Illinois Journal of Mathematics
Volume 55, Number 4, Winter 2011, Pages 1551–1567
S 0019-2082

FIXED POINTS IN ABSOLUTELY IRREDUCIBLE REAL REPRESENTATIONS

HAIBO RUAN

ABSTRACT. It has been an open question whether any bifurcation problem with absolutely irreducible group action would lead to bifurcation of steady states. A positive proposal is also known as the "Ize-conjecture". Algebraically speaking, this is to ask whether every absolutely irreducible real representation has an odd dimensional fixed point subspace corresponding to some subgroups. Recently, Reiner Lauterbach and Paul Matthews have found counter examples to this conjecture and interestingly, all of the representations are of dimension $4k$, for $k \in \mathbb{N}$. A natural question arises: what about the case $4k+2$?

In this paper, we give a partial answer to this question and prove that in any 6-dimensional absolutely irreducible real representation of a finite solvable group, there exists an odd dimensional fixed point subspace with respect to subgroups.

1. Introduction

In the course of investigating dynamical systems with symmetries, one becomes interested in studying properties of fixed point sets in the underlying representation spaces. Very often the algebraic property of fixed point spaces gives a strong indication to the topological property of the appearing dynamics.[1] As an example, in the context of equivariant bifurcations, an odd dimensional fixed point subspace leads to a bifurcation of steady states.

Received July 28, 2010; received in final form October 21, 2011.

Supported by Grant DFG LA 525/11-1.

2010 *Mathematics Subject Classification.* Primary 20C30, 19A22. Secondary 37G40, 37C25.

[1] In this paper, by "fixed point spaces" we mean fixed point spaces with respect to subgroups; that is, not only with respect to the whole group. Unless otherwise stated, the group is a compact Lie group and the subgroup is a closed subgroup.

It has been a fundamental question to the bifurcation theorists whether any bifurcation problem with absolutely irreducible group action always leads to bifurcation of steady states. From an algebraic viewpoint, this is to ask whether every absolutely irreducible real representation has an odd dimensional fixed point space. From a topological viewpoint, since the existence of odd dimensional fixed point spaces gives precisely the kind of obstruction that keeps the antipodal map from being *equivariantly* homotopic to the identity map, this is to ask whether the antipodal map is never equivariantly homotopic to the identity map on an absolutely irreducible real representation (cf. Lemma 2.16, Remark 2.20 and see also [1], [9], [10], [12]). However, the question was neither considered in the representation theory nor answered in the equivariant homotopy theory.

In [4], Mike Field quoted a private communication with Jorge Ize, who claimed that every absolutely irreducible real representation has an odd dimensional fixed point space.[2] However, counter examples were found by Reiner Lauterbach and Paul Matthews recently in [11], where they showed that there are three infinite series of groups of order $48 + 32m$, for $m \in \mathbb{N}$, which act absolutely irreducibly on $\mathbb{R}^4$, and have no odd dimensional fixed point spaces. Further counter examples were also found in dimensions $4k$ for $k \leq 5$. Nevertheless, no counter examples could be found in dimensions $4k+2$. Whether it is true in general that every $(4k + 2)$-dimensional absolutely irreducible real representation has an odd dimensional fixed point space remains open.

In this paper, we give a partial answer to this question and show that every 6-dimensional absolutely irreducible real representation of a finite solvable group has an odd dimensional fixed point space (cf. Theorem 3.2). The reason for the choice of finite solvable groups is that on one hand, they contain properly Abelian groups, nilpotent groups and monomial groups, for all of which the statement naturally holds; on the other hand, a finite solvable group has a composition series all of whose factors are of prime order, which allows us to inductively use results from character theory about reduced characters of prime index normal subgroups.

The proof is essentially an algebraic proof, with inspiration coming from character theory and equivariant degree theory. Notice that since a finite solvable group has a composition series all of whose factors are cyclic groups of prime order, we can assume the group G has a normal subgroup N of prime index p such that the restricted character χ_N is not absolutely irreducible. Moreover, we show that χ_N is either irreducible of complex type or splits into p distinct absolutely irreducible real representations of N (cf. Corollary 2.10).

[2] Jorge Ize believed this was true, since he thought that he had a topological proof of it. However, he realized that his proof was incomplete (cf. [7]).

In the case $p = 2$, we first prove a technical lemma on the characters of degree 6 and then reduce the problem to examining *a special type of groups*, namely finite solvable groups G which have an index 2 subgroup N such that all elements in $G \setminus N$ have order 4. In the case χ_N is irreducible of complex type, we show that if the set $S := \{g^2 : g \in G \setminus N\}$ consists of at least 2 conjugacy classes, then G has an order 8 subgroup, which produces a 1-dimensional fixed point space.

The remaining question is what happens if the set S forms a *single* conjugacy class in G. This case is analyzed by I. Martin Isaacs in his recent work [6], where he shows that if such a group has an irreducible character of degree $2m$ for an odd integer m, then the square-free part of m is not divisible by any prime $p \equiv 3 \pmod 4$. In particular, it excludes the possibility of G having degree 6 irreducible characters, and thus closes the proof for the case χ_N being irreducible of complex type. The splitting case was similar.

In the case $p > 2$, we adopt the concept of the *basic degree* from the equivariant degree theory (cf. [1], [9], [10], [12]) and associate to each G-representation V an element $a(V)$ in the Burnside ring $(A(G), \cdot)$ of G and call it the *characteristic* of G-representations. Without referring to the equivariant degree theory, we show that it is invariant with respect to G-isomorphisms and multiplicative with respect to product of G-representations. Moreover, V has no odd-dimensional fixed point spaces if and only if $a(V) = e$, where e denotes the multiplicative identity in $A(G)$ (cf. Lemma 2.16). With this algebraic tool at hand, we can show that the splitting case of χ_N is impossible without allowing odd dimensional fixed point spaces in V. The other case of χ_N being irreducible of complex type cannot happen if $p > 2$.

2. Preliminaries

2.1. Group representations and characters of representations. Let $\mathbb{F}$ be a field, V be a finite-dimensional $\mathbb{F}$-vector space, G be a compact Lie group and $\mathrm{GL}(V)$ be the space of invertible linear maps on V. Denote by $\mathbb{R}, \mathbb{C}, \mathbb{H}$ the field of real, complex and quaternion numbers, respectively.

