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EXISTENCE OF NONSTATIONARY PERIODIC SOLUTIONS FOR Γ-SYMMETRIC LOTKA-VOLTERRA TYPE SYSTEMS

Norimichi Hirano

Department of Mathematics
Yokohama National University
Tokiwadai, Hodogayaku, Yokohama, Japan

Wieslaw Krawcewicz and Haibo Ruan

Department of Mathematical Sciences
University of Texas at Dallas
Richardson, TX 75080, USA
and
Department of Mathematics
Universität Hamburg
20146 Hamburg, Germany

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Abstract. In this paper we present a general framework for applications of the twisted equivariant degree (with one free parameter) to an autonomous Γ-symmetric system of functional differential equations in order to detect and classify (according to their symmetric properties) its periodic solutions. As an example we establish the existence of multiple non-constant periodic solutions of delay Lotka-Volterra equations with Γ-symmetries. We also include some computational examples for several finite groups Γ.

1. **Introduction.**

Many natural phenomena exhibit certain symmetric properties which are reflected in group symmetries of the corresponding mathematical models and expressed as equivariant dynamical systems. The investigation of the impact of symmetries in dynamical systems on symmetric properties of the actual dynamics is a difficult problem. There is a large variety of effective topological methods and techniques widely used to study symmetric variational problems (see [3] and references therein). However, in the case of symmetric non-variational differential equations, there are only a few topological methods that are traditionally used in order to find periodic solutions (for example, see [8]).

During the last 15 years, the Equivariant Degree Theory was developed and took a firm position among the main topological tools in equivariant nonlinear analysis (cf. [2, 6, 7, 13]). This method, in contrast to the traditional ones, can

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be computerized and applied in a standard way to different kinds of nonlinear problems admitting symmetry groups of arbitrarily large size. The equivariant degree method also proved its effectiveness even in non-symmetric problems (cf. [4, 5, 2]); the classical Leray-Schauder degree is ineffective to detect periodic solutions of autonomous dynamical systems due to their S^1-equivariant nature. The information produced by this method provides a topological classification of solutions according to their symmetry properties (cf. [2]).

The important feature of the equivariant degree is that it can be used systematically in a setting where the Leray-Schauder degree is typically applied. The only difference is that it includes an additional parameter, which is related to the unknown period of periodic solutions. In this paper, we follow the original idea by Hirano and Rybicki (cf. [5]), who applied the S^1-equivariant degree to a non-symmetric system of Lotka-Volterra equations with delay.

The Lotka-Volterra equations with delay play an important role in population dynamics. They can be traced to a model of a single population—the so-called Verhulst logistic equation (cf. [12, 11])

$$\dot{v} = \alpha v \left(1 - \frac{v}{K}\right), \tag{1}$$

describing an exponentially growing population at low densities and saturating towards the carrying capacity K (of resources) at high densities, and to the system

$$\begin{cases} \dot{x} = x(\alpha - \beta y), \\ \dot{y} = -y(\gamma - \delta x), \end{cases} \tag{2}$$

describing the predator-prey model—the so-called *Lotka-Voltera equations*. In this system, which was proposed independently by Alfred J. Lotka (1925) and Vito Volterra (1926), $x = x(t)$ stands for the prey density, $y = y(t)$ is the predator density, α is the intristic growth of prey, γ is the diminishing rate of predator, and β and δ reflect predation impacts. However, the system (2) is unrealistic since in absence of predators, the prey population grows exponentially. These equations can be easily adjusted by including the competition within the prey species (i.e. by assuming logistic growth of prey: $\dot{x} = x(\alpha - \eta x)$). One can also assume competition within the predators. The modified equation becomes

$$\begin{cases} \dot{x} = x(\alpha - \eta x - \beta y), \\ \dot{y} = y(-\gamma + \delta x - \mu y). \end{cases} \tag{3}$$

Suppose there are n species $v_1, v_2, \dots, v_n$ interacting with each other (e.g. predation, competition, symbiosis, cooperation, etc.). The *Lotka-Volterra Model* for this multi-species ecosystem is described by the system

$$\begin{cases} \dot{v}_1 = v_1(r_1 - a_{11}v_1 - a_{12}v_2 - \cdots - a_{1n}v_n), \\ \dot{v}_2 = v_2(r_2 - a_{21}v_1 - a_{22}v_2 - \cdots - a_{2n}v_n), \\ \vdots \qquad\qquad \vdots \\ \dot{v}_n = v_n(r_n - a_{n1}v_1 - a_{n2}v_2 - \cdots - a_{nn}v_n). \end{cases} \tag{4}$$

If we use the coordinate-wise vector-multiplication $x \cdot y = (x_1y_1, \dots, x_ny_n)^T$, for $x = (x_1, \dots, x_n)^T$, $y = (y_1, \dots, y_n)^T$, then (4) can be written as

$$\dot{v}(t) = v(t) \cdot (r - Av(t)), \quad r := (r_1, \dots, r_n)^T, \tag{5}$$

where

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \tag{6}$$

is called the *community matrix*. Here, a_{jj} describes the self-inhibiting nature of the j-th species, and $a_{ij} < 0$ (resp. $a_{ij} > 0$) is the competing (resp. cooperating) coefficient between species i and j. In what follows, we will assume that $a_{ij} = a_{ji}$.

By taking into account a delayed response to the remaining resources, George E. Hutchinson derived from (1) the following model for a single species

$$\dot{v}(t) = \alpha v(t) \left(1 - \frac{v(t-\tau)}{K}\right), \tag{7}$$

where $\tau > 0$ is a presumed delay constant. Therefore, for n interacting species v_1, v_2, $\dots$, v_n, by introducing to (5) a delay $\tau > 0$, one obtains the following *Lotka-Volterra model with delay* for a multi-species ecosystem

$$\begin{cases} \dot{v}_1(t) = v_1(t)(r_1 - a_{11}v_1(t-\tau) - \cdots - a_{1n}v_n(t-\tau)), \\ \dot{v}_2(t) = v_2(t)(r_2 - a_{21}v_1(t-\tau) - \cdots - a_{2n}v_n(t-\tau)), \\ \vdots \qquad\qquad \vdots \\ \dot{v}_n(t) = v_n(t)(r_n - a_{n1}v_1(t-\tau) - \cdots - a_{nn}v_n(t-\tau)). \end{cases} \tag{8}$$

This system can be written as

$$\dot{v}(t) = v(t) \cdot (r - Av(t-\tau)), \quad r := (r_1, \dots, r_n)^T. \tag{9}$$

Following Richard Levins (cf. [14]), one can associate to (9) the loop diagram shown on Figure 1, representing the interacting species in this ecosystem. Notice that if the interacting communities are identical, then the system (9) is symmetric. Such

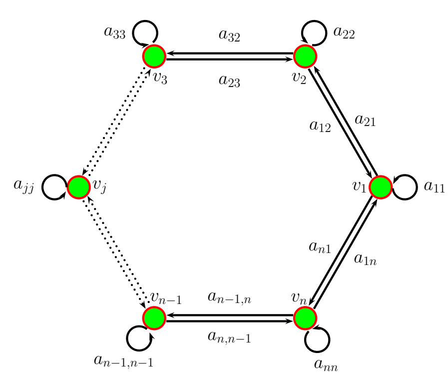

FIGURE 1. System with dihedral symmetries.

a community ecosystem can be described by the following equations,

$$\dot{v}(t) = \alpha A v(t) \cdot \left(\bar{1} - \frac{v(t-\tau)}{K} \right), \tag{10}$$

where $v(t) \in \mathbb{R}^n$ and $\bar{1} = (1, 1, \dots, 1)^T$. By applying the standard transformation

$$v(t) = K(1 + u(t)), \tag{11}$$

to the system (10), one obtains an equivalent system

$$\dot{u}(t) = -\alpha\, A u(t-\tau) \cdot \big(\bar{1} + u(t)\big), \tag{12}$$

where $u(t) = \frac{v(t)}{K} - 1$ represents a population saturation index with respect to the available resources. This model is our main example, for which we establish the existence of multiple symmetric periodic solutions.

The equation (12) is certainly complex and its analysis is very difficult in general. Establishing the existence of multiple periodic solutions exhibiting various symmetric properties in such a system, can be helpful in providing some explanations for complexity of its dynamics. This information can also be used to clarify the appearance of patterns in synchronized fluctuations of populations. It is hard to observe anything that is exactly symmetric in population ecology, although, when dealing with models of limited accuracy, one can assume that the considered populations are approximatively identical. This simplification allows us to exploit the symmetries, in order to better understand the dynamics of such systems.

2. **Twisted G-Equivariant Degree.** In this section we provide a short explanation of the *twisted equivariant degree* and its properties, which are used in this paper to establish existence and multiplicity results for non-constant periodic solutions of Γ-symmetric Lotka-Volterra equation. For more details, including the theoretical foundations of the equivariant degree, we refer to [2].

Notation. Consider the group $G := \Gamma \times S^1$ with Γ being a finite group and $S^1 := \{z \in \mathbb{C} : |z| = 1\}$. Assume that $\mathbb{H}$ stands for a Hilbert G-representation. For an element $x \in \mathbb{H}$, we denote by $G_x = \{g \in G : gx = x\}$ the *isotropy group* of x and by $G(x) := \{gx : g \in G\}$ the *orbit* of x. For a subgroup H of G, we denote by (H) the *conjugacy* class of H. Since for two elements x' and x'' in the same orbit $G(x)$, their isotropy groups $G_{x'}$ and $G_{x''}$ are conjugate, the conjugacy class (G_x) is called the *orbit type of* x. The set of all conjugacy classes (H) in G admits a partial order: $(H_1) \le (H_2)$ if and only if $H_1 \subset gH_2g^{-1}$ for some $g \in G$.

Twisted Subgroups of $\Gamma \times S^1$. Let $K \subset \Gamma$ be a subgroup, $l \in \mathbb{N}$, and $\varphi : K \to S^1$ a homomorphism. The subgroup $K^{\varphi,l} \subset \Gamma \times S^1$, given by

$$K^{\varphi,l} := \{(\gamma, z) \in K \times S^1 : \varphi(\gamma) = z^l\},$$

is called a *twisted* (by φ) l*-folded* (or simply *twisted*) subgroup of $\Gamma \times S^1$. For $l = 1$, we put $K^\varphi := K^{\varphi,1}$.

In the case $\mathbb{H} := H^1(S^1; V)$, where V is an orthogonal Γ-representation V (see Subsection 3.3), the isotropy group G_x of a non-constant 2π-periodic function $x \in \mathbb{H}$ is a twisted l-folded subgroup of $\Gamma \times S^1$.

Values of Twisted Degree. Put $\Phi_1(G) := \{(H) : H$ is a twisted subgroup of $G\}$ and let $A_1(G) := \mathbb{Z}[\Phi_1(G)]$ be a free $\mathbb{Z}$-module generated by the symbols in $\Phi_1(G)$. Suppose that $\mathfrak{f} : \mathbb{R} \oplus \mathbb{H} \to \mathbb{H}$ is a G-equivariant completely continuous field and $\Omega \subset \mathbb{R}\oplus\mathbb{H}$ an open bounded G-invariant set such that $\mathfrak{f}$ is Ω-*admissible*, i.e. $\mathfrak{f}(x) \neq 0$ for $x \in \partial\Omega$. Then, it is possible to assign to the pair $(\mathfrak{f}, \Omega)$ an element $G\text{-Deg}\,(\mathfrak{f}, \Omega) \in A_1(G)$, called *twisted G-equivariant degree.*

The twisted G-equivariant degree satisfies all the standard properties expected from a reasonable degree theory. More precisely, we have the following (cf. [2]):

Theorem 2.1. *There exists a (non-trivial) function $G\text{-Deg}$, associating to each pair $(\mathfrak{f}, \Omega)$, where $\mathfrak{f} : \mathbb{R} \oplus \mathbb{H} \to \mathbb{H}$ is an Ω-admissible completely continuous field, an element $G\text{-Deg}\,(\mathfrak{f}, \Omega) \in A_1(G)$ of the form*

$$G\text{-Deg}\,(\mathfrak{f}, \Omega) = n_1(H_1) + n_2(H_2) + \cdots + n_k(H_k),$$

and satisfying the following properties:

- Existence: *If $G\text{-Deg}\,(\mathfrak{f}, \Omega) \neq 0$, i.e. $n_i \neq 0$ for some $i \in \{1, 2, \ldots, k\}$, then there exists $x \in \Omega$ such that $\mathfrak{f}(x) = 0$ and $G_x \supset H_i$.*
- Additivity: *Let Ω_1 and Ω_2 be two disjoint open bounded G-invariant subsets of Ω such that $\mathfrak{f}^{-1}(0) \cap \Omega \subset \Omega_1 \cup \Omega_2$. Then*
$$G\text{-Deg}\,(\mathfrak{f}, \Omega) = G\text{-Deg}\,(\mathfrak{f}, \Omega_1) + G\text{-Deg}\,(\mathfrak{f}, \Omega_2).$$
- Homotopy: *Let $\mathfrak{h} : [0, 1] \times \mathbb{H} \to \mathbb{H}$ be an Ω-admissible homotopy (i.e. $\mathfrak{h}(t, x) \neq 0$ for $x \in \partial\Omega$, $t \in [0, 1]$) of G-equivariant completely continuous fields. Then*
$$G\text{-Deg}\,(\mathfrak{h}_t, \Omega) = \text{constant},$$
where $\mathfrak{h}_t := \mathfrak{h}(\cdot, x)$.

