^1$) with the related modules (cf. [**1, 22**]). Some additional partial results describing the multiplicative structure of the Euler ring are also presented. In Section 4, we discuss the Euler ring homomorphisms induced by Lie group homomorphisms. The obtained results are applied in Section 5 to establish the complete multiplication table for the Euler ring $U(O(2) \times S^1)$. Section 6 is devoted to several methods for computations of the equivariant gradient degree in the case $G = \Gamma \times S^1$, which are applied to establish a computational database for $G = O(2) \times S^1$. These results are applied in the last section to a periodic-Dirichlet mixed boundary value problem for an elliptic asymptotically linear variational equation with $O(2)$-symmetries.

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2. Equivariant Topology Preliminaries

2.1. G-actions. In what follows G always stands for a compact Lie group and all subgroups of G are assumed to be closed.

For a subgroup $H \subset G$, denote by $N(H)$ the normalizer of H in G, and by $W(H) = N(H)/H$ the Weyl group of H in G. In the case when we are dealing with different Lie groups, we also write $N_G(H)$ (resp. $W_G(H)$) instead of $N(H)$ (resp. $W(H)$). The following simple fact will be essentially used in what follows.

Lemma 2.1. (see [**1**], Lemma 2.55) *Given subgroups $L \subset H \subset G$, one has*

$$\dim W(H) \leq \dim W(L).$$

Denote by (H) the conjugacy class of H in G. We also use the notations:

$$\Phi(G) := \{(H) : H \text{ is a subgroup of } G\},$$
$$\Phi_n(G) := \{(H) \in \Phi(G) : \dim W(H) = n\}.$$

The set $\Phi(G)$ has a natural partial order defined by

$$(H) \leq (K) \iff \exists_{g\in G}\ gHg^{-1} \subset K. \tag{2}$$

A topological space X equipped with a left (resp. right) G-action is called a *G-space* (resp. *space-G*) (if an action is not specified, it is assumed to be a left one). For a G-space X and $x \in X$, put:

$G_x := \{g \in G : \ gx = x\}$ – the isotropy of x;
(G_x) – the orbit type of x in X;
$G(x) := \{gx : \ g \in G\}$ – the orbit of x.

Also, for a subgroup $H \subset G$, we adopt the following notations:

$X_H := \{x \in X : \ G_x = H\}$;
$X^H := \{x \in X : \ G_x \supset H\}$;
$X_{(H)} := \{x \in X : \ (G_x) = (H)\}$;
$X^{(H)} := \{x \in X : \ (G_x) \geq (H)\}$.

As is well-known (see, for instance, [**12**]), $W(H)$ acts on X^H and this action is free on X_H. All the above notions and notations can be reformulated for a space-G in an obvious way.

The orbit space for a G-space X is denoted by X/G and for the space-G by $G\backslash X$.

Let G_1 and G_2 be compact Lie groups. Assume that X is a G_1-space and space-G_2 simultaneously, and $g_1(xg_2) = (g_1x)g_2$ for all $x \in X$, $g_i \in G_i$, $i = 1, 2$. In this case, we use the notation G_1-space-G_2. Clearly, X/G_1 is a space-G_2 while $G_2\backslash X$ is a G_1-space. For the double orbit spaces $G_2\backslash(X/G_1)$ and $(G_2\backslash X)/G_1$, we use the same notation $G_2\backslash X/G_1$ (this is justified by the fact that both double orbit spaces are homeomorphic).

In particular, assume $X := G$ and $H \subset G$ (resp. $L \subset G$) is a subgroup acting on G by left (resp. right) translations. Then G is an H-space-L. Moreover, G/H (resp. $L\backslash G$) can be identified with the set of left cosets $\{Hg\}$ (resp. right cosets $\{gL\}$), L (resp. H) acts on G/H (resp. $L\backslash G$) by the formula $(Hg)l := H(gl)$, $l \in L$ (resp. $h(gL) := (hg)L$, $(h \in H)$). In addition, $L\backslash G/H$ can be identified with the set of the corresponding double cosets. Similar observations can be applied to G replaced by a G-invariant subset of G.

For the equivariant topology background frequently used in this paper, we refer to [**12, 23, 4, 1**].

2.2. The sets $N(L,H)$, $N(L,H)/H$ and $N(L,H)/N(H)$. Take subgroups $L \subset H$ of G and put

$$N(L,H) := \{g \in G : gLg^{-1} \subset H\}. \tag{3}$$

Lemma 2.2. *Let L, H be as in Lemma 2.1. Then the space $N(L,H) \subset G$ has two natural actions: left $N(H)$-action and right $N(L)$-action. Thus, $N(L,H)$ is an $N(H)$-space-$N(L)$.*

Proof. Since G is an $N(H)$-space-$N(L)$, it is enough to show that $N(L,H)$ is invariant with respect to these two actions. Suppose that $l \in N(L)$ and $h \in N(H)$. Then, for $g \in N(L,H)$, one has $gLg^{-1} \subset H$; thus $hgL(hg)^{-1} \subset hHh^{-1} = H$, and consequently $hg \in N(L,H)$. Similarly, $glLl^{-1}g^{-1} = gLg^{-1} \subset H$; thus $gl \in N(L,H)$. □

Next, take $N(L,H)/H$ (which is correctly defined by Lemma 2.2). By the same lemma and observations from Subsection 2.1, $N(L,H)/H$ is a space-$N(L)$ (in fact, space-$W(L)$).

Lemma 2.3. (see [**4**, Corollary 5.7]) *Let L, H be as in Lemma 2.1 and take a G-space G/H. Then:*

(i) *$(G/H)^L$ is $W(L)$-equivariantly diffeomorphic to $N(L,H)/H$;*
(ii) *$(G/H)^L$ contains finitely many $W(L)$-orbits.*

Using Lemma 2.3 and the same argument as in the proof of Proposition 2.52 from [**1**], one can easily establish the following

Lemma 2.4. *Let L and H be as in Lemma 2.1 Then the set $N(L,H)/H$ is composed of connected components M_i, $i = 1, 2, \ldots, k$, which are smooth manifolds (possibly of different dimensions) such that* $\dim W(H) \le \dim M_i \le \dim W(L)$.

Given subgroups $L \subset H \subset G$, let M_i, $i = 1, 2, \ldots, k$, be the connected components of $N(L,H)/H$ (cf. Lemma 2.4). Put

$$\operatorname{Dim} N(L,H)/H := \max\{\dim M_i : i = 1, 2, \ldots, k\}.$$

Lemma 2.5. *Assume that $L' \subset L \subset H$ are three subgroups of G. Then*

$$\operatorname{Dim} \mathrm{N(L,H)/H} \le \operatorname{Dim} \mathrm{N(L',H)/H}.$$

Proof. Notice that $N(L,H) \subset N(L',H)$, therefore

$$\frac{N(L,H)}{H} \subset \frac{N(L',H)}{H},$$

and the conclusion follows. □

Consider now the set $N(L,H)/N(H)$ and put $n(L,H) := |N(L,H)/N(H)|$, where the symbol $|X|$ stands for the cardinality of X (cf. [**21, 26**]).

Lemma 2.6. (see [**1**], Proposition 2.52) *Let L, H be as in Lemma 2.1 and assume, in addition, that* $\dim W(L) = \dim W(H)$. *Then $n(L,H)$ is finite.*

The numbers $n(L,H)$, whenever they are finite, play an important role in the computation of multiplication tables of Burnside rings and the corresponding modules (and, therefore, may be used to establish partial results on the multiplication structure of the Euler ring $U(G)$). However, the main assumption providing the finiteness of $n(L,H)$ is not satisfied for arbitrary $L \subset H \subset G$. Below we introduce a notion close in spirit to $n(L,H)$.

Definition 2.7. Given subgroups $L \subset H \subset G$, we say that L is $\mathfrak{N}$-*finite* in H if the space $N(L,H)/H$ is finite. For a given subgroup H, denote by $\mathfrak{N}(H)$ the set of all conjugacy classes (L) such that there exists $\tilde{L} \in (L)$ which is $\mathfrak{N}$-finite in H. For $(L) \in \mathfrak{N}(H)$ $(L \subset H)$, put

$$m(L,H) := \left| \frac{N(L,H)}{H} \right|,$$

where $|X|$ stands for the number of elements in the set X.

Remark 2.8. Let L, H be as in Lemma 2.1. (i) Take a subgroup $L' \subset L$. If L' is $\mathfrak{N}$-finite in H, then L is $\mathfrak{N}$-finite in H (cf. Lemma 2.5).

(ii) It follows immediately from Lemma 2.3 that if $W(L)$ is finite, then L is $\mathfrak{N}$-finite.

(iii) Finally, if $W(L)$ and $W(H)$ are finite, then

$$m(L,H) = n(L,H) \cdot |W(H)|.$$

We complete this subsection with the following simple but important observation.

Proposition 2.9. *Let L, H be as in Lemma 2.1. Then G contains a finite sequence of elements $g_1 = e$ (where e is the unit element in G), g_2, g_3, …, g_n such that*

$$N(L,H) = N(H)g_1N(L) \sqcup N(H)g_2N(L) \sqcup \cdots \sqcup N(H)g_nN(L),$$

where $N(H)g_jN(L)$ denotes a double coset, for $j = 1, 2, \ldots, n$, and $\sqcup$ stands for the disjoint union.

Proof. Since the $N(H)$-space-$N(L)$ $N(L,H)/H$ consists of a finite number of $N(L)$-orbits (cf. Lemma 2.3), the space $N(L,H)/N(H)$ also consists of a finite number of $N(L)$-orbits. Consequently, the set $N(L)\backslash N(L,H)/N(H)$ is finite and the result follows. □

2.3. Euler characteristic in the Alexander-Spanier cohomology ring. For a topological space Y, denote by $H_c^*(Y)$ the ring of Alexander-Spanier cohomology with compact support (see [**34**]). If $H_c^*(Y)$ is finitely generated, then the Euler characteristic $\chi_c(Y)$ is correctly defined in a standard way. The following well-known fact (see, for instance, [**34**, Chap. 6, Sect. 6, Lemma 11]) is a starting point for our discussions.

Lemma 2.10. *Let X be a compact CW-complex and A a closed subcomplex. Then $H_c^*(X\backslash A)) \cong H^*(X, A; \mathbb{R})$, where $H^*(\cdot)$ stands for a usual cellular cohomology ring.*

From Lemma 2.10 immediately follows

Lemma 2.11.

(i) *Let X, A be as in Lemma 2.10. Then $\chi_c(X \setminus A)$ is correctly defined and*

$$\chi(X) = \chi_c(X \setminus A) + \chi(A) = \chi(X, A) + \chi(A).$$

(Here $\chi(\cdot)$ stands for the Euler characteristic with respect to the cellular cohomology groups.)

(ii) *Let X, A be as in Lemma 2.10, Y a compact CW-complex, $B \subset Y$ a closed subcomplex and $p : X \setminus A \to Y \setminus B$ a locally trivial fibre bundle with path-connected base and the fibre F a compact manifold. Then (cf.* [**34**, Chap. 9, Sect. 3, Thm. 1]*;* [**11**, Statement 5.2.10]*),*

$$\chi_c(X \setminus A) = \chi(F)\chi_c(Y \setminus B).$$

In what follows, we will use two facts on the Euler characteristic related to the tori actions (cf. Lemma 2.12 and Corollary 2.15).

Lemma 2.12. (see [**23, 24**]) *Let X be a compact G-ENR-space with $G = T^n$ an n-dimensional torus and $n > 0$. Then*

$$\chi(X) = \chi(X^G).$$

In particular, if $X^G = \emptyset$, then $\chi(X) = 0$.

Lemma 2.13. *Suppose G is abelian, Δ the diagonal in $G \times G$ and X, Y two G-spaces. Then $A := (G \times G)/\Delta$ acts on $(X \times Y)/\Delta$ without A-fixed points iff*

$$\text{(p)} \qquad \text{for any} \quad x \in X \quad \text{and} \quad y \in Y,\ G_x \cdot G_y \neq G,$$

where "$G_x \cdot G_y$" stands for a subgroup of G generated by G_x and G_y.

Proof. Notice that since G is abelian and Hausdorff, Δ is a closed normal subgroup in $G \times G$. Thus, $A = (G \times G)/\Delta \cong G$ and A acts on $(X \times Y)/\Delta$.

Observe that if $\Delta(x, y) \in (X \times Y)/\Delta$ is an A-fixed point, then for any $g_1, g_2 \in G$ (i.e., for any coset $\Delta(g_1, g_2) \in A$), we have that $\Delta(g_1x, g_2y) = \Delta(x, y)$, which is equivalent to the existence of some $g_o \in G$ such that $(g_og_1x, g_og_2y) = (x, y)$. Put $t_o := g_og_1$ and $t := g_1^{-1}g_2$. Then, from above, we conclude that $((X \times Y)/\Delta)^A \neq \emptyset$ iff for any $t \in G$, there exists $t_o \in G$ such that $(t_ox, t_oty) = (x, y)$ for some $(x, y) \in X \times Y$. In terms of isotropies of x, y, the latter requires for any $t \in G$, we can write $t \in t_o^{-1}G_y$ for some $t_o \in G_x$, which is equivalent to $G \subset G_x^{-1} \cdot G_y$ for some $x \in X$, $y \in Y$, i.e., $G_x \cdot G_y = G$. □

From Lemma 2.12 and Lemma 2.13 immediately follows

Corollary 2.14. *Under the assumptions of Lemma 2.13, if X and Y are compact G-ENR-spaces with $G = T^n$ $(n > 0)$, then condition* (p) *implies* $\chi((X \times Y)/\Delta) = 0$.

Corollary 2.15. *Under the assumptions of Lemma 2.13, if X and Y are compact G-ENR-spaces with $G = T^n$ $(n > 0)$ satisfying*

$$\text{(p1)} \qquad \dim G_x + \dim G_y < \dim G \quad \text{for any} \quad x \in X \quad \text{and} \quad y \in Y,$$

then $\chi((X \times Y)/\Delta)) = 0$. In particular, it holds for $G = S^1$, $X^{S^1} = Y^{S^1} = \emptyset$.

Proof. Notice that (p1) implies condition (p) in Lemma 2.13. □

Remark 2.16. In what follows, given G-spaces X and Y, we assume $X \times Y$ to be a G-space equipped with the diagonal action (if not specified otherwise).

We complete this section with the standard fact on the Euler characteristic of the space G/H.

Definition 2.17. A subgroup $H \subset G$ is said to be *of maximal rank* if H contains a maximal torus $T^n \subset G$.

Proposition 2.18. *Let $H \subset G$ be a subgroup of G.*

(i) *If H is not of maximal rank, then $\chi(G/H) = 0$.*
(ii) *If H is of maximal rank, then $W_H(T^n)$ is finite and*

$$\chi(G/H) = |W_G(T^n)|/|W_H(T^n)|.$$

In particular, $\chi(G/T^n) = |W_G(T^n)|$.

Proof. (i) If H is not of maximal rank, then G/H admits an action of a torus T^k $(0 < k < n)$ without T^k-fixed-points, and the result follows from Lemma 2.12.

(ii) Assume H is of maximal rank. Then, for the proof of the finiteness of $W_H(T)$, we refer to [**5**, Chap. IV, Thm. (1.5)]. Next, we have a fibre bundle $G/T^n \to G/H$ with the fibre H/T^n. Then, by Lemma 2.11(ii), $\chi(G/T^n) = \chi(H/T^n) \cdot \chi(G/H)$. On the other hand, by Lemma 2.12 and Lemma 2.3(i),

$$\chi(H/T^n) = \chi((H/T^n)^{T^n}) = \chi(N_H(T^n)/T^n) = |W_H(T^n)|, \tag{4}$$

from which the statement follows. □

3. Euler Ring, Burnside Ring, Twisted Subgroups and Related Modules

3.1. Euler ring.

Definition 3.1. (cf. [**12**]) Consider the free $\mathbb{Z}$-module generated by $\Phi(G)$

$$U(G) := \mathbb{Z}[\Phi(G)].$$

Define a ring multiplication on generators (H), $(K) \in \Phi(G)$ as follows:

$$(H) * (K) = \sum_{(L)\in\Phi(G)} n_L(L), \tag{5}$$

where

$$n_L := \chi_c((G/H \times G/K)_L/N(L)) \tag{6}$$

for χ_c the Euler characteristic taken in Alexander-Spanier cohomology with compact support (cf. [**34**]). The $\mathbb{Z}$-module $U(G)$ equipped with the multiplication (5), (6) is called the *Euler ring* of the group G (cf. [**12, 11**]).