Definition 2.1. (i) An $\mathbb{F}$-*representation* of G is a homomorphism of G to $\mathrm{GL}(V)$ for some V. The *dimension* of the representation is defined as the dimension of V. An $\mathbb{F}$-representation of G is called *irreducible*, if $\{0\}$ and V are the only G-invariant subspaces in V.

(ii) An irreducible $\mathbb{R}$-representation is said to be of *real type, complex type or quaternionic type*, if the set of commuting linear maps is isomorphic to $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$. An irreducible $\mathbb{R}$-representation of real type is also called *absolutely irreducible.*

Definition 2.2. Let $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$ and $\rho : G \to \mathrm{GL}(V)$ be an $\mathbb{F}$-representation. The *character* of the representation ρ is the function $\chi : G \to \mathbb{F}$ defined by

$$\chi(g) := \mathrm{Tr}\big(\rho(g)\big),$$

where $\mathrm{Tr}(\rho(g))$ is the trace of the linear map $\rho(g)$ on V. With respect to the Haar integral on G, the *inner product* of two characters χ, ψ is defined by

$$\langle \chi, \psi \rangle := \int_G \chi(g) \cdot \overline{\psi(g)}\, dg.$$

The *Frobenius–Schur indicator* of χ is defined by

$$\nu(\chi) := \int_G \chi\big(g^2\big)\, dg.$$

We call a character of an irreducible $\mathbb{F}$-representation an *irreducible* $\mathbb{F}$*-character.*

REMARK 2.3. Observe that for a finite group G, a subgroup $H \subset G$ and a character χ of G, we have

$$\langle \chi_H, \chi_H \rangle \leq |G : H| \langle \chi, \chi \rangle, \tag{1}$$

where χ_H is the restricted character on H, $|G : H|$ is the index of H in G, and the equality is achieved if and only if $\chi(g) = 0$ for all $g \in G \setminus H$.

Recall the following result on Frobenius–Schur indicators.

THEOREM 2.4 (Cf. [2]). *Let V be an irreducible $\mathbb{R}$-representation and χ be its character. Then, V is of real type if and only if $\langle \chi, \chi \rangle = 1$ and $\nu(\chi) = 1$; of complex type if and only if $\langle \chi, \chi \rangle = 2$ and $\nu(\chi) = 0$; and of quaternionic type if and only if $\langle \chi, \chi \rangle = 4$ and $\nu(\chi) = -2$.*

In what follows, we denote by $\mathrm{Irr}(G, \mathbb{R})$ and $\mathrm{Irr}(G, \mathbb{C})$ the set of all irreducible $\mathbb{R}$-characters and $\mathbb{C}$-characters, respectively. Let $\mathrm{Irr}_{\mathbb{F}}(G, \mathbb{R}) \subset \mathrm{Irr}(G, \mathbb{R})$ be the set of all irreducible $\mathbb{R}$-characters of $\mathbb{F}$-type, for $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$.

Recall the first orthogonality relation of irreducible characters.

THEOREM 2.5 (Cf. [5]). *Let $\chi, \psi \in \mathrm{Irr}(G, \mathbb{C})$. Then,*

$$\langle \chi, \psi \rangle = \begin{cases} 1, & \text{if } \chi = \psi \\ 0, & \text{otherwise.} \end{cases}$$

The following results will be used to exclude certain "singular" types of groups in the proof of our main result (cf. Theorem 3.2).

THEOREM 2.6 (I. M. Isaacs (cf. [6])). *Let G be a finite solvable group and $N \subset G$ be an index 2 subgroup such that $g^4 = 1$ for all $g \in G \setminus N$. Assume that all elements of the form g^2 for some $g \in G \setminus N$, are conjugate. Then G has a normal 2-complement*[3] *and if $\chi \in \mathrm{Irr}(G, \mathbb{C})$ has degree $2m$, where m is odd, then the square-free part of m is not divisible by any prime $p \equiv 3 \bmod 4$.*

[3] A subgroup H of G is called a *normal 2-complement*, if H is normal in G and for every 2-Sylow subgroup P of G, $HP = G$ and $H \cap P$ is trivial.

THEOREM 2.7 (I. M. Isaacs (cf. [6])). *Let G be a finite solvable group and $N \subset G$ be an index 2 subgroup such that $g^4 = 1$ for all $g \in G \setminus N$. Assume that the set $\{g^2 : g \in G \setminus N\}$ is a union of at most 3 conjugacy classes of G. Then if $\theta \in \mathrm{Irr}(N, \mathbb{C})$ is real valued and nonlinear of odd degree m, then $m = 3$ and θ is invariant in G.*

2.2. Normal subgroups and restricted characters. Let G be a finite group and denote by $N \triangleleft G$ a normal subgroup N in G.

DEFINITION 2.8. Let $N \triangleleft G$ and $g \in G$. For a character φ of an N-representation, define the *conjugate character* $\varphi^g : N \to \mathbb{C}$ by $\varphi^g(h) := \varphi(ghg^{-1})$.

Recall the following result on restricted characters of prime index normal subgroups, which plays an important role in the proof of our main result.

THEOREM 2.9 (Cf. [5]). *Let $N \triangleleft G$ with $|G : N| = p$, a prime. Suppose $\chi \in \mathrm{Irr}(G, \mathbb{C})$. Then either*

(a) *χ_N is irreducible or*
(b) *$\chi_N = \sum_{i=1}^{p} \theta_i$, where the θ_i are distinct, conjugate and irreducible.*

Now suppose that χ is a character of an absolutely irreducible $\mathbb{R}$-representation and $N \triangleleft G$ is a normal subgroup of prime index p. Then, χ is automatically a character of an irreducible $\mathbb{C}$-representation. By Theorem 2.9, either χ_N is irreducible or it splits into p distinct irreducible characters of N.