Remark 1. (cf. [2])

(a) Suppose that V is an orthogonal Γ-representation, $\mathcal{O} \subset V$ an open bounded Γ-invariant set, and $\mathfrak{g} : V \to V$ a continuous $\mathcal{O}$-admissible and Γ-equivariant map. Then it is also possible to define the Γ-equivariant degree (with no free parameter) of $\mathfrak{g}$ in Ω, denoted by $\Gamma\text{-Deg}(\mathfrak{g}, \mathcal{O})$, which takes the values in $A(\Gamma) = \mathbb{Z}[\Phi(\Gamma)]$, $\Phi(\Gamma) = \{(H) : H$ is a subgroup of $\Gamma\}$. This Γ-equivariant degree also satisfies similar existence, additivity and homotopy properties, as the twisted degree, including the so-called *multiplicativity property.* More precisely, the $\mathbb{Z}$-module $A(\Gamma)$ has a natural structure of a ring, which is called the *Burnside ring* of Γ, and the Γ-degree of a product of two maps is the product of their Γ-degrees in the Burnside ring $A(\Gamma)$.

(b) The $\mathbb{Z}$-module $A_1(G)$ has a natural structure of an $A(\Gamma)$-module.

(c) It is possible to provide a full list of axioms for the twisted G-equivariant degree, which uniquely defines the function $G\text{-Deg}$ and its properties.

In addition, we also have the following important property (cf. [2]):

Theorem 2.2. *Let $\Omega \subset \mathbb{R} \oplus \mathbb{H}$ be an open bounded G-invariant set, $\mathfrak{f} : \mathbb{R} \oplus \mathbb{H} \to \mathbb{H}$ an Ω-admissible and G-equivariant completely continuous field. Assume also that V is an orthogonal Γ-representation, $\mathcal{O} \subset V$ an open bounded and Γ-invariant set, and $\mathfrak{g} : V \to V$ a continuous $\mathcal{O}$-admissible and Γ-equivariant map. Then we have*

- (Multiplicativity Property) *The map* $\mathfrak{f} \times \mathfrak{g} : \mathbb{R} \oplus \mathbb{H} \oplus V \to \mathbb{H} \oplus V$, *defined by* $\mathfrak{f} \times \mathfrak{g}(\lambda, x, y) = (\mathfrak{f}(\lambda, x), \mathfrak{g}(y))$, *is an* $\Omega \times \mathcal{O}$*-admissible* G*-equivariant completely continuous field and*

$$G\text{-Deg}\,(\mathfrak{f} \times \mathfrak{g}, \Omega \times \mathcal{O}) = \Gamma\text{-Deg}(\mathfrak{g}, \mathcal{O}) \cdot G\text{-Deg}\,(\mathfrak{f}, \Omega),$$

where the multiplication is taken in the $A(\Gamma)$*-module* $A_1(G)$.

In this paper we use the conventions and notations that were introduced in the monograph [2]. In particular, we refer to [2] for the notion of the so-called *basic degrees* and symbols used to denote the conjugacy classes of subgroups in $G = \Gamma \times S^1$ for various finite groups Γ. The effective computations of the twisted G-equivariant degree can be done using a special Maple™ package that is available online (see [15]).

3. Framework for Delayed Differential Equations.

3.1. Statement of the Problem. Assume that Γ is a compact Lie group and let V be an orthogonal Γ-representation. For a given constant $\tau > 0$, denote by $C_{V,\tau}$ the Banach space of continuous functions from $[-\tau, 0]$ to V equipped with the supremum norm $\|\phi\|_\infty := \sup\{|\phi(\theta))| : -\tau \le \theta \le 0\}$, $\phi \in C_{V,\tau}$. Given a continuous function $x : \mathbb{R} \to V$ and $t \in \mathbb{R}$, define $x_t \in C_{V,\tau}$ by $x_t(\theta) := x(t + \theta)$, $\theta \in [-\tau, 0]$. Notice that $C_{V,\tau}$ is a natural isometric Banach representation of Γ.

Assume that:

(A1) $\mathscr{A} : C_{V,\tau} \to V$ is a bounded Γ-equivariant linear operator with $\mathscr{B} := \mathscr{A}|_V : V \to V$ being an isomorphism;

(A2) $\mathscr{R} : C_{V,\tau} \to V$ is a continuously differentiable Γ-equivariant map, such that $\mathscr{R}(0) = 0$ and $D\mathscr{R}(0) = 0$.

We are interested in the following problem:

Problem: *Find a continuously differentiable function* $u : \mathbb{R} \to V$ *satisfying the following autonomous functional differential equation*

$$\begin{cases} \dot{u}(t) = \mathscr{A}(u_t) + \mathscr{R}(u_t), \\ u_0 = u_p, \end{cases} \tag{13}$$

where $p > 0$ *is the unknown period of* $u(t)$.

3.2. Normalization of Period. By normalization of the period in (13) we understand the following change of variable $x(t) = u(\lambda t)$, where $\lambda = \frac{p}{2\pi}$ is considered to be a *new* parameter. We obtain the following equation, which is equivalent to (13)

$$\begin{cases} \dot{x}(t) = \lambda \left[\mathscr{A}(x_{t,\lambda}) + \mathscr{R}(x_{t,\lambda}) \right], \\ x_0 = x_{2\pi}, \end{cases} \tag{14}$$

where $x : \mathbb{R} \to V$, $x_{t,\lambda} \in C_{V,\tau}$ is defined by $x_{t,\lambda}(\theta) := x\left(t + \frac{\theta}{\lambda}\right)$, $\theta \in [-\tau, 0]$.

3.3. **Choice of Functional Space.** By using the standard identification of $\mathbb{R}/2\pi\mathbb{Z}$ with S^1, we consider the first Sobolev space of 2π-periodic V-valued functions

$$\mathbb{H} := H^1(S^1; V). \tag{15}$$

The space $\mathbb{H}$ is equipped with the following inner product and norm

$$\langle u, v\rangle_{H^1} := \int_0^{2\pi} \dot{u}(t) \bullet \dot{v}(t)dt + \int_0^{2\pi} u(t) \bullet v(t)dt, \quad \|u\|_{H^1} := \sqrt{\langle u, u\rangle_{H_1}}, \quad u, v \in \mathbb{H}.$$

Define a $\Gamma \times S^1$-action on $\mathbb{H}$ by

$$((\gamma, e^{is})u)(t) := \gamma u(t + s), \quad \gamma \in \Gamma, \;\; e^{is} \in S^1 \;\; u \in \mathbb{H}.$$

Then, $\mathbb{H}$ becomes a Hilbert isometric G-representation for $G := \Gamma \times S^1$.

3.4. **Setting in Functional Spaces.** Under the assumptions (A1) and (A2), the existence result for the equation (13) can be obtained by means of the G-equivariant twisted degree with one free parameter using the standard homotopy argument and *a priori* bounds for the equations

$$\begin{cases} \dot{x}(t) = \alpha\lambda[\mathscr{A}(x_{t,\lambda}) + \mathscr{R}(x_{t,\lambda})], \\ x_0 = x_{2\pi}, \end{cases} \tag{14_α}$$

and

$$\begin{cases} \dot{x}(t) = \alpha\lambda[\mathscr{A}(x_{t,\lambda}) + \rho\mathscr{R}(x_{t,\lambda})], \\ x_0 = x_{2\pi}, \end{cases} \tag{$14_{\alpha\rho}$}$$

where $\rho \in [0, 1]$, $\alpha \in (0, 1]$ and $\lambda \in [\lambda_1, \lambda_2]$, $\lambda_2 > \lambda_1 > 0$.

More precisely, we rewrite the equation ($14_{\alpha\rho}$) in functional spaces as

$$Lx = \alpha\lambda[N_{\mathscr{A}}(\lambda, j(x)) + \rho N_{\mathscr{R}}(\lambda, j(x))], \tag{16}$$

where

$$\begin{aligned}
L &: \mathbb{H} \to L^2(S^1; V), & Lx &= \dot{x}, & (17)\\
j &: \mathbb{H} \to C(S^1; V), & j(u) &= u, & (18)\\
N_{\mathscr{A}} &: \mathbb{R}_+ \times C(S^1; V) \to L^2(S^1; V), & N_{\mathscr{A}}(\lambda, x)(t) &= \mathscr{A}(x_{t,\lambda}), & (19)\\
N_{\mathscr{R}} &: \mathbb{R}_+ \times C(S^1; V) \to L^2(S^1; V), & N_{\mathscr{R}}(\lambda, x)(t) &= \mathscr{R}(x_{t,\lambda}). & (20)
\end{aligned}$$

Put

$$K : \mathbb{H} \to L^2(S^1; V), \quad Kx = \frac{1}{2\pi}\int_0^{2\pi} x(t)dt.$$

Since $L + K$ is an isomorphism, ($14_{\alpha\rho}$) is equivalent to

$$x - \alpha\lambda(L + K)^{-1}[N_{\mathscr{A}}(\lambda, j(x)) + \rho N_{\mathscr{R}}(\lambda, j(x)) + Kx] = 0, \quad x \in \mathbb{H}.$$

3.5. **Establishing** *A Priori* **Bounds.** In order to establish *a priori* bounds on solutions, one has to use specific properties of the system ($14_{\alpha\rho}$), which may not be satisfied on the whole space $\mathbb{H}$ but only on an open convex G-invariant set $\mathcal{C} \subset \mathbb{H}$. We assume that $0 \in \mathcal{C}$ and for every solution $x \in \mathcal{C}$ to (14_α)

$$\int_0^{2\pi} x(t)dt = 0.$$

Clearly, if such x is non-zero, then it is a non-constant periodic function. In addition, we fix two appropriate values $\lambda_2 > \lambda_1 > 0$.

The specific properties, needed to establish the *a priori* bounds in our example, are:

(P1) there exists $\alpha_o \in (0,1)$ such that for all $0 \le \alpha \le \alpha_o$, $\rho \in [0,1]$ and $\lambda \in [\lambda_1, \lambda_2]$ the system (14_α) has no non-zero solution in $\mathcal{C}$.

and

(P2) there exist an open bounded G-invariant set $\widetilde{\mathcal{U}} \subset \mathcal{C}$, $\varepsilon > 0$ and $\mathcal{U} := \{x \in E : \operatorname{dist}(x, \widetilde{\mathcal{U}}) < \varepsilon\}$, such that
$$0 \in \widetilde{\mathcal{U}} \subset \overline{\mathcal{U}} \subset \mathcal{C},$$
and every non-zero solution in $\mathcal{C}$ to ($14_{\alpha\rho}$), with $\alpha \in (0,1]$ and $\lambda \in [\lambda_1, \lambda_2]$, belongs to $\widetilde{\mathcal{U}}$.

At the moment we do not specify exactly what is the set $\widetilde{\mathcal{U}}$. However, we expect that $\widetilde{\mathcal{U}}$ has "good" properties, for example it is a star-shaped open set around the origin in $\mathbb{H}$.

In order to control the solutions near the origin, we assume that:

(P3) There exists $m_1 > 0$ such that for $\alpha = 1$ and $\rho \in [0,1]$, equation ($14_{\alpha\rho}$) has no non-zero solution in $\overline{B} := \{x \in \mathbb{H} : \|x\|_{H^1} \le m_1\} \subset \widetilde{\mathcal{U}}$.

Finally, we also need the following:

(P4) For $\alpha = 1$ and $\rho = 0$, the linearized system ($14_{\alpha\rho}$) does not have non-zero solutions in $\mathbb{H}$;
(P5) For $\lambda = \lambda_i$, $i = 1, 2$, the system ($14_{\alpha\rho}$) has no non-zero solution in $\mathcal{U}$.

Define
$$\Omega_{\lambda_1,\lambda_2} := \{(\lambda, x) : \lambda_1 < \lambda < \lambda_2, \ \ x \in \mathcal{U} \setminus \overline{B}\}.$$

3.6. **Control Function** β**.** Choose α_1 with $0 < \alpha_1 < \alpha_o$, to be sufficiently small and take a continuous function $\xi : [0,\infty) \to [\alpha_1, 1]$ such that (see Figure 2)
$$\xi(t) = \begin{cases} 1, & \text{if} \quad t = 0, \\ \text{strictly decreasing} & \text{if} \quad 0 \le t \le \varepsilon, \\ \alpha_1, & \text{if} \quad t > \varepsilon, \end{cases} \tag{21}$$

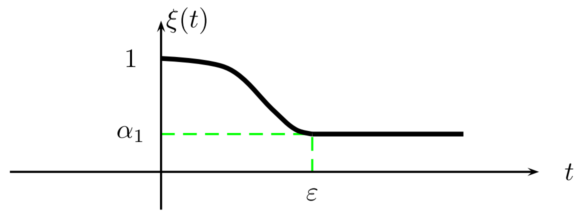

FIGURE 2. The bump function $\xi : [0, +\infty) \to [\alpha_o, 1]$

and define $\beta : \mathbb{H} \to \mathbb{R}_+$ by

$$\beta(x) = \xi(\text{dist}\,(x, \widetilde{\mathcal{U}})). \tag{22}$$

Next replace α in ($14_{\alpha\rho}$) by $\beta(x)$, i.e. consider the equation

$$\begin{cases} \dot{x}(t) = \beta(x)\lambda[\mathscr{A}(x_{t,\lambda}) + \rho\mathscr{R}(x_{t,\lambda})] \\ x(0) = x(2\pi). \end{cases} \tag{$14_{\beta\rho}$}$$

Notice that for $\rho = 1$, ($14_{\beta\rho}$) has exactly the same solution set in $\Omega_{\lambda_1,\lambda_2}$ as (14).