Proposition 3.2. (General Recurrence Formula) *Given (H), $(K) \in \Phi(G)$, one has the following recurrence formula for the coefficients n_L in (5)*

$$n_L = \chi((G/H \times G/K)^L/N(L)) - \sum_{(\widetilde{L})>(L)} n_{\widetilde{L}}\, \chi((G/\widetilde{L})^L/N(L)). \tag{7}$$

Proof. Let $X := G/H \times G/K$. The projection $X_{(\widetilde{L})} \to X_{(\widetilde{L})}/G$ is a fibre bundle with fibre $G/\widetilde{L}$, which implies that $X^L_{(\widetilde{L})}/N(L) \to X_{(\widetilde{L})}/G$ is a fibre bundle with fibre $((G/\widetilde{L})^L)/N(L)$. By Lemma 2.11 (ii),

$$\chi_c(X^L_{(\widetilde{L})}/N(L)) = \chi((G/\widetilde{L})^L/N(L)) \cdot \chi_c(X_{(\widetilde{L})}/G).$$

Therefore (see Lemma 2.11(i)),

$$\begin{aligned}\chi(X^L/N(L)) &= \sum_{(\widetilde{L})\geq(L)} \chi_c(X^L_{(\widetilde{L})}/N(L))\\ &= \sum_{(\widetilde{L})\geq(L)} \chi((G/\widetilde{L})^L/N(L))\cdot\chi_c(X_{(\widetilde{L})}/G)\\ &= \sum_{(\widetilde{L})\geq(L)} \chi((G/\widetilde{L})^L/N(L))\cdot\chi_c(X_{\widetilde{L}}/N(\widetilde{L}))\\ &= \sum_{(\widetilde{L})\geq(L)} \chi((G/\widetilde{L})^L/N(L))\cdot n_{\widetilde{L}}\\ &= n_L + \sum_{(\widetilde{L})>(L)} \chi((G/\widetilde{L})^L/N(L))\cdot n_{\widetilde{L}},\end{aligned}$$

and the result follows. □

3.2. Burnside ring. The $\mathbb{Z}$-module $A(G) = A_0(G) := \mathbb{Z}[\Phi_0(G)]$ equipped with a similar multiplication as in $U(G)$ but restricted only to generators from $\Phi_0(G)$, is called a *Burnside ring*, i.e., for (H), $(K) \in \Phi_0(G)$

$$(H)\cdot(K) = \sum_{(L)} n_L(L), \qquad (H),(K),(L)\in\Phi_0(G),$$

where $n_L := \chi((G/H\times G/K)_L/N(L)) = |(G/H\times G/K)_L/N(L)|$ (here χ stands for the usual Euler characteristic). In this case, formula (7) can be expressed as

$$n_L = \frac{n(L,K)|W(K)|n(L,H)|W(H)| - \sum_{(\widetilde{L})>(L)} n(L,\widetilde{L})n_{\widetilde{L}}|W(\widetilde{L})|}{|W(L)|},$$

where $(H),(K),(L)$ and $(\widetilde{L})$ are taken from $\Phi_0(G)$.

Observe that since the ring $A(G)$ is a $\mathbb{Z}$-submodule of $U(G)$, it may *not* be a subring of $U(G)$, in general. Indeed, consider the following

EXAMPLE 3.3. *Let* $G = O(2)$*. Then* $(D_n)\cdot(SO(2)) = 0$ *while* $(D_n)*(SO(2)) = (\mathbb{Z}_n)$.

To see a connection between the rings $U(G)$ and $A(G)$, take the natural projection $\pi_0 : U(G)\to A(G)$ defined on generators $(H)\in\Phi(G)$ by

$$\pi_0((H)) = \begin{cases}(H) & \text{if } (H)\in\Phi_0(G),\\ 0 & \text{otherwise.}\end{cases} \tag{8}$$

LEMMA 3.4. *The map* π_0 *defined by (8) is a ring homomorphism, i.e.,*

$$\pi_0((H)*(K)) = \pi_0((H))\cdot\pi_0((K)), \qquad (H),(K)\in\Phi(G).$$

PROOF. Assume $(H)\notin\Phi_0(G)$ and

$$(H)*(K) = \sum_{(R)\in\Phi(G)} m_R(R). \tag{9}$$

Then, for any (R) occurring in (9), one has $(R)\leq(H)$, hence (see Lemma 2.1) $\dim W(R)>0$, which means that $\pi_0((R)) = 0$ and $\pi_0((H)*(K)) = 0 = \pi_0((H))\cdot\pi_0((K)) = 0\cdot\pi_0(K)$.

Thus, without loss of generality, assume $(H), (K) \in \Phi_0(G)$ and

$$(H) * (K) = \sum_{(L) \in \Phi_0(G)} n_L (L) + \sum_{(\tilde{L}) \notin \Phi_0(G)} m_{\tilde{L}} (\tilde{L}).$$

Then

$$\pi_0((H) * (K)) = \sum_{(L) \in \Phi_0(G)} n_L \pi_0((L)) = \sum_{(L) \in \Phi_0(G)} n_L (L)$$

and

$$(H) \cdot (K) = \sum_{(L) \in \Phi_0(G)} n'_L (L).$$

However,

$$\begin{aligned} n_L &= \chi_c((G/H \times G/K))_L/N(L)) = \chi((G/H \times G/K))_L/N(L)) \\ &= |(G/H \times G/K)_L/N(L)| = n'_L, \end{aligned} \tag{10}$$

and the result follows. ☐

On the other hand, the following result (its proof is an almost direct consequence of Lemma 2.12) is well-known (cf. [**12**, Proposition 1.14, p. 231]).

Proposition 3.5. *Let $(H) \in \Phi_n(G)$ with $n > 0$. Then (H) is a nilpotent element in $U(G)$, i.e., there is an integer k such that $(H)^k = 0$ in $U(G)$.*

Combining Proposition 3.5 with Lemma 3.4 and the fact that the multiplication table for $A(G)$ contains only non-negative coefficients (cf. formula (10)), yields

Proposition 3.6. (cf. [**18**]) *Let π_0 be defined by (8). Then $\mathfrak{N} = \operatorname{Ker} \pi_0 = \mathbb{Z}[\Phi(G) \setminus \Phi_0(G)]$ is a maximal nilpotent ideal in $U(G)$ and $A(G) \cong U(G)/\mathfrak{N}$.*

Summing up, the Burnside ring multiplication structure in $A(G)$ can be used to describe (partially) the Euler ring multiplication structure in $U(G)$.

3.3. Twisted subgroups and related modules. In this subsection, we assume that Γ is a compact Lie group and $G = \Gamma \times S^1$. In this case, there are exactly two sorts of subgroups $H \subset G$, namely,

(a) $H = K \times S^1$ with K a subgroup of Γ;

(b) the so-called *φ-twisted l-folded* subgroups $K^{\varphi,l}$ (in short, twisted subgroups) defined as follows: if K is a subgroup of Γ, $\varphi : K \to S^1$ a homomorphism and $l = 1, \dots,$ then

$$K^{\varphi,l} := \{(\gamma, z) \in K \times S^1 : \varphi(\gamma) = z^l\}.$$

Clearly, if a subgroup $H \subset G$ is twisted, then any subgroup conjugate to H is twisted as well. This allows us to speak about twisted conjugacy classes in $\Phi(G)$.

Proposition 3.7. *Let $G = \Gamma \times S^1$, where Γ is a compact Lie group. Given a twisted subgroup $K^{\varphi,l} \subset G$, for some $l \in \mathbb{N}$ and a homomorphism $\varphi : K \to S^1$, the following holds*

$$\dim \left(N_G(K^{\varphi,l})\right) = \dim \left(N_\Gamma(K) \cap N_\Gamma(\operatorname{Ker} \varphi)\right) + 1. \tag{11}$$

Proof. For the homomorphism $\varphi : K \to S^1$, put $L := \operatorname{Ker} \varphi$. Also, for simplicity, write $N(K^{\varphi,l})$ for $N_G(K^{\varphi,l})$, and $N(K)$ (resp. $N(L)$) for $N_\Gamma(K)$ (resp. $N_\Gamma(L)$).

Notice that $N(K^{\varphi,l}) = N_o \times S^1$, where

$$N_o := \{\gamma \in N(K) : \varphi(\gamma k \gamma^{-1}) = \varphi(k), \ \forall k \in K\}.$$

Hence, it is sufficient to show that $\dim N_o = \dim\big(N(K) \cap N(L)\big)$. By direct verification, $N_o \subset N(K) \cap N(L)$, hence $\dim N_o \le \dim (N(K) \cap N(L))$. To prove the opposite inequality, consider two cases.

Case 1. φ is surjective.

Since G is compact, without loss of generality one can assume that $N(K) \cap N(L)$ is connected (otherwise, one can pass to the connected component of unit $e \in G$).

Fix an element $\gamma \in N(K) \cap N(L)$. Since $\gamma \in N(K)$, formula $h_\gamma(k) := \gamma k \gamma^{-1}$ defines an automorphism $h_\gamma : K \to K$. Since $\gamma \in N(L)$, h_γ induces a homomorphism on the factor group K/L, which will be denoted by $\bar{h}_\gamma$. Then we have the commutative diagram shown in Figure 1.

$$\begin{array}{ccc} K & \xrightarrow{\varphi} & K/L \simeq S^1 \\ \Big\downarrow{\scriptstyle h_\gamma} & & \Big\downarrow{\scriptstyle \bar{h}_\gamma} \\ K & \xrightarrow{\varphi} & K/L \simeq S^1 \end{array}$$

FIGURE 1. Commutative diagram for surjective φ.

Take a path connecting $\gamma \in N(L) \cap N(K)$ to e. This path determines a homotopy of automorphisms $h_\gamma, h_e : K \to K$ which, in turn, determines a homotopy between the induced automorphisms $\bar{h}_\gamma, \bar{h}_e : S^1 \to S^1$. However, any automorphism $S^1 \to S^1$ is of the form $z \to z^k$. Since any two continuous maps of the circle $z \to z^k$ and $z \to z^l$ are homotopic iff $k = l$, the automorphism $\bar{h}_\gamma$ is of the form $z \to z^1$ which means that $\bar{h}_\gamma \equiv \mathrm{Id}$. By the commutative diagram in Figure 1, this implies $\varphi(\gamma k \gamma^{-1}) = \varphi(k)$ for all $k \in K$, i.e., $\gamma \in N_o$, thus $\dim (N(K) \cap N(L)) \le \dim N_o$.

Case 2. φ is not surjective.

Take any element γ in the connected component of $e \in N(K) \cap N(L)$, and denote by σ_γ a path from γ to e. Define $\varphi_\sigma : [0,1] \times K \to S^1$ by $\varphi_\sigma(t,k) := \varphi\big(\sigma_\gamma(t) k (\sigma_\gamma(t))^{-1}\big)$. Since φ is not surjective, φ_σ has a discrete image in S^1. Hence, when restricted on a connected component, φ_σ is constant, so we have $\varphi(\gamma k \gamma^{-1}) = \varphi(k)$ for all k in the same connected component of K. In particular, for any element γ in the connected component of $e \in N(K)$, we have $\varphi(\gamma k \gamma^{-1}) = \varphi(k)$ for all $k \in K$, i.e., $\gamma \in N_o$, which implies $\dim N(K) \cap N(L) \le \dim N_o$. $\square$

LEMMA 3.8. *Let Γ be a compact Lie group, $G = \Gamma \times S^1$ and $H = K^{\varphi,l} \subset G$ a twisted subgroup. Then:*

(i) $1 \le \dim W_G(H) \le 1 + \dim W_\Gamma(K)$;

(ii) *any subgroup $\tilde{H} \subset H$ is twisted.*

PROOF. (i) By definition of twisted subgroup, $\dim K = \dim K^{\varphi,l}$. Hence, by Proposition 3.7,

$$\begin{aligned} \dim W_G(K^{\varphi,l}) &= \dim N_G(K^{\varphi,l}) - \dim K^{\varphi,l} = \dim N_G(K^{\varphi,l}) - \dim K \\ &= \dim (N_\Gamma(K) \cap N_\Gamma(\operatorname{Ker}\varphi)) - \dim K + 1. \end{aligned} \tag{12}$$

Since $K \subset N_\Gamma(K) \cap N_\Gamma(\operatorname{Ker}\varphi)$ and $\dim (N_\Gamma(K) \cap N_\Gamma(\operatorname{Ker}\varphi)) \le \dim N_\Gamma(K)$,

$$\begin{aligned} 1 = \dim K - \dim K + 1 &\le \dim (N_\Gamma(K) \cap N_\Gamma(\operatorname{Ker}\varphi)) - \dim K + 1 \\ &\le \dim N_\Gamma(K) - \dim K + 1 = \dim W_\Gamma(K) + 1. \end{aligned}$$

Therefore, by (12),

$$1 \le \dim W_G(H) \le \dim W_\Gamma(K) + 1.$$

(ii) It is obvious that $\tilde{H}$ is twisted by the same homomorphism φ. $\square$

Lemma 3.8 immediately implies

COROLLARY 3.9. *Let G and H be as in Lemma 3.8.*

(a) *If $\dim W_\Gamma(K) = 0$, then $\dim W_G(H) = 1$.*
(b) *$\Phi_0(G) = \{(\widehat{H}) : \widehat{H} \subset G,\ \widehat{H} = \widehat{K} \times S^1$ with $\dim W_\Gamma(\widehat{K}) = 0\}$ which means that*

$$A(G) \cong A(\Gamma). \tag{13}$$

PROOF. Statement (a) follows directly from Lemma 3.8(i). To show (b), observe that by Lemma 3.8(i), $(H) \notin \Phi_0(G)$, thus $\Phi_0(G)$ is composed of conjugacy classes of product subgroups $\widehat{H} = \widehat{K} \times S^1$ only. Since $\dim W_G(\widehat{H}) = 0$ if and only if $\dim W_\Gamma(\widehat{K}) = 0$, the statement (b) follows. $\square$

The identification (13) will be systematically used in this paper.

Being motivated by Corollary 3.9(a), put

$$\Phi_1^t(G) := \{(H) \in \Phi(G) :\ H = K^{\varphi,l} \text{ for some } K \subset \Gamma \text{ with } \dim W_\Gamma(K) = 0\}.$$

COROLLARY 3.10. *Let G and H be as in Lemma 3.8. Assume $(H) \in \Phi_1^t(G)$. Then, for every $(\widetilde{H}) \in \Phi_1(G)$ such that $(H) < (\widetilde{H})$, $(\widetilde{H}) \in \Phi_1^t(G)$.*

PROOF. Notice that $\widetilde{H}$ cannot be a subgroup of type $\widetilde{K} \times S^1$, since it would imply $\dim W_\Gamma(\widetilde{K}) = 1$ and $(K) \le (\widetilde{K})$, which would be a contradiction to Lemma 2.1(a). Thus, $\widetilde{H} = \widetilde{K}^{\psi,m}$, where $\psi : \widetilde{K} \to S^1$ is a homomorphism and $K \subset \widetilde{K}$. Since $\dim W_\Gamma(K) = 0$, then, by Lemma 2.1(a), $\dim W_\Gamma(\widetilde{K}) = 0$, which implies $(\widetilde{H}) \in \Phi_1^t(G)$. $\square$

REMARK 3.11. Let G and $H = K^{\varphi,l}$ be as in Lemma 3.8 and $\dim W_G(H) = 1$. In such a case, it is not clear in general if $(H) \in \Phi_1^t(G)$. However, if the homomorphism $\varphi : K \to S^1$ is not surjective, then $\dim W_\Gamma(K) = 0$, and consequently $(H) \in \Phi_1^t(G)$. Indeed, by assumption, $\operatorname{Ker}\varphi$ is a normal subgroup of K such that $\operatorname{Ker}\varphi$ is a component (i.e., a union of connected components) of K, thus $\dim \operatorname{Ker}\varphi = \dim K$. Since $\dim W_\Gamma(\operatorname{Ker}\varphi) \ge \dim W_\Gamma(K)$ (cf. Lemma 2.1), we have that $\dim N_\Gamma(\operatorname{Ker}\varphi) \ge \dim N_\Gamma(K)$. Denote by M_o the connected component of $e \in N_\Gamma(K)$. Choose $g \in M_o$ and let $\sigma : [0,1] \to M_o$ be a path connecting g with e. Put $g_t := \sigma(t)$. Since $\operatorname{Ker}\varphi$ is a component of K, then by the homotopy argument, $g_t^{-1}\operatorname{Ker}\varphi g_t = \operatorname{Ker}\varphi$, which implies $M_o \subset N_\Gamma(\operatorname{Ker}\varphi)$. Thus, $\dim N_\Gamma(K) = \dim (N_\Gamma(K) \cap N_\Gamma(\operatorname{Ker}\varphi))$, and by Proposition 3.7,

$$\begin{aligned} 1 = \dim W_G(H) &= \dim (N_\Gamma(K) \cap N_\Gamma(\operatorname{Ker}\varphi)) - \dim K^{\varphi,l} + 1 \\ &= \dim N_\Gamma(K) - \dim K + 1 = \dim W_\Gamma(K) + 1. \end{aligned}$$

Consequently, $\dim W_\Gamma(K) = 0$.

In the sequel, we use the following notations:

$$\begin{aligned} \Phi_1^*(G) &:= \{(H) \in \Phi_1(G) : (H) \notin \Phi_1^t(G)\}, \\ \Phi_k^*(G) &:= \{(H) \in \Phi(G) :\ \dim W_G(H) = k\}, \quad k \ge 2, \end{aligned}$$

and

$$A_1^t(G) := \mathbb{Z}[\Phi_1^t(G)], \quad A_k^*(G) := \mathbb{Z}[\Phi_k^*(G)], \;\; k \geq 1, \quad A^*(G) := \bigoplus_{k\geq 1} A_k^*(G).$$

PROPOSITION 3.12. *Let $G := \Gamma \times S^1$, where Γ is a compact Lie group. If (H), $(\widetilde{H}) \in \Phi_1^t(G)$ then*

$$(H) * (\widetilde{H}) = \sum_{(L)\in\Phi(G)} n_L(L), \tag{14}$$

where $n_L = 0$ for all $(L) \in \Phi_1^t(G)$.