In the case (a), χ_N is irreducible as a $\mathbb{C}$-character. Since χ_N is real-valued, it is then absolutely irreducible as an $\mathbb{R}$-character. In the case (b), χ_N splits into p distinct irreducible characters $\theta_1, \dots, \theta_p$. Consider the complex conjugation on χ. Since χ is real valued, it fixes χ and thus permutes θ_i's. If p is odd, the permutation fixes at least one of these characters, say θ_r. This implies that θ_r is real-valued. Since every θ_i is conjugate to θ_r, θ_i is real-valued for $i = 1, \dots, p$. Consequently, each θ_i is absolutely irreducible as an $\mathbb{R}$-character. Otherwise, $p = 2$. Then, either the complex conjugation fixes θ_i, for $i = 1, 2$, in which case each θ_i is absolutely irreducible, or the complex conjugation permutes θ_1 to θ_2, which implies that χ_N is irreducible but not absolutely irreducible as an $\mathbb{R}$-character.

In summary, we have shown the following corollary.

COROLLARY 2.10. *Let $N \triangleleft G$ with $|G : N| = p$, a prime. Suppose $\chi \in \mathrm{Irr}_{\mathbb{R}}(G, \mathbb{R})$. Then one of the following holds*:

(a) $\chi_N \in \mathrm{Irr}_{\mathbb{R}}(N, \mathbb{R})$;
(b) $\chi_N \in \mathrm{Irr}_{\mathbb{C}}(N, \mathbb{R})$;
(c) $\chi_N = \sum_{i=1}^{p} \varphi_i$, *where the φ_i are distinct, conjugate and absolutely irreducible.*

2.3. An elementary lemma on $\chi(g^2)$. Let $\rho\colon G\to \mathrm{GL}(6,\mathbb{R})$ be a 6-dimensional real representation of a group G and χ be the corresponding character. We call a group element g *rotational*, if it acts on $\mathbb{R}^6$ by rotations of 3 complex planes; that is, $\rho(g)$ has 3 pairs of complex conjugate eigenvalues of unit modulus.

The following lemma provides an upper bound of the values of $\chi(g^2)$, for rotational elements g such that $\chi(g)=0$.

LEMMA 2.11. *Let G be a group, V be a real representation of G of dimension 6 and χ be the character of V. If g is rotational and $\chi(g)=0$, then $\chi(g^2)\le 2$, where "$=$" is achieved if and only if g is of order 4.*

Proof. Let $M_g\in \mathrm{GL}(6,\mathbb{R})$ be the matrix representation of g, and $s_1,\bar{s}_1,s_2,\bar{s}_2,s_3,\bar{s}_3$ be the eigenvalues of M_g. Write $s_m=x_m+iy_m$ for $m=1,2,3$. Then, we have

$$x_m^2+y_m^2=1, \tag{2}$$

and

$$\sum_{m=1}^{3} x_m=\frac{1}{2}\sum_{m=1}^{3}(s_m+\bar{s}_m)=\frac{1}{2}\chi(g)=0.$$

Let $x:=(x_1,x_2,x_3)^T\in\mathbb{R}^3$. Then, x lies in the cube with vertices $(\pm1,\pm1,\pm1)^T$, as well as on the plane $x_1+x_2+x_3=0$; that is, x lies in the regular hexagon with vertices $(0,-1,1)^T$, $(1,-1,0)^T$, $(1,0,-1)^T$, $(0,1,-1)^T$, $(-1,1,0)^T$ and $(-1,0,1)^T$.

Thus, $\|x\|^2\le 1^2+1^2=2$, which combined with (2) implies that

$$\chi(g^2)=\sum_{m=1}^{3}\left(s_m^2+\bar{s}_m^2\right)=2\sum_{m=1}^{3}\left(x_m^2-y_m^2\right)=2\big(2\|x\|^2-3\big)\le 2.$$

Notice that the equality is achieved precisely when x is a vertex of the hexagon, in which case, g is of order 4. □

2.4. Burnside ring. Let G be a compact Lie group, V be a real representation of G and χ be the corresponding character. By $H\subset G$, we mean H is a closed subgroup in G.

DEFINITION 2.12. For $H\subset G$, define the *fixed point subspace with respect to H* by

$$V^H:=\{x\in V: hx=x\ \forall h\in H\}.$$

Notice that

$$\dim V^H=\langle\chi_H,1_H\rangle=\frac{1}{|H|}\sum_{h\in H}\chi(h), \tag{3}$$

where 1_H stands for the trivial character of H.

For a G-space X,[4] denote by

$$X_H := \{x \in X : hx = x \Leftrightarrow h \in H\}.$$

It is known that for a finite-dimensional G-representation V, V_H is an open dense subset of V^H (cf. [8]). For $x \in X$, define the *isotropy subgroup* of x by $G_x := \{g \in G : gx = x\}$. Then, $x \in X_H$ if and only if $G_x = H$.

Let $H \subset G$ be a closed subgroup. Denote by

$$N(H) := \{g \in G : gHg^{-1} = H\}$$

the normalizer of H and by $W(H) := N(H)/H$ the Weyl group of H in G. The conjugacy class of H in G is denoted by (H). We write

$$(H_1) < (H_2), \quad \text{if } H_1 \subsetneq gH_2g^{-1} \quad \text{for some } g \in G.$$

Set

$$\Phi_0(G) := \{(H) : H \subset G \text{ s.t.} \dim W(H) = 0\}.$$

Recall the definition of the Burnside ring of G (cf. [3]).

Definition 2.13. Let G be a compact Lie group. The *Burnside ring* of G, which will be denoted by $A(G)$, is the free $\mathbb{Z}$-module generated by the set $\Phi_0(G)$ with the following multiplication:

$$(H) \cdot (K) = \sum_{(L)} n_L(H,K)(L), \quad (H),(K) \in \Phi_0(G), \tag{4}$$

where

$$n_L(H,K) = \left|(G/H \times G/K)_L/W(L)\right|. \tag{5}$$

In the definition of $n_L(H,K)$, $G/H \times G/K$ is considered as a G-space with the action given by

$$\varphi\big(g,(g_1H, g_2K)\big) = (gg_1H, gg_2K)$$

for all $g, g_1, g_2 \in G$. Thus, L is an isotropy subgroup of an element in $G/H \times G/K$ if and only if $(L) = (H \cap gKg^{-1})$ for some $g \in G$. We call (L) to be *nontrivial*, if $n_L \neq 0$.

Remark 2.14. (i) The multiplicative identity in $A(G)$ is (G).