3.7. **Admissible Homotopy.** Define

$$\mathfrak{F}_\rho(\lambda, x) := x - \beta(x)\lambda(L + K)^{-1}[N_{\mathscr{A}}(\lambda, j(x)) + \rho N_{\mathscr{R}}(\lambda, j(x)) + Kx], \tag{23}$$

where $\rho \in [0, 1]$, $\lambda \in [\lambda_1, \lambda_2]$, $x \in \mathbb{H}$. Let us observe that:

- If $x \in \partial\mathcal{U}$ then $\beta(x) = \alpha_1 < \alpha_o$. Consequently, by (P1), $\mathfrak{F}_\rho(\lambda, x) \neq 0$.
- If $x \in \partial\overline{B}$ then, by (P3) and (P4), $\mathfrak{F}_\rho(\lambda, x) \neq 0$.
- If $x \in \mathcal{U}$ and $\rho \in [0, 1]$ then, by (P5), $\mathfrak{F}_\rho(\lambda_i, x) \neq 0$ for $i = 1, 2$.

Therefore, $\mathfrak{F}_\rho$, $\rho \in [0, 1]$, is $\Omega_{\lambda_1,\lambda_2}$-admissible homotopy.

3.8. **Existence Result.** Under the assumptions (P1)–(P5), the G-equivariant twisted degree $G\text{-Deg}\,(\mathfrak{F}_\rho, \Omega_{\lambda_1,\lambda_2})$ is well defined and does not depend on $\rho \in [0, 1]$.

Definition 3.1. We introduce the following notation

$$回 := G\text{-Deg}\,(\mathfrak{F}_0, \Omega_{\lambda_1,\lambda_2}),$$

we will call 回 the *G-equivariant topological invariant** for the system (14).

We have the following result

Theorem 3.2. *Under the assumptions (P1)–(P5), if the G-equivariant topological invariant*

$$回 = \sum_{(H)} n_H(H)$$

is non-zero, i.e. there is a coefficient $n_{H_o} \neq 0$ with $H_o = K_o^{\varphi,l}$, then there exists $(\lambda, x) \in \Omega_{\lambda_1,\lambda_2}$ such that $\mathfrak{F}_1(\lambda, x) = 0$ with $G_x \supset H_o$. In other words, there exists a non-constant 2π-periodic solution to (14) for some $\lambda \in [\lambda_1, \lambda_2]$, and consequently, there is a p-periodic solution to (13) with $p = 2\pi\lambda$. In addition, if $H = K^{\varphi,l}$ is

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* We use here the Chinese symbol 回 (*huí*), which means '*return*', i.e. it returns the topological information about the solution set.

a dominating type† *in* $\mathbb{H}$*, then there exists a non-constant periodic solution* $x(t)$ *to (13) with the isotropy group* K^{φ}*.*

3.9. **Computations of the Equivariant Topological Invariant.** To compute 回 $= G\text{-Deg}\,(\mathfrak{F}_0, \Omega_{\lambda_1,\lambda_2})$, we consider the "linearized" map

$$\mathfrak{F}_0(\lambda, x) := x - \beta(x)\lambda(L+K)^{-1}\left[N_{\mathscr{A}}(\lambda, j(x)) + Kx\right],$$

where $(\lambda, x) \in \overline{\Omega_{\lambda_1,\lambda_2}}$.

Isotypical Decomposition and Related Transformations. Consider the S^1-isotypical decomposition of the space $\mathbb{H}$

$$\mathbb{H} = V \oplus \overline{\bigoplus_{l=1}^{\infty} \mathbb{H}_l} =: V \oplus \mathbb{H}^*, \quad V = \mathbb{H}^{S^1}, \tag{24}$$

where each of the S^1-isotypical components $\mathbb{H}_l$, $l > 0$, can be identified with (cf. [2])

$$\mathbb{H}_l := \{e^{ilt}z : z \in V^c\}.$$

Put

$$\overline{\mathfrak{F}}_0(\lambda, \cdot) := \mathfrak{F}_0(\lambda, \cdot)|_V, \quad \text{and} \quad {\mathfrak{F}^*}_0(\lambda, \cdot) := \mathfrak{F}_0(\lambda, \cdot)|_{\mathbb{H}^*}.$$

The map $\mathfrak{F}_0(\lambda, \cdot)$ is the product ${\mathfrak{F}^*}_0(\lambda, \cdot) \times \overline{\mathfrak{F}}_0(\lambda, \cdot)$. Since $\lambda > 0$, by the homotopy and multiplicativity properties of the twisted equivariant degree, we can replace $\overline{\mathfrak{F}}_0(\lambda, \cdot)$ with $-\mathscr{B} : V \to V$ (cf. (A1)). Thus,

$$G\text{-Deg}\,(\mathfrak{F}_0, \Omega_{\lambda_1,\lambda_2}) = \Gamma\text{-Deg}(-\mathscr{B}, \mathcal{B}) \cdot G\text{-Deg}\,(\mathfrak{F}_0^*, \Omega^*_{\lambda_1,\lambda_2}),$$

where $\Omega^*_{\lambda_1,\lambda_2} = \Omega_{\lambda_1,\lambda_2} \cap (\lambda_1, \lambda_2) \times \mathbb{H}^*$ and $\mathcal{B}$ denotes the unit ball in V.

Consider the negative spectrum $\sigma_-(-\mathscr{B})$ of the operator $-\mathscr{B}$ and let

$$V := V_0 \oplus V_1 \oplus \cdots \oplus V_r,$$

be the Γ-isotypical decomposition of V, with V_j modelled on the irreducible Γ-representation $\mathcal{V}_j$. For an eigenvalue $\mu \in \sigma_-(-\mathscr{B})$, denote by $E(\mu) \subset V$ the associated with μ eigenspace. Then, $\widehat{m}_j(\mu) := \dim\,(E(\mu) \cap V_j)/\dim \mathcal{V}_j$ is called the $\mathcal{V}_j$*-multiplicity* of μ. By the multiplicativity property of the Γ-equivariant degree (cf. [2]), one can show that

$$\Gamma\text{-Deg}(-\mathscr{B}, \mathcal{B}) = \prod_{\mu \in \sigma_-(-\mathscr{B})} \prod_{j=0}^{r} \left(\deg_{\mathcal{V}_j}\right)^{\widehat{m}_j(\mu)},$$

where $\deg_{\mathcal{V}_j}$ denotes the basic degree for the representation $\mathcal{V}_j$ (cf. [2]).

In order to compute $G\text{-Deg}\,(\mathfrak{F}_0^*, \Omega^*_{\lambda_1,\lambda_2})$, by the homotopy and excision properties of the twisted equivariant degree, we conveniently deform the involved maps and the admissible sets. In particular, we can assume that the set $\Omega^*_{\lambda_1,\lambda_2}$ is exactly

*† An orbit type $(K^{\varphi,l})$ in $\mathbb{H}$ is called an *dominating orbit type*, if it is a maximal element among all the twisted l-folded orbit types.

$(\lambda_1,\lambda_2)\times(B_2^*\setminus B_{\frac{1}{2}}^*)$, where $B_r^*:=\{x^*\in\mathbb{H}^*:\|x\|_{H^1}<r\}$, and the function β is given by

$$\beta(x)=\begin{cases}1 & \text{if } \|x\|_{H^1}\le 1,\\ 2-\alpha_1-(1-\alpha_1)\|x\|_{H^1} & \text{if } 1<\|x\|_{H^1}<2,\\ \alpha_1 & \text{if } \|x\|_{H^1}\ge 2.\end{cases}$$

By equivariance of $\mathfrak{F}_0^*(\lambda,\cdot)$, we have $\mathfrak{F}_0^*(\lambda,\cdot)(\mathbb{H}_l)\subset\mathbb{H}_l$. A function $x\in\mathbb{H}_l$ can be represented as $x(t)=e^{itl}z$ for some $z\in V^c$. This implies that $\mathbb{H}_l$ can be identified with V^c. We define the maps $\mathcal{A}_l(\lambda):\mathbb{H}_l\to\mathbb{H}_l$ by

$$\begin{aligned}\mathcal{A}_l(\lambda)(e^{ilt}z)&=e^{ilt}z-\beta(z)\lambda L^{-1}[\mathscr{A}(e^{il(t+\frac{\theta}{\lambda})})]\\&=e^{ilt}\left[z-\frac{\beta(z)}{il}\mathscr{A}(e^{\frac{il\theta}{\lambda}}z)\right].\end{aligned}$$

By the Splitting Lemma (cf. Lemma 4.21 in [2]), we have

$$G\text{-Deg}\,(\mathfrak{F}_0^*,\Omega^*_{\lambda_1,\lambda_2})=\sum_{l>0}G\text{-Deg}\,(\mathcal{A}_l,\Omega^*_{\lambda_1,\lambda_2}\cap\mathbb{H}_l).$$

Consider the linear operators $\mathscr{A}_l(\lambda):V^c\to V^c$ defined by

$$\mathscr{A}_l(\lambda)z:=\mathscr{A}(e^{\frac{il\theta}{\lambda}}z),\quad z\in V^c.$$

In order to simplify the computations, assume:

(B1) For every eigenvalue $\mu_{l,r}\in\sigma(\mathscr{A}_l(\lambda))$, $l=1,2,3,\dots$, the eigenspace $\widetilde{E}_{l,r}:=\widetilde{E}_l(\mu_r(\lambda))$ doesn't depend on $\lambda\in[\lambda_1,\lambda_2]$, and $\mathbb{H}_l=\bigoplus_r\widetilde{E}_{l,r}$.

Using the G-invariant decomposition $\mathbb{H}_l\simeq\bigoplus_r\widetilde{E}_{l,r}$, we can write

$$\mathscr{A}_l(\lambda)z=\sum_r\mu_{l,r}(\lambda)z_r,\qquad\text{where } z=\sum_r z_r,\ \ z_r\in\widetilde{E}_{l,r}.$$

Put

$$\widetilde{\mathcal{A}}_{l,r}(\lambda,z_r):=z_r-\frac{\beta(z_r)}{il}\mu_{l,r}(\lambda)z_r,\quad z_r\in\widetilde{E}_{l,r},$$

and define the sets

$$\mathcal{U}_{l,r}:=\{z\in\widetilde{E}_{l,r}:\frac{1}{2}<\|z\|<2\},\quad\Omega_{l,r}:=(\lambda_1,\lambda_2)\times\mathcal{U}_{l,r}.$$

By the Splitting Lemma and standard properties of the twisted equivariant degree, we obtain

$$G\text{-Deg}\,(\mathfrak{F}_0^*,\Omega^*_{\lambda_1,\lambda_2})=\sum_{l>0}\sum_{r=1}^{k}G\text{-Deg}\,(\widetilde{\mathcal{A}}_l,\Omega_{l,r}).$$

Define $\varphi_{l,r}:(\lambda_1,\lambda_2)\times\mathbb{R}_+\to\mathbb{C}$ by

$$\varphi_{l,r}(\lambda,t):=1-(1-\alpha_1)\frac{\gamma-t}{il}\mu_{l,r}(\lambda),\quad\gamma:=\frac{2-\alpha_1}{1-\alpha_1}.$$

Then

$$\widetilde{\mathcal{A}}_{l,r}(\lambda,z)=\varphi_{l,r}(\lambda,\|z\|)z,\quad z\in\widetilde{E}_{l,r}.$$

Reduction to Basic Maps. By the homotopy property of the twisted equivariant degree, we may assume that the functions $\varphi_{l,r} : (\lambda_1, \lambda_2) \times \left(\frac{1}{2}, 2\right) \to \mathbb{C}$ are continuously differentiable and the sets $\varphi_{l,r}^{-1}(0)$ are composed of a finite number of regular points.

Lemma 3.3. *Let $U \subset \mathbb{R} \times \mathbb{R}_+$ be an open bounded set satisfying the conditions*

(a) if $(\lambda, t) \in U$ then $t > 0$;
(b) $\varphi : \mathbb{R}^2 \to \mathbb{C}$ is a continuously differentiable and U-admissible map;
(c) the set $\Lambda := \varphi^{-1}(0) \cap U$ is composed of regular points of φ.

