Improving Condensation Methods for Eigenvalue Problems via Rayleigh Functional

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March 31, 1993

Abstract

The dynamic analysis of structures using the finite element method leads to very large eigenvalue problems. Condensation is often required to reduce the number of degrees of freedom of these problems to manageable size. The (statically or dynamically) condensed problem can be considered as linearization of a matrix eigenvalue problem which is nonlinear with respect to the eigenparameter. As a consequence only a few natural frequencies can be obtained with sufficient accuracy. We use the Rayleigh functional of the nonlinear problem to improve the eigenvalue approximations considerably. Two examples are presented to demonstrate the efficiency of the method.

1 Introduction

In the dynamic analysis of a structure using the finite element method usually a very large number of degrees of freedom (several hundreds or thousands) is required to describe the structure accurately. In this situation it is often necessary to reduce the size of the involved matrices before proceeding to the calculation of the required natural frequencies and modal shapes.

To accomplish this reduction Irons [?] and Guyan [?] independently proposed the static condensation (or reduction or economization) where only a moderate number of degrees of freedom (called masters) are retained and the other ones (called slaves) are eliminated. For static problems the condensation method is exact. For dynamic problems, however, a linearization with respect to the eigenparameter introduces errors (a posteriori error bounds were given by Geradin [?], a priori bounds by Thomas [?] and the second author [?]).

Several attempts have been made to increase the accuracy of the eigenvalues of the condensed problem. Johnson et al. [?] considered a quadratic approximation of the exactly reduced problem. Leung [?], [?] and Paz [?] studied dynamic condensation, i.e. Guyan's method for a shifted eigenvalue problem, which yields good approximations of those eigenvalues which are close to the shift parameter (cf. Schwarz [?] for a numerical example). Suarez and Singh [?] proposed an iterative approach for the dynamic condensation which guarantees a good approximation to the lower (or the upper) part of the spectrum. On the other hand, their method seems to be very costly because in every iteration step a condensed eigenvalue problem has to be solved.

In this paper we take advantage of the structure of the exactly condensed eigenvalue problem $\boldsymbol{T}(\lambda)\boldsymbol{u} = \mathbf{o}$ which is nonlinear with respect to the eigenparameter. For $\boldsymbol{T}$ a Rayleigh functional p can be defined which has properties similar to the Rayleigh quotient for linear eigenvalue problems: The lower part of the spectrum of $\boldsymbol{T}$ can be characterized as min-max-values of p. Furthermore, a first order error in an eigenvector approximation $\hat{\boldsymbol{u}}$ yields a second order error of the eigenvalue estimate $p(\hat{\boldsymbol{u}})$ for the corresponding eigenvalue. Because the eigenvectors $\boldsymbol{u}_j$ of the condensed (linear) eigenvalue problem are usually good approximations of the master portions of the eigenvectors of the original problem the eigenvalue approximations from the condensed problem can be improved considerably if they are replaced by the values of the Rayleigh functional $p(\boldsymbol{u}_j)$.

The proposed method is idealy suited for the application of substructuring methods: The condensation process can be performed substructurewise (which was already pointed out by Leung [?]) and the data needed for the evaluation of the Rayleigh functional can be calculated substructurewise too. Moreover, in this case the effort to evaluate the Rayleigh functional can be decreased substantially since usually the substructures are much stiffer than the entire structure. Last but not least most of the computations can be done in parallel. This will be discussed in detail in a forthcoming paper.

2 The Condensed Eigenvalue Problem

Consider the generalized eigenvalue problem

$$\boldsymbol{K}\boldsymbol{x} = \lambda \boldsymbol{M}\boldsymbol{x}, \tag{1}$$

where $\boldsymbol{K} \in \mathbb{R}^{(n,n)}$ and $\boldsymbol{M} \in \mathbb{R}^{(n,n)}$ are real symmetric and positive definite matrices. In the finite element analysis of structures undergoing small vibrations $\boldsymbol{K}$ and $\boldsymbol{M}$ are the stiffness and mass matrices, respectively, $\boldsymbol{x}$ is the vector of modal displacements and $\lambda = \omega^2$ is given by the angular frequency ω.