(ii) For $L, H \subset G$, set $N(L,H) := \{g \in G : gLg^{-1} \subset H\}$ and

$$n(L,H) := \left|\frac{N(L,H)}{N(H)}\right|. \tag{6}$$

Notice that the number $n(L,H)$ represents the number of different subgroups $\tilde{H}$ in the conjugacy class H such that $L \subset \tilde{H}$. It is proved that the coefficients

[4] By a *G-space* we mean a set X together with a *G-action*, i.e. a map $\varphi : G \times X \to X$ such that (i) $\varphi(e,x) = x$ for all $x \in X$, where e is the identity of G; (ii) $\varphi(g_2, \varphi(g_1,x)) = \varphi(g_2g_1, x)$ for all $g_1, g_2 \in G$ and $x \in X$.

$n_L(H,K)$ in (4) can be computed recurrently from the following relation (cf. [1], [3])

$$n(L,H)\big|W(H)\big|n(L,K)\big|W(K)\big| = \sum_{(L')\geq(L)} n\big(L,L'\big)n_{L'}(H,K)\big|W\big(L'\big)\big|. \tag{7}$$

(iii) In the case $(H)=(K)$, we have

$$(H)\cdot(H) = \big|W(H)\big|(H) + R, \tag{8}$$

where R is the remaining part containing terms (L) such that $(L)<(H)$.

2.5. A characteristic of a representation. Let G be a finite group and V be a representation of G. Denote by $a(V)$ the G-equivariant degree of the antipodal map in the unit ball of V. It was known in the equivariant degree theory that $a(V)$ takes a value in the Burnside ring of G and the assignment $V\mapsto a(V)$ is invariant with respect to G-isomorphisms and multiplicative with respect to product of G-representations (cf. [1], [9], [10], [12]).

However, to avoid topological jargon and to make the exposition self-contained, we give an algebraic definition of $a(V)$ and an algebraic proof of its multiplicative property (cf. Proposition 2.18). Moreover, we show that V has no odd-dimensional fixed point spaces if and only if $a(V)=e$, where e denotes the multiplicative identity in $A(G)$ (cf. Lemma 2.16).

Definition 2.15. Let G be a finite group and V be a representation of G. The following defined element $a(V)\in A(G)$ is called the *characteristic* of V:

$$a(V)=\sum_{(H)} n_H(H),$$

where n_H is defined recurrently by

$$n_H = \frac{(-1)^{\dim V^H} - \sum_{(H)<(H')} n(H,H')n_{H'}|W(H')|}{|W(H)|}, \tag{9}$$

where the numbers $n(H,H')$ are defined by (6).

As an immediate consequence of the definition, the characteristic of a representation satisfies the following property.

Lemma 2.16. *Let V be a representation of a finite group G. Then, $a(V)=(G)$ if and only if all the fixed point subspaces in V are even dimensional.*

Proof. Write $a(V)=\sum n_H(H)$. Assume that all the fixed point subspaces in V are even dimensional. Then, $n_G=(-1)^{\dim V^G}=1$. We prove $n_H=0$ for every (H) with $(H)<(G)$ by induction. Let (H) be a maximal conjugacy class with $(H)<(G)$. Then,

$$n_H = \frac{(-1)^{\dim V^H} - n(H,G)n_G|W(G)|}{|W(H)|} = \frac{1-1\cdot1\cdot1}{|W(H)|} = 0.$$

Suppose that $n_H = 0$ for all (H) such that $(H_o) < (H) < (G)$. Then,

$$\begin{aligned} n_{H_o} &= \Bigg((-1)^{\dim V^{H_o}} - n(H,G) n_G \big|W(G)\big| \\ &\qquad - \sum_{(H_o)<(H)<(G)} n(H_o,H) n_H \big|W(H)\big| \Bigg) \\ &\qquad \Big/ \big|W(H_o)\big| \\ &= \frac{1 - 1\cdot 1\cdot 1 - \sum 0}{|W(H)|} = 0. \end{aligned}$$

Thus, $a(V) = (G)$.

Assume that V has an odd-dimensional fixed point subspace with respect to a subgroup. Let (H_o) be a maximal conjugacy class corresponding to an odd dimensional fixed point subspace. If $(H_o) = (G)$, then $n_G = (-1)^{\dim V^G} = -1$ and thus $n(V) \neq (G)$. Otherwise, if $(H_o) < (G)$, then $n_G = 1$ and $n_H = 0$ for every (H) such that $(H_o) < (H) < (G)$. Consequently, we have

$$\begin{aligned} n_{H_o} &= \Bigg((-1)^{\dim V^{H_o}} - n(H_o,G) n_G \big|W(G)\big| \\ &\qquad - \sum_{(H_o)<(H)<(G)} n(H_o,H) n_H \big|W(H)\big| \Bigg) \\ &\qquad \Big/ \big|W(H_o)\big| \\ &= \frac{-1 - 1\cdot 1\cdot 1 - \sum 0}{|W(H)|} = -\frac{2}{|W(K)|} \neq 0. \end{aligned}$$

Thus, $a(V) \neq (G)$. ☐

Similarly, we have the following corollary.

Corollary 2.17. *Let V be a representation of a finite group G such that V^G is even dimensional. If (H) is a maximal conjugacy class such that $(H) < (G)$ with a nonzero coefficient n_H in $a(V)$, then $n_H = -\frac{2}{|W(H)|}$.*

We show that the characteristic considered as a function defined on the space of all representations of G satisfies a multiplicative property.

Proposition 2.18. *Let V, W be two representations of a finite group G. Then, we have*

$$a(V \times W) = a(V) \cdot a(W),$$

where "·" denotes the multiplication in the Burnside ring $A(G)$ of G.

Proof. Write $a(V) = \sum_{(H)} n_H(H)$ and $a(W) = \sum_{(K)} m_K(K)$. Let

$$a(V) \cdot a(W) = \sum_{(H),(K)} n_H m_K (H) \cdot (K) := \sum_{(L)} p_L(L),$$
$$a(V \times W) = \sum_{(L)} q_L(L),$$

where $n_H, m_K, p_L, q_L \in \mathbb{Z}$.