Put

$$T := \max\{|t| : \exists_\lambda \ \ (\lambda, t) \in \Lambda\} + 1, \quad \tau := \frac{1}{2} \max\{|t| : \exists_\lambda \ \ (\lambda, t) \in \Lambda\}.$$

Consider an irreducible G-representation $\mathcal{V}_{j,l}$, $l > 0$, and define the set

$$\Omega := \{(\lambda, v) \in \mathbb{R} \oplus \mathcal{V}_{j,l} : (\lambda, |v|) \in U, \ \ \tau < |v| < T\},$$

and the G-equivariant map $\widetilde{\mathcal{A}} : \mathbb{R} \oplus \mathcal{V}_{j,l} \to \mathcal{V}_{j,l}$ by

$$\widetilde{\mathcal{A}}(\lambda, v) = \varphi(\lambda, |v|) \cdot v.$$

Then $\widetilde{\mathcal{A}}$ is Ω-admissible G-equivariant map and

$$G\text{-Deg}\,(\widetilde{\mathcal{A}}, \Omega) = \sum_{(\lambda,t)\in\Lambda} \text{sign det}\, D\varphi(\lambda, t) \deg_{j,l}.$$

Proof: For every point $(\lambda_o, t_o) \in \Lambda$ we define a small neighborhood Ω_o of the zero set $\{(\lambda_o, v) : |v| = t_o\}$ in the space $\mathbb{R} \oplus \mathcal{V}_{j,l}$ by

$$\Omega_o := \{(\lambda, v) : |\lambda - \lambda_o| < \varepsilon_i, \ 0 < t_o - \delta < |v| < t_o + \delta\},$$

where δ is chosen to be sufficiently small. Then

$$G\text{-Deg}\,(F, \Omega) = \sum_{(\lambda_o,t_o)\in\Lambda} G\text{-Deg}\,(F, \Omega_o).$$

Since for every (λ_o, t_o) the map $\widetilde{\mathcal{A}}$ can be approximated on Ω_o by $(\lambda, v) \mapsto D\varphi(\lambda_o, t_o)(\lambda - \lambda_o, |v| - t_o)^T \cdot z$, which is clearly homotopic to

$$(\lambda, v) \mapsto J_{i,\pm}(\lambda - \lambda_o, |v| - t_o)^T \cdot v,$$

where

$$J_{i,+} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad \text{if sign det}\, D\varphi(\lambda_i, t_i) = 1,$$
$$J_{i,-} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}, \quad \text{if sign det}\, D\varphi(\lambda_i, t_i) = -1,$$

so the result follows. □

In this way we obtain the following computational formula for 回.

Theorem 3.4. *Under the above assumptions we have*

$$回 = \prod_{\mu\in\sigma_-(-\mathscr{B})} \prod_{j=0}^{r} \Big(\deg_{\mathcal{V}_j}\Big)^{\widehat{m}_j(\mu)} \cdot \sum_{l>0} \sum_{r=1}^{k} \sum_{j=0}^{s} \sum_{(\lambda,t)\in\Lambda_{l,r}} m_j(\mu_{l,r}(\lambda)) \text{sd}(\lambda, t) \cdot \deg_{\mathcal{V}_{j,l}}, \tag{25}$$

where $\mathrm{sd}(\lambda, t) := \text{sign}\det D\varphi(\lambda, t)$, $V^c = U_0 \oplus \cdots \oplus U_s$ *is the* Γ*-isotypical (complex) decomposition of* V^c *with* U_j *modelled on* $\mathcal{U}_j$*, and* $m_j(\mu_{l,r}(\lambda)) = \dim(\widetilde{E}_{l,r} \cap U_j)/\dim \mathcal{U}_j$ *is the* $\mathcal{U}_j$*-multiplicity of* $\mu_{l,r}(\lambda)$*.*

4. **Symmetric Lotka-Volterra Systems.** Let Γ be a finite group and assume that $V := \mathbb{R}^n$ is an orthogonal Γ-representation such that Γ acts on V by permuting the coordinates of vectors $x = (x_1, x_2, \dots, x_n)^T \in V$.

Consider the following Γ-symmetric Lotka-Volterra type system

$$\dot{u}(t) = u(t) \cdot \big(r - Au(t-\tau)\big), \tag{26}$$

where $u : \mathbb{R} \to V$, $\tau > 0$ and

$$u = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix}, \quad r = \begin{pmatrix} r_1 \\ r_2 \\ \vdots \\ r_n \end{pmatrix} \in V, \quad A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ & \dots & & \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix},$$

and '$\cdot$' stands for the component-wise vector multiplication.

By an appropriate transformation (see (11)), the problem (26) is equivalent to

$$\dot{u}(t) = -Au(t-\tau) \cdot \big(b + u(t)\big), \tag{27}$$

where $b = A^{-1}r$. Let p be the unknown period of a solution u to (27). By a change of variable, letting $\lambda = \frac{p}{2\pi}$, $x(t) = u(\lambda t)$, we have that x is a 2π-periodic solution to the problem

$$\dot{x}(t) = -\lambda Ax(t - \frac{\tau}{\lambda}) \cdot \big(b + x(t)\big). \tag{28}$$

In what follows we assume that the following conditions hold:

(H0) A and b have positive entries, i.e. $a_{i,j}, b_i > 0$, for $1 \le i, j \le n$.

(H1) A is symmetric, positive definite (i.e. $A = A^T$ and $\langle Ax, x\rangle > 0$ for all $x \in \mathbb{R}^n \setminus \{0\}$) and A is Γ-equivariant. In particular, the matrix $B := \text{diag}(b)A$ (i.e. $Bx = Ax \cdot b$), where $\text{diag}(b)$ denotes the diagonal matrix $[d_{ij}]$ with $d_{jj} = b_j$, $j = 1, \dots, n$, has only real positive eigenvalues $\mu_1, \cdots, \mu_n$ (not necessarily distinct).

(H2) The vector $b = [b_1, \dots, b_n]^T \in V$ is Γ-invariant, i.e. $\gamma b = b$ for all $\gamma \in \Gamma$.

We make also the following assumption

(H3) For every $\mu \in \sigma(B)$

$$\mu\tau \ne 2n\pi + \frac{\pi}{2}, \qquad \text{for all } \ n \in \mathbb{Z}. \tag{29}$$

Under the assumptions (H0)–(H2) the equation (26) is Γ-symmetric.

We are interested in finding non-constant periodic solutions to (26). This problem is equivalent to finding non-constant 2π-periodic solutions to (28) for some $\lambda > 0$.

Define $\mathscr{A}, \mathscr{R} : C_{V,\tau} \to V$ by

$$\mathscr{A}(u) := -Au(-\tau) \cdot b, \quad \mathscr{R}(u) := -Au(-\tau) \cdot u(0),$$

where $u \in C_{V,\tau}$. Notice that $\mathscr{A}$ and $\mathscr{R}$ satisfy the assumptions (A1)–(A3).

Reformulation in Functional Spaces. We use the functional setting that was presented in Subsection 3.3, i.e. $\mathbb{H}$ is given by (15) and the operators L, j, $N_{\mathscr{A}}$ and $N_{\mathscr{R}}$ are given by (17)–(20).

Following the framework presented in Section 3, we consider the following parameterized problems

$$\begin{cases} \dot{x}(t) = -\lambda A x(t - \frac{\tau}{\lambda}) \cdot \big(b + \rho x(t)\big), \\ x_0 = x_{2\pi}, \end{cases} \tag{28_ρ}$$

and

$$\begin{cases} \dot{x}(t) = -\alpha\lambda A x(t - \frac{\tau}{\lambda}) \cdot \big(b + \rho x(t)\big), \\ x_0 = x_{2\pi}, \end{cases} \tag{$28_{\alpha\rho}$}$$

where $\alpha \in (0, 1]$ and $\rho \in [0, 1]$.

Define the map $\mathcal{N} : [0, 1] \times \mathbb{R}_+ \times \mathbb{H} \to L^2(S^1; V)$ by

$$\mathcal{N}(\rho, \lambda, x)(t) = Aj(x(t - \frac{\tau}{\lambda})) \cdot (b + \rho j(x(t))).$$

Then, (28_ρ) is equivalent to

$$Lx = \lambda[N_{\mathscr{A}}(\lambda, j(x)) + \rho N_{\mathscr{R}}(\lambda, j(x))] = -\lambda\mathcal{N}(\rho, \lambda, x),$$

which can be written as

$$x - \lambda(L + K)^{-1}\mathcal{N}(\rho, \lambda, x) = 0.$$

4.1. *A Priori* **Bounds.** Define a partial order in $V = \mathbb{R}^n$ by

$$x \succ y \quad \Longleftrightarrow \quad x_i > y_i, \text{ for all } 1 \le i \le n,$$

where $x = (x_1, \dots, x_n)^T$ and $y = (y_1, \dots, y_n)^T$ are two vectors from $\mathbb{R}^n$. Introduce the following set

$$\mathcal{C} = \{x \in \mathbb{H} : -b \prec x(t) \quad \text{for all} \quad t \in [0, 2\pi]\}.$$

We begin with the following important observation

Proposition 1. *Consider λ, $\alpha > 0$ and $\tau \ge 0$. Then for every periodic solution $x \in \mathcal{C}$ of*

$$\dot{x}(t) = -\alpha\lambda A x(t - \tau/\lambda) \cdot \big(b + x(t)\big), \tag{28_α}$$

we have

$$\int_0^{2\pi} x(t)dt = 0. \tag{30}$$

In particular, the equation (28_α) does not have any non-zero constant solution.

Proof: Let $x \in \mathcal{C}$ be a solution to (28_α), $x(t) = [x_1(t), \dots, x_n(t)]^T$. Then for $k = 1, 2, \dots, n$

$$\dot{x}_k(t) = -\alpha\lambda \sum_j a_{kj} x_j(t - \tau/\lambda) \cdot (b_k + x_k(t)), \tag{31}$$

which leads to

$$\frac{\dot{x}_k(t)}{b_k + x_k(t)} = -\alpha\lambda \sum_j a_{kj} x_j(t - \tau/\lambda). \tag{32}$$

By integrating (32) from 0 to 2π, we obtain (by periodicity of $x(t)$) that

$$\sum_j a_{kj} \int_0^{2\pi} x_j(t-\tau/\lambda)dt = \sum_j a_{kj} \int_0^{2\pi} x_j(t)dt = 0, \quad k = 1, 2, \dots, n.$$

Since the matrix A is invertible, one can easily deduce (30). □

We also need the following lemmas:

Lemma 4.1. *(i) For $\lambda_1, \lambda_2 \in \mathbb{R}^+$ with $\lambda_1 < \lambda_2$, there exist a positive number R, and positive Γ-invariant vectors d_1, $d_2 \succ 0$ such that for each $\lambda \in [\lambda_1, \lambda_2]$, $\alpha \in (0, 1]$, $\tau \geq 1$, all solutions $x \in \mathcal{C}$ of the problem (28_α) satisfy $\|x\|_{H^1} < R$ and*

$$-b \prec -d_1 \prec x(t) \prec d_2, \quad t \in [0, 2\pi].$$

In addition, there exists $m_o > 0$ such that $\|\dot{x}\|_\infty < m_0$ and $\|\ddot{x}\|_\infty < m_0$, where $\|x\|_\infty := \sup\{|x(t)| : t \in \mathbb{R}\}$.

(ii) There exists $\alpha_o \in (0, 1)$ such that there is no non-zero solution in $\mathcal{C}$ to (28_α) for $\alpha \in (0, \alpha_o]$ and $\lambda \in [\lambda_1, \lambda_2]$.

Proof: (i) Let $x \in \mathcal{C}$ be a solution to (28_α), $x(t) = (x_1(t), \dots, x_n(t))^T$. Then for $k = 1, 2, \dots, n$ we have the relations (31) and (32) which lead to

$$\ln(b_k + x_k(t)) - \ln(b_k + x_k(s)) = -\alpha\lambda \int_s^t \sum_j a_{kj} x_j(w - \tau/\lambda)dw,$$

where we assume $s \leq t$. Consequently, if s is such that $x_k(s) = 0$ then

$$b_k + x_k(t) = b_k \exp\left(-\alpha\lambda_2 \int_s^t \sum_j a_{kj} x_j(w - \tau/\lambda)dw\right), \quad \text{for all} \;\; t \in \mathbb{R}.$$

By the assumptions (H0) and (H1),

$$x_k(t) < d_2^k := b_k \exp\left(2\pi\alpha\lambda_2 \sum_j a_{kj} b_j\right) - b_k \quad \text{for all } t \in \mathbb{R}, \tag{33}$$

and by (H2), the vector $d_2 := (d_2^1, \dots, d_2^n)^T$ is Γ-invariant. On the other hand,

$$-b_k < -d_1^k := b_k \exp\left(-2\pi\alpha\lambda_1 \sum_j a_{kj} d_2^j\right) - b_k < x_k(t), \quad \text{for all } t \in \mathbb{R},$$

and agian by (H2), the vector $d_1 := (d_1^1, \dots, d_1^n)^T$ is Γ-invariant. By differentiating (28_α) we obtain

$$\ddot{x}(t) = -\alpha\lambda\left(A\dot{x}(t - \tau/\lambda) \cdot \big(b + x(t)\big) + Ax(t - \tau/\lambda) \cdot \dot{x}(t)\right). \tag{34}$$

The above obtained upper and lower bounds of $x_k(t)$, combined with (31) and (34), imply that there exists $m_o > 0$ such that

$$|\dot{x}_k(t)| < m_o \quad \text{and} \quad |\ddot{x}_k(t)| < m_o,$$

for all $k = 1, \dots, n$ and $t \in \mathbb{R}$. Consequently,

$$\|\dot{x}\|_\infty < m_o \text{ and } \quad \|\ddot{x}\|_\infty < m_o.$$

Therefore,

$$\|x\|^2_{H^1} = \int_0^{2\pi} \dot{x}(t) \bullet \dot{x}(t)dt + \int_0^{2\pi} x(t) \bullet x(t)dt \le 2\pi\|\dot{x}\|^2_\infty + 2\pi \sum_{k=1}^n d_2^k =: R^2.$$