In order to reduce the size of the eigenvalue problem we apply the condensation method introduced by Irons [?] and Guyan [?]. To this end the degrees of freedom are partitioned into masters and slaves, and eq. (**??**) is rewritten in block form

$$\begin{bmatrix} \boldsymbol{K}_{mm} & \boldsymbol{K}_{ms} \\ \boldsymbol{K}_{sm} & \boldsymbol{K}_{ss} \end{bmatrix} \begin{Bmatrix} \boldsymbol{x}_m \\ \boldsymbol{x}_s \end{Bmatrix} = \lambda \begin{bmatrix} \boldsymbol{M}_{mm} & \boldsymbol{M}_{ms} \\ \boldsymbol{M}_{sm} & \boldsymbol{M}_{ss} \end{bmatrix} \begin{Bmatrix} \boldsymbol{x}_m \\ \boldsymbol{x}_s \end{Bmatrix} \tag{2}$$

where $\boldsymbol{x}_s \in \mathbb{R}^s$ are the slave variables to be eliminated and $\boldsymbol{x}_m \in \mathbb{R}^m$ are the master variables to be retained.

Solving the second part of eq. (**??**) for $\boldsymbol{x}_s$ one obtains for $\lambda = \hat\lambda$

$$\boldsymbol{x}_s = -(\boldsymbol{K}_{ss} - \hat\lambda \boldsymbol{M}_{ss})^{-1}(\boldsymbol{K}_{sm} - \hat\lambda \boldsymbol{M}_{sm})\boldsymbol{x}_m. \tag{3}$$

We therefore project the eigenvalue problem (**??**) onto the m dimensional subspace

$$\left\{ \boldsymbol{P}\boldsymbol{x}_m \,:\, \boldsymbol{P} = \begin{pmatrix} \boldsymbol{I}_m \\ -(\boldsymbol{K}_{ss} - \hat\lambda \boldsymbol{M}_{ss})^{-1}(\boldsymbol{K}_{sm} - \hat\lambda \boldsymbol{M}_{sm}) \end{pmatrix},\ \boldsymbol{x}_m \in \mathbb{R}^m \right\}$$

and obtain the condensed eigenvalue problem

$$\boldsymbol{P}^t(-\boldsymbol{K} + \lambda \boldsymbol{M})\boldsymbol{P}\boldsymbol{x}_m = \mathbf{o},$$

where

$$\begin{aligned} \boldsymbol{P}^t(-\boldsymbol{K} + \lambda\boldsymbol{M})\boldsymbol{P} \;&=\; -\boldsymbol{K}_{mm} + \lambda \boldsymbol{M}_{mm} \\ &+(\boldsymbol{K}_{ms} - \hat\lambda \boldsymbol{M}_{ms})(\boldsymbol{K}_{ss} - \hat\lambda \boldsymbol{M}_{ss})^{-1}(\boldsymbol{K}_{sm} - \lambda \boldsymbol{M}_{sm}) \\ &+(\boldsymbol{K}_{ms} - \lambda \boldsymbol{M}_{ms})(\boldsymbol{K}_{ss} - \hat\lambda \boldsymbol{M}_{ss})^{-1}(\boldsymbol{K}_{sm} - \hat\lambda \boldsymbol{M}_{sm}) \\ &-(\boldsymbol{K}_{ms} - \hat\lambda \boldsymbol{M}_{ms})(\boldsymbol{K}_{ss} - \hat\lambda \boldsymbol{M}_{ss})^{-1}(\boldsymbol{K}_{ss} - \lambda \boldsymbol{M}_{ss})(\boldsymbol{K}_{ss} - \hat\lambda \boldsymbol{M}_{ss})^{-1}(\boldsymbol{K}_{sm} - \hat\lambda \boldsymbol{M}_{sm}). \end{aligned}$$

For $\hat\lambda = 0$ this is the statically condensed eigenvalue problem (cf. Irons [?], Guyan [?]). For fixed $\hat\lambda \neq 0$ this is the dynamically condensed problem (cf. Leung [?], Paz [?], Schwarz [?]), which may be viewed as the static condensation of the shifted problem

$$(\boldsymbol{K} - \hat\lambda \boldsymbol{M})\boldsymbol{x} = \mu \boldsymbol{M}\boldsymbol{x}, \quad \mu := \lambda - \hat\lambda. \tag{4}$$