It is sufficient to show that $p_L = q_L$ for every $(L) \in \Phi_0(G)$, which we prove by induction. If $(L) = (G)$, then

$$\begin{aligned} p_G &\overset{(7)}{=} n_G m_G n(G,G)\big|W(G)\big|n(G,G)\big|W(G)\big| = n_G m_G \\ &\overset{(9)}{=} (-1)^{\dim V^G} \cdot (-1)^{\dim W^G} = (-1)^{\dim (V\times W)^G} = q_G. \end{aligned}$$

Suppose that $p_{L'} = q_{L'}$ for all $(L') > (L)$. Then, we have

$$\begin{aligned} (-1)^{\dim (V\times W)^L} &\overset{(9)}{=} \sum_{(L')\geq(L)} q_{L'}\big|W\big(L'\big)\big|n\big(L,L'\big) \\ &= \sum_{(L')>(L)} p_{L'}\big|W\big(L'\big)\big|n\big(L,L'\big) + q_L\big|W(L)\big|n(L,L). \end{aligned}$$

Thus, to conclude $p_L = q_L$, it is sufficient to show

$$(-1)^{\dim (V\times W)^L} = \sum_{(L')\geq(L)} p_{L'}\big|W\big(L'\big)\big|n\big(L,L'\big). \tag{10}$$

Notice that $p_{L'} = \sum_{(H),(K)} n_H m_K p_{L'}(H,K)$, where $p_{L'}(H,K)$ is the (L')-coefficient in $(H) \cdot (K)$ defined by (5) and $(H),(K)$ are such that $(H \cap gKg^{-1}) = (L')$ for some $g \in G$. Thus, the right-hand side of (10) is equal to

$$\begin{aligned} &\sum_{(L')\geq(L)} \sum_{(H),(K)} n_H m_K p_{L'}(H,K)\big|W\big(L'\big)\big|n\big(L,L'\big) \\ &\quad= \sum_{(H),(K)} n_H m_K \sum_{(L')\geq(L)} p_{L'}(H,K)\big|W\big(L'\big)\big|n\big(L,L'\big) \\ &\quad\overset{(7)}{=} \sum_{(H),(K)} n_H m_K n(L,H)\big|W(H)\big|n(L,K)\big|W(K)\big|, \end{aligned} \tag{11}$$

where $(H),(K)$ are such that $(H \cap gKg^{-1}) \geq (L)$ for some $g \in G$. Notice that $(H \cap gKg^{-1}) \geq (L)$ for some $g \in G$ if and only if $(H) \geq (L)$ and $(K) \geq (L)$. Thus, the right-hand side of (11) is equal to

$$\begin{aligned} &\sum_{(H)\geq(L)} n_H n(L,H)\big|W(H)\big| \sum_{(K)\geq(L)} m_K n(L,K)\big|W(K)\big| \\ &\quad\overset{(9)}{=} (-1)^{\dim V^L} \cdot (-1)^{\dim W^L} = (-1)^{\dim (V\times W)^L}, \end{aligned}$$

which proves (10). Therefore, $p_L = q_L$ for all (L) and the statement follows. □

COROLLARY 2.19. *Let V be a representation of a finite group G. Then, we have*

$$a(V) \cdot a(V) = (G). \tag{12}$$

Proof. Note that $\dim(V \times V)^L$ is always even dimensional whenever non-empty, for all $L \subset G$. By Lemma 2.16, this implies that $a(V \times V) = (G)$. Then, it follows from the Proposition 2.18 that $a(V) \cdot a(V) = (G)$. □

REMARK 2.20. The definition of the characteristic of a real representation is in fact an algebraic version of the definition of the *primary equivariant degree* of the map $-\mathrm{Id} : V \to V$ on the unit ball of V (cf. [1] and references therein). It is proved in the setting of equivariant topology that the primary equivariant degree has a multiplicative property (cf. [12]). But as shown above, important algebraic properties of the primary equivariant degree can be recovered by using (9) directly. These properties turn out to be sufficient for our purpose.

3. Proof

Let G be a finite group. Let V be an absolutely irreducible real representation of G. We will show that every 6-dimensional absolutely irreducible $\mathbb{R}$-representation of a solvable finite group has at least one odd dimensional fixed point space.

DEFINITION 3.1. Let G be a finite group and V be an absolutely irreducible real representation of G. Let χ be the $\mathbb{R}$-character of G afforded by V. If the restricted character χ_H is not absolutely irreducible for every proper subgroup $H \subset G$, then (G, V) is called *minimal*.

THEOREM 3.2. *Let G be a finite solvable group and V be an absolutely irreducible real representation of G of dimension 6. Then, V has an odd-dimensional fixed point subspace.*

Proof. Let χ be the corresponding character. Without loss of generality, we may assume that χ is faithful and (G, V) is minimal. Otherwise, if χ is not faithful, then one considers $G/\ker(\chi)$ for $\ker(\chi) := \{g \in G : \chi(g) = \chi(1)\}$; or if (G, V) is not minimal, then one considers $H \subset G$, on which χ remains absolutely irreducible. Then, the problem is reduced to a (solvable) group of order lower than G.

Since G is a finite solvable group, it has a normal subgroup of prime index. Let $N \subset G$ be a normal subgroup of prime index p. Denote by χ_N the restricted character of N. As we assume (G, V) to be minimal, χ_N is not absolutely irreducible. Thus, by Corollary 2.10, we have either $\chi_N \in \mathrm{Irr}_{\mathbb{C}}(N, \mathbb{R})$, or $\chi_N = \sum_{i=1}^{p} \lambda_i$ for distinct $\lambda_i \in \mathrm{Irr}_{\mathbb{R}}(N, \mathbb{R})$ that are conjugate in G.

Without loss of generality, we assume

$$\chi(\sigma) \in \{2, -2, -6\} \quad \text{for any involution } \sigma \in G \tag{13}$$

since any involution having 1 as an eigenvalue of odd multiplicity, induces an odd dimensional fixed point subspace in V.

Case I: Suppose $p = 2$.