(ii) Suppose for contradiction that there exist sequences $\{\alpha_n\} \subset (0, \alpha_o]$ and $\{x^m\} \in \mathcal{C}$ such that x^m is a non-zero solution to (28_α) for $\alpha = \alpha_m$, $\lambda = \lambda_m \in [\lambda_1, \lambda_2]$ and $\lim_{m\to\infty} \alpha_m = 0$. Then (33) holds for $x_k(t) = x_k^m(k)$ with $m = 1, 2, \dots,$ and therefore,

$$\lim_{m\to\infty} \|x^m\|_\infty = 0.$$

Since

$$\dot{x}^m(t) = -\alpha_m \lambda A x^m(t - \tau/\lambda_m) \cdot \big(b + x^m(t)\big), \tag{35}$$

we have

$$\|\dot{x}^m\|_\infty \le \alpha_m \lambda_2 |A| \|x^m\|_\infty (|b|_\infty + |d_2|_\infty), \tag{36}$$

where $|A| = \sum_{ij} a_{ij}$ and $|y|_\infty = \max\{|y_j| : j = 1, \dots, n\}$ for $y \in \mathbb{R}^n$. Define $u^m(t)$ by

$$u_k^m(t) = \frac{x_k^m(t)}{\|x^m\|_\infty}, \quad t \in \mathbb{R}.$$

Clearly, $u^m \in \mathbb{H}$ and by (36),

$$\|\dot{u}^m\|_\infty \le \alpha_m \lambda_2 |A|(|b|_\infty + |d_2|_\infty),$$

which implies that $\lim_{m\to\infty} \|\dot{u}^m\|_\infty = 0$. Since

$$\|u^m\|_\infty \le 2\pi \|\dot{u}^m\|_\infty,$$

it follows that $\lim_{m\to\infty} \|u^m\|_\infty = 0$, which is a contradiction with $\|u^m\|_\infty = 1$. $\square$

Lemma 4.2. *Assume that for fixed values of $\lambda \in \mathbb{R}^+$ and $\alpha \in (0, 1]$, the linearized equation*

$$\dot{x}(t) = -\alpha\lambda A x(t - \tau/\lambda) \cdot b \tag{37}$$

has a non-zero solution in $\mathbb{H}$. Then

(i) there exist k, $n \in \mathbb{Z}$, $n \ge 0$, $k > 0$ such that

$$\begin{cases} \lambda = \frac{k\tau}{2\pi n + \pi/2} =: \lambda_{k,n}, \\ \alpha = \frac{k}{\lambda\mu}, \end{cases} \tag{38}$$

where μ is an eigenvalue of the matrix $B := \operatorname{diag}(b)A$.

(ii) In particular, for $\alpha = 1$ the equation (37) has no non-zero solution in $\mathbb{H}$.

Proof: The equation (37) can be written as

$$\dot{x}(t) = -\alpha\lambda B x(t - \tau/\lambda). \tag{39}$$

Clearly, (39) allows a non-zero solution u in $\mathbb{H}$ if and only if, there is $k \in \mathbb{N}$ such that $x = e^{ikt} \cdot z$, for some $z \in V^c$, is a solution to (39), which leads to the equation

$$ik + \alpha\lambda\mu e^{-il\frac{\tau}{\lambda}} = 0,$$

for some $\mu \in \sigma(B)$. One can easily verify that such a case is possible if and only if, the relations in (38) are satisfied for some $n \in \mathbb{Z}$. On the other hand, if $\alpha = 1$ then (38) implies that $\mu\tau = 2\pi n + \pi/2$, which contradicts the assumption (H3). $\square$

Lemma 4.3. *Assume that $\lambda \in \mathbb{R}^+$, $\rho \in [0,1]$ and $\alpha \in (0,1]$ are fixed.*

(i) If zero is not an isolated solution in $\mathbb{H}$ to the equation ($28_{\alpha\rho}$), then there exist integers $k > 0$ and $n \geq 0$ such that λ and α satisfy the relations in (38) for an eigenvalue μ of the matrix $B := \operatorname{diag}(b)A$.

(ii) If λ_1, $\lambda_2 \in \mathbb{R}^+$ with $\lambda_1 < \lambda_2$, then there exists $m_1 > 0$ such that for all $\lambda \in [\lambda_1, \lambda_2]$, the equation ($28_\rho$) has no non-zero solution $x \in \mathbb{H}$ such that $\|x\|_{H^1} \leq m_1$.

Proof: Define $\mathfrak{F}_\alpha : [\lambda_1, \lambda_2] \times \mathbb{H} \to \mathbb{H}$ by

$$\mathfrak{F}_\alpha(\rho, \lambda, x) := x - \alpha\lambda(L+K)^{-1}\mathcal{N}(\rho, \lambda, x), \quad x \in \mathbb{H}.$$

By the implicit function theorem, if $(\lambda, 0)$ is not an isolated solution to ($28_{\alpha\rho}$) for some $\rho \in [0,1]$, then $D_x\mathfrak{F}_\alpha(\rho, \lambda, 0) : \mathbb{H} \to \mathbb{H}$ is not an isomorphism, which implies that the equation (37) has a non-zero solution. Consequently, by Lemma 4.2, α and λ satisfy the relations in (38). Therefore, if $\alpha = 1$, again by Lemma 4.2 and the implicit function theorem, there exists $m_1 > 0$ such that the equation (28_ρ) has no non-zero solution x for $\lambda \in [\lambda_1, \lambda_2]$ and $\rho \in [0,1]$. □

The following fact is well-known, but for the sake of completeness we include its elementary proof.

Lemma 4.4. *For any $\rho \in [0,1]$ and $\lambda > 0$, the following equation*

$$\begin{cases} \dot{x}(t) = -\lambda A x(t) \cdot \big(b + \rho x(t)\big), \\ x_0 = x_{2\pi}. \end{cases} \tag{40}$$

has no non-zero solution.

Proof: Assume first that $\rho \in (0,1]$. Suppose that x is a non-zero 2π-periodic solution to (40). By integrating (40) from 0 to 2π, we obtain

$$\int_0^{2\pi} Ax(t)\cdot x(t)dt = 0 \quad \iff \quad \sum_{j=1}^n a_{kj}\int_0^{2\pi} x_j(t)x_k(t)dt = 0, \; k = 1, 2, \ldots, n. \tag{41}$$

On the other hand, A is positively definite, i.e. $Ax(t) \bullet x(t) > 0$ for $x(t) \neq 0$, which implies that

$$\int_0^{2\pi} Ax(t) \bullet x(t)dt > 0.$$

But this is a contradiction, because by summing up the equations in (41), we obtain

$$\int_0^{2\pi} Ax(t) \bullet x(t)dt = \sum_{k=1}^n\sum_{j=1}^n a_{kj}\int_0^{2\pi} x_j(t)x_k(t)dt = 0.$$

Suppose now that $\rho = 0$, then the equation (40) becomes $\dot{x}(t) = -\lambda Bx(t)$. Consequently, if x is a 2π-periodic solution to (40) for $\rho = 0$, then it also satisfies the equation

$$\frac{d}{dt}(x(t)\cdot x(t)) = 2\dot{x}(t)\cdot x(t) = -2\lambda Bx(t)\cdot x(t),$$

which leads to

$$\int_0^{2\pi} Bx(t)\cdot x(t)dt = 0.$$

Be a similar argument as above, we obtain again that $x(t) = 0$. □

4.2. **Sets and Deformations.** Fix $\lambda_1, \lambda_2 \in \mathbb{R}^+$ with $\lambda_1 < \lambda_2$ and assume $d_2 \succ \frac{b_1+d_1}{2}$. We define the following $\Gamma \times S^1$-invariant sets

$$\begin{aligned}
\mathcal{D} &:= \left\{ x \in \mathbb{H} : -\frac{b+d_1}{2} \prec x(t) \prec 2d_2,\ t \in [0, 2\pi] \right\},\\
\widetilde{\mathcal{D}} &:= \{x \in \mathbb{H} : -d_1 \prec x(t) \prec d_2,\ t \in [0, 2\pi]\},\\
\overline{B} &:= \{x \in \mathbb{H} : \|x\|_{H^1} \le m_1\},\\
B_R &:= \{x \in \mathbb{H} : \|x\|_{H^1} < R\},
\end{aligned}$$

where R, d_1 and d_2 are specified in Lemma 4.1 and m_1 in Lemma 4.3. We can choose $m_1 > 0$ to be sufficiently small so that

$$\overline{B} \subsetneq \widetilde{\mathcal{D}} \subsetneq \overline{\mathcal{D}} \subsetneq \mathcal{C}.$$

and define

$$\widetilde{\mathcal{U}} := \widetilde{\mathcal{D}} \cap B_R.$$

Choose $\varepsilon > 0$ to be sufficiently small such that the set

$$\mathcal{U} := \{x \in \mathbb{H} : \operatorname{dist}(x, \widetilde{\mathcal{U}}) < \varepsilon\},$$

satisfies

$$\overline{\mathcal{U}} \subset \mathcal{D}.$$

Next, we choose λ_1 and λ_2 to be

$$\lambda_1 := \frac{\tau}{2j_1\pi}, \quad \lambda_2 := \frac{\tau}{2j_2\pi}, \quad j_1 > j_2,\ j_1,\ j_2 \in \mathbb{N}, \tag{42}$$

and put

$$\Omega_{\lambda_1,\lambda_2} := (\lambda_1, \lambda_2) \times (\mathcal{U} \setminus \overline{B}) \subset \mathbb{R}^+ \times \mathbb{H}.$$

Control Function β. Take the function $\xi : [0, \infty) \to [\alpha_1, 1]$ given by (21), where we assume that $\alpha_1 > 0$ was chosen to be sufficiently small such that

$$\alpha_1\lambda_2\mu_{\max} \in (0, 1),\ \alpha_1 < \alpha_o, \quad \text{where } \ \mu_{\max} := \max\{\mu : \mu \in \sigma(B)\},$$

and $\beta : \mathbb{H} \to \mathbb{R}_+$ given by

$$\beta(x) = \xi(\operatorname{dist}(x, \widetilde{\mathcal{U}})),$$

(cf. (22), see Figure 3). Then replace the parameter α in the equation ($28_{\alpha\rho}$) with $\beta(x)$, i.e. consider the following equation:

$$\begin{cases} \dot{x}(t) = -\beta(x)\lambda A x(t - \frac{\tau}{\lambda}) \cdot \big(b + \rho x(t)\big),\\ x_0 = x_{2\pi}, \end{cases} \tag{$28_{\beta\rho}$}$$

where $\rho \in [0, 1]$.

Admissible Homotopies. We consider the compact field $\mathfrak{F}_\rho : \mathbb{R}^+ \times \mathbb{H} \to \mathbb{H}$ by (23) and the set

$$\mathcal{S} := \{(\rho, \lambda, x) \in [0, 1] \times \mathbb{R}^+ \times \mathcal{D} : \mathfrak{F}_\rho(\lambda, x) = 0\},$$

which is the solution set of ($28_{\beta\rho}$) in $\mathcal{D}$.

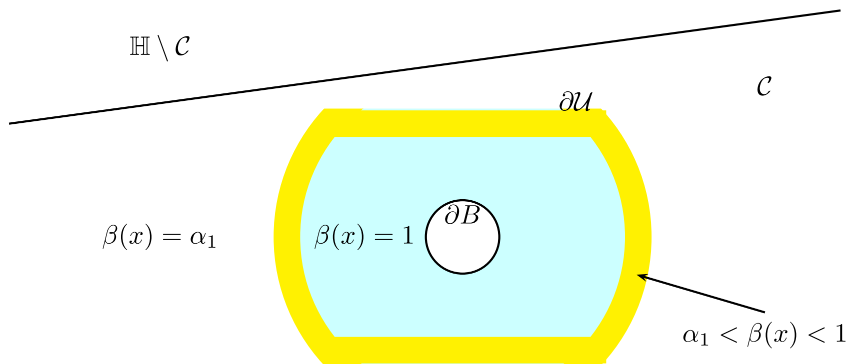

FIGURE 3. The sets $\mathcal{U}\setminus\overline{B}$, $\partial\mathcal{U}$ and ∂B.

Lemma 4.5. *Under the above assumptions, for fixed $\lambda_1,\lambda_2\in\mathbb{R}^+$ with $\lambda_1<\lambda_2$, we have*

$$\begin{aligned}\mathcal{S}\cap\big([0,1]\times(\lambda_1,\lambda_2)\times\partial(\mathcal{U}\setminus B)\big)&=\emptyset,\\ \mathcal{S}\cap\big([0,1]\times\{\lambda_1\}\times\overline{(\mathcal{U}\setminus B)}\big)&=\emptyset,\\ \mathcal{S}\cap\big([0,1]\times\{\lambda_2\}\times\overline{(\mathcal{U}\setminus B)}\big)&=\emptyset.\end{aligned}$$

Proof: Let $(\rho,\lambda,x)\in\mathcal{S}$, i.e. x is a solution to ($28_{\beta\rho}$). Assume first that $\rho\in(0,1]$. Suppose that $(\lambda,x)\in\partial\mathcal{U}$. By multiplying ($28_{\beta\rho}$) with ρ and using the substitution $u(t):=\rho x(t)$, we obtain

$$\dot u(t)=-\lambda\beta(x)Au(t-\tau/\lambda)\cdot\big(b+u(t)\big),$$

which means u is a solution to (28_α) with $\alpha=\beta(x)$. If $(\lambda,x)\in\partial\mathcal{U}$, then by definition of the function β, we have $\beta(x)=\alpha_1\le\alpha_o$. However, by Lemma 4.1(ii), this implies that $u=0$, which is the contradiction.