Finally, in case of exact condensation, i.e. $\hat{\lambda} = \lambda$, one obtains the nonlinear matrix eigenvalue problem

$$\boldsymbol{T}(\lambda)\boldsymbol{x}_m = \mathbf{o}, \tag{5}$$

where

$$\boldsymbol{T}(\lambda) := -\boldsymbol{K}_{mm} + \lambda \boldsymbol{M}_{mm} + (\boldsymbol{K}_{ms} - \lambda \boldsymbol{M}_{ms})(\boldsymbol{K}_{ss} - \lambda \boldsymbol{M}_{ss})^{-1}(\boldsymbol{K}_{sm} - \lambda \boldsymbol{M}_{sm}). \tag{6}$$

Note that (**??**) is equivalent to the original linear problem (**??**), if (**??**) and the slave eigenvalue problem (**??**) have no eigenvalues in common.

We write the nonlinear eigenvalue problem (**??**), (**??**) in a more convenient form (cf. Ruge [**?**], Voss [**?**], Berkan, Dokainish [**?**]). Let $\boldsymbol{\Phi} \in \mathbb{R}^{(s,s)}$ be the modal matrix of the slave eigenvalue problem

$$\boldsymbol{K}_{ss}\boldsymbol{\phi} = \omega \boldsymbol{M}_{ss}\boldsymbol{\phi} \tag{7}$$

and $\boldsymbol{\Omega} := \text{diag}\,[\omega_i]$ be the diagonal matrix containing the slave eigenvalues. With

$$\boldsymbol{\Phi}^t \boldsymbol{M}_{ss}\boldsymbol{\Phi} = \boldsymbol{I} \quad \text{and} \quad \boldsymbol{\Phi}^t \boldsymbol{K}_{ss}\boldsymbol{\Phi} = \boldsymbol{\Omega} \tag{8}$$

the matrix (**??**) can be rewritten as

$$\boldsymbol{T}(\lambda) = -\boldsymbol{K}_0 + \lambda \boldsymbol{M}_0 + \boldsymbol{S}\boldsymbol{D}(\lambda)\boldsymbol{S}^t, \tag{9}$$

where

$$\begin{aligned}
\boldsymbol{K}_0 &:= \boldsymbol{K}_{mm} - \boldsymbol{K}_{ms}\boldsymbol{K}_{ss}^{-1}\boldsymbol{K}_{sm}, \\
\boldsymbol{M}_0 &:= \boldsymbol{M}_{mm} - \boldsymbol{K}_{ms}\boldsymbol{K}_{ss}^{-1}\boldsymbol{M}_{sm} - \boldsymbol{M}_{ms}\boldsymbol{K}_{ss}^{-1}\boldsymbol{K}_{sm} + \boldsymbol{K}_{ms}\boldsymbol{K}_{ss}^{-1}\boldsymbol{M}_{ss}\boldsymbol{K}_{ss}^{-1}\boldsymbol{K}_{sm}, \\
\boldsymbol{S} &:= \boldsymbol{M}_{ms}\boldsymbol{\Phi} - \boldsymbol{K}_{ms}\boldsymbol{\Phi}\boldsymbol{\Omega}^{-1}, \\
\boldsymbol{D}(\lambda) &:= \text{diag}\left[\frac{\lambda^2}{\omega_i - \lambda}\right].
\end{aligned}$$

Notice that the static condensation yields the linear eigenvalue problem

$$(\boldsymbol{K}_0 - \lambda \boldsymbol{M}_0)\boldsymbol{x}_m = \mathbf{o}. \tag{10}$$

Hence static condensation can be considered as the linearization of the nonlinear eigenvalue problem (**??**) at $\hat{\lambda} = 0$ (and analogously dynamic condensation as linearization at $\hat{\lambda} \neq 0$). Moreover, the reduced stiffness and mass matrices $\boldsymbol{K}_0$ and $\boldsymbol{M}_0$ are projections of the positive definite matrices $\boldsymbol{K}$ and $\boldsymbol{M}$, and therefore are symmetric and positive definite.

Usually in the literature the statically condensed problem (**??**) is solved to get approximations for some of the smallest eigenvalues of (**??**) and for the master portions $\boldsymbol{x}_m$ of the corresponding eigenvectors. The slave portions $\boldsymbol{x}_s$ can then be obtained from eq. (**??**). It is well known that only very few eigenmodes can be obtained with sufficient accuracy.