PROOF. Let $(L) \in \Phi_1^t(G)$. Since $(H), (\widetilde{H}) \in \Phi_1^t(G)$, we have $\dim W(L) = \dim W(H) = \dim W(\widetilde{H}) = 1$. By Lemma 2.4, $N(L,H)/H$ and $N(L,\widetilde{H})/\widetilde{H}$ are one-dimensional manifolds. Consider the space

$$\begin{aligned} X &:= \left(G/H \times G/\widetilde{H}\right)^L = (G/H)^L \times \left(G/\widetilde{H}\right)^L \\ &\simeq N(L,H)/H \times N(L,\widetilde{H})/\widetilde{H} \qquad \text{(by Lemma 2.3(i))}. \end{aligned}$$

We claim that

$$\chi(X/N(L)) = \chi\left(\left(G/H \times G/\widetilde{H}\right)^L /N(L)\right) = 0. \tag{15}$$

Put $\mathcal{G} := N(L) = N_o \times S^1$, where $L = K^{\varphi,l}$. Notice that $N_o \subset \Gamma$ is a closed subgroup such that $K \subset N_o \subset N_\Gamma(K)$. Identify $\{e\} \times S^1$ with S^1 and consider the composition $\eta : S^1 \hookrightarrow N(L) \to W(L)$ which maps S^1 onto the connected component of $e \in W(L)$. The homomorphism η induces S^1-actions on $(G/H)^L$ and $(G/\widetilde{H})^L$, and consequently a T^2-action on X. By Lemma 2.3 (ii), the space $N(L,H)/H$ consists of finitely many $N(L)$-orbits (in fact, $W(L)$-orbits), namely $N(L,H)/H = W(L)(p_1) \cup \cdots \cup W(L)(p_k)$, each of dimension one. Since $(L) \in \Phi_1^t(G)$, so $\dim S^1 = \dim W(L)(p_i) = 1$, hence the S^1-isotropies in $W(L)(p_i)$ are finite subgroups. Similarly, the S^1-isotropies in each $W(L)$-orbit of $N(L,\widetilde{H})/\widetilde{H}$ are also finite. Therefore, all the isotropy groups in X, with respect to the T^2-action, are finite, which implies that T^2-orbits in X are two-dimensional and clearly connected.

On the other hand, since $\dim X = 2$, each connected component X_i of X is a T^2-orbit. Let us show that all points in X_i have the same $N(L)$-isotropy (and, therefore, $W(L)$-isotropy). Indeed, consider $x = (Hg, \widetilde{H}\widetilde{g}) \in X_i$, then $T^2(x) = X_i$. Notice that for any twisted subgroup $\widehat{H}$, the conjugate $g^{-1}\widehat{H}g$ is also twisted and since $z := (e,z) \in \{e\} \times S^1 \subset N(\widehat{H})$, one has $z^{-1}\widehat{H}z = \widehat{H}$. Hence, for $(z,\widetilde{z}) \in T^2$,

$$\mathcal{G}_x = g^{-1}Hg \cap \widetilde{g}^{-1}\widetilde{H}\widetilde{g} = z^{-1}(g^{-1}Hg)z \cap \widetilde{z}^{-1}(\widetilde{g}^{-1}\widetilde{H}\widetilde{g})\widetilde{z} = \mathcal{G}_{(z,\widetilde{z})x}.$$

Therefore, $\mathcal{G}_x$ is the same for all $x \in X_i$, so $N(L)(X_i)/N(L)$ is a one-dimensional compact manifold of dimension 1. Consequently, so is $X/N(L) = X/W(L)$, thus $\chi(X/N(L)) = 0$.

Consider the set Λ of all $(\widetilde{L}) \in \Phi_1^t(G)$ which are the orbit types in $G/H \times G/\widetilde{H}$. If $(\widetilde{L}) \in \Lambda$ is a maximal orbit type in $G/H \times G/\widetilde{H}$, then

$$\left(G/H \times G/\widetilde{H}\right)^{\widetilde{L}} = \left(G/H \times G/\widetilde{H}\right)_{\widetilde{L}}$$

and

$$n_{\widetilde{L}} = \chi_c\left(\left(G/H \times G/\widetilde{H}\right)_{\widetilde{L}}/N(\widetilde{L})\right) = \chi\Big(\left(G/H \times G/\widetilde{H}\right)^{\widetilde{L}}/N(\widetilde{L})\Big) = 0. \tag{16}$$

However, if $(L) \in \Phi_1^t(G)$ is not a maximal orbit type in $G/H \times G/\widetilde{H}$, then we can assume for the induction that $n_{\widetilde{L}} = 0$ for all $(\widetilde{L}) \in \Lambda$ with $(\widetilde{L}) > (L)$. Then, by applying the recurrence formula (7) and (15), one obtains $n_L = 0$. □

EXAMPLE 3.13. Let $G := O(2) \times S^1$. Then

$$\begin{aligned}
\Phi_0(G) &= \{(O(2) \times S^1), (SO(2) \times S^1), (D_n \times S^1),\ n = 1, 2, \dots\},\\
\Phi_1^t(G) &= \{(O(2) \times \mathbb{Z}_l), (SO(2) \times \mathbb{Z}_l), (D_n \times \mathbb{Z}_l,),\\
&\qquad (O(2)^{-,l}), (SO(2)^{\varphi_k,l}), (D_n^{z,l}), (D_{2n}^{d,l}),\ n, l = 1, 2, \dots\},\\
\Phi_1(G) &= \Phi_1^t(G) \cup \{(\mathbb{Z}_m \times S^1),\ m = 1, 2, \dots\},\\
\Phi_2(G) &= \{(\mathbb{Z}_n \times \mathbb{Z}_l), (\mathbb{Z}_n^{\varphi_k,l}), (\mathbb{Z}_{2n}^{d,l}),\ n, l = 1, 2, \dots\}
\end{aligned}$$

(we refer to [**1**] for conventions).

(a) Consider (H), $(\widetilde{H}) \in \Phi_1^t(G)$. By Proposition 3.12, we have that $n_L = 0$ in (14) for $(L) \in \Phi_1^t(G)$.

(b) Using the argument similar to the one used in the proof of Proposition 3.12, one can show that if H and K are subgroups of G with $\dim W(H) \geq 1$ and $\dim W(K) = 2$, then

$$(H) * (K) = 0.$$

Indeed, assume that for some $(L) \in \Phi(G)$ one has that the coefficient n_L in $(H)*(K)$ is different from zero. Then $(L) \leq (K)$ which, by assumption and Lemma 2.1, implies $\dim W(L) = 2$. In particular,

$$N(L) \supset SO(2) \times S^1 = T^2. \tag{17}$$

Consider the space

$$\left(\frac{G}{H} \times \frac{G}{K}\right)^L = \left(\frac{G}{H}\right)^L \times \left(\frac{G}{K}\right)^L = \frac{N(L,K)}{K} \times \frac{N(L,H)}{H}.$$

Combining (17) with Proposition 2.9 implies that $N(L,H)$ and $N(L,K)$ contain T^2. Therefore, $X := N(L,H)/H$ and $Y := N(L,K)/K$ admit T^2-actions. Since for $x = Hg \in X$ the isotropy $T^2_x = g^{-1}Hg \cap T^2$, thus $\dim T^2_x \leq \dim H \leq 1$. Similarly, for $y = Kg \in Y$, $\dim T^2_y \leq \dim g^{-1}Kg = 0$, thus $\dim T^2_x + \dim T^2_y < \dim T^2 = 2$. Consequently, by Corollary 2.15, $\chi((X \times Y)/T^2) = 0$. If $N(L) = T^2$, then $\chi((X \times Y)/N(L)) = 0$. Another possibility for $N(L)$ may be $N(L) = O(2) \times S^1$. Then one can use the fibre bundle $(X \times Y)/T^2 \to (X \times Y)/N(L)$ to conclude that $\chi((X \times Y)/N(L)) = 0$ as well. If (L) is a maximal orbit type in $X \times Y$, then the last equality implies $n_L = 0$. If (L) is not maximal, one can use the same induction argument as in the proof of Proposition 3.12 to show that $n_L = 0$.

As it was established in [**1**, Theorem 6.6], there is a natural $A(\Gamma)$-module structure on $A_1^t(G)$. Namely,

PROPOSITION 3.14. *Let Γ be a compact Lie group and $G = \Gamma \times S^1$. Then there exists a "multiplication function" $\circ : A(\Gamma) \times A_1^t(G) \to A_1^t(G)$ defined on generators*

$(R) \in \Phi_0(\Gamma)$ and $(K^{\varphi,l}) \in \Phi_1^t(G)$ as follows:

$$(R) \circ (K^{\varphi,l}) = \sum_{(L)} n_L (L^{\varphi,l}),$$

where the summation is taken over all subgroups L such that $W(L)$ is finite and $L = \gamma^{-1} R \gamma \cap H$ for some $\gamma \in \Gamma$, and

$$n_L = \left| \left(\frac{G}{R \times S^1} \times \frac{G}{K^{\varphi,l}} \right)_{(L^{\varphi,l})} /G \right|.$$

One can use Corollary 3.9 to establish a relation between the $A(\Gamma)$-module structure on $A_1^t(G)$ provided by Proposition 3.14 and the ring structure in $U(G)$. To this end, consider the natural projection $\pi_1 : U(G) \to A_1^t(G)$ defined by

$$\pi_1(H) = \begin{cases} (H) & \text{if } (H) \in \Phi_1^t(G), \\ 0 & \text{otherwise.} \end{cases}$$

Then one immediately obtains

Proposition 3.15. *Let G be as in Proposition 3.14. If $(\widetilde{H}) \in \Phi_0(G)$ with $\widetilde{H} = K \times S^1$ and $(H) \in \Phi_1^t(G)$, then*

$$\pi_1((\widetilde{H}) * (H)) = (K) \circ (H).$$

Remark 3.16. Propositions 3.15 and 3.12 indicate that the multiplication table in the $\mathbb{Z}$-module decomposition $U(G) = A(G) \oplus A_1^t(G) \oplus A^*(G)$ can be described by the following table:

$*$	$A(G) \cong A(\Gamma)$	$A_1^t(G)$	$A^*(G)$
$A(G) \cong A(\Gamma)$	$A(G)$-multip $+T_*$	$A(\Gamma)$-module multip $+T_*$	T_*
$A_1^t(G)$	$A(\Gamma)$-module multip $+T_*$	T_*	T_*
$A^*(G)$	T_*	T_*	T_*

where T_* stands for an element from $A^*(G)$.

3.4. The Euler ring $U(T^n)$. In this subsection we present the computations for the Euler ring $U(T^n)$, where T^n is an n-dimensional torus. The following statement was observed by S. Rybicki.

Proposition 3.17. *If (H), $(K) \in \Phi(T^n)$, and $L = H \cap K$, then*

$$(H) * (K) = \begin{cases} (L) & \textit{if } \dim H + \dim K - \dim L = \dim T^n, \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. Put $G := T^n$. Since every compact abelian connected Lie group is a torus and H and K are normal in G, the groups G/H and G/K are tori. Take $L = H \cap K$. Since G is abelian, L is the only one isotropy in $(G/H \times G/K)^L$ with respect to the $N(L) = G$-action. Hence,

$$(H) * (K) = \chi \left((G/H \times G/K)^L /G \right) (L).$$

Next, $N(L, H) = G$, therefore

$$(G/H \times G/K)^L /G = (G/H \times G/K) /G.$$

Put $M := (G/H \times G/K)/G$. Observe that M is a compact connected G-manifold of precisely one orbit type (L). Thus, it is of dimension $N := \dim G/H + \dim G/K - \dim G + \dim L = \dim G - \dim K - \dim H + \dim L$. If $N := 0$, then $\chi(M) = 1$, and if $N > 0$, then there is an action of a torus on M without G-fixed-points, so $\chi(M) = 0$ (cf. Lemma 2.12). $\square$

EXAMPLE 3.18. As an example, describe the Euler ring structure in $U(T^2)$. Obviously,

$$(T^2) * (K) = (K), \quad \text{for all } (K) \in U(T^2). \tag{18}$$

Next, it follows from Proposition 3.17 that if $(H) * (K)$ is non-trivial for some $(H), (K) \neq (T^2)$, then $\dim H = \dim K = 1$ and $K \neq H$.

For simplicity, identify T^2 with $SO(2) \times S^1$. Then $\dim H = 1$ for $(H) \in \Phi(T^2)$ implies $H = \mathbb{Z}_n \times S^1$, $n = 1, 2, \dots$ or H is twisted. The twisted 1-folded subgroups of $SO(2) \times S^1$ are: $SO(2)^{\varphi_m}$, $\varphi_m : SO(2) \to S^1$, $\varphi_m(z) = z^m$, $m = 0, \pm 1, \pm 2, \dots$, (notice $SO(2)^{\varphi_0} = SO(2)$); $\mathbb{Z}_n^{\varphi_m}$, $m = 0, 1, 2, \dots, n-1$ (in the case n is even and $m = \frac{n}{2}$, we write $\mathbb{Z}_n^d$ instead of $\mathbb{Z}_n^{\varphi_{\frac{n}{2}}}$ and put $\mathbb{Z}_n$ instead of $\mathbb{Z}_n^{\varphi_0}$). Taking into account (18), the full multiplication table for $U(T^2)$ is presented in Table 1.

	$(\mathbb{Z}_m \times S^1)$	$(SO(2) \times \mathbb{Z}_{l_1})$	$(\mathbb{Z}_m \times \mathbb{Z}_{l_1})$	$(SO(2)^{\varphi_m, l_1})$	$(\mathbb{Z}_m^{\varphi_k, l_1})$
$(\mathbb{Z}_n \times S^1)$	0	$(\mathbb{Z}_n \times \mathbb{Z}_{l_1})$	0	$(\mathbb{Z}_n^{\varphi_m, l_1})$	0
$(SO(2) \times \mathbb{Z}_{l_2})$	$(\mathbb{Z}_m \times \mathbb{Z}_{l_2})$	0	0	$(\mathbb{Z}_m \times \mathbb{Z}_l)$	0
$(\mathbb{Z}_n \times \mathbb{Z}_{l_2})$	0	0	0	0	0
$(SO(2)^{\varphi_n, l_2})$	$(\mathbb{Z}_m^{\varphi_n, l_2})$	$(\mathbb{Z}_n \times \mathbb{Z}_l)$	0	$(\mathbb{Z}_{m-n}^{\varphi_n, l})$	0
$(SO(2)^{\varphi_{-m}, l_2})$	$(\mathbb{Z}_m \times \mathbb{Z}_{l_2})$	$(\mathbb{Z}_m \times \mathbb{Z}_l)$	0	$(\mathbb{Z}_{2m}^{d, l})$	0
$(\mathbb{Z}_n^{\varphi_{k'}, l_2})$	0	0	0	0	0

TABLE 1. Multiplication Table for the $U(T^2)$

4. Euler Ring Homomorphism

4.1. General case. Let $\psi : G' \to G$ be a homomorphism between compact Lie groups. Then the formula $g'x := \psi(g')x$ defines a (left) G'-action on G (a similar procedure can be applied to right actions). In particular, for any subgroup $H \subset G$, the map ψ induces the G'-action on G/H with

$$G'_{gH} = \psi^{-1}(gHg^{-1}). \tag{19}$$

In this way, ψ induces a map $\Psi : U(G) \to U(G')$ defined on generators by

$$\Psi((H)) := \sum_{(\tilde{H}) \in \Phi(G')} \chi_c((G/H)_{(\tilde{H})}/G')(\tilde{H}). \tag{20}$$

LEMMA 4.1. *The map Ψ defined by (20) is the Euler ring homomorphism.*

PROOF. Combining formulae (6), (20) and Lemma 2.11 one obtains (cf. also [**4**, p. 88])

$$\begin{aligned}\Psi((H)*(K)) &= \Psi(\sum_{(L)}\chi_c((G/H\times G/K)_{(L)}/G)\cdot(L))\\ &= \sum_{(L)}\chi_c((G/H\times G/K)_{(L)}/G)\cdot\Psi(L)\\ &= \sum_{(L)}\chi_c((G/H\times G/K)_{(L)}/G)\sum_{(L')}\chi_c((G/L)_{(L')}/G')\cdot(L')\\ &= \sum_{(L')}\sum_{(L)}\chi_c((G/H\times G/K)_{(L)}/G)\chi_c((G/L)_{(L')}/G')\cdot(L').\end{aligned}$$

On the other hand,

$$\begin{aligned}&\Psi(H)*\Psi(K)\\ &=\sum_{(H')}\chi_c((G/H)_{(H')}/G')\cdot(H')*\sum_{(K')}\chi_c((G/K)_{(K')}/G')\cdot(K')\\ &=\sum_{(H'),(K')}\chi_c((G/H)_{(H')}/G')\chi_c((G/K)_{(K')}/G')\cdot(H')*(K')\\ &=\sum_{(H'),(K')}\chi_c((G/H)_{(H')}/G')\chi_c((G/K)_{(K')}/G')\cdot\sum_{(L')}\chi_c((G'/H'\times G'/K')_{(L')}/G')\cdot(L')\\ &=\sum_{(L')}\sum_{(H'),(K')}\chi_c((G/H)_{(H')}/G')\chi_c((G/K)_{(K')}/G')\chi_c((G'/H'\times G'/K')_{(L')}/G')\cdot(L').\end{aligned}$$

Put

$$n_{L'} := \sum_{(L)}\chi_c((G/H\times G/K)_{(L)}/G)\chi_c((G/L)_{(L')}/G'),$$

$$m_{L'} := \sum_{(H'),(K')}\chi_c((G/H)_{(H')}/G')\chi_c((G/K)_{(K')}/G')\chi_c((G'/H'\times G'/K')_{(L')}/G').$$