In the case $\rho=0$, we are dealing with the linear system

$$\dot x(t)=-\beta(x)\lambda Ax(t-\tau/\lambda)\cdot b,\quad \lambda\in[\lambda_1,\lambda_2]. \tag{43}$$

If $(\lambda,x)\in[\lambda_1,\lambda_2]\times\mathcal{D}$ is a solution to (43) such that $(\lambda,x)\in\partial\mathcal{U}$ then $\beta(x)=\alpha_1$, which means x is a non-zero solution to (37) with $\alpha=\lambda\alpha_1$ and $\lambda\le\lambda_2$. However, by Lemma 4.2(i) (see (38)) there exist positive integers k and $\mu\in\sigma(B)$ such that $\lambda\alpha_1\mu=k$, which is impossible because $\lambda\alpha_1\mu<\alpha_1\lambda_1\mu_{\max}<1\le k$ for all $k\in\mathbb{N}$. In this way we obtain that $\mathcal{S}\cap\big([0,1]\times(\lambda_1,\lambda_2)\times\partial(\mathcal{U}\setminus\overline{B})\big)=\emptyset$.

Consider now $\lambda_o=\lambda_1$ or λ_2, where $\lambda_1=\frac{\tau}{2j_1\pi}<\lambda_2=\frac{\tau}{2j_2\pi}$, for some $j_1,j_2\in\mathbb{N}$, and assume that $x\in\partial B$ is a solution to the equation ($28_{\beta\rho}$) for $\rho\in[0,1]$. Since $\beta(x)=1$, the equation ($28_{\beta\rho}$) becomes (40) and by Lemma 4.4, $x=0$, which is again a contradiction. This completes the proof. □

4.3. **Computation of the Equivariant Topological Invariant.** By Lemma 4.5, we obtain that $\mathfrak{F}_\rho$ for $\rho\in[0,1]$, is an $\Omega_{\lambda_1,\lambda_2}$-admissible homotopy. Consequently, we have the G-equivariant topological invariant

$$\boxed{\omega}:=G\text{-Deg}\,(\mathfrak{F}_1,\Omega_{\lambda_1,\lambda_2})=G\text{-Deg}\,(\mathfrak{F}_0,\Omega_{\lambda_1,\lambda_2}).$$

By using the decomposition (24) of the space $\mathbb{H}$ and making small adjustments related to the function $\beta(x)$, we can represent the map $\mathfrak{F}_0 : \mathbb{R}_+ \times \mathbb{H} \to \mathbb{H}$ in the following form

$$\mathfrak{F}_0(\lambda, x) = \big(\lambda\beta(\bar{x})B\bar{x}, x^* + \lambda\beta(x^*)L^{-1}\mathcal{N}(0, \lambda, j(x^*))\big), \tag{44}$$

where $x = \bar{x} + x^* \in V \oplus \mathbb{H}^*$. Our goal is to compute G-Deg $(\mathfrak{F}_0, \Omega_{\lambda_1,\lambda_2})$.

For convenience, define $\overline{\mathfrak{F}}_0 : \mathbb{R}_+ \times V \to V$ and $\mathfrak{F}_0^* : \mathbb{R}_+ \times \mathbb{H}^* \to \mathbb{H}^*$ by

$$\overline{\mathfrak{F}}_0(\lambda, \bar{x}) := \lambda\beta(\bar{x})B\bar{x}, \quad \bar{x} \in V$$
$$\mathfrak{F}_0^*(\lambda, x^*) := x^* + \lambda\beta(x^*)L^{-1}\mathcal{N}(0, \lambda, x^*),$$

and define

$$\Omega^*_{\lambda_1,\lambda_2} := \Omega_{\lambda_1,\lambda_2} \cap \mathbb{H}^*, \quad \mathcal{B} := \{\bar{x} \in V : \|x\| < 1\}.$$

Then, since $\mathfrak{F}_0$ is a G-homotopic to the product map $\mathfrak{F}_0^* \times \overline{\mathfrak{F}}_0$, by the multiplicativity property (cf. Theorem 2.2), we have

$$\boxed{\square} = G\text{-Deg}\,(\mathfrak{F}_0, \Omega_{\lambda_1,\lambda_2}) = \Gamma\text{-Deg}(\overline{\mathfrak{F}}_0, \mathcal{B}) \cdot G\text{-Deg}\,(\mathfrak{F}_0^*, \Omega^*_{\lambda_1,\lambda_2}).$$

Since $\beta(x) > 0$, the map $\overline{\mathfrak{F}}_0$ is G-homotopic to the map given by $\bar{x} \mapsto B\bar{x}$. However, $B := \operatorname{diag}(b)A$ has all the eigenvalues positive, therefore it can be deformed Γ-equivariantly, using for example a path in $GL^\Gamma(V)$ to Id. Thus,

$$\Gamma\text{-Deg}(\overline{\mathfrak{F}}_0, \mathcal{B}) = \Gamma\text{-Deg}(\mathrm{Id}\,, \mathcal{B}) = (\Gamma),$$

and consequently

$$\boxed{\square} = G\text{-Deg}\,(\mathcal{F}_2, \Omega^*_{\lambda_1,\lambda_2}).$$

In order to compute G-Deg $(\mathfrak{F}_0^*, \Omega^*_{\lambda_1,\lambda_2})$, consider the S^1-isotypical decomposition (24) of the space $\mathbb{H}^*$ and the restrictions $A_l := \mathfrak{F}_0^*|_{\mathbb{H}_l} = \mathfrak{F}_0|_{\mathbb{H}_l}$, $l = 1, 2, \dots$. By a homotopy argument, we can assume that

$$\mathcal{A}_l(\lambda, e^{ilt}z) = e^{itl}\left[z + \frac{\beta(z)}{il}e^{-\frac{il\tau}{\lambda}}Bz\right], \quad z \in V,$$

i.e., the map $\mathcal{A}_l$ can be represented as

$$\mathcal{A}_l(\lambda, z) := z - \frac{i\beta(z)}{l}e^{-\frac{il\tau}{\lambda}}Bz, \quad z \in V.$$

Therefore, by the additivity property of the twisted equivariant degree (cf. Theorem 2.1) we have

$$G\text{-Deg}\,(\mathfrak{F}_0, \Omega_{\lambda_1,\lambda_2}) = \sum_{l>0} G\text{-Deg}\,(\mathcal{A}_l, \Omega^*_{\lambda_1,\lambda_2} \cap \mathbb{H}_l),$$

where the above sum contains only finitely many non-zero terms.

Assume for simplicity that the matrix B is fully digonalizable and consider $\mu \in \sigma(B)$. Then for $z \in \widetilde{E}_l(\mu) \subset \mathbb{H}_l \simeq V^c$, where $\widetilde{E}_l(\mu) := \widetilde{E}(\mu)$ denotes, with respect to this identification, the Γ-invariant (complex) eigenspace of μ for $B : V^c \to V^c$, and

$$\mathcal{A}_l(\lambda, z) = z - \frac{i\beta(z)\mu}{l}e^{-\frac{il\tau}{\lambda}}z, \quad z \in \widetilde{E}(\mu).$$

Therefore,

$$G\text{-Deg}\,(\mathfrak{F}_0, \Omega_{\lambda_1,\lambda_2}) = \sum_{l>0} \sum_{\mu \in \sigma(B)} G\text{-Deg}\,(\mathcal{A}_l, \Omega^*_{\lambda_1,\lambda_2} \cap \widetilde{E}(\mu)).$$

By applying the homotopy and excision properties of the twisted equivariant degree to the set $\Omega^*_{\lambda_1,\lambda_2} \cap \widetilde{E}(\mu)$ and the function β we can simplify further these computations. More precisely, for each $l = 1, 2, \dots$ and $\mu \in \sigma(B)$ we put

$$\mathcal{U}_{l,\mu} := \left\{ z \in \widetilde{E}_l(\mu) : \frac{1}{2} < |z| < 2 \right\},$$
$$\Omega_{l,\mu} := (\lambda_1, \lambda_2) \times \mathcal{U}_{l,\mu},$$

and define the function $\beta : \widetilde{E}_l(\mu) \to [\alpha_1, 1]$ by

$$\beta(z) = \begin{cases} 1 & \text{if } \ |z| \le 1, \\ 2 - \alpha_1 - (1-\alpha_1)|z| & \text{if } \ 1 < |z| < 2, \\ \alpha_1 & \text{if } \ |z| \ge 2. \end{cases}$$

Then,

$$G\text{-Deg}\,(\mathfrak{F}_0, \Omega_{\lambda_1,\lambda_2}) = \sum_{l>0} \sum_{\mu \in \sigma(B)} G\text{-Deg}\,(\mathcal{A}_l, \Omega_{l,\mu}).$$

Suppose that $A_l(\lambda, z) = 0$ for some $0 \ne z \in \widetilde{E}(\mu)$, $\mu \in \sigma(B)$. Then by (38),

$$\begin{cases} \beta(z) = \frac{l}{\lambda\mu} < 1, \\ \lambda := \lambda_{l,m} = \frac{l\tau}{2\pi m + \pi/2}, \quad \text{for some } \ m \in \mathbb{N}. \end{cases}$$

Denote by $n(\mu)$ a positive integer such that

$$\frac{\pi}{2} + 2n(\mu)\pi < \mu\tau < \frac{\pi}{2} + 2(n(\mu)+1)\pi. \tag{45}$$

Denote

$$\Lambda_{l,\mu} := \{m : lj_2 \le m < lj_1, \ \ n(\mu) \ge m\}. \tag{46}$$

Then, one can easily verify that we have the following set of zeros of A_l in $\Omega_{l,\mu}$

$$Z_{l,\mu} := \{(\lambda_{l,m}, z) : m \in \Lambda_{l,\mu}, \ \beta(z) = \frac{1}{\lambda_{l,m}\mu}\} = \{(\lambda_{l,m}, z) : \lambda_{l,m} \in \Lambda_{l,\mu}, \ |z| = t_{l,m,\mu}\},$$

where

$$t_{l,m,\mu} := \frac{(2-\alpha_1)\lambda_{l,m}\mu - 1}{\lambda_{l,m}\mu(1-\alpha_1)}.$$

In particular, for $z \in \widetilde{E}(\mu)$, $\lambda \in [\lambda_1, \lambda_2]$, on a small neighborhood $\mathcal{U}_{l,m,\mu}$ of $\{(\lambda_{l,m}, z) : |z| = t_{l,m,\mu}\}$ in $\Omega_{l,\mu}$ we have

$$\begin{aligned} \mathcal{A}_l(\lambda, z) =& (1 - \frac{i\mu\lambda\beta(z)}{l} e^{-i\frac{l\tau}{\lambda}})z \\ =& (1 - \frac{\mu\lambda\beta(z)}{l} \sin\frac{k\tau}{\lambda} - i\frac{\mu\lambda\beta(z)}{l}\cos\frac{l\tau}{\lambda})\mathrm{Id}\,. \end{aligned}$$

For a small neighbourhood of $\lambda = \lambda_{l,m} = \frac{l\tau}{2m\pi + \frac{\pi}{2}}$, (i.e. $\frac{l\tau}{\lambda} = 2m\pi + \frac{\pi}{2}$) the above map is homotopic to

$$(\lambda, z) \mapsto (1 - \frac{\mu\lambda_{l,m}\beta(z)}{l} - i\frac{\mu\lambda_{l,m}\beta(z)}{l}(\lambda_{l,m} - \lambda))z =: \varphi(\lambda, |z|)z,$$

where

$$\varphi(\lambda,t)=1-\frac{\mu\lambda_{l,m}\xi(t)}{l}+i\frac{\mu\lambda_{l,m}\xi(t)}{l}(\lambda-\lambda_{l,m}),\quad \xi(t)=2-\alpha_1-(1-\alpha_1)t.$$

It is easy to compute

$$D\varphi(t_{l,m,\mu})=\begin{bmatrix}0 & \frac{\mu\lambda_{l,m}}{l}(1-\alpha_1)\\ \frac{\mu\lambda_{l,m}}{l}\xi(t_{l,m,\mu}) & 0\end{bmatrix},$$

and since $\det D\varphi(t_{l,m,\mu})<0$, we immediately obtain (cf. [2])

$$G\text{-Deg}\,(\mathcal{A}_l,\mathcal{U}_{l,m,\mu})=-\sum_{j=0}^{s}m_j(\mu)\deg_{\mathcal{V}_{j,l}},$$

where $V^c:=U_0\oplus\cdots\oplus U_s$ is a (complex) Γ-isotypical decomposition of V^c, with U_j modelled on $\mathcal{U}_j$, $m_j(\mu)=\dim(\widetilde{E}(\mu)\cap U_j)/\dim\mathcal{U}_j$ being the $\mathcal{U}_j$-multiplicity of μ, and $\deg_{\mathcal{V}_{j,l}}$ standing for the basic degree for the representation $\mathcal{V}_{j,l}$ (see [2] for more information and the precise description of the convention used here).

In this way, the following result provides us with a computational formula for 回.