Arbitrarily exact results can be obtained by safeguarded inverse iteration for the exactly condensed problem (**??**), which was introduced by Leung in [**?**] and modified in [**?**], or by the improvement of Rothe [**?**] which applies to more general nonlinear eigenvalue problems. The drawback of these methods is that they are very time consuming because the computation of every single mode contains the evaluation of the matrix $\boldsymbol{T}(\lambda)$ for different values of λ.

We will now describe a more efficient way which uses the structure of the nonlinear eigenvalue problem (**??**), (**??**) to improve the approximations obtained from the statically condensed problem considerably. First, let

$$\underline{\omega} := \min_{i=1,\dots,s} \omega_i$$

denote the smallest eigenvalue of the slave eigenvalue problem (**??**) and let J be the open interval $J := (0\,,\,\underline{\omega})$.

For any fixed vector $\boldsymbol{u} \in \mathbb{R}^m \setminus \{\mathbf{o}\}$ we consider the real valued function

$$f(\cdot,\boldsymbol{u})\;:\;J \to \mathbb{R} \quad , \quad \lambda \mapsto f(\lambda,\boldsymbol{u}) := \boldsymbol{u}^t\boldsymbol{T}(\lambda)\boldsymbol{u},$$

i.e.

$$\begin{aligned} f(\lambda,\boldsymbol{u}) &= -\boldsymbol{u}^t\boldsymbol{K}_0\boldsymbol{u} + \lambda\boldsymbol{u}^t\boldsymbol{M}_0\boldsymbol{u} + \sum_{i=1}^{s}\frac{\lambda^2}{\omega_i-\lambda}(\boldsymbol{S}^t\boldsymbol{u})_i^2 \\ &= -\boldsymbol{u}^t\boldsymbol{K}_0\boldsymbol{u} + \lambda\boldsymbol{u}^t\boldsymbol{M}_0\boldsymbol{u} + \sum_{i=1}^{s}\frac{\lambda^2}{\omega_i-\lambda}\left(\boldsymbol{\phi}_i^t\boldsymbol{M}_{sm}\boldsymbol{u} - \frac{1}{\omega_i}\boldsymbol{\phi}_i^t\boldsymbol{K}_{sm}\boldsymbol{u}\right)^2. \end{aligned} \tag{11}$$

Since $\boldsymbol{M}_0$ is positive definite

$$\frac{\partial}{\partial\lambda}f(\lambda,\boldsymbol{u}) = \boldsymbol{u}^t\boldsymbol{M}_0\boldsymbol{u} + \sum_{i=1}^{s}\frac{\lambda(2\omega_i-\lambda)}{(\omega_i-\lambda)^2}(\boldsymbol{S}^t\boldsymbol{u})_i^2 \;\geq\; \boldsymbol{u}^t\boldsymbol{M}_0\boldsymbol{u} \;>\; 0. \tag{12}$$

Hence the mapping $\lambda \mapsto f(\lambda,\boldsymbol{u})$ increases monotonically on J and the nonlinear equation $f(\lambda,\boldsymbol{u}) = 0$ has at most one solution in J. Therefore $f(\lambda,\boldsymbol{u}) = 0$ implicitly defines a functional

$$p\;:\;\mathbb{R}^m \supset D(p) \to J \quad , \quad f(p(\boldsymbol{u}),\boldsymbol{u}) = 0,$$

where the domain of definition is given by

$$D(p) := \{\boldsymbol{u} \in \mathbb{R}^m \setminus \{\mathbf{o}\}\;:\;f(\lambda,\boldsymbol{u}) = 0 \text{ is solvable in } J\}.$$

For linear eigenvalue problems $\boldsymbol{T}(\lambda)\boldsymbol{u} := (\boldsymbol{K} - \lambda\boldsymbol{M})\boldsymbol{u} = \mathbf{o}$ this functional is nothing else but the Rayleigh quotient. We therefore call p the Rayleigh functional of the nonlinear eigenvalue problem (**??**).