We need to show that for all G'-orbit types (L') in $G/H\times G/K$

$$n_{L'} = m_{L'}. \tag{21}$$

Consider $u_{L'} := \chi_c((G/H\times G/K)_{(L')}/G') = \chi_c((G/H\times G/K)_{L'}/N(L'))$. If (L') is a maximal orbit type, then

$$\begin{aligned}u_{L'} &= \chi_c(G/H\times G/K)_{L'}/N(L') = \chi_c(G/H\times G/K)^{L'}/N(L')\\ &= \sum_{(L)}\chi_c(G/H\times G/K)^{L'}_{(L)}/N(L'),\end{aligned}$$

where the union is taken over all (L)-orbit types occuring in $(G/H\times G/K)^{L'}$ (considered as an $N(\psi(L'))$-space) (cf. (19)). Using the fibre bundle $G/L \hookrightarrow (G/H\times G/K)_{(L)} \to (G/H\times G/K)_{(L)}/G$, we obtain that $(G/H\times G/K)^{L'}_{(L)}/N(L') \to (G/H\times G/K)_{(L)}/G$ is a fibre bundle with the fibre $(G/L^{L'})/N(L')$. Thus,

$$
\begin{aligned}
u_{L'} &= \chi((G/H \times G/K)^{L'}/N(L')) = \sum_{(L)} \chi_c((G/H \times G/K)^{L'}_{(L)}/N(L')) \\
&= \sum_{(L)} \chi_c((G/H \times G/K)_{(L)}/G)\chi((G/L^{L'})/N(L')) \\
&= \sum_{(L)} \chi_c((G/H \times G/K)_{(L)}/G)\chi_c(((G/L)_{L'})/N(L')) = n_{L'}.
\end{aligned}
$$

In the case (L') is not a maximal orbit type, assume, by induction, that $u_{\widetilde{L}'} = n_{\widetilde{L}'}$ for all $(\widetilde{L}') > (L')$. Then

$$
\begin{aligned}
u_{L'} &= \chi_c((G/H \times G/K)_{L'}/N(L')) \\
&= \chi((G/H \times G/K)^{L'}/N(L')) - \sum_{(\widetilde{L}')>(L')} \chi_c((G/H \times G/K)_{\widetilde{L}'}/N\widetilde{L}') \\
&= \chi((G/H \times G/K)^{L'}/N(L')) - \sum_{(\widetilde{L}')>(L')} u_{\widetilde{L}'} \\
&= \sum_{(L)} \chi_c((G/H \times G/K)_{(L)}/G)\chi((G/L^{L'})/N(L')) - \sum_{(\widetilde{L}')>(L')} u_{\widetilde{L}'} \\
&= \sum_{(\widetilde{L}')\geq(L')} \sum_{(L)} \chi_c((G/H \times G/K)_{(L)}/G)\chi((G/L_{\widetilde{L}'})/N\widetilde{L}') - \sum_{(\widetilde{L}')>(L')} u_{\widetilde{L}'} \\
&= \sum_{(\widetilde{L}')\geq(L')} n_{\widetilde{L}'} - \sum_{(\widetilde{L}')>(L')} u_{\widetilde{L}'} = n_{L'} + \sum_{(\widetilde{L}')>(L')} (n_{\widetilde{L}'} - u_{\widetilde{L}'}) = n_{L'}
\end{aligned}
$$

On the other hand, in the case (L') is a maximal orbit type,

$$
\begin{aligned}
(G/H \times G/K)_{L'}/N(L') &= (G/H \times G/K)^{L'}/N(L') \\
&= \bigcup_{(H'),(K')} ((G/H)_{(H')} \times (G/K)_{(K')})^{L'}/N(L'),
\end{aligned}
$$

where the union is taken over all (H')-orbit types (resp. (K')-orbit types) occuring in $(G/H)^{L'}$ (resp. in $(G/K)^{L'}$), considered as an $N(L')$-space. By using the fibre bundles $G'/H' \hookrightarrow (G/H)_{(H')} \to (G/H)_{(H')}/G'$ and $G'/K' \hookrightarrow (G/K)_{(K')} \to (G/K)_{(K')}/G'$, we obtain the product bundle $G'/H' \times G'/K' \hookrightarrow (G/H)_{(H')} \times (G/K)_{(K')} \to (G/H)_{(H')}/G' \times (G/K)_{(K')}/G'$. Therefore,

$$
((G/H)_{(H')} \times (G/K)_{(K')})^{L'}/N(L') \to (G/H)_{(H')}/G' \times (G/K)_{(K')}/G'
$$

is a fibre bundle with the fibre $(G'/H' \times G'/K')^{L'}/N(L')$. Consequently,

$$
\begin{aligned}
u_{L'} &= \chi((G/H \times G/K)^{L'}/N(L')) = \sum_{(H'),(K')} \chi(((G/H)_{(H')} \times (G/K)_{(K')})^{L'}/N(L')) \\
&= \sum_{(H'),(K')} \chi_c((G/H)_{(H')}/G' \times (G/K)_{(K')}/G')\chi((G'/H' \times G'/K')^{L'}/N(L')) \\
&= \sum_{(H'),(K')} \chi_c((G/H)_{(H')}/G' \times (G/K)_{(K')}/G')\chi((G'/H' \times G'/K')_{L'}/N(L')) \\
&= \sum_{(H'),(K')} \chi_c((G/H)_{(H')}/G')\chi_c((G/K)_{(K')}/G')\chi((G'/H' \times G'/K')_{L'}/N(L')) \\
&= m_{L'}.
\end{aligned}
$$

In the case (L') is not a maximal orbit type, by applying induction over the orbit types in the same way as above,

$$
\begin{aligned}
\chi_c((G/H \times G/K)_{L'}/N(L')) &= \chi((G/H \times G/K)^{L'}/N(L')) - \sum_{(\tilde{L}')>(L')} u_{\tilde{L}'} \\
&= \sum_{(H'),(K')} \big(\chi_c((G/H)_{(H')}/G')\chi_c((G/K)_{(K')}/G') \\
&\qquad \cdot \chi((G'/H' \times G'/K')^{L'}/N(L'))\big) - \sum_{(\tilde{L}')>(L')} u_{\tilde{L}'} \\
&= \sum_{(\tilde{L}')\geq(L)} m_{\tilde{L}'} - \sum_{(\tilde{L}')>(L')} u_{\tilde{L}'} = m_{L'} + \sum_{(\tilde{L}')>(L)} (m_{\tilde{L}'} - u_{\tilde{L}'}) \\
&= m_{L'}.
\end{aligned}
$$

Therefore, the statement follows. □

REMARK 4.2. The result stated in Lemma 4.1 was obtained in [**12**], with a proof containing several omissions. We present here an alternative proof for completeness.

EXAMPLE 4.3. Consider the simplest example of Euler ring homomorphism. Namely, assume that G is a compact Lie group and denote by $\mathbb{Z}_1$ the trivial subgroup $\{e\} \subset G$. The inclusion $\psi_o : \mathbb{Z}_1 \hookrightarrow G$ induces the Euler ring homomorphism

$$\Psi_o : U(G) \to U(\mathbb{Z}_1) \simeq \mathbb{Z}.$$

By (20), we have that for $(H) \in \Phi(G)$,

$$\Psi_o(H) = \chi(G/H)(\mathbb{Z}_1).$$

It follows from Proposition 2.18,

$$
\Psi_o(H) = \begin{cases} 0, & \text{if } (H) \text{ is not of maximal rank,} \\ |W_G(T^n)|/|W_H(T^n)| \cdot (\mathbb{Z}_1), & \text{if } (H) \text{ is of maximal rank.} \end{cases} \tag{22}
$$

4.2. Euler ring homomorphism $\Psi : U(G) \to U(T^n)$. Below we specify the homomorphism $\psi : \tilde{G} \to G$ to the case $\tilde{G} = T^n$ – a maximal torus in G and $\psi : T^n \to G$ – the natural embedding. Then the homomorphism Ψ takes the form

$$\Psi(H) = \sum_{(K)\in\Phi(T^n)} \chi_c((G/H)_{(K)}/T^n) \cdot (K), \tag{23}$$

with $K = H' \cap T^n$, $H' \in (H)$. Observe, by the way, that since all the maximal tori in a compact Lie group are conjugate (see, for instance, [**5**, p. 159]), the homomorphism (23) is independent of a choice of a maximal torus in G.

We will show that Ψ can be used to find additional coefficients for the multiplication formulae in $U(G)$. To compute Ψ, we start with the following

PROPOSITION 4.4. *Let T^n be a maximal torus in G and the homomorphism Ψ is defined by (23). Then*

$$\Psi(T^n) = |W(T^n)|(T^n) + \sum_{(T')} n_{T'}(T'),$$

where $T' = gT^n g^{-1} \cap T^n$ for some $g \in G$ and $(T') \neq (T^n)$.

PROOF. By Proposition 2.18, the Weyl group $W(T^n)$ is finite and the coefficient of $\Psi(T^n)$ corresponding to (T^n) can be computed as follows (cf. (23)):

$$\begin{aligned}\chi_c((G/T^n)_{(T^n)}/T^n) &= \chi((G/T^n)^{(T^n)}/T^n) = \chi((G/T^n)^{T^n}/T^n) \\ &= \chi\left(\left(\frac{G}{T^n}\right)^{T^n}\right) = \chi\left(\frac{N(T^n,T^n)}{T^n}\right) = |W(T^n)|.\end{aligned}$$

□

Proposition 4.4 tells us what is precisely the coefficient of $\Psi(T^n)$ related to T^n. In general, to compute a coefficient related to an arbitrary (K) in (23), one can use the following

PROPOSITION 4.5. (RECURRENCE FORMULA) *Let T^n be a maximal torus in G, $\psi : T^n \to G$ a natural embedding, and $\Psi : U(G) \to U(T^n)$ the induced homomorphism of the Euler rings. For $(H) \in \Phi(G)$, put*

$$\Psi(H) = \sum_{(K)} n_K(K),$$

where (K)'s stand for the orbit types in the T^n-space G/H, i.e., $K = H' \cap T^n$ with $H' = gHg^{-1}$ for some $g \in G$. Then, for $K = H' \cap T^n$,

$$n_K = \chi\left(\frac{N(K,H')}{H'}/T^n\right) - \sum_{(\widetilde{K})>(K)} n_{\widetilde{K}}. \tag{24}$$

PROOF. Put $X := G/H$. Then

$$X^{(K)}/T^n = \bigcup_{(\widetilde{K})\geq(K)} X_{(\widetilde{K})}/T^n,$$

which (since T^n is abelian) implies

$$\chi(X^{(K)}/T^n) = \sum_{(\widetilde{K})\geq(K)} \chi_c(X_{(\widetilde{K})}/T^n) = \sum_{(\widetilde{K})\geq(K)} \chi_c(X_{\widetilde{K}}/T^n).$$

Therefore,

$$\chi_c(X_K/T^n) = \chi(X^K/T^n) - \sum_{(\widetilde{K})>(K)} \chi_c(X_{\widetilde{K}}/T^n).$$

To complete the proof, it remains to observe that $X^K/T^n = \frac{N(H'\cap T^n,H')}{H'}/T^n$ (see Lemma 2.3(i)) from which (24) follows directly. □

EXAMPLE 4.6. Consider the natural embedding $\psi : T^2 := SO(2)\times S^1 \to O(2)\times S^1$, which induces the homomorphism of Euler rings $\Psi : U(O(2)\times S^1) \to U(T^2)$. Using Proposition 4.5 one can verify by direct computations that:

$$\begin{aligned}
\Psi(O(2)\times S^1) &= (SO(2)\times S^1), & \Psi(SO(2)\times S^1) &= 2(SO(2)\times S^1)\\
\Psi(D_n\times S^1) &= (\mathbb{Z}_n\times S^1), & \Psi(\mathbb{Z}_m\times S^1) &= 2(\mathbb{Z}_m\times S^1)\\
\Psi(O(2)\times \mathbb{Z}_l) &= (SO(2)\times \mathbb{Z}_l), & \Psi(SO(2)\times \mathbb{Z}_l) &= 2(SO(2)\times \mathbb{Z}_l)\\
\Psi(D_n\times \mathbb{Z}_l) &= (\mathbb{Z}_n\times \mathbb{Z}_l), & \Psi(\mathbb{Z}_m\times \mathbb{Z}_l) &= 2(\mathbb{Z}_m\times \mathbb{Z}_l),\\
\Psi(O(2)^{-,l}) &= (SO(2)\times \mathbb{Z}_l), & \Psi(SO(2)^{\varphi_m,l}) &= (SO(2)^{\varphi_m,l}) + (SO(2)^{\varphi_{-m},l})\\
\Psi(D_n^{z,l}) &= (\mathbb{Z}_n\times \mathbb{Z}_l), & \Psi(D_{2k}^{d,l}) &= (\mathbb{Z}_{2k}^{d,l})\\
\Psi(\mathbb{Z}_n^{\varphi_m,l}) &= (\mathbb{Z}_n^{\varphi_m,l}) + (\mathbb{Z}_n^{\varphi_{-m},l}), & \Psi(\mathbb{Z}_{2k}^{d,l}) &= 2(\mathbb{Z}_{2k}^{d,l})
\end{aligned}$$

where all the symbols used follow the convention established in [**1**].

We conclude this section with a brief explanation of how to use the homomorphism $\Psi : U(G) \to U(T^n)$ to compute the multiplication structure in $U(G)$.

The knowledge of the Burnside Ring $A(G)$ (cf. Lemma 3.4 (see also [**12**], [**1**])), the module $A_1^t(G)$ (cf. Propositions 3.14 and 3.15, Remark 3.16 (see also [**1**])), Proposition 3.12 as well as some ad hoc computations of certain coefficients in the multiplication table for $U(G)$ (cf. Example 3.13) may provide some partial information on the structure of $U(G)$. Thus, taking some $(H), (K) \in \Phi(G)$, one can express $(H) * (K)$ as follows

$$(H) * (K) = \sum_{(L)} n_L(L) + \sum_{(L')} x_{L'}(L'), \tag{25}$$

where n_L are "known" coefficients while $x_{L'}$ are "unknown". On the other hand, Proposition 3.17 allows in principle to completely evaluate the ring $U(T^n)$ (cf. Example 3.18). Since we also know the homomorphism Ψ (cf. Propositions 4.4–4.5), one has that

$$\Psi((H)) * \Psi((K)) = \sum_{(L'')} n_{L''}(L'') \in U(T^n), \tag{26}$$

where all the coefficients $n_{L''}$ are "known". Applying the homomorphism Ψ to (25) and comparing the coefficients of the obtained expression with those obtained in (26) (related to the same conjugacy classes) leads to a linear system over $\mathbb{Z}$ from which, in principle, it is possible to determine some unknown coefficients in (25). However, it might happen that the number of equations in the above linear system is less than the number of unknowns. Summing up, the more partial information on $U(G)$ we have, there is a better chance to compute the remaining coefficients. In the next section, we will illustrate the described strategy by computing the multiplication table for $U(O(2)\times S^1)$.

5. Euler Ring Structure for $U(O(2)\times S^1)$

As an example, we apply the above obtained results to the group $G := O(2)\times S^1$, and using the Euler ring homomorphism $\Psi : U(O(2)\times S^1) \to U(T^2)$ (based on the known structure of the Euler ring $U(T^2)$, see Table 1), we compute the Euler ring structure for $U(O(2)\times S^1)$.

(H)	$(SO(2)\times S^1)$	$(D_m\times S^1)$	$(\mathbb{Z}_m\times S^1)$
$(SO(2)\times S^1)$	$2(SO(2)\times S^1)$	$(\mathbb{Z}_m\times S^1)$	$2(\mathbb{Z}_m\times S^1)$
$(D_n\times S^1)$	$(\mathbb{Z}_n\times S^1)$	$\begin{cases}2(D_k\times S^1)-(\mathbb{Z}_k\times S^1)\\ k=\gcd(m,n)\end{cases}$	$\begin{cases}(\mathbb{Z}_k\times S^1)\\ k=\gcd(m,n)\end{cases}$
$(\mathbb{Z}_n\times S^1)$	$2(\mathbb{Z}_n\times S^1)$	$\begin{cases}(\mathbb{Z}_k\times S^1)\\ k=\gcd(m,n)\end{cases}$	0
$(O(2)\times\mathbb{Z}_l)$	$(SO(2)\times\mathbb{Z}_l)$	$(D_m)\times\mathbb{Z}_l)$	$(\mathbb{Z}_m\times\mathbb{Z}_l)$
$(SO(2)\times\mathbb{Z}_l)$	$2(SO(2)\times\mathbb{Z}_l)$	$(\mathbb{Z}_m\times\mathbb{Z}_l)$	$2(\mathbb{Z}_m\times\mathbb{Z}_l)$
$(D_n\times\mathbb{Z}_l)$	$(\mathbb{Z}_n\times\mathbb{Z}_l)$	$\begin{cases}2(D_k\times\mathbb{Z}_l)-(\mathbb{Z}_k\times\mathbb{Z}_l),\\ k=\gcd(n,m)\end{cases}$	0
$(\mathbb{Z}_n\times\mathbb{Z}_l)$	$2(\mathbb{Z}_n\times\mathbb{Z}_l)$	0	0
$(O(2)^{-,l})$	$(SO(2)\times\mathbb{Z}_l$	$(D_m^{z,l})$	$(\mathbb{Z}_m\times\mathbb{Z}_l)$
$(SO(2)^{\varphi_k,l})$	$2(SO(2)^{\varphi_k,l})$	$(\mathbb{Z}_m^{\varphi_k,l})$	$2(\mathbb{Z}_m^{\varphi_k,l})$
$(D_n^{z,l})$	$(\mathbb{Z}_n\times\mathbb{Z}_l)$	$\begin{cases}2(D_k^{z,l})-(\mathbb{Z}_k\times\mathbb{Z}_l),\\ k=\gcd(m,n)\end{cases}$	0
$\begin{cases}(D_{2n}^{d,l})\\ 2k=\gcd(m,2n),\\ \quad 2k\nmid n\end{cases}$	$(\mathbb{Z}_{2k}^{d,l})$	$2(D_{2k}^{d,l})-(\mathbb{Z}_{2k}^{d,l})$	0
$\begin{cases}(D_{2n}^{d,l})\\ k=\gcd(m,2n),\\ \quad k\mid n\end{cases}$	$(\mathbb{Z}_{2k}^{d,l})$	$(D_k\times\mathbb{Z}_l)+(D_k^{z,l})-(\mathbb{Z}_k\times\mathbb{Z}_l)$	0
$(\mathbb{Z}_n^{\varphi_k,l})$	$2(\mathbb{Z}_n^{\varphi_k,l})$	0	0
$(\mathbb{Z}_{2n}^{d,l})$	$2(\mathbb{Z}_{2n}^{d,l})$	0	0

TABLE 2. Multiplication Table for $U(O(2)\times S^1)$

Let us illustrate the computations of the ring structure of $U(O(2)\times S^1)$ with two examples.