Proposition 2. *Under the above assumptions we have*

$$回=-\sum_{l>0}\sum_{\mu\in\sigma(B)}\sum_{m\in\Lambda(l,\mu)}\sum_{j=0}^{s}m_j(\mu)\deg_{\mathcal{V}_{j,l}}.$$

On the other hand, by Theorem 3.2, we have

Theorem 4.6. *Under the assumptions (H0)–(H3), if the G-equivariant topological invariant*

$$回=\sum_{(H)}n_H(H)$$

is nonzero, i.e. there exist a coefficient $n_H\neq 0$ with $H=K^{\varphi,l}$. Moreover, there exists $(\lambda,x)\in\Omega_{\lambda_1,\lambda_2}$ such that $\mathfrak{F}_1(\lambda,x)=0$ with $G_x\supset H$. In other words, there exists a nonconstant 2π-periodic solution to (28) for some $\lambda\in[\lambda_1,\lambda_2]$, and consequently, there is a p-periodic solution to (26) with $p=2\pi\lambda$. In addition, if $H=K^{\varphi,l}$ is a dominating type in $\mathbb{H}$, then there exists a non-constant periodic solution $x(t)$ to (26) with the isotropy group K^{φ}.

Therefore, as an immediate consequence, we obtain the following generalization of the result obtained in [5] (without assumption of simplicity on the eigenvalues of the matrix B):

Corollary 1. *Suppose that $\Gamma=\{e\}$. Under the assumptions (H0)–(H3), if there exist an eigenvalue $\mu\in\sigma(B)$ and $n\in\mathbb{N}\cup\{0\}$ such that*

$$\frac{\pi}{2}+2n\pi<\mu\tau<\frac{\pi}{2}+2(n+1)\pi,$$

then the G-equivariant topological invariant

$$回=\sum_{(H)}n_H(H)$$

is nonzero, and consequently, there exists a p-periodic solution to (26).

5. **Examples.** In the following examples we consider the system (27), symmetric with respect to the group Γ being Q_8, D_8 and S_4. We assume that $b = [1, 1, \dots, 1]^T$ and consider the matrix A with concrete numerical values of its entries. We also specify the numerical value of the delay $\tau > 0$. The spectrum of A is denoted by $\{\mu_i : 1 \le i \le r\}$ and the eigenspace $\widetilde{E}(\mu_i)$ corresponding to μ_i, as it turns out, is of a single Γ-isotypical type such that $\widetilde{E}(\mu_i) \equiv \mathcal{V}_{j(i)}$, i.e. the $\mathcal{V}_j$-multiplicity $\widehat{m}_j(\mu)$ of the eigenvalue of μ is one. Similarly, for the matrix $A : V^c \to V^c$ we denote by $E(\mu_i)$ the (complex) eigenspace, which in our cases is $E(\mu_i) = m_{j(i)}(\mu_i) \cdot \mathcal{U}_{j(n)}$, where $m_{j(i)}(\mu_i)$ is the $\mathcal{U}_{j(i)}$-multiplicity of μ_i. The number $m_{j(i)}(\mu_i)$ is always one, except for the case $\Gamma = Q_8$, where the considered real eigenspace is of quaternionic type, so this number is equal to 2. We choose the values of $j_1 = 2$ and $j_2 = 1$, and put (cf. (46))

$$\mathfrak{m}_{l,j(i)} := m_{j(i)}(\mu_i)|\Lambda_{l,\mu_i}|,$$

where $E(\mu_i) = m_{j(i)}(\mu_i) \cdot \mathcal{U}_{j(i)}$ and $|X|$ denotes the number of elements in the set X. Then, using this notation, our computational formula for the G-equivariant topological invariant can be written as

$$回 = -\sum_{l>0} \sum_{\mu \in \sigma(B)} \sum_{m \in \Lambda(l,\mu)} \sum_{j=0}^{s} m_j(\mu) \deg_{\mathcal{V}_{j,l}} = -\sum_{l>0} \sum_{j=1}^{s} \mathfrak{m}_{j,l} \deg_{\mathcal{V}_{j,l}}. \tag{47}$$

For the computation of the numbers $n(\mu_i)$ specified by (45) we will use Table 1.

n	1	2	3	4	5	6	7	8	9	10
$\frac{\pi}{2} + 2n\pi$	7.9	14.1	20.4	26.7	33.0	39.27	45.6	51.8	58.1	64.4

TABLE 1. Values of $\frac{\pi}{2} + 2n\pi$.

The final results are formulated using the so-called basic degrees $\deg_{\mathcal{V}_{j,l}}$. We will only list the basic degrees $\deg_{\mathcal{V}_{j,1}}$. The degrees $\deg_{\mathcal{V}_{j,l}}$ can be determined by taking the *l-folding homomorphism* of $\deg_{\mathcal{V}_{j,1}}$, i.e. $\deg_{\mathcal{V}_{j,l}} = \Psi_l(\deg_{\mathcal{V}_{j,1}})$, for $\Psi_l : A_1^t(G) \to A_1^t(G)$ defined (on generators) by $(H^{\varphi,k}) \mapsto (H^{\varphi,kl})$ (cf. [2]).

For each non-zero coefficient in G-Deg$(\mathfrak{F}_0, \Omega_{\lambda_1,\lambda_2})$ of $(H^{\varphi,l})$, where (H^φ) is a dominating orbit type, there exist at least $|\Gamma/H|$ different non-constant p-periodic solutions with the least symmetry $(H^{\varphi,k})$ for some integer $k \ge 1$. However, the k-folding in the isotropy group $(H^{\varphi,k})$ of $x \in \mathbb{H}^*$ means that x is a p/k-periodic solution with exact symmetry (H^φ). In this way we are able to predict the exact symmetries of certain periodic solutions.

5.1. **Quaternionic Group Q_8.** The quaternionic group Q_8 can be described as a subgroup of S_8 generated by

$$i := (1324)(5867), \quad j := (1526)(3748).$$

We consider the space $V := \mathbb{R}^8$ on which Q_8 acts by permuting the coordinates of vectors $x \in V$. Consider the matrix

$$A := \begin{bmatrix} a & c & b & b & d & d & e & e \\ c & a & b & b & d & d & e & e \\ b & b & a & c & e & e & d & d \\ b & b & c & a & e & e & d & d \\ d & d & e & e & a & c & b & b \\ d & d & e & e & c & a & b & b \\ e & e & d & d & b & b & a & c \\ e & e & d & d & b & b & c & a \end{bmatrix}$$

The matrix A commutes with the Q_8-action on V. The matrix A has the following eigenvalues and eigenspaces:

$$\begin{aligned}
&\mu_1 := a - 2e + c - 2b + 2d, && \widetilde{E}(\mu_1) \simeq \mathcal{V}_2 \\
&\mu_2 := a - 2e + c + 2b - 2d, && \widetilde{E}(\mu_2) \simeq \mathcal{V}_1, \\
&\mu_3 := a - c, && \widetilde{E}(\mu_3) \simeq \mathcal{V}_4 \ \text{(quaternionic type)}, \\
&\mu_4 := a + 2e + c - 2b - 2d, && \widetilde{E}(\mu_4) \simeq \mathcal{V}_3, \\
&\mu_5 := a + 2e + c + 2b + 2d, && \widetilde{E}(\mu_5) \simeq \mathcal{V}_0.
\end{aligned}$$

For definiteness, we choose the positive entries of A being $a = 8$, $b = 1$, $c = 3$, $d = 2$, $e = 1.5$ and $\tau = 4$, so

$$\tau\mu_1 = 40, \ \ \tau\mu_2 = 24, \ \ \tau\mu_3 = 20, \ \ \tau\mu_4 = 32, \ \ \tau\mu_5 = 80,$$

and we can easily determine the values $n(\mu_i)$ from Table 1, i.e.

$$n(\mu_1) = 6, \ \ n(\mu_2) = 3, \ \ n(\mu_3) = 2, \ \ n(\mu_4) = 4, \ \ n(\mu_5) = 12.$$

Then we have

$$\begin{gathered}
\mathfrak{m}_{0,1} = 1, \ \ \mathfrak{m}_{0,2} = 2, \ \ \mathfrak{m}_{0,3} = 3, \ \ \mathfrak{m}_{0,4} = 4, \ \ \mathfrak{m}_{0,5} = 5, \ \ \mathfrak{m}_{0,6} = 6, \\
\mathfrak{m}_{0,7} = 7, \ \ \mathfrak{m}_{0,8} = 5, \ \ \mathfrak{m}_{0,9} = 4, \ \ \mathfrak{m}_{0,10} = 3, \ \ \mathfrak{m}_{0,11} = 2, \ \ \mathfrak{m}_{0,12} = 1, \\
\mathfrak{m}_{1,1} = 1, \ \ \mathfrak{m}_{1,2} = 2, \ \ \mathfrak{m}_{1,3} = 1, \ \ \mathfrak{m}_{2,1} = 1, \ \ \mathfrak{m}_{2,2} = 2, \ \ \mathfrak{m}_{2,3} = 3, \\
\mathfrak{m}_{2,4} = 3, \ \ \mathfrak{m}_{2,5} = 2, \ \ \mathfrak{m}_{2,6} = 1, \ \ \mathfrak{m}_{3,1} = 1, \ \ \mathfrak{m}_{3,2} = 2, \ \ \mathfrak{m}_{3,3} = 2, \\
\mathfrak{m}_{3,4} = 1, \ \ \mathfrak{m}_{4,1} = 2, \ \ \mathfrak{m}_{4,1} = 2.
\end{gathered}$$

By applying formula (47) we obtain

$$\begin{aligned}
回 = &- \deg_{\mathcal{V}_{0,1}} - 2\deg_{\mathcal{V}_{0,2}} - 3\deg_{\mathcal{V}_{0,3}} - 4\deg_{\mathcal{V}_{0,4}} - 5\deg_{\mathcal{V}_{0,5}} - 6\deg_{\mathcal{V}_{0,6}} - 7\deg_{\mathcal{V}_{0,7}} \\
&- 5\deg_{\mathcal{V}_{0,8}} - 4\deg_{\mathcal{V}_{0,9}} - 3\deg_{\mathcal{V}_{0,10}} - 2\deg_{\mathcal{V}_{0,11}} - \deg_{\mathcal{V}_{0,12}} - \deg_{\mathcal{V}_{1,1}} - 2\deg_{\mathcal{V}_{1,2}} \\
&- \deg_{\mathcal{V}_{1,3}} - \deg_{\mathcal{V}_{2,1}} - 2\deg_{\mathcal{V}_{2,2}} - 3\deg_{\mathcal{V}_{2,3}} - 3\deg_{\mathcal{V}_{2,4}} - 2\deg_{\mathcal{V}_{2,5}} - \deg_{\mathcal{V}_{2,6}} \\
&- \deg_{\mathcal{V}_{3,1}} - 2\deg_{\mathcal{V}_{3,2}} - 2\deg_{\mathcal{V}_{3,3}} - \deg_{\mathcal{V}_{3,4}} - 2\deg_{\mathcal{V}_{4,1}} - 2\deg_{\mathcal{V}_{4,1}},
\end{aligned}$$

where

$$\begin{aligned}
\deg_{\mathcal{V}_{0,1}} &= (\mathbf{Q_8}), \quad \deg_{\mathcal{V}_{k,1}} = (\mathbf{Q_8^{k-}}), \ \ k = 1, 2, 3 \\
\deg_{\mathcal{V}_{4,1}} &= (\mathbb{Z}_\mathbf{4}^{\mathbf{1+}}) + (\mathbb{Z}_\mathbf{4}^{\mathbf{2+}}) + (\mathbb{Z}_\mathbf{4}^{\mathbf{3+}}) - (\mathbb{Z}_2^-)
\end{aligned}$$

The dominating orbit types in $\mathbb{H}^*$ are (Q_8), (Q_8^{k-}) and $(\mathbb{Z}_4^{k+})$ for $k = 1, 2, 3$. Consequently, we obtain

- there is at least 1 non-constant periodic solution with orbit type (Q_8) ,

- there is at least 1 non-constant periodic solution with orbit type (Q_8^{1-}),
- there are at least 1 non-constant periodic solution with orbit type (Q_8^{2-}),
- there are at least 1 non-constant periodic solution with orbit type (Q_8^{3-}),
- there are at least 2 non-constant periodic solutions with orbit type $(\mathbb{Z}_4^{1+})$,
- there are at least 2 non-constant periodic solutions with orbit type $(\mathbb{Z}_4^{2+})$,
- there are at least 2 non-constant periodic solutions with orbit type $(\mathbb{Z}_4^{3+})$.

In summary, there exist at least 10 nonconstant periodic solutions of (26).