In accordance with the linear theory, it is possible to characterize some of the eigenvalues of (**??**) as min-max-values of p. The following result is a direct consequence of Theorem 2.1 and Theorem 2.9 in Voss and Werner [**?**]:

Theorem 1 *Let $\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n$ be the eigenvalues of the linear eigenvalue problem (**??**) and choose $k \in \mathbb{N}$ such that $\lambda_k < \underline{\omega} \leq \lambda_{k+1}$.*

Let H_j be the set of all j-dimensional subspaces of $\mathbb{R}^m$ and denote by $\tilde{H}_j := \{V \in H_j\;:\;V \setminus \{\mathbf{o}\} \subset D(p)\}$ the set of j dimensional subspaces of $\mathbb{R}^m$ which are contained in the domain of definition of the Rayleigh functional p.

Then $\tilde{H}_k \neq \emptyset$ and for $j = 1,\dots,k$ the following min-max-characterization holds:

$$\lambda_j = \min_{V\in\tilde{H}_j}\;\max_{\boldsymbol{u}\in V\setminus\{\mathbf{o}\}} p(\boldsymbol{u}). \tag{13}$$

We take advantage of the approximation properties of the Rayleigh functional which are similar to those of the Rayleigh quotient in the linear case: A first order error of an eigenvector approximation $\bar{\boldsymbol{u}}$ yields a second order error of the value of the Rayleigh functional $p(\bar{\boldsymbol{u}})$ for the corresponding eigenvalue (if $\bar{\boldsymbol{u}} \in D(p)$). We will show that the eigenvectors $\bar{\boldsymbol{x}}_j$ of the statically condensed problems (**??**) corresponding to eigenvalues $\lambda_j \in J$ are contained in $D(p)$. Hence $p(\bar{\boldsymbol{x}}_j)$ is defined and can be expected to be a better approximation to λ_j than $\bar{\lambda}_j$.

Let $\bar{\lambda} \in J$ be an eigenvalue of the (linear) condensed problem (**??**) and let $\bar{\boldsymbol{x}}$ be the corresponding eigenvector. Then (cf. Fig. 1)

$$f(0, \bar{\boldsymbol{x}}) = -\bar{\boldsymbol{x}}^t \boldsymbol{K}_0 \bar{\boldsymbol{x}} < 0 \tag{14}$$

and

$$f(\bar{\lambda}, \bar{\boldsymbol{x}}) = -\bar{\boldsymbol{x}}^t \boldsymbol{K}_0 \bar{\boldsymbol{x}} + \bar{\lambda} \bar{\boldsymbol{x}}^t \boldsymbol{M}_0 \bar{\boldsymbol{x}} + \bar{\boldsymbol{x}}^t \boldsymbol{S}\boldsymbol{D}(\bar{\lambda}) \boldsymbol{S}^t \bar{\boldsymbol{x}} = \bar{\boldsymbol{x}}^t \boldsymbol{S}\boldsymbol{D}(\bar{\lambda}) \boldsymbol{S}^t \bar{\boldsymbol{x}} > 0. \tag{15}$$

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Fig. 1. Rayleigh functional evaluation at $\bar{\boldsymbol{x}}$

The following theorem gives error bounds for the approximations $p(\bar{\boldsymbol{x}}_j)$ of λ_j. The proof can be found in the Appendix.

Theorem 2 *Let $\lambda_1 \le \lambda_2 \le \ldots \le \lambda_n$ be the eigenvalues of (**??**) and let*

$$\bar{\lambda}_1 \le \bar{\lambda}_2 \le \ldots \le \bar{\lambda}_\ell < \underline{\omega}$$

*be the eigenvalues of the linear condensed eigenvalue problem (**??**) which are contained in J. Denote by $\bar{\boldsymbol{x}}_1, \ldots, \bar{\boldsymbol{x}}_\ell$ a set of corresponding eigenvectors. Then*

$$\bar{\lambda}_1 \frac{1}{1 + \bar{\lambda}_1 / \underline{\omega}} \;\le\; \lambda_1 \;\le\; p(\bar{\boldsymbol{x}}_1) \;\le\; \bar{\lambda}_1 \tag{16}$$

and for $j = 2, \ldots, \ell$

$$\bar{\lambda}_j \frac{1}{1 + \bar{\lambda}_j / \underline{\omega}} \;\le\; \lambda_j \;,\; p(\bar{\boldsymbol{x}}_j) \;\le\; \bar{\lambda}_j. \tag{17}$$

The inclusion (**??**) shows that $p(\bar{\boldsymbol{x}}_1)$ is always a better approximation to λ_1 than the smallest eigenvalue $\bar{\lambda}_1$ of the condensed problem. For $j \ge 2$ the inequality $\lambda_j \le p(\bar{\boldsymbol{x}}_j)$ does not hold in general (cf. the examples in Section 4.). Thus we can not deduce from (**??**) that $p(\bar{\boldsymbol{x}}_j)$ improves the eigenvalue approximation $\bar{\lambda}_j$ to λ_j. However, in all computed examples we observed a considerable improvement as long as $\bar{\lambda}_j$ was not too close to the smallest eigenvalue $\underline{\omega}$ of the slave eigenvalue problem.