EXAMPLE 5.1.
(i) Consider, for example, two orbit types $(D_m\times S^1)$, $(D_n^{z,l})\in\Phi(O(2)\times S^1)$. The $O(2)\times S^1$-space

$$\frac{O(2)\times S^1}{D_m\times S^1}\times\frac{O(2)\times S^1}{D_n^{z,l}}$$

is composed of two orbit types: $(D_k^{z,l})$ and $(\mathbb{Z}_k\times\mathbb{Z}_l)$, where $k=\gcd(n,m)$. Since (see [**1**, Table 6.13])

$$(D_m\times S^1)\circ(D_n^{z,l})=2(D_k^{z,l}),$$

we know that (cf. Proposition 3.15)

$$(D_m\times S^1)*(D_n^{z,l})=2(D_k^{z,l})+x(\mathbb{Z}_k\times\mathbb{Z}_l),\tag{27}$$

where x is an unknown integer. Using the ring homomorphism $\Psi: U(O(2)\times S^1)\to U(T^2)$, we obtain (see Example 4.6 and Table 1)

$$\begin{aligned}\Psi\big((D_m\times S^1)*(D_n^{z,l})\big)&=\Psi((D_m\times S^1))*\Psi((D_n^{z,l}))\\&=(\mathbb{Z}_m\times S^1)*(\mathbb{Z}_n\times\mathbb{Z}_l)=0.\end{aligned}$$

On the other hand, by applying Ψ to (27),

$$\Psi\big((D_m\times S^1)*(D_n^{z,l})\big)=2\Psi(D_k^{z,l})+x\Psi(\mathbb{Z}_k\times\mathbb{Z}_l)=2(\mathbb{Z}_k\times\mathbb{Z}_l)+2x(\mathbb{Z}_k\times\mathbb{Z}_l),$$

which implies that $x = -1$, so

$$(D_m \times S^1) * (D_n^{z,l}) = 2(D_k^{z,l}) - (\mathbb{Z}_k \times \mathbb{Z}_l).$$

(ii) Similarly, take $(D_m \times S^1)$ and $(D_{2n}^{d,l}) \in \Phi(O(2) \times S^1)$ and consider the orbit types occurring in the $O(2) \times S^1$-space

$$\frac{O(2) \times S^1}{D_m \times S^1} \times \frac{O(2) \times S^1}{D_{2n}^{d,l}}.$$

We have two cases: (a) $(D_{2k}^{d,l})$ and $(\mathbb{Z}_{2k}^{d,l})$, where $2k = \gcd(m, 2n)$ and $2k \nmid n$; (b) $(D_k^{z,l})$, $(D_k \times \mathbb{Z}_l)$ and $(\mathbb{Z}_k \times \mathbb{Z}_l)$, where $k = \gcd(m, 2n)$ and $k|n$. Notice that

$$(D_m \times S^1) \circ (D_{2n}^{d,l}) = \begin{cases} 2(D_{2k}^{d,l}) & \text{if } 2k = \gcd(m, 2n),\ 2k \nmid n, \\ (D_k^{z,l}) + (D_k \times \mathbb{Z}_l) & \text{if } k = \gcd(m, 2n),\ k|n, \end{cases}$$

thus

$$(D_m \times S^1) * (D_{2n}^{d,l}) = \begin{cases} 2(D_{2k}^{d,l}) + x(\mathbb{Z}_{2k}^{d,l}) & \text{if } 2k = \gcd(m, 2n),\ 2k \nmid n \\ (D_{2k}^{z,l}) + (D_k \times \mathbb{Z}_l) + x(\mathbb{Z}_k \times \mathbb{Z}_l) & \text{if } k = \gcd(m, 2n),\ k|n. \end{cases}$$

Then, in the case $2k = \gcd(m, 2n)$, $2k \nmid n$, by applying the homomorphism Ψ, we obtain

$$\begin{aligned} 0 &= (\mathbb{Z}_m \times S^1) * (\mathbb{Z}_{2n}^{d,l}) = \Psi\big((D_m \times S^1) * (D_{2n}^{d,l})\big) \\ &= 2\Psi(D_{2k}^{d,l}) + x\Psi(\mathbb{Z}_{2k}^{d,l}) = 2(\mathbb{Z}_{2k}^{d,l}) + 2x(\mathbb{Z}_{2k}^{d,l}), \end{aligned}$$

which implies again $x = -1$. Similarly, in the case $k = \gcd(m, 2m)$, $k|n$, we have

$$\begin{aligned} 0 &= (\mathbb{Z}_m \times S^1) * (\mathbb{Z}_{2n}^{d,l}) = \Psi\big((D_m \times S^1) * (D_{2n}^{d,l})\big) \\ &= \Psi(D_k^{z,l}) + \Psi(D_k \times \mathbb{Z}_l) + x\Psi(\mathbb{Z}_k \times \mathbb{Z}_l) \\ &= (\mathbb{Z}_k \times \mathbb{Z}_l) + (\mathbb{Z}_k \times \mathbb{Z}_l) + 2x(\mathbb{Z}_k \times \mathbb{Z}_l), \end{aligned}$$

which implies $x = -1$ (here $k = \gcd(m, 2n)$).

The multiplication table for $U(O(2) \times S^1)$ is mainly presented in Table 2. In addition, we have the following non-zero products

$$\begin{aligned} (SO(2)^{\varphi_n, l_1}) * (O(2) \times \mathbb{Z}_{l_2}) &= 2(\mathbb{Z}_n \times \mathbb{Z}_l), \\ (SO(2)^{\varphi_n, l_1}) * (SO(2) \times \mathbb{Z}_{l_2}) &= 2(\mathbb{Z}_n \times \mathbb{Z}_l), \\ (SO(2)^{\varphi_n, l_1}) * (O(2)^{-,l}) &= 2(\mathbb{Z}_n \times \mathbb{Z}_l), \\ (SO(2)^{\varphi_n, l_1}) * (SO(2)^{\varphi_m, l_2}) &= (\mathbb{Z}_{n-m}^{\varphi_n, l}) + (\mathbb{Z}_{n+m}^{\varphi_m, l}), \quad n > m \\ (SO(2)^{\varphi_n, l_1}) * (SO(2)^{\varphi_{-n}, l_2}) &= 2(\mathbb{Z}_{2n}^{d,l}), \end{aligned}$$

where $l = \gcd(l_1, l_2)$. All other products (except for those containing $(O(2) \times S^1)$, which is the unit element in $U(O(2) \times S^1)$) are zero.

6. Equivariant Gradient Degree

Let V be an orthogonal G-representation, $\Omega \subset V$ a bounded invariant open subset and $f : V \to V$ an Ω-*admissible* (i.e., f has no zeros on $\partial\Omega$) G-gradient map). In this setting, K. Gęba assigned to f the so-called G-gradient degree (denoted $\nabla_G\text{-deg}\,(f, \Omega)$) taking its values in the Euler ring $U(G)$ (see [**17**]; cf. Definitions 6.1, 6.2 and formulae (28), (29)).

This degree contains a complete topological information on the symmetric properties of zeros of f. However, the computation of $\nabla_G\text{-deg}\,(f,\Omega)$ is a complicated task, in general. In this section, we establish several formulae useful for effective computations of the equivariant gradient degree.

6.1. Construction of G-gradient degree and basic properties. Let us recall the construction of the G-gradient degree (cf. [**17**]) and present some of its properties.

Definition 6.1.

(i) A map $f : V \to V$ is called *G-gradient* if there exists a G-invariant C^1-function $\varphi : V \to \mathbb{R}$ such that $f = \nabla\varphi$. Similarly, one can define a G-gradient homotopy.

(ii) Denote by τM the tangent bundle of M. Take $x \in V$ and put $H := G_x$, $W_x := \tau_x V_{(H)} \ominus \tau_x G(x)$. The orbit $G(x)$ is called a *regular zero orbit* of f if $f(x) = 0$ and $Kf(x) := Df(x)|_{W_x}$ is an isomorphism. Also, define the *index* of the regular zero orbit $G(x)$ by $i(G(x)) := \operatorname{sign}\det Kf(x)$.

(iii) For an open G-invariant subset U of $V_{(H)}$ such that $\overline{U} \subset V_{(H)}$, and a small $\varepsilon > 0$, put

$$\mathcal{N}(U,\varepsilon) := \{y \in V \;:\; y = x + v, x \in U, v \perp \tau_x V_{(H)}, \|v\| < \varepsilon\},$$

and call it a *tubular neighborhood of type* (H). A G-gradient map $f : V \to V$, $f := \nabla\varphi$, is called (H)*-normal*, if there exists a tubular neighborhood $\mathcal{N}(U,\varepsilon)$ of type (H) such that $f^{-1}(0) \cap \Omega_{(H)} \subset \mathcal{N}(U,\varepsilon)$ and for $y \in \mathcal{N}(U,\varepsilon)$, $y = x + v$, $x \in U, v \perp \tau_x V_{(H)}$,

$$\varphi(y) = \varphi(x) + \frac{1}{2}\|v\|^2.$$

The following notion of ∇_G-generic map plays an essential role in the construction of the G-gradient degree presented in [**17**].

Definition 6.2. A G-gradient Ω-admissible map f is called ∇_G*-generic* in Ω if there exists an open G-invariant subset $\Omega_o \subset \Omega$ such that (i) $f|_{\Omega_o}$ is of class C^1; (ii) $f^{-1}(0) \cap \Omega \subset \Omega_o$; (iii) $f^{-1}(0) \cap \Omega_o$ is composed of regular zero orbits; (iv) for each (H) with $f^{-1}(0) \cap \Omega_{(H)} \neq \emptyset$, there exists a tubular neighborhood $\mathcal{N}(U,\varepsilon)$ such that f is (H)-normal on $\mathcal{N}(U,\varepsilon)$.

As it was shown in [**17**], any G-gradient Ω-admissible map is G-gradiently homotopic (by an Ω-admissible homotopy) to a map which is ∇_G-generic in Ω. Define a G-gradient degree for a G-gradient admissible map f by

$$\nabla_G\text{-deg}\,(f,\Omega) := \nabla_G\text{-deg}\,(f_o,\Omega) = \sum_{(H)\in\Phi(G)} n_H \cdot (H), \tag{28}$$

where f_o is the ∇_G-generic approximation of f and

$$n_H := \sum_{(G_{x_i})=(H)} i(G(x_i)), \tag{29}$$

with $G(x_i)$'s being isolated orbits of type (H) in $f_o^{-1}(0) \cap \Omega$.

We refer to [**17**] for the verification that $\nabla_G\text{-deg}\,(f,\Omega)$ is well-defined and satisfies the standard properties expected from a degree: *Existence*, *Additivity*, *Homotopy* (G-Gradient) and *Suspension* (cf. [**1**]). In addition, (Multiplicativity) *Let V and W be two orthogonal G-representations, $f : V \to V$ (resp. $\widetilde{f} : W \to W$) a G-gradient Ω-admissible (resp. $\widetilde{\Omega}$-admissible) map, where $\Omega \subset V$ and $\widetilde{\Omega} \subset W$. Then*

$$\nabla_G\text{-deg}\,(f \times \widetilde{f}, \Omega \times \widetilde{\Omega}) = \nabla_G\text{-deg}\,(f, \Omega) * \nabla_G\text{-deg}\,(\widetilde{f}, \widetilde{\Omega}),$$

where the multiplication '$*$' is taken in the Euler ring $U(G)$.

(Functoriality) *Let V be an orthogonal G-representation, $f : V \to V$ a G-gradient Ω-admissible map, and $\psi : G_o \to G$ a homomorphism of Lie groups. Then ψ induces a G_o-action on V such that f is an Ω-admissible G_o-gradient map, and the following equality holds*

$$\Psi[\nabla_G\text{-deg}\,(f, \Omega)] = \nabla_{G_o}\text{-deg}(f, \Omega), \tag{30}$$

where $\Psi : U(G) \to U(G_o)$ is the homomorphism of Euler rings induced by ψ.

Remark 6.3. Suppose that $V := \mathbb{R}^n$ is a Euclidean space and $f : V \to V$ an Ω-admissible gradient map. Then one can consider V to be the representation of the trivial group $\mathbb{Z}_1$. It is easy to notice that in such a case $\nabla_{\mathbb{Z}_1}\text{-deg}(f, \Omega) = n_o(\mathbb{Z}_1) \in U(\mathbb{Z}_1) \simeq \mathbb{Z}$ is exactly the Brouwer degree $\deg(f, \Omega) = n_o$.

Example 6.4. Let V be an orthogonal G-representation, $f : V \to V$ a G-gradient Ω-admissible map. Consider the trivial homomorphism $\psi_o : \mathbb{Z}_1 \to G$ (see Example 4.3). Then, by the Functoriality property,

$$\Psi_o[\nabla_G\text{-deg}\,(f, \Omega)] = \deg(f, \Omega),$$

where $\deg(f, \Omega)$ stands for the usual Brouwer degree. Put

$$\nabla_G\text{-deg}\,(f, \Omega) = \sum_{(H)} n_H(H).$$

Then, by applying (22), we obtain that

$$\deg(f, \Omega) = \sum_{(H) \in \Phi_m(G)} n_H\, |W_G(T^n)|/|W_H(T^n)|, \tag{31}$$

where $\Phi_m(G) := \{(H) \in \Phi(G) : H \text{ of maximal rank}\}$ and T^n is a maximal torus in G.

Remark 6.5.

(i) Formula (31) is nicely compatible with the corresponding result from [**7**], where the case of an arbitrary equivariant continuous f (in general, non-gradient) was considered (see also [**33**]; for the case of maps equivariant with respect to two different actions, see [**26**]).

(ii) In a standard way, the notion of G-gradient degree can be extended to compact G-equivariant gradient vector fields on Hilbert representations. In what follows, we will use the same notation for this extended degree.

Remark 6.6. Suppose that $G = S^1$ and consider an Ω-admissible G-gradient map $f : V \to V$. Then

$$\nabla_{S^1}\text{-deg}(f, \Omega) = n_G(S^1) + \sum_{k=1}^{\infty} n_k(\mathbb{Z}_k)$$

where S^1 is the only subgroup of maximal rank. Therefore, by (31), $\deg(f, \Omega) = n_G$. Thus, it is clear that the Brouwer degree ignores all the coefficients n_k,

$k = 1, 2, \dots$, containing the information about the solutions $x \in \Omega$ to $f(x) = 0$, with the isotropies $G_x \neq S^1$. For variational problems related to finding periodic solutions to an autonomous differential equation, these isotropies correspond exactly to non-constant periodic solutions. This justifies a commonly 'well-known' fact that the Leray-Schauder degree is "blind" to non-constant periodic solutions in such systems.

We complete this subsection with the following

LEMMA 6.7. *Let V be an orthogonal G-representation, $\Omega \subset V$ an open bounded G-invariant set and $f : V \to V$ a G-gradient Ω-admissible map. Then, for every orbit type (L) in Ω, the map $f^L := f|_{V^L} : V^L \to V^L$ is an Ω^L-admissible $W(L)$-equivariant gradient map. Moreover, if*

$$\nabla_G\text{-deg}\,(f, \Omega) = \sum_{(K)\in\Phi(G)} n_K(K), \quad \text{and} \quad \nabla_{W(L)}\text{-deg}\,(f^L, \Omega^L) = \sum_{(H)\in\Phi(W(L))} m_H(H),$$

then

$$n_L = m_{\mathbb{Z}_1}, \tag{32}$$

where $\mathbb{Z}_1 := \{e\}$ and "e" stands for the identity element in $W(L)$.