By Theorem 2.4 and Corollary 2.10, we have $\langle \chi_N, \chi_N \rangle = 2$, independent of the reducibility of χ_N. Combined with $\langle \chi, \chi \rangle = 1$, by Remark 2.3, we have

$$\chi(g) = 0 \quad \forall g \in G \setminus N. \tag{14}$$

Thus, by (13), there exist no involutions in $G \setminus N$. On the other hand, since $|G : N| = 2$, every $g \in G \setminus N$ satisfies that $g^{2k} \in N$ and $g^{2k+1} \in G \setminus N$ for all $k \in \mathbb{N}$. Consequently, g^{2k+1} is not an involution for all $k \in \mathbb{N}$. Thus, we have

(C_1) In the case I, we have $\mathrm{Ord}(g) = 4k$, $k \geq 1$, for $g \in G \setminus N$.

Case I.A: Suppose $\chi_N \in \mathrm{Irr}_{\mathbb{C}}(N, \mathbb{R})$.

By Theorem 2.4, we have $\nu(\chi_N) = 0$ and $\nu(\chi) = 1$. It follows that

$$\frac{1}{|N|} \sum_{g \in G \setminus N} \chi(g^2) = 2. \tag{15}$$

Let $\hat{\chi}_N$ be the character of V when considered as a complex representation. Since $\chi_N \in \mathrm{Irr}_{\mathbb{C}}(N, \mathbb{R})$, the $\mathbb{C}$-representation $\hat{\chi}_N$ is reducible. Moreover, by Theorem 2.9, we have

$$\hat{\chi}_N = \lambda_1 + \lambda_2,$$

for two distinct irreducible $\mathbb{C}$-representations $\lambda_1, \lambda_2 \in \mathrm{Irr}_{\mathbb{C}}(N, \mathbb{C})$ of degree 3. Let W be the underlying representation space of λ_1. By (6.6)(ii) in [2], we have

$$\mathbb{C} \otimes V = W \oplus \overline{W},$$

where $\overline{W}$ is the complex conjugate representation of W. Thus,

(C_2) In the case I.A, the matrix representation M_n for every element $n \in N$, when considered as a complex matrix, is similar to a diagonal matrix

$$M_n := \mathrm{diag}[\varepsilon_1, \varepsilon_2, \varepsilon_3, \bar{\varepsilon}_1, \bar{\varepsilon}_2, \bar{\varepsilon}_3],$$

where $\varepsilon^r = 1$, r is the order of n and $\bar{\varepsilon}_i$ is the complex conjugate of ε_i.

Case I.A.a: There exists $g \in G \setminus N$ such that either $\chi(g^2) > 2$ or $\mathrm{Ord}(g) > 4$.

We first prove

(C_3) In the case I.A.a, if there exists $g_o \in G \setminus N$ such that $\mathrm{Ord}(g_o) > 4$ and $\chi(g_o^2) \geq 2$, then V has an odd-dimensional fixed point subspace.

Proof. Set $h_o := g_o^2$ and denote by M_{h_o} (resp. M_{g_o}) the matrix representation of h_o (resp. g_o). We show that M_{h_o} has exactly 2 pairs of nonreal complex eigenvalues.

If M_{h_o} has 1 pair of nonreal complex eigenvalues, then by (C_2), M_{h_o} is similar to $\mathrm{diag}[\delta_1, \delta_2, r, \delta_1, \delta_2, \bar{r}]$, for $\delta_1, \delta_2 \in \{\pm 1\}$. But $\chi(h_o) \geq 2$, thus we have $\delta_1 = \delta_2 = 1$. Consequently, M_{g_o} is similar to $\mathrm{diag}[\varepsilon_1, \varepsilon_2, s, \varepsilon_3, \varepsilon_4, \bar{s}]$, for $\varepsilon_i \in \{-1, 1\}$. Observe that $\sum_{i=1}^4 \varepsilon_i \in \{\pm 4, \pm 2, 0\}$ and $s + \bar{s} \in (-2, 2) \setminus \{0\}$. Therefore, $\chi(g_o) \neq 0$, which is a contradiction to (14).

Suppose that M_{h_o} has 3 pairs of nonreal complex eigenvalues and write M_{h_o} as $\mathrm{diag}[r_1, r_2, r_3, \bar{r}_1, \bar{r}_2, \bar{r}_3]$. Thus, $M_{g_o} = \mathrm{diag}[s_1, s_2, s_3, \bar{s}_1, \bar{s}_2, \bar{s}_3]$, where $s_i^2 = r_i$. By (14) and Lemma 2.11, we conclude that $\mathrm{Ord}(g_o) = 4$, which is a contradiction to our assumption.

Therefore, we showed that M_{h_o} has exactly 2 pairs of nonreal complex eigenvalues. Thus, by (C_2), M_{h_o} is similar to $\mathrm{diag}[\delta, r_1, r_2, \delta, \bar{r}_1, \bar{r}_2]$, where $\delta \in \{\pm 1\}$. But $\chi(h_o) \geq 2$, so we have $\delta = 1$. Let $H := \langle h_o \rangle$ and $H' := \langle g_o \rangle$. Then, it follows that $\dim V^H = 2$. On the other hand, we have (cf. (3))

$$\begin{aligned}\dim V^{H'} &= \frac{1}{|H'|} \sum_{g \in H'} \chi(g) \overset{(14)}{=} \frac{1}{|H'|} \sum_{g \in H} \chi(g)\\ &= \frac{1}{2|H|} \sum_{g \in H} \chi(g) = \frac{1}{2} \dim V^H.\end{aligned}$$

Thus, $\dim V^{H'} = 1$, which is an odd-dimensional fixed point subspace in V.

$\square_{\text{Proof of } (\mathrm{C}_3)}$

We now show that under the assumption of case I.A.a, there exists an odd-dimensional fixed point subspace in V. Assume that there exists $g_o \in G \setminus N$ such that $\chi(g_o^2) > 2$. By (13), g_o^2 is not an involution. By (C_1), $\mathrm{Ord}(g_o) > 4$. Thus, (C_3) applies and we can conclude the existence of an odd-dimensional fixed point subspace in V.

Otherwise, assume that $\chi(g^2) \leq 2$ for all $g \in G \setminus N$ but there exists $g_o \in G \setminus N$ such that $\mathrm{Ord}(g_o) > 4$. Then, by (15), we have $\chi(g^2) = 2$ for all $g \in G \setminus N$. In particular, $\chi(g_o^2) = 2$. Thus, (C_3) applies and the statement follows.