5.2. **Dihedral Group $\Gamma = D_8$.** Consider the dihedral group $D_8 = \{1, \gamma, \gamma^2, \dots, \gamma^7, \kappa, \kappa\gamma, \gamma^2, \dots, \kappa\gamma^7\} \subset O(2)$, where γ can be identified with $e^{\frac{\pi i}{4}}$ (i.e. γ is a complex linear operator $\gamma(z) = e^{\frac{\pi i}{4}} z$) and $\kappa := \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$. We consider the space $V := \mathbb{R}^8$, where $\gamma \in D_8$ acts on a vector $(x_1, x_2, \dots, x_8)$ by sending x_k to $x_{k+1 \pmod n}$ and $\kappa \in D_8$ acts by reversing the order of the components of x. Consider the following D_8-equivariant matrix A

$$A := \begin{bmatrix} d & c & d & 0 & 0 & 0 & 0 & 0 \\ 0 & d & c & d & 0 & 0 & 0 & 0 \\ 0 & 0 & d & c & d & 0 & 0 & 0 \\ 0 & 0 & 0 & d & c & d & 0 & 0 \\ 0 & 0 & 0 & 0 & d & c & d & 0 \\ 0 & 0 & 0 & 0 & 0 & d & c & d \\ d & 0 & 0 & 0 & 0 & 0 & d & c \end{bmatrix}$$

. The matrix A has the following eigenvalues and the corresponding eigenspaces

$$\begin{aligned} \mu_1 &:= c + 2d, & \widetilde{E}(\mu_1) &\simeq \mathcal{V}_0 \\ \mu_2 &:= c + \sqrt{2}d, & \widetilde{E}(\mu_2) &\simeq \mathcal{V}_1, \\ \mu_3 &:= c, & \widetilde{E}(\mu_3) &\simeq \mathcal{V}_2, \\ \mu_4 &:= c - \sqrt{2}d, & \widetilde{E}(\mu_4) &\simeq \mathcal{V}_3, \\ \mu_5 &:= c - 2d, & \widetilde{E}(\mu_5) &\simeq \mathcal{V}_5. \end{aligned}$$

For definiteness, choose $c = 9$, $d = 3$ and $\tau = 4$, so

$$\tau\mu_1 = 60, \;\; \tau\mu_2 \approx 52.97, \;\; \tau\mu_3 = 36, \;\; \tau\mu_4 \approx 19.03, \;\; \tau\mu_5 = 12.$$

To determine the numbers $n(\mu_i)$, for $i = 0, 1, 2, 3, 5$, we list the approximate values of $\frac{\pi}{2} + 2n\pi$ and use the Table 1. Thus, we have

$$n(\mu_1) = 9, \;\; n(\mu_2) = 8, \;\; n(\mu_3) = 5, \;\; n(\mu_4) = 2, \;\; n(\mu_5) = 1.$$

Let $j_1 = 2$ and $j_2 = 1$. Then,

$$\begin{gathered} \mathfrak{m}_{0,1} = 1, \;\; \mathfrak{m}_{0,2} = 2, \;\; \mathfrak{m}_{0,3} = 3, \;\; \mathfrak{m}_{0,4} = 4, \;\; \mathfrak{m}_{0,5} = 5, \;\; \mathfrak{m}_{0,6} = 4, \\ \mathfrak{m}_{0,7} = 3, \;\; \mathfrak{m}_{0,8} = 2, \;\; \mathfrak{m}_{0,9} = 1, \;\; \mathfrak{m}_{1,1} = 1, \;\; \mathfrak{m}_{1,2} = 2, \;\; \mathfrak{m}_{1,3} = 3, \\ \mathfrak{m}_{1,3} = 4, \;\; \mathfrak{m}_{1,5} = 4, \;\; \mathfrak{m}_{1,6} = 3, \;\; \mathfrak{m}_{1,7} = 2, \;\; \mathfrak{m}_{1,8} = 1, \;\; \mathfrak{m}_{2,1} = 1, \\ \mathfrak{m}_{2,2} = 2, \;\; \mathfrak{m}_{2,3} = 3, \;\; \mathfrak{m}_{2,4} = 2, \;\; \mathfrak{m}_{2,5} = 1, \;\; \mathfrak{m}_{3,1} = 1, \;\; \mathfrak{m}_{3,2} = 2, \\ \mathfrak{m}_{3,3} = 1, \;\; \mathfrak{m}_{5,1} = 1. \end{gathered}$$

By applying formula (47) we obtain

$$\begin{aligned}
\text{回} = &-\deg_{\mathcal{V}_{0,1}} -2\deg_{\mathcal{V}_{0,2}} -3\deg_{\mathcal{V}_{0,3}} -4\deg_{\mathcal{V}_{0,4}} -5\deg_{\mathcal{V}_{0,5}} -4\deg_{\mathcal{V}_{0,6}} -3\deg_{\mathcal{V}_{0,7}}\\
&-2\deg_{\mathcal{V}_{0,8}} -\deg_{\mathcal{V}_{0,9}} -\deg_{\mathcal{V}_{1,1}} -2\deg_{\mathcal{V}_{1,2}} -3\deg_{\mathcal{V}_{1,3}} -4\deg_{\mathcal{V}_{1,4}} -4\deg_{\mathcal{V}_{1,5}}\\
&-3\deg_{\mathcal{V}_{1,6}} -2\deg_{\mathcal{V}_{1,7}} -\deg_{\mathcal{V}_{1,8}} -\deg_{\mathcal{V}_{2,1}} -2\deg_{\mathcal{V}_{2,2}} -3\deg_{\mathcal{V}_{2,3}} -2\deg_{\mathcal{V}_{2,4}}\\
&-\deg_{\mathcal{V}_{2,5}} -\deg_{\mathcal{V}_{3,1}} -2\deg_{\mathcal{V}_{3,2}} -\deg_{\mathcal{V}_{3,3}} -\deg_{\mathcal{V}_{5,1}},
\end{aligned}$$

where

$$\begin{aligned}
\deg_{\mathcal{V}_{0,1}} &= (\mathbf{D_8}),\\
\deg_{\mathcal{V}_{1,1}} &= (\mathbb{Z}_{\mathbf{8}}^{\mathbf{t_1}}) + (\widetilde{D}_2^d) + (D_2^d) - (\mathbb{Z}_2^-),\\
\deg_{\mathcal{V}_{2,1}} &= (\widetilde{\mathbf{D}}_{\mathbf{4}}^{\mathbf{d}}) + (D_4^d) + (\mathbb{Z}_{\mathbf{8}}^{\mathbf{t_2}}) - (\mathbb{Z}_4^d),\\
\deg_{\mathcal{V}_{3,1}} &= (\mathbb{Z}_{\mathbf{8}}^{\mathbf{t_3}}) + (\widetilde{D}_2^d) + (D_2^d) - (\mathbb{Z}_2^-),\\
\deg_{\mathcal{V}_{5,1}} &= (\mathbf{D_8^d}).
\end{aligned}$$

The dominating orbit types in $\mathbb{H}^*$ are (D_8), (D_8^d), $(\mathbb{Z}_8^{t_1})$, $(\mathbb{Z}_8^{t_2})$, $(\mathbb{Z}_8^{t_3})$ and $(\widetilde{D}_4^d)$. Consequently, we obtain

- there is at least 1 non-constant periodic solution with orbit type (D_8),
- there is at least 1 non-constant periodic solution with orbit type (D_8^d),
- there are at least 2 non-constant periodic solutions with orbit type $(\widetilde{D}_4^d)$,
- there are at least 2 non-constant periodic solutions with orbit type $(\mathbb{Z}_8^{t_1})$,
- there are at least 2 non-constant periodic solutions with orbit type at lest $(\mathbb{Z}_8^{t_2})$,
- there are at least 2 non-constant periodic solutions with orbit type at least $(\mathbb{Z}_8^{t_3})$.

In summary, there exist at least 10 nonconstant periodic solutions of (26).

5.3. **Octahedral Group S_4.** Assume that the octahedral group S_4 acts on $V := \mathbb{R}^8$ by permuting the coordinates in such a way that $(1234) \in S_4$ corresponds to the permutation $(1234)(5678) \in S_8$ and $(12) \in S_4$ corresponds to $(17)(28)34)(56) \in S_8$ (i.e. S_4 acts on V in the same way as it permutes the vertices of a regular cube). Consider the matrix

$$A := \begin{bmatrix}
a & b & c & b & b & c & d & c\\
b & a & b & c & c & b & c & d\\
c & b & a & b & d & c & b & c\\
b & c & b & a & c & d & c & b\\
b & c & d & c & a & b & c & b\\
c & b & c & d & b & a & b & c\\
d & c & b & c & c & b & a & b\\
c & d & c & b & b & c & b & a
\end{bmatrix}$$

. The matrix A commutes with the S_4-action on V and has the following eigenvalues and eigenspaces:

$$\begin{aligned}
\mu_1 &:= 3b+a+3c+d, & \widetilde{E}(\mu_1) &\simeq \mathcal{V}_0\\
\mu_2 &:= -3b+a+3c-d, & \widetilde{E}(\mu_2) &\simeq \mathcal{V}_1,\\
\mu_3 &:= -b+a-c+d, & \widetilde{E}(\mu_3) &\simeq \mathcal{V}_4,\\
\mu_4 &:= b+a-c-d, & \widetilde{E}(\mu_4) &\simeq \mathcal{V}_3.
\end{aligned}$$

For definiteness, choose $a=6$, $b=1$, $c=2$, $d=2.5$ and $\tau=4$, so

$$\tau\mu_1=70,\ \ \tau\mu_2=26,\ \ \tau\mu_3=22,\ \ \tau\mu_4=10,$$

and we can easily determine the values $n(\mu_i)$ from Table 1, i.e.

$$n(\mu_1)=10,\ \ n(\mu_2)=3,\ \ n(\mu_3)=3,\ \ n(\mu_4)=1.$$

As before, we choose $j_2=1$ and $j_1=2$. Then we have

$$\begin{gathered}
\mathfrak{m}_{0,1}=1,\ \ \mathfrak{m}_{0,2}=2,\ \ \mathfrak{m}_{0,3}=3,\ \ \mathfrak{m}_{0,4}=4,\ \ \mathfrak{m}_{0,5}=5,\ \ \mathfrak{m}_{0,6}=5,\\
\mathfrak{m}_{0,7}=4,\ \ \mathfrak{m}_{0,8}=3,\ \ \mathfrak{m}_{0,9}=2,\ \ \mathfrak{m}_{0,10}=1,\ \ \mathfrak{m}_{0,11}=2,\ \ \mathfrak{m}_{0,12}=1,\\
\mathfrak{m}_{1,1}=1,\ \ \mathfrak{m}_{1,2}=2,\ \ \mathfrak{m}_{1,3}=1,\ \ \mathfrak{m}_{3,1}=1,\ \ \mathfrak{m}_{3,2}=2,\ \ \mathfrak{m}_{3,3}=1,\\
\mathfrak{m}_{4,1}=1.
\end{gathered}$$

By applying formula (47) we obtain

$$\begin{aligned}
回 = &-\deg_{\mathcal{V}_{0,1}} -2\deg_{\mathcal{V}_{0,2}} -3\deg_{\mathcal{V}_{0,3}} -4\deg_{\mathcal{V}_{0,4}} -5\deg_{\mathcal{V}_{0,5}} -5\deg_{\mathcal{V}_{0,6}} -4\deg_{\mathcal{V}_{0,7}}\\
&-3\deg_{\mathcal{V}_{0,8}} -2\deg_{\mathcal{V}_{0,9}} -\deg_{\mathcal{V}_{0,10}} -\deg_{\mathcal{V}_{1,1}} -2\deg_{\mathcal{V}_{1,2}} -\deg_{\mathcal{V}_{1,3}} -\deg_{\mathcal{V}_{3,1}}\\
&-2\deg_{\mathcal{V}_{3,2}} -\deg_{\mathcal{V}_{3,3}} -\deg_{\mathcal{V}_{4,1}},
\end{aligned}$$

where

$$\begin{aligned}
\deg_{\mathcal{V}_{0,1}} &= (\mathbf{S_4}),\\
\deg_{\mathcal{V}_{1,1}} &= (\mathbf{S_4^-}),\\
\deg_{\mathcal{V}_{3,1}} &= (\mathbb{Z}_{\mathbf{4}}^{\mathbf{c}})+(\mathbf{D_4^d})+(\mathbf{D_2^d})+(D_3)+(\mathbb{Z}_{\mathbf{3}}^{\mathbf{t}})-(\mathbb{Z}_2^-)-(D_1),\\
\deg_{\mathcal{V}_{4,1}} &= (\mathbb{Z}_{\mathbf{4}}^{\mathbf{c}})+(\mathbf{D_4^z})+(D_2^d)+(D_3^z)+(\mathbb{Z}_{\mathbf{3}}^{\mathbf{t}})-(\mathbb{Z}_2^-)-(D_1^z).
\end{aligned}$$

The dominating orbit types in $\mathbb{H}^*$ are (S_4), (S_4^-), (D_4^d), (D_2^d), $(\mathbb{Z}_4^c)$, $(\mathbb{Z}_3^t)$ and (D_4^z). Consequently, we obtain

- there is at least 1 non-constant periodic solution with orbit type (S_4),
- there is at least 1 non-constant periodic solution with orbit type (S_4^-),
- there are at least 3 non-constant periodic solutions with orbit type (D_4^d),
- there are at least 6 non-constant periodic solutions with orbit type (D_2^d),
- there are at least 6 non-constant periodic solutions with orbit type at least $(\mathbb{Z}_4^c)$,
- there are at least 8 non-constant periodic solutions with orbit type at least $(\mathbb{Z}_3^t)$,
- there are at least 3 non-constant periodic solutions with orbit type at least (D_4^z).

In summary, there exist at least 28 nonconstant periodic solutions of (26).

Remark 2. One can consider other symmetry groups in (26), such as D_3, D_4, D_5, D_6, D_7, D_9, D_{10}, D_{11}, D_{12}, A_4 or A_5, for which there have been developed computational database (including Maple$^{\text{TM}}$ routines for the twisted equivariant degree). As it is clear from the formula (47) and the above examples, the similar existence results for all these groups can be easily obtained.

Remark 3. Let us point out that the results obtained in this paper can be easily translated into algorithms and computational routines allowing a development of a special software that could be used to instantly analyze this type of Lotka-Volterra symmetric systems, based on the spectral equivariant information provided by the matrix A. Taking into account that these systems are of special interest in mathematical biology, this computational tool could be of great benefit.

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E-mail address: `hirano0918@yahoo.co.jp`
E-mail address: `wieslaw@utdallas.edu`
E-mail address: `haibo.ruan@math.uni-hamburg.de`