PROOF. By the homotopy property of G-gradient degree, without loss of generality, one can assume that f is ∇_G-generic in Ω. Therefore, f^L is ∇_G-generic in Ω^L, and formula (32) follows from the construction of G-gradient degree. □

6.2. Equivariant gradient degree of linear maps. In several cases (important from the application viewpoint), using the standard linearization techniques, one can reduce the computation of $\nabla_G\text{-deg}\,(f, \Omega)$ to $\nabla_G\text{-deg}\,(A, \mathcal{B})$, where $A : V \to V$ is a G-equivariant linear symmetric isomorphism and $\mathcal{B}$ is the unit ball in V. By suspension and the homotopy property,

$$\nabla_G\text{-deg}\,(A, \mathcal{B}) = \nabla_G\text{-deg}\,(-\mathrm{Id}\,, \mathcal{B}_-),$$

where $\mathcal{B}_-$ stands for the unit ball in the negative eigenspace E_- of A, i.e., we consider the negative spectrum $\sigma_-(A)$ of A and put $E_- := \bigoplus_{\mu\in\sigma_-(A)} E(\mu)$, where $E(\mu)$ is the eigenspace corresponding to μ. Consider the complete list of all irreducible G-representations $\mathcal{P}_i$, $i = 0, 1, \dots$, and let

$$V = V_0 \oplus V_1 \oplus \cdots \oplus V_r$$

be the G-isotypical decomposition of V, where the isotypical component V_i is modeled on the irreducible G-representation P_i, $i = 0, 1, \dots, r$. Since for every $\mu \in \sigma_-(A)$ the eigenspace $E(\mu)$ is G-invariant,

$$E(\mu) = E_0(\mu) \oplus E_1(\mu) \oplus \cdots \oplus E_r(\mu),$$

where $E_i(\mu) := E(\mu) \cap V_i$, $i = 0, 1, \dots, r$. Put

$$m_i(\mu) = \dim E_i(\mu)/\dim \mathcal{P}_i, \quad i = 0, 1, 2, \dots, r. \tag{33}$$

The number $m_i(\mu)$ is called a *$\mathcal{P}_i$-multiplicity* of the eigenvalue μ. By applying the multiplicativity property, we obtain

$$\nabla_G\text{-deg}\,(-\mathrm{Id}\,, \mathcal{B}_-) = \prod_{\mu\in\sigma_-(A)} \prod_{i=0}^{r} \nabla_G\text{-deg}\,(-\mathrm{Id}\,, B_i)^{m_i(\mu)},$$

where B_i stands for the unit ball in $\mathcal{P}_i$.

The operator $-\mathrm{Id}$ on irreducible G-representations plays an important role in computations of the equivariant gradient degree for linear maps, which motivates the following

DEFINITION 6.8. For an irreducible G-representation $\mathcal{P}_i$, put

$$\mathrm{Deg}_{\mathcal{P}_i} := \nabla_G\text{-deg}\,(-\mathrm{Id}\,, B_i), \tag{34}$$

and call $\mathrm{Deg}_{\mathcal{P}_i}$ the $\mathcal{P}_i$*-basic gradient degree* (or simply *basic gradient degree*).

Consequently, we obtain the following computational formula

$$\nabla_G\text{-deg}\,(A, \mathcal{B}) = \prod_{\mu\in\sigma_-(A)} \prod_{i=0}^{r} (\mathrm{Deg}_{\mathcal{P}_i})^{m_i(\mu)}, \tag{35}$$

where $\mathrm{Deg}_{\mathcal{P}_i}$ is defined by (34) and $m_i(\mu)$ is defined by (33).

Observe, however, that for arbitrary G, the computation of $\mathrm{Deg}_{\mathcal{P}_i}$ is still difficult. In this section, we develop a method for the computation of $\mathrm{Deg}_{\mathcal{P}_i}$ in the case $G = \Gamma \times S^1$, where Γ is a compact Lie group. The main ingredients of the method are:

(i) for each $(L) \in \Phi(G)$, the n_L-coefficient of $\mathrm{Deg}_{\mathcal{P}_i}$ can be computed via the $W(L)$-gradient degree of the restriction to V^L (cf. Lemma 6.7);

(ii) if $(L) \in \Phi_1^t(G)$, then the computation of the related $W(L)$-gradient degree can be done using a canonical passage to the so-called orthogonal degree (cf. formulae (39)–(45));

(iii) the computation of basic gradient degree related to the maximal torus-action usually is simple, therefore the remaining (non-twisted) coefficients n_L can be computed using the homomorphism $\Psi : U(G) \to U(T^n)$ and the information obtained for the twisted orbit types.

6.3. Orthogonal degree for one-dimensional bi-orientable compact Lie groups. In this subsection, G stands for a one-dimensional compact Lie group which is *bi-orientable* (i.e., admitting an orientation which is invariant with respect to both left and right translations), V denotes an orthogonal G-representation and $\Omega \subset V$ stands for an open bounded invariant subset.

It turns out that one can associate to any G-gradient Ω-admissible map $f : V \to V$ (in fact, more generally, to any orthogonal map (see Definition 6.9)) a G-equivariant map $\tilde{f} : \mathbb{R} \oplus V \to V$ in such a way that the primary degree of $\tilde{f}$ (see [**1**] for details) is intimately connected to $\nabla_G\text{-deg}\,(f, \Omega)$. Observe that in the case $G = \Gamma \times S^1$ with Γ finite, a similar construction was suggested in [**29, 1**] (see also [**30**]). Since our exposition is parallel to [**29, 1**], we only briefly sketch the main points starting with the following

DEFINITION 6.9. A G-equivariant map $f : V \to V$ is called G*-orthogonal* on Ω, if f is continuous and for all $v \in \Omega$, the vector $f(v)$ is orthogonal to the orbit $G(v)$ at v. Similarly, one can define the notion of a G*-orthogonal homotopy* on Ω.

It is easy to see that any G-gradient map is orthogonal, however (see [**1**, Example 8.4]), one can easily construct an orthogonal map which is not G-gradient[(*)].

[(*)] Observe that autonomous systems of ODEs admitting the first integral lead to S^1-orthogonal maps. An S^1-equivariant degree with rational values was constructed for such systems by Dancer and Toland (cf. [**10**]). One can easily show that the values of this degree can be obtained from the corresponding S^1-orthogonal degree.

To associate with an orthogonal map a G-equivariant map (and the corresponding primary degree) some preliminaries (related to G-orbits) are needed.

Take the maximal torus T^1 of G (=the connected component of $e \in G$), choose an orientation on T^1 invariant with respect to left and right translations and identify T^1 with S^1. The chosen orientation on $T^1 = S^1$ can be extended invariantly on the whole group G. We assume the orientation to be fixed throughout this subsection.

Next, take a vector $v \in V$ and define the diffeomorphism

$$\mu_v : G/G_v \to G(v), \quad \mu_v(gG_v) := gv. \tag{36}$$

Take the decomposition

$$V = V^{S^1} \oplus V', \quad V' := (V^{S^1})^{\perp}. \tag{37}$$

If $v \in V^{S^1}$, then $\dim G_x = 1$ so that the orbit $G(x) \cong G/G_x$ is finite and, therefore, admits the "natural" orientation.

If $v \notin V^{S^1}$, then G_v is a finite subgroup of G, and by bi-orientability of G, both (left and right) actions of G_v preserve the fixed orientation of G. Therefore, G/G_v has a natural orientation, induced from G. Consequently, the orientation obtained by (36) (again by bi-orientability of G_u) does not depend on a choice of the point v from the orbit $G(v)$ (cf. [**1**], Remark 2.43).

Summing up, in both cases ($v \in V^{S^1}$ and $v \notin V^{S^1}$), $G(v)$ admits a "natural" orientation, although exhibits different algebraic and topological properties. Hence, given an orthogonal map f, the orbits of $f^{-1}(0)$ belonging to V^{S^1} and those belonging to $V \setminus V^{S^1}$ contribute in equivariant homotopy properties of f in different ways, and one needs to "separate" these contributions. To this end (as well as to make the construction of the "orthogonal degree" compatible with the suspension and other properties of the primary degree), we use the following concept.

Definition 6.10. Let $f : V \to V$ be G-orthogonal on Ω. Then f is called S^1-*normal* on Ω if

$$\exists_{\delta>0}\ \forall_{x\in\Omega^{S^1}}\ \forall_{u\perp V^{S^1}} \quad \|u\| < \delta \implies f(x+u) = f(x) + u. \tag{38}$$

Similarly, one can define the notion of G-orthogonal S^1-normal homotopy on Ω.

Literally following the proof of Theorem 8.7 from [**1**], one can establish

Proposition 6.11. *Let $f : V \to V$ be a G-orthogonal Ω-admissible map. Then there exists an Ω-admissible G-orthogonal S^1-normal (on Ω) map $f_o : V \to V$ which is G-orthogonally homotopic to f (on Ω). In addition, a similar result for G-orthogonal S^1-normal homotopies is also true.*

We are now in a position to define an orthogonal degree. Consider $v \in V$ and the map $\varphi_v : G \to G(v)$ given by

$$\varphi_v(g) = gv, \quad g \in G. \tag{39}$$

Clearly, φ_v is smooth and $D\varphi_v(1) : \tau_1(G) = \tau_1(S^1) \to \tau_v(G(v))$. Since the total space of the tangent bundle to S^1 can be written as

$$\tau(S^1) = \{(z,\gamma) \in \mathbb{C} \times S^1 : z \perp \gamma\} = \{(it\gamma,\gamma) \in \mathbb{C} \times S^1 : t \in \mathbb{R}\},$$

the vector

$$\tau(v) := D\varphi_v(1)(i) = \lim_{t\to 0} \frac{1}{t}\left[e^{it}v - v\right] \tag{40}$$

is tangent to the orbit $G(v)$ (here, $e^{it}v$ stands for the result of the action of $e^{it} \in S^1$ on $v \in V$). In the case $v \notin V^{S^1}$, we have $\tau(v) \neq 0$.

Next, take a G-orthogonal Ω-admissible map $f : V \to V$. By Proposition 6.11, there exists a map $f_o : V \to V$ which is G-orthogonal S^1-normal on Ω and G-orthogonally homotopic to f. Consider decomposition (37). Since f_o is S^1-normal, there exists $\delta > 0$ such that for all $x \in \Omega \cap V^{S^1}$ and $u \in V'$,

$$f_o(x+u) = f_o(x) + u, \quad \text{provided} \quad \|u\| < \delta.$$

Take the set

$$U_\delta := \{(t,v) \in (-1,1) \times \Omega : v = x+u,\ x \in V^{S^1},\ u \in V',\ \ \|u\| > \delta\}, \tag{41}$$

and define $\widetilde{f_o} : \mathbb{R} \oplus V \to V$ by

$$\widetilde{f_o}(t,v) := f_o(v) + t\tau(v), \quad (t,v) \in \mathbb{R} \oplus V, \tag{42}$$

where $\tau(v)$ is given by (40). It is clear that $\widetilde{f_o}$ is G-equivariant and U_δ-admissible.

Set $\bar{f}_o := f_o|_{V^{S^1}}$. Obviously, $\bar{f}_o : V^{S^1} \to V^{S^1}$ is G-equivariant (in fact, G/S^1-equivariant) and Ω^{S^1}-admissible.

Put

$$A_1(G) := \mathbb{Z}[\Phi_1(G)], \quad \Phi_1^+(G) := \{(H) \in \Phi_1(G) :\ W(H) \text{ is bi-orientable}\},$$
$$A_1^+(G) := \mathbb{Z}[\Phi_1^+(G)].$$

Define the *orthogonal G-equivariant degree* $G\text{-Deg}^{\,o}(f,\Omega)$ of the map f to be an element of $A_0(G) \oplus A_1^+(G) \subset A_0(G) \oplus A_1(G) =: U(G)$ given by

$$G\text{-Deg}^{\,o}(f,\Omega) := \Big(\mathrm{Deg}^0_G(f,\Omega), \mathrm{Deg}^1_G(f,\Omega)\Big), \tag{43}$$

where $\mathrm{Deg}^0_G(f,\Omega) \in A_0(G)$ is

$$\mathrm{Deg}^0_G(f,\Omega) := G\text{-deg}(\bar{f}_o, \Omega^{S^1}), \tag{44}$$

and $\mathrm{Deg}^1_G(f,\Omega) \in A_1(G)$ is

$$\mathrm{Deg}^1_G(f,\Omega) := G\text{-Deg}\,(\widetilde{f_o}, U_\delta). \tag{45}$$

Here (and everywhere below) "G-deg" stands for the G-equivariant degree without free parameter while "G-Deg" denotes the primary G-equivariant degree (cf. [**1**]).

One can show (cf. [**1**]) that formula (43) is independent of a choice of a G-orthogonal S^1-normal approximation f_o. Moreover, the orthogonal degree defined by (43) satisfies all the properties described in Theorem 8.8 from [**1**].

We complete this subsection with the following result connecting the orthogonal and G-gradient degree in the case G is a compact one-dimensional bi-orientable Lie group.

Proposition 6.12. *Let $f : V \to V$ be a G-gradient Ω-admissible map. Then*

$$\nabla_G\text{-deg}\,(f,\Omega) = \Big(\mathrm{Deg}^0_G(f,\Omega), -\mathrm{Deg}^1_G(f,\Omega)\Big),$$

where $\mathrm{Deg}^0_G(f,\Omega) \in A_0(G)$ is defined by (44) and $\mathrm{Deg}^1_G(f,\Omega) \in A_1^+(G)$ is defined by (45).

Proof. The proof of this proposition follows the same scheme as the one of Theorem 4.3.2 from [**29**] and, therefore, is omitted here. $\square$

6.4. Computations for $\Gamma \times S^1$-gradient degree. In this subsection, we assume $G = \Gamma \times S^1$, where Γ is an *arbitrary* compact Lie group. Let V be an orthogonal G-representation and $\Omega \subset V$ a bounded open invariant subset. Our goal is to establish (by means of the results and constructions from Subsections 6.1 and 6.3) some formulae useful for the computations of G-gradient degree. As an example, basic gradient degrees for $G = O(2) \times S^1$ are computed (cf. Definition 6.8 and formula (35)).

Take a G-gradient Ω-admissible map $f : V \to V$. For every orbit type $(L) \in \Phi_1^t(G)$ in Ω, associate to Ω^L and $f^L : V^L \to V^L$ the set

$$U_\delta^L := \{(t,v) \in (-1,1) \times \Omega^L : v = x + u,\ x \in (V^L)^{S^1},\ u \in V',\ \|u\| > \delta\}, \tag{46}$$

and the one-parameter map $\widetilde{f}_o^L : \mathbb{R} \oplus V^L \to V^L$ given by

$$\widetilde{f}_o^L(t,v) := f_o^L(v) + t\tau(v), \quad (t,v) \in \mathbb{R} \oplus V^L, \tag{47}$$

where f_o^L is an S^1-normal approximation of f^L on Ω^L (cf. Definition 6.10) and $\tau(v)$ is the tangent vector to the orbit $W(L)(v)$ given by formula (40) with V replaced with V^L. It is clear that $\widetilde{f}_o^L$ is $W(L)$-equivariant and U_δ^L-admissible. Combining Lemma 6.7 and Proposition 6.12 with properties of twisted orbit types yields

Proposition 6.13. *Let $f : V \to V$ be a G-gradient Ω-admissible map, $(L) \in \Phi_1^t(G)$ an orbit type in Ω and $\widetilde{f}_o^L : \mathbb{R} \oplus V^L \to V^L$ (resp. U_δ) be defined by (47) (resp. (46)). Assume*

$$\nabla_G\text{-deg}\,(f,\Omega) = \sum_{(K) \in \Phi(G)} n_K (K),$$

and

$$-W(L)\text{-Deg}\,(\widetilde{f}_o^L, U_\delta^L) = \sum_{(H) \in \Phi_1^+(W(L))} m_H (H).$$

Then

$$n_L = m_{\mathbb{Z}_1},$$

where $\mathbb{Z}_1 := \{e\}$ and "e" stands for the identity element in $W(L)$.

Next, we apply Proposition 6.13 to the case when f is the linear symmetric isomorphism and Ω is the unit ball in V. In view of formula (35), it is enough to consider basic gradient degrees (cf. (34)).

Following [**1**], we differ in $\{\mathcal{P}_k\}$, $k = 0,1,2,\dots$, between two sorts of irreducible $\Gamma \times S^1$-representations:

(i) those, where S^1 acts trivially (denoted by $\mathcal{V}_i$, $i \geq 0$,) which can be identified with irreducible Γ-representations;

(ii) those, where S^1 acts non-trivially defined as follows: if $\{\mathcal{U}_j\}$, $j \geq 0$, is the complete list of all complex irreducible Γ-representations, then, with each $\mathcal{U}_j$ and $l = 1,2,\dots$, associate the real irreducible G-representation $\mathcal{V}_{j,l}$ with the G-action given by

$$(\gamma,z)w = z^l \cdot (\gamma w), \quad (\gamma,z) \in \Gamma \times S^1,\ w \in \mathcal{U}_j. \tag{48}$$

In the case (i), put

$$\deg_{\mathcal{V}_i} := G\text{-deg}(-\mathrm{Id}, B_i), \tag{49}$$

where B_i is the unit ball in $\mathcal{V}_i$. In the case (ii), consider the set $\mathcal{O} \subset \mathbb{R} \oplus \mathcal{V}_{j,l}$ given by

$$\mathcal{O} = \left\{ (t, v) : \frac{1}{2} < \|v\| < 2, \;\; -1 < t < 1 \right\}, \tag{50}$$

and define $b : \overline{\mathcal{O}} \to \mathcal{V}_{j,l}$ by

$$b(t, v) = (1 - \|v\| + it) \cdot v, \quad (t, v) \in \overline{\mathcal{O}}. \tag{51}$$

Put

$$\deg_{\mathcal{V}_{j,l}} := G\text{-}\mathrm{Deg}^{\,t}(b, \mathcal{O}) \in A_1^t(G), \tag{52}$$

where "G-Deg$^{\,t}$" stands for the so-called twisted degree (see [**1**]). We refer to [**1**], where effective computational formulae for both $\deg_{\mathcal{V}_i}$ and $\deg_{\mathcal{V}_{j,l}}$ (as well as many concrete examples) are presented.