By (C_1) and (15), the complementary case of case I.A.a in case I.A is the following

Case I.A.b: For all $g \in G \setminus N$, we have $\mathrm{Ord}(g) = 4$ and $\chi(g^2) = 2$.

In what follows, we write σ as -1, if $\chi(\sigma) = -6$. Since χ is assumed to be faithful, such element is unique.

Consider the set $S := \{g^2 : g \in G \setminus N\}$. By Theorem 2.6, S is composed of at least 2 distinct conjugacy classes of G. We show that

(C_4) In the case I.A, there exists a $g_o \in G \setminus N$ such that g_o commutes with an involution $\sigma \in G$ with $\sigma \notin \{-1, g_o^2, -g_o^2\}$.

Proof. Let c_1, c_2 be 2 distinct conjugacy classes of G, containing g_1^2, g_2^2 respectively, for some $g_i \in G \setminus N$ such that $\chi(g_i^2) = 2$, $i = 1, 2$.

Assume that one of the c_i's has even size, say $|c_1| = 2k$, $k \in \mathbb{N}$. Then, c_1 contains an element σ, which is different from g_1^2, such that $g_1 \sigma = \sigma g_1$. Indeed, consider the conjugation action on c_1 by g_1, which will be denoted by ρ_1. Notice that $\rho_1 \in \mathrm{Aut}(c_1) \simeq S_{2k}$. Since ρ_1 fixes at least one element in c_1, namely g_1^2, we have that $\rho_1 \in S_{2k-1}$. On the other hand, since $\mathrm{Ord}(g_1)=4$, we have $\mathrm{Ord}(\rho_1)\in\{1,2,4\}$. Consequently, ρ_1 fixes at least another element in c_1, say σ, which is not g_1^2. Since $\chi(\sigma)=\chi(g_1^2)=2$, we also have that $\sigma\notin\{-1,-g_1^2\}$. Thus, (C_4) holds.

Otherwise, assume both c_1 and c_2 have odd sizes, say $|c_1|=2r+1$, $|c_2|=2s+1$, for $r,s\in\mathbb{N}$. Since $\chi(g_i^2)=2$, c_i does not contain -1, for $i=1,2$. In particular, $|c_i|\geq 3$, for $i=1,2$. Consider the conjugation action on c_1 by g_2, which is denoted by ρ_2. Then, $\rho_2\in\mathrm{Aut}(c_1)\simeq S_{2r+1}$. Again, since $\mathrm{Ord}(g_2)=4$, we have $\mathrm{Ord}(\rho_2)\in\{1,2,4\}$. Therefore, ρ_2 fixes at least one element in c_1, say σ, which is clearly different from g_2^2. Also, since $\chi(\sigma)=\chi(g_1^2)=2$, we have that $\sigma\notin\{-1,-g_2^2\}$. Thus, (C_4) holds. $\square_{\text{Proof of } \mathrm{C}_4}$

Let g_o,σ be given by (C_4). Consider the group

$$H' := H\cup\sigma H=\{1,g_o,g_o^2,g_o^3,\sigma,\sigma g_o,\sigma g_o^2,\sigma g_o^3\}.$$

By (14), $\chi(g_o)=\chi(g_o^3)=\chi(\sigma g_o)=\chi(\sigma g_o^3)=0$. Thus, we have

$$\dim V^{H'}=\frac{1}{8}\big(\chi(1)+\chi(g_o^2)+\chi(\sigma)+\chi(\sigma g_o^2)\big)=:\frac{1}{8}(6+2+x+y).$$

Since $-1\notin H'$, by (13), $x,y\in\{2,-2\}$. Consequently, from $\dim V^{H'}\in\mathbb{Z}$, it follows that $x+y=0$ and thus $\dim V^{H'}=1$.

Case I.B: Suppose that $\chi_N=\lambda+\lambda^g$, where λ, λ^g are distinct conjugate absolutely irreducible N-representations, and $g\in G\setminus N$.

Since $\nu(\lambda)=\nu(\lambda^g)=\nu(\chi)=1$, we have

$$\sum_{g\in G\setminus N}\chi(g^2)=0. \tag{16}$$

Case I.B.a: There exists $g_o\in G\setminus N$ such that $\mathrm{Ord}(g_o)>4$.

Let H' be the group generated by g_o and $H:=H'\cap N$. Notice that a half of the elements in H' has character 0 (cf. (14)). Moreover, we have

$$\chi_N(h)=\lambda(h)+\lambda^g(h)=\lambda(h)+\lambda^{g_o}(h)=2\lambda(h)\quad\forall h\in H. \tag{17}$$

Denote by W the underlying representation space of λ. Then, (17) implies that H acts faithfully on W. Now that H is a cyclic group of order larger than 2, which acts faithfully on a 3-dimensional representation, so $\dim W^H=1$. On the other hand, we have

$$\begin{aligned}\dim(V)^{H'}&=\frac{1}{2|H|}\sum_{h\in H}\chi(h)\\&=\frac{1}{2|H|}\sum_{h\in H}2\lambda(h)=\dim W^H.\end{aligned}$$

Consequently, the group H' has a 1-dimensional fixed point space in V.

Case I.B.b: For all $g\in G\setminus N$, we have $\mathrm{Ord}(g)=4$.

Combined with (14), we have that every element $g \in G \setminus N$ is similar to one of the following three diagonal matrices

$$\begin{aligned} &\mathrm{Diag}[1,1,-1,-1,i,-i], \quad \text{in which case } \chi\big(g^2\big)=2;\\ &\mathrm{Diag}[1,-1,i,-i,i,-i], \quad \text{in which case } \chi\big(g^2\big)=-2;\\ &\mathrm{Diag}[i,-i,i,-i,i,-i], \quad \text{in which case } \chi\big(g^2\big)=-6. \end{aligned}$$

Notice that in the case $\chi(g^2)=-2$, the group generated by g has a 1-dimensional fixed point space. Thus, without loss of generality, we assume

$$\chi\big(g^2\big) \in \{2,-6\} \quad \forall g \in G \setminus N.$$

Taking into account of (16), we have that $-1 \in G$. Moreover,

$$\big\{g^2 : g \in G \setminus N\big\} = \{-1\} \cup \big\{g^2 : \chi\big(g^2\big)=2, g \in G \setminus N\big\}.$$

Denote by $S' := \{g^2 : \chi(g^2)=2, g \in G \setminus N\}$. Then, by Theorem 2.7, S' is a union of at least 3 conjugacy classes. Let c_1 and c_2 be two of the conjugacy classes in S'. By the argument used in the proof of the claim (C_4), we have

(C_5) In the case I.B.b, there exists a $g_o \in G \setminus N$ such that $\chi(g_o^2)=2$ and g_o commutes with an involution $\sigma \in G$ with $\sigma \notin \{-1, g_o^2, -g_o^2\}$.