THEOREM 6.14. *Let Γ be a compact Lie group, $G = \Gamma \times S^1$ and $\mathcal{V}_i$ (resp. $\mathcal{V}_{j,l}$) an irreducible orthogonal G-representation with the trivial S^1-action (resp. the G-action defined by (48)). Then*

(a) $\mathrm{Deg}_{\,\mathcal{V}_i} = \deg_{\mathcal{V}_i} + T_*$;
(b) $\mathrm{Deg}_{\,\mathcal{V}_{j,l}} = (G) - \deg_{\mathcal{V}_{j,l}} + T_*$,

where $\deg_{\mathcal{V}_i}$ (resp. $\deg_{\mathcal{V}_{j,l}}$) is given by (49) (resp. by (50)–(52)) and $T_* \in A^*(G)$ (see Subsection 3.3).

PROOF. (a) This formula follows directly from the construction of G-gradient degree. Indeed, assume

$$\mathrm{Deg}_{\,\mathcal{V}_i} := \nabla_G\text{-}\deg(-\mathrm{Id}\,, B_i) = \sum_{(L) \in \Phi(G)} n_L\,(L) \quad \text{and} \quad \deg_{\mathcal{V}_i} = \sum_{(K) \in \Phi_0(G)} m_K\,(K).$$

Since every ∇_G-generic approximation of $-\mathrm{Id}$ is regular normal (cf. [**1**]), one can easily observe that for $(K) \in \Phi_0(G)$, one has $n_K = m_K$.

(b) This statement is a consequence of Proposition 6.13. Indeed, let

$$\deg_{\mathcal{V}_{j,l}} = \sum_{(R) \in \Phi_1^t(G)} m_R\,(R) \text{ and } \mathrm{Deg}_{\,\mathcal{V}_{j,l}} := \nabla_G\text{-}\deg(-\mathrm{Id}\,, B_{j,l}) = \sum_{(L) \in \Phi(G)} n_L\,(L), \tag{53}$$

and put $V := V_{j,l}$. Since for $(L) \in \Phi_0(G)$, $V_{(L)} = \{0\}$ if $(L) = (G)$ and $V_{(L)} = \emptyset$ otherwise,

$$n_L = \begin{cases} 1 & \text{if} \quad (L) = (G), \\ 0 & \text{for all } (L) \in \Phi_0(G) \text{ such that } (L) \neq (G). \end{cases} \tag{54}$$

To compute the n_L-coefficients of $\mathrm{Deg}_{\,\mathcal{V}_{j,l}}$ for $(L) \in \Phi_1^t(G)$, observe that the map $-\mathrm{Id}$ is not S^1-normal on V. Take the function $\eta_\delta : \mathbb{R} \to \mathbb{R}$ given by

$$\eta_\delta(\rho) := \begin{cases} 0 & \text{if } \rho < \delta, \\ \frac{\rho - \delta}{\delta} & \text{if } \delta \le \rho \le 2\delta, \\ 1 & \text{if } \rho > 2\delta, \end{cases} \tag{55}$$

where $\delta > 0$ is chosen to be sufficiently small, and correct $-\mathrm{Id}$ to the S^1-normal map $f_o : V \to V$ by

$$f_o(v) := \eta_\delta(\|v\|)(-v) + (1 - \eta_\delta(\|v\|))v = 1 - 2\eta_\delta(\|v\|)v, \quad v \in V.$$

Next, define the map $\widetilde{f}_o : \mathbb{R} \oplus V \to V$ by formula (42). Combining a linear change of variables on V with homotopy and excision property of the twisted degree yields

$$\deg_{\mathcal{V}_{j,l}} = G\text{-Deg}^{\,t}(\widetilde{f}_o, U_\delta), \tag{56}$$

where U_δ is defined by (41).

Take $(L) \in \Phi_1^t(G)$ and put $\widetilde{f}_o^L := \widetilde{f}_o|_{V^L}$. Obviously, the primary degree

$$W(L)\text{-Deg}\,(\widetilde{f}_o^L, U_\delta^L) = \sum_{(K)\in\Phi_1^+(W(L))} \hat{m}_K\,(K) \tag{57}$$

is correctly defined (cf. (46)). Then (cf. (53) and (56)), Proposition 4.4 from [**1**] yields

$$m_L = \hat{m}_{\mathbb{Z}_1}, \tag{58}$$

where $\mathbb{Z}_1 := \{e\}$ and "e" stands for the identity element in $W(L)$.

On the other hand, consider the $W(L)$-equivariant map $-\text{Id}\,|_{V^L}$. By identifying S^1 with the connected component of e in $W(L)$, the above construction utilizing (55) can be applied to the map $-\text{Id}\,|_{V^L}$, i.e., put

$$f_*^L(v) := \eta_\delta(\|v\|)(-v) + (1 - \eta_\delta(\|v\|))v = 1 - 2\eta_\delta(\|v\|)v, \quad v \in V^L,$$

and define $\widetilde{f}_*^L : \mathbb{R} \oplus V^L \to V^L$ by

$$\widetilde{f}_*^L(t, v) := f_*^L(v) + t\tau(v) \qquad (v \in V^L).$$

Then $\widetilde{f}_o^L$ and $\widetilde{f}_*^L$ are homotopic by a U_δ^L-admissible homotopy and

$$W(L)\text{-Deg}\,(\widetilde{f}_o^L, U_\delta^L) = W(L)\text{-Deg}\,(\widetilde{f}_*^L, U_\delta^L).$$

Therefore, by Proposition 6.13, $\hat{m}_{\mathbb{Z}_1} = -n_L$ and (see (58))

$$m_L = -n_L. \tag{59}$$

By combining (54) and (59), the conclusion follows. □

Example 6.15. As the simplest example illustrating Theorem 6.14, we compute the gradient basic degrees in the case of n-dimensional torus $G = T^n$ $(n \geq 1)$. Take an irreducible T^n-representation and denote it by $\mathcal{V}_o$. If $\mathcal{V}_o$ is the trivial (one-dimensional) representation, then Theorem 6.14(a) together with formula (5.15) from [**1**] imply $\text{Deg}_{\,\mathcal{V}_o} = -(T^n)$. If $\mathcal{V}_o$ is non-trivial, then $\dim \mathcal{V}_o = 2$ and there are precisely two orbit types (T^n) and $(H) = (\mathbb{Z}_k \times T^{n-1})$ in $\mathcal{V}_o$ (for some subgroup $\mathbb{Z}_k$ depending on $\mathcal{V}_o$). Combining Theorem 6.14(b) and the Functoriality property with formula (4.20) from [**1**] yields

$$\text{Deg}_{\,\mathcal{V}_o} = (T^n) - (H).$$

Remark 6.16. The computations of G-gradient basic degrees can be effectively completed by using the Functoriality property of the gradient degree for the homomorphism $T^n \hookrightarrow G$ (i.e., the induced by it Euler ring homomorphism $\Psi : U(G) \to U(T^n)$), formula (35), Theorem 6.14, and the known basic gradient degrees for irreducible T^n-representations, which are used to establish relations between the unknown coefficients and the values of the gradient basic degrees in a form of simple linear equations.

EXAMPLE 6.17. Our next computations are related to basic gradient degrees for $G = O(2) \times S^1$.

Following [**1**, Subsection 5.7.6], denote by $\mathcal{V}_0 \simeq \mathbb{R}$ the trivial representation of $O(2)$, by $\mathcal{V}_{\frac{1}{2}} \simeq \mathbb{R}$ the one-dimensional irreducible real representation, where $O(2)$ acts on $\mathbb{R}$ through the homomorphism $O(2) \to O(2)/SO(2) \simeq \mathbb{Z}_2$, and by $\mathcal{V}_m \simeq \mathbb{C}$, $m = 1, 2, \dots$, the two-dimensional irreducible real representation of $O(2)$, where the action of $O(2)$ is given by

(i) $uz = u^m \cdot z$, for $u \in SO(2)$ and $z \in \mathcal{V}_m$ (here "$\cdot$" stands for complex multiplication);

(ii) $\kappa \cdot z = \bar{z}$.

It is well-known that the above list of irreducible $O(2)$-representations is complete. Observe that all the orbit types occurring in irreducible $O(2)$-representations belong to $\Phi_0(O(2))$. Therefore, combining Theorem 6.14(a) with the results for $\deg_{\mathcal{V}_i}$ obtained in [**1**, Subsection 5.7.6], yield:

$$\operatorname{Deg}_{\mathcal{V}_0} = -(O(2)), \quad \operatorname{Deg}_{\mathcal{V}_{\frac{1}{2}}} = (O(2)) - (SO(2))$$
$$\operatorname{Deg}_{\mathcal{V}_i} = (O(2)) - (D_i), \quad i = 1, 2, 3, \dots .$$

Observe that all the irreducible representations of $O(2)$ are of real type. Therefore, the irreducible representations of $O(2) \times S^1$ can be obtained by taking complexifications of the representations $\mathcal{V}_i$ and applying formula (48) (see [**1**, Subsection 5.7.6], for details). We need to compute $\operatorname{Deg}_{\mathcal{V}_{j,l}}$, $j = 0, \frac{1}{2}, 1, 2, 3, \dots$, $l \in \mathbb{N}$.

Clearly, $\mathcal{V}_{0,l}$ (resp. $\mathcal{V}_{\frac{1}{2},l}$), contains precisely two orbit types: $(O(2) \times S^1)$ and $(O(2) \times \mathbb{Z}_l)$ (resp. $(O(2) \times S^1)$ and $(O(2)^{-,l})$. Combining Theorem 6.14(b) with the corresponding results from [**1**, Subsection 5.7.6], yields

$$\operatorname{Deg}_{\mathcal{V}_{0,l}} = (O(2) \times S^1) - (O(2) \times \mathbb{Z}_l), \qquad \operatorname{Deg}_{\mathcal{V}_{\frac{1}{2},l}} = (O(2) \times S^1) - (O(2)^{-,l}).$$

Next, each $\mathcal{V}_{j,l}$, $j = 1, 2, \dots$, $l \in \mathbb{N}$, contains the following orbit types: $(O(2) \times S^1) \in \Phi_0(G)$; $(SO(2)^{\varphi_j,l})$, $(D^{d_{2j},l}) \in \Phi_1^t(G)$; $(\mathbb{Z}^{d_{2j},l}) \in A^*(G)$. Combining Theorem 6.14(b) with the corresponding results from [**1**, Subsection 5.7.6], yields

$$\operatorname{Deg}_{\mathcal{V}_{j,l}} = (O(2) \times S^1) - (SO(2)^{\varphi_j,l}) - (D_{2j}^{d,l}) + k \cdot (\mathbb{Z}_{2j}^{d,l}).$$

To compute k, we use the ring homomorphism $\Psi : U(O(2) \times S^1) \to U(T^2)$.

The irreducible T^2-representations ($T^2 = SO(2) \times S^1$) are obtained from the complex $SO(2)$-irreducible representations $\mathcal{U}_j$, $j = 0, \pm 1, \pm 2, \dots$, by defining the S^1-action on $\mathcal{U}_j$ by l-folding. In order to avoid confusion, we denote these T^2-irreducible representations by $\mathcal{V}_{(j,l)}$, $j = 0, \pm 1, \pm 2, \dots$, $l = 1, 2, \dots$. It is easy to compute the T^2-basic gradient degrees for these representations, namely,

$$\operatorname{Deg}_{\mathcal{V}_{(j,l)}} := \nabla_{T^2}\text{-deg}(-\mathrm{Id}, B_{(j,l)}) = (T^2) - (SO(2)^{\varphi_j,l}),$$

where $B_{(j,l)}$ stands for the unit ball in $\mathcal{V}_{(j,l)}$. Observe that $\mathcal{V}_{j,l}$, considered as a T^2-representation, is equivalent to the sum $\mathcal{V}_{(j,l)} \oplus \mathcal{V}_{(-j,l)}$ and (cf. Example 4.6)

$$\Psi(O(2) \times S^1) = (SO(2) \times S^1), \qquad \Psi(D_{2j}^{d,l}) = (\mathbb{Z}_{2j}^{d,l}),$$
$$\Psi(SO(2)^{\varphi_j,l}) = (SO(2)^{\varphi_j,l}) + (SO(2)^{\varphi_{-j},l}), \quad \Psi(\mathbb{Z}_{2j}^{d,l}) = 2(\mathbb{Z}_{2j}^{d,l}).$$

By applying the Functoriality property and (35),

$$\Psi(\operatorname{Deg}_{\mathcal{V}_{j,l}}) = \operatorname{Deg}_{\mathcal{V}_{(j,l)}} * \operatorname{Deg}_{\mathcal{V}_{(-j,l)}},$$

where

$$\Psi(\mathrm{Deg}_{\mathcal{V}_{j,l}}) = (T^2) - (SO(2)^{\varphi_j,l}) - (SO(2)^{\varphi_{-j},l}) - (\mathbb{Z}_{2j}^{d,l}) + 2k(\mathbb{Z}_{2j}^{d,l}) \tag{60}$$

and

$$\mathrm{Deg}_{\mathcal{V}_{(j,l)}} * \mathrm{Deg}_{\mathcal{V}_{(-j,l)}} = (T^2) - (SO(2)^{\varphi_j,l}) - (SO(2)^{\varphi_{-j},l}) + (\mathbb{Z}_{2j}^{d,l}). \tag{61}$$

By combining (60) with (61), we obtain that $k = 1$, thus

$$\mathrm{Deg}_{\,\mathcal{V}_{j,l}} = (O(2) \times S^1) - (SO(2)^{\varphi_j,l}) - (D_{2j}^{d,l}) + (\mathbb{Z}_{2j}^{d,l}). \tag{62}$$

7. Application: Periodic-Dirichlet Mixed Boundary Value Problem for an Elliptic Asymptotically Linear Equation with $O(2)$-Symmetries

7.1. Statement of the problem. Suppose that $\mathcal{O} \subset \mathbb{R}^2 \simeq \mathbb{C}$ is the unit disc and take $\Omega := (0, 2\pi) \times \mathcal{O}$. Consider the following elliptic periodic-Dirichlet BVP

$$\begin{cases} -\frac{\partial^2 u}{\partial t^2} - \triangle_x u(t,x) = f(u(t,x)), \;\; (t,x) \in \Omega, \\ u(t,x) = 0 \quad \text{a.e. for} \quad x \in \partial\mathcal{O}, \; t \in (0,2\pi), \\ u(0,x) = u(2\pi,x) \quad \text{a.e. for} \quad x \in \mathcal{O}, \\ \frac{\partial u}{\partial t}(0,x) = \frac{\partial u}{\partial t}(2\pi,x) \quad \text{a.e. for} \quad x \in \mathcal{O}, \end{cases} \tag{63}$$

where $(t,x) \in (0,2\pi) \times \mathcal{O}$, $u \in H^2(\Omega;\mathbb{R})$. A solution to (63), which is not constant with respect to the t-variable, will be called a *non-stationary periodic solution* to (63).

We assume that $f : \mathbb{R} \to \mathbb{R}$ is a C^1-function satisfying the conditions:

(B1) $f(0) = 0$ and $f'(0) = a > 0$;

(B2) f is asymptotically linear at infinity, i.e., there exists $b \in \mathbb{R}$ such that

$$\lim_{|t|\to\infty} \frac{f(t) - bt}{t} = 0; \tag{64}$$

(B3) there are $2 < p < 6$ and α, $\beta > 0$ such that

$$|f'(t)| \le \alpha + \beta |t|^{p-2}, \qquad \text{for all} \;\; t \in \mathbb{R}. \tag{65}$$

Put $F(t) := \int_0^t f(\tau)d\tau$. Notice that under the assumptions (B2) and (B3), there exist α_i, $\beta_i > 0$, $i = 0,1,2$, such that the function $F : \mathbb{R} \to \mathbb{R}$ satisfies the conditions

$$|F^{(i)}(t)| \le \alpha_i + \beta_i |t|^{p-i}, \quad t \in \mathbb{R}. \tag{66}$$

Indeed, since f is asymptotically linear (cf. (B2)), we have that for some α_1' and $\beta_1' > 0$, $|F'(t)| = |f(t)| \le \alpha_1' + \beta_1'|t|$ for all $t \in \mathbb{R}$; thus clearly there exist α_1 and β_1 such that $|f(t)| \le \alpha_1 + \beta_1 |t|^{p-1}$. On the other hand, there exist α_0, $\beta_0 > 0$ such that

$$|F(t)| = \left| \int_0^t f(\tau)d\tau \right| \le \alpha_1'|t| + \frac{1}{2}\beta_1'|t|^2 \le \alpha_0 + \beta_0 |t|^p.$$

Consider the Laplace operator $-\triangle_x$ on $\mathcal{O}$ with the Dirichlet boundary condition. Then the operator $-\triangle_x$ has the spectrum

$$\sigma(-\triangle_x) := \{\mu_{k,j} : \mu_{k,j} = z_{k,j}^2, \;\; k = 1,2,\ldots, \;\; j = 0,1,2,\ldots, J_j(z_{k,j}) = 0\},$$

where $z_{k,j}$ denotes the k-th zero of the j-th Bessel function J_j. The corresponding to $\mu_{j,k}$ eigenfunctions (expressed in polar coordinates) are

$$\text{for } j=0,\quad \varphi_{k,0}(r) := J_0(\sqrt{\mu_{k,0}}r),$$
$$\text{for } j>0,\quad \varphi^c_{k,j}(r,\theta) := J_j(\sqrt{\mu_{k,j}}r)\cos(j\theta),\quad \varphi^s_{k,j}(r,\theta) := J_j(\sqrt{\mu_{k,j}}r)\sin(j\theta).$$

The space $\operatorname{span}\{\varphi^c_{k,j}, \varphi^s_{k,j}\}$ is equivalent to the j-th irreducible $O(2)$-representation $\mathcal{V}_j$ $(j>0)$, and the space $\operatorname{span}\{\varphi_{k,0}\}$ is equivalent to the trivial irreducible $O(2)$-representation $\mathcal{V}_0$. We need additional assumptions:

(B4) $a,\, b \notin \{l^2+\mu_{k,l},\ \mu_{k,j} \in \sigma(-\triangle_x),\ l=0,1,2,\dots\}$.