Thus, a similar argument leads to the fact that the group generated by g_o has a 1-dimensional fixed point space in V.

Case II: Suppose $p>2$.

Case II.A: Suppose $\chi_N \in \mathrm{Irr}_{\mathbb{C}}(N,\mathbb{R})$.

Then, by Theorem 2.4, we have $\nu(\chi_N)=0$ and thus $\langle \chi_N, \chi_N\rangle = 2$.

On the other hand, complexify the real representation V as a complex one and denote by $\hat{\chi}$ the corresponding character. Though $\chi(g)=\hat{\chi}(g)$ for $g \in G$, we keep the seperate notations. Since $\chi_N \in \mathrm{Irr}_{\mathbb{C}}(N,\mathbb{R})$ is irreducible of complex type, $\hat{\chi}_N$ becomes reducible when considered as a complex representation. By Theorem 2.9, we have $\hat{\chi}_N = \sum_{i=1}^{p} \lambda_i$, where $\lambda_i \in \mathrm{Irr}(N,\mathbb{C})$ are distinct and irreducible. By Theorem 2.5, we have $\langle \hat{\chi}_N, \hat{\chi}_N\rangle = \sum_{i=1}^{p} \langle \lambda_i, \lambda_i\rangle = p$. Consequently,

$$2 = \langle \chi_N, \chi_N\rangle = \langle \hat{\chi}_N, \hat{\chi}_N\rangle = p,$$

which contradicts the assumption $p>2$.

Case II.B: Suppose that $\chi_N = \sum_{i=1}^{p} \lambda_i$, where $\lambda_i \in \mathrm{Irr}_{\mathbb{R}}(N,\mathbb{R})$, $i=1,2,\dots,$ p, are distinct absolutely irreducible N-representations which are conjugate in G.

By the conjugacy relation, we have $\lambda_i(1)=\lambda_j(1)$ for $i,j \in \{1,2,\dots,p\}$. Set $r := \lambda_1(1)$. Then, $6 = \chi_N(1) = p \cdot r$. Since $p>2$, we conclude that $p=3$.

Write $\chi_N = \lambda_1+\lambda_2+\lambda_3$. Then, λ_i is afforded by a 2-dimensional real vector space, which will be denoted by V_i, for $i=1,2,3$. Consider the characteristic $a(V_i)$ of V_i, for $i=1,2,3$ (cf. Definition 2.15). Then, by (12), we have

$$a(V_i) \cdot a(V_i) = (N). \tag{18}$$

Assume that V has only even dimensional fixed point subspaces. Then, by Lemma 2.16 and Proposition 2.18, we obtain $(N)=a(V_1)\cdot a(V_2)\cdot a(V_3)$. Since $(a(V_i))^2=(N)$, we have

$$a(V_1)=a(V_2)\cdot a(V_3), \tag{19}$$

$$a(V_2)=a(V_1)\cdot a(V_3), \tag{20}$$

$$a(V_3)=a(V_1)\cdot a(V_2). \tag{21}$$

Write

$$\begin{aligned} a(V_1)&:=(N)+n_H(H)+R_1,\\ a(V_2)&:=(N)+m_K(K)+R_2,\\ a(V_3)&:=(N)+l_L(L)+R_3, \end{aligned}$$

where (H), (K), (L) are maximal conjugacy classes such that $n_H, m_K, l_L \neq 0$ and R_i denotes the remainder, for $i=1,2,3$. Then,

$$\begin{aligned} a(V_2)\cdot a(V_3) &:= (N)+m_K(K)+l_L(L)+m_Kl_L(K)(L)\\ &\quad\; + m_K(K)R_3+l_L(L)R_2+R_2+R_3+R_2R_3\\ &\overset{(19)}{=} (N)+n_H(H)+R_1. \end{aligned} \tag{22}$$

Comparing the leading terms, we have either $(H)=(K)$ or $(H)=(L)$. Similarly, by (20)–(21), we have either $(K)=(H)$ or $(K)=(L)$; and either $(L)=(K)$ or $(L)=(H)$. It follows that

$$(H)=(K)=(L). \tag{23}$$

Substitute (23) in (22), we have

$$\begin{aligned} &(N)+(m_K+l_L)(H)+m_Kl_L(H)(H)\\ &\qquad + m_K(H)R_3+l_L(H)R_2+R_2+R_3+R_2R_3\\ &\quad =(N)+n_H(H)+R_1, \end{aligned} \tag{24}$$

where $(H)\cdot(H)=|W(H)|(H)+R$ (cf. (8)). Thus, by comparing the (H)-coefficients in (24), we obtain

$$m_K+l_L+m_kl_L\big|W(H)\big|=n_H.$$

Similar analysis of (20)–(21) leads to

$$\begin{aligned} n_H+l_L+n_Hl_L\big|W(H)\big|&=m_K,\\ n_H+m_K+n_Hm_K\big|W(H)\big|&=l_L. \end{aligned}$$

Therefore, $n_H=m_K=l_L=-\frac{1}{|W(H)|}$, which is a contradiction to Corollary 2.17. Consequently, V must have an odd dimensional fixed point subspace. $\square$

Acknowledgments. The author would like to thank Reiner Lauterbach for his suggestion of the problem and his constant encouragement. The author would also like to thank I. Martin Isaacs for his continuous support and his work on characters of groups with fixed-point-free automorphisms of order 4, without which the proof would not be completed at this time.

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Haibo Ruan, Department Mathematik, Universität Hamburg, D-20146 Hamburg, Germany

E-mail address: `haibo.ruan@math.uni-hamburg.de`