(B5) The system

$$\begin{cases} -\triangle_x u = f(u) \\ u|_{\partial\mathcal{O}} = 0 \end{cases} \tag{67}$$

has a unique solution $u \equiv 0$.

Remark 7.1. Condition (B1) assures the existence of the (stationary) zero solution and (B2) reflects the asymptotically linear character of the problem in question. Condition (B3) is the standard one required to assure that the associated functional (68) on the Sobolev space is twice differentiable (see conditions (66), cf. [**28**]). The non-resonance condition (B4) is imposed to simplify the computations (see also [**16, 15**], where degenerate systems are discussed). Finally, condition (B5) prevents the interaction between stationary and non-stationary periodic solutions, allowing the equivariant gradient degree to detect the existence of non-stationary periodic solutions. This condition is discussed in detail in Subsection 7.3.

7.2. Setting in functional spaces. By using the standard identification $\mathbb{R}/2\pi \simeq S^1$, assume that $\Omega := S^1\times\mathcal{O}$ with $\partial\Omega = S^1\times S^1$. Put $W := H^1_0(\Omega) := \{u \in H^1(\Omega;\mathbb{R}) : u|_{\partial\Omega}\equiv 0\}$, which is a Hilbert G-representation for $G=O(2)\times S^1$, with the inner product

$$\langle u,v\rangle := \int_\Omega \nabla u(y)\bullet\nabla v(y)\,dy.$$

Associate to the problem (63) the functional $\Psi : W\to\mathbb{R}$ given by

$$\Psi(u) := \frac{1}{2}\int_\Omega |\nabla u(y)|^2 dy - J(u), \tag{68}$$

where $J : W\to\mathbb{R}$ is given by

$$J(u) := \int_\Omega F(u(y))dy.$$

By conditions (66), J is of class C^2 and for $h\in W$,

$$DJ(u)h = \int_\Omega f(u(y))h(y)dy.$$

Thus, Ψ is also C^2-differentiable with respect to u and

$$D\Psi(u)h = \int_\Omega \nabla u(y)\nabla h(y)dy - DJ(u)h,\quad h\in W.$$

Consequently, by the standard argument, if $D\Psi(u) \equiv 0$, then u is a solution to (63). In particular,

$$\nabla\Psi(u) = 0 \iff u \text{ is a solution to (63)},$$

where

$$\nabla\Psi(u) = u - \nabla J(u). \tag{69}$$

Consider the following operators:

$$j : H_0^1(\Omega) \hookrightarrow L^p(\Omega), \quad j(u) = u,$$
$$N_f : L^p(\Omega) \to L^{\frac{p}{p-1}}(\Omega), \quad N_f(u)(y) = f(u(y)),$$

where j is a compact operator. Then

$$\nabla J(u) = R \circ N_f \circ j(u),$$

where $R : L^{\frac{p}{p-1}}(\Omega) \to H_0^1(\Omega)$ is defined by $R := \iota \circ \tau$ with $\iota : \big(H_0^1(\Omega)\big)^* \to H_0^1(\Omega)$ which is the isomorphism given by the Riesz representation theorem, and $\tau : L^{\frac{p}{p-1}}(\Omega) \to \big(H_0^1(\Omega)\big)^*$ the (continuous) map defined by

$$\tau(\psi)(v) := \int_\Omega \psi(x)v(x)dx, \quad \psi \in L^{\frac{p}{p-1}}(\Omega), \quad v \in H_0^1(\Omega).$$

In other words, R is the inverse of the Laplacian $-\triangle$. Therefore (cf. (69)),

$$\mathfrak{F}(u) := \nabla\Psi(u) = u - R \circ N_f \circ j(u), \quad u \in W,$$

is a completely continuous $O(2) \times S^1$-equivariant gradient field on W, and the problem (63) is equivalent to the equation

$$\mathfrak{F}(u) = 0. \tag{70}$$

7.3. Example of a function f satisfying (B1)–(B5). It is very easy to construct a function f satisfying (B1)-(B4). In order for f to satisfy (B5), we will "play" with the constants a and b (see (B1) and (B2)). To this end, observe that a functional setting similar to the one presented in Subsection 7.2, can also be established for the boundary problem (67). Namely, we can reformulate it as the equation

$$\mathfrak{F}_x(u) = 0, \quad u \in H_0^1(\mathcal{O}), \tag{71}$$

where

$$\mathfrak{F}_x(u) := \nabla\Psi_x(u) = u - R_x \circ N_f \circ j(u),$$

with R_x the inverse of the Laplacian $-\triangle_x$. Suppose that

(B5′) $0 < a < b$ and $[a, b] \cap \sigma(-\triangle_x) = \emptyset$.

Denote simply by μ_m, $m = 1, 2, \dots$, the elements of $\sigma(-\triangle_x)$, $\mu_1 < \mu_2 < \dots$. Assume, in addition, that

(B5″) $\eta := \sup\{|f'(t)| : t \in \mathbb{R}\} < \dfrac{1}{2} \inf\left\{\dfrac{\mu_m\mu_1}{|\mu_m - b|} : m = 1, 2, \dots\right\}$.

Clearly, condition (B2) implies $b \leq \eta$.

Proposition 7.2. *Let f be a C^1-function satisfying conditions (B1)–(B4), (B5′) and (B5″). Then the boundary value problem (67) has a unique solution $u \equiv 0$.*

PROOF. We claim that the derivative $D\mathfrak{F}_x(u) : H_0^1(\mathcal{O}) \to H_0^1(\mathcal{O})$ is an isomorphism for all $u \in H_0^1(\mathcal{O})$, and $D\mathfrak{F}_x(u)$ can be connected by a continuous path to $D\mathfrak{F}_x(\infty) := \text{Id } - bR_x \circ j$ in $GL_c(H_0^1(\mathcal{O}))$. To this end, observe that

$$D\mathfrak{F}_x(u)(v) = v - bR_x j(v) - R_x[N_{f'}(u)j(v) - bj(v)],$$

and put

$$A := \text{Id } - bR_x \circ j, \quad B := R_x[N_{f'}(u) \cdot j - b \cdot j].$$

Since $|f'(t)|$ is bounded, the Nemitsky operator $N_f : L^2(\mathcal{O}) \to L^2(\mathcal{O})$ is Gâteaux differentiable (cf. [**25**]) and its (Gâteaux) derivative is $\mathcal{D}N_f(u)(v) = f'(u) \cdot v$. Consider the embedding $j' : H_0^1(\mathcal{O}) \hookrightarrow L^2(\mathcal{O})$ and the inverse Laplacian $R'_x : L^2(\mathcal{O}) \to H_0^1(\mathcal{O})$. Then $\|j'\| = \frac{1}{\sqrt{\mu_1}}$, $\|R'_x\| = \frac{1}{\sqrt{\mu_1}}$ and $\|(f'(u) - b) \cdot \text{Id }\|_{L^2} \leq 2\eta$. Therefore,

$$\|Bv\| \leq \|R'_x\| \|(f'(u) - b) \cdot \text{Id }\|_{L^2} \|j'\| \|v\| \leq 2\eta \|R'_x\| \|j'\| \|v\| = \frac{2\eta}{\mu_1} \|v\|.$$

Since (by (B5$'$)) A is invertible with $\|A^{-1}\| = \sup\{\frac{|\mu_m - b|}{\mu_m} : m = 1, 2, \dots\}$ and, by (B5$''$),

$$\|A^{-1}B\| \leq \|A^{-1}\| \|B\| \leq \sup\left\{\frac{|\mu_m - b|}{\mu_m} : m = 1, 2, \dots\right\} \frac{2\eta}{\mu_1} < 1,$$

the linear operator

$$A - \lambda B = A(\text{Id } - \lambda A^{-1}B), \quad \text{for all } \lambda \in [0, 1]$$

is invertible. Consequently, $D\mathfrak{F}_x(u) = A - B$ is also invertible and by the homotopy property of the Leray-Schauder degree, we obtain

$$\deg(D\mathfrak{F}_x(u), B_1(0)) = \deg(A, B_1(0)), \quad \text{for all } u \in H_0^1(\mathcal{O}), \tag{72}$$

where $B_r(0)$ stands for the open ball at 0 of radius r. Therefore, every solution $u \in H_0^1(\mathcal{O})$ to the problem (71) is a regular point of $\mathfrak{F}_x$. Consequently, each solution to (71) is isolated. Since $D\mathfrak{F}_x(\infty) : H_0^1(\mathcal{O}) \to H_0^1(\mathcal{O})$ is an isomorphism, there can only be finitely many solutions to the equation (71), and for every solution u, the Leray-Schauder degree $\deg(\mathfrak{F}_x, B_u)$ is well-defined. On an isolating neighborhood B_u of u, by using the linearization of $\mathfrak{F}_x$ on B_u and (72),

$$\deg(\mathfrak{F}_x, B_u) = \deg(D\mathfrak{F}_x(0), B_1(0)) = \deg(D\mathfrak{F}_x(\infty), B_1(0)) \neq 0.$$

However, this implies (by additivity property of the Leray-Schauder degree) that there can be no solution u to (71) (and consequently to (67)) other than $u \equiv 0$. □

REMARK 7.3. By choosing a and b to satisfy (B5$'$) and (B5$''$), one can easily verify that the function

$$f(u) = bu - (b - a)\frac{u}{1 + u^2}, \quad u \in \mathbb{R}$$

satisfies conditions (B1)–(B5).

7.4. Equivariant invariant and the isotypical decomposition of W. By conditions (B1), (B2) and (B4), there exist R, $\varepsilon > 0$ such that $u = 0$ is the only solution to equation (70) in $\overline{B_\varepsilon(0)} \subset W$, and (70) has no solutions $u \in W$ such that $\|u\| \geq R$. Define the *equivariant invariant* ω for the problem (63) by

$$\omega := \deg_0 - \deg_\infty, \tag{73}$$

where

$$\deg_0 := \nabla_{O(2)\times S^1}\text{-deg}\,(\mathfrak{F}, B_\varepsilon(0)), \quad \deg_\infty := \nabla_{O(2)\times S^1}\text{-deg}\,(\mathfrak{F}, B_R(0)).$$

The spectrum σ of $-\triangle$ on Ω (with the boundary conditions listed in (63)) is

$$\sigma = \{\lambda_{k,j,l} : \lambda_{k,j,l} := l^2 + \mu_{k,j}, \ \ \mu_{k,j} \in \sigma(-\triangle_x), \ l = 0, 1, 2, \dots\}.$$

Denote by $E_{k,j,l}$ the eigenspace of $-\triangle$ in W corresponding to the eigenvalue $\lambda_{k,j,l}$. Observe that for j, $l > 0$,

$$E_{k,j,l} = \operatorname{span}\{\cos lt \cdot \varphi^c_{k,l}(x), \cos lt \cdot \varphi^s_{k,j}(x), \sin lt \cdot \varphi^c_{k,j}(x), \sin lt \cdot \varphi^s_{k,j}(x)\},$$

and $E_{k,j,l}$ is equivalent to the irreducible orthogonal $O(2) \times S^1$-representation $\mathcal{V}_{j,l}$. If $j = 0$ and $l > 0$, then

$$E_{k,0,l} = \operatorname{span}\{\cos lt \cdot \varphi_{k,0}(x), \sin lt \cdot \varphi_{k,0}(x)\},$$

and it is equivalent to the irreducible orthogonal $O(2) \times S^1$-representation $\mathcal{V}_{0,l}$. If $j > 0$ and $l = 0$,

$$E_{k,j,0} = \operatorname{span}\{\varphi^c_{k,j}(x), \varphi^s_{k,j}(x)\},$$

is equivalent to the j-th irreducible $O(2)$-representation $\mathcal{V}_j$. For $j = l = 0$, we have that

$$E_{k,0,0} = \operatorname{span}\{\varphi_{k,0}(x)\},$$

is equivalent to the trivial $O(2) \times S^1$-representation $\mathcal{V}_{0,0}$. The $O(2) \times S^1$-isotypical components of the space W are

$$W_{j,l} := \overline{\bigoplus_k E_{k,j,l}}, \quad j, l = 0, 1, 2, \dots$$

7.5. Computation of the equivariant invariant. Assume, in addition to conditions (B1)–(B5) that

(B6) $0 < a < b$ and there exists (k_o, j_o, l_o), $l_o \geq 1$, such that

$$\sigma(-\triangle) \cap (a, b) = \{\lambda_{k_o,j_o,l_o}\}.$$

Put $p = 0$ or ∞ and denote by σ^-_p the negative spectrum of $D\mathfrak{F}(p)$, i.e.,

$$\sigma^-_p := \{\lambda \in \sigma(D\mathfrak{F}(p)) : \lambda < 0\}.$$

By assumption (B6),

$$\sigma^-_\infty = \sigma^-_0 \cup \{\lambda_o\}, \quad \lambda_o := \lambda_{k_o,j_o,l_o} \tag{74}$$

The linear operator $D\mathfrak{F}(p)$ is G-homotopic (in the class of gradient maps) to

$$A_p = (-\mathrm{Id}\,) \times \mathrm{Id}\, : E_p \oplus E_p^\perp \to E_p \oplus E_p^\perp, \quad E_p := \bigoplus_{\lambda_{k,j,l} \in \sigma^-_p} E_{k,j,l},$$

and consequently (cf. (35) and (62)),

$$\begin{aligned}\deg_p = \nabla_G\text{-deg}\,(A_p, B_1(0)) &= \prod_{\lambda_{k,j,l} \in \sigma^-_p} \nabla_G\text{-deg}\,(-\mathrm{Id}\,, B_1(E_{k,j,l}))\\ &= \prod_{\lambda_{k,j,l} \in \sigma^-_p} \mathrm{Deg}_{\mathcal{V}_{j,l}}.\end{aligned}$$

Therefore (by (74)),

$$\omega = \deg_0 - \deg_\infty = \prod_{\lambda_{k,j,l}\in\sigma_0^-} \mathrm{Deg}_{\mathcal{V}_{j,l}} * \Big((G) - \mathrm{Deg}_{\mathcal{V}_{j_o,l_o}}\Big)$$
$$= \prod_{\lambda_{k,j,l}\in\sigma_0^-} \mathrm{Deg}_{\mathcal{V}_{j,l}} * \Big((SO(2)^{\varphi_{j_o},l_o}) + (D_{2j_o}^{d,l_o}) - (\mathbb{Z}_{2j_o}^{d,l_o})\Big).$$

Notice that the element $\mathfrak{a} := \prod_{\lambda_{k,j,l}\in\sigma_0^-} \mathrm{Deg}_{\mathcal{V}_{j,l}}$ is invertible (cf. [**19**]), therefore $\omega \neq 0$. Moreover, by using the multiplication table for $U(O(2)\times S^1)$ and the list of basic gradient degrees for irreducible $O(2)\times S^1$-representations, one can easily conclude that

$$\mathfrak{a} * (SO(2)^{\varphi_{j_o},l_o}) = (SO(2)^{\varphi_{j_o},l_o}) + x^*, \qquad \text{and} \qquad \mathfrak{a} * (D_{2j_o}^{d,l_o}) = (D_{2j_o}^{d,l_o}) + y^*,$$

where x^* and y^* denote the terms in $U(G)$, which do not contain $(SO(2)^{\varphi_{j_o},l_o})$ and $(D_{2j_o}^{d,l_o})$.

Consequently, we can formulate the following existence result.

THEOREM 7.4. *Under the assumptions (B1)–(B6), equation (63) has at least two $O(2)\times S^1$-orbits of non-stationary periodic solutions with the orbit types at least $(SO(2)^{\varphi_{j_o},l_o})$ and $(D_{2j_o}^{d,l_o})$, respectively.*

Let us point out that the periodic solutions corresponding to the orbit types $(SO(2)^j)$ are commonly called *rotating waves* or *spiral vortices* while those with the orbit type (D_{2j}^d) are called *ribbons* or *stationary waves*. Therefore, it seems appropriate to call the periodic solutions with the orbit type $(SO(2)^{\varphi_{j_o},l_o})$ the l_o-folded *rotating waves* or *spiral vortices* and those with the orbit type $(D_{2j_o}^{d,l_o})$ the l_o-folded *ribbons* or *stationary waves.*

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