3 Computational Aspects

In general (e. g. if the slave variables are selected by the automatic technique introduced by Henshell and Ong [**?**]) the evaluation of the Rayleigh functional $p(\bar{\boldsymbol{x}})$ for a given vector $\bar{\boldsymbol{x}} \in \mathbb{R}^m$ is very costly. If we use the form of $f(\cdot, \bar{\boldsymbol{x}})$ given in eq. (**??**) the complete solution of the slave eigenvalue problem is needed the dimension of which is still very large. If we employ the form of $\boldsymbol{T}(\lambda)$ given in eq. (**??**) every evaluation of the function $f(\lambda, \bar{\boldsymbol{x}}) = \bar{\boldsymbol{x}}^t \boldsymbol{T}(\lambda) \bar{\boldsymbol{x}}$ involves the solution of a linear system of dimension s.

In the finite element analysis of structures undergoing small vibrations the slave variables can be chosen such that the matrices $\boldsymbol{K}_{ss}$ and $\boldsymbol{M}_{ss}$ are block diagonal. Here, the slave eigenvalue problem decouples to a number of independent problems of manageable size.

Suppose that the structure is decomposed into r substructures and that the vibration problem is discretized (by finite elements or finite differences) in correspondence to the substructure decomposition (i.e. $k_{ij} = 0$ and $m_{ij} = 0$ whenever i and j denote indices of interior nodes of different substructures). We choose the degrees of freedom which are on the boundaries of the substructures as masters and the interior degrees of freedom as slaves (cf. Fig. 3).

If the variables are numbered in an appropriate way the stiffness and mass matrices have the following block form

$$\boldsymbol{K} = \begin{bmatrix} \boldsymbol{K}_{mm} & \boldsymbol{K}_{ms1} & \boldsymbol{K}_{ms2} & \dots & \boldsymbol{K}_{msr} \\ \boldsymbol{K}_{sm1} & \boldsymbol{K}_{ss1} & \boldsymbol{O} & \dots & \boldsymbol{O} \\ \boldsymbol{K}_{sm2} & \boldsymbol{O} & \boldsymbol{K}_{ss2} & \dots & \boldsymbol{O} \\ \dots & \dots & \dots & \dots & \dots \\ \boldsymbol{K}_{smr} & \boldsymbol{O} & \boldsymbol{O} & \dots & \boldsymbol{K}_{ssr} \end{bmatrix}, \tag{18}$$

$$\boldsymbol{M} = \begin{bmatrix} \boldsymbol{M}_{mm} & \boldsymbol{M}_{ms1} & \boldsymbol{M}_{ms2} & \dots & \boldsymbol{M}_{msr} \\ \boldsymbol{M}_{sm1} & \boldsymbol{M}_{ss1} & \boldsymbol{O} & \dots & \boldsymbol{O} \\ \boldsymbol{M}_{sm2} & \boldsymbol{O} & \boldsymbol{M}_{ss2} & \dots & \boldsymbol{O} \\ \dots & \dots & \dots & \dots & \dots \\ \boldsymbol{M}_{smr} & \boldsymbol{O} & \boldsymbol{O} & \dots & \boldsymbol{M}_{ssr} \end{bmatrix}, \tag{19}$$

where the coupling of the interior degrees of freedom of the j-th substructure and the master variables is given by the matrices $\boldsymbol{K}_{smj} = \boldsymbol{K}^t_{msj}$ and $\boldsymbol{M}_{smj} = \boldsymbol{M}^t_{msj}$, respectively. Note that only those columns of $\boldsymbol{K}_{smj}$ and $\boldsymbol{M}_{smj}$ which correspond to master variables on the boundary of the j-th substructure can be different from the nullvector.

From the blockstructure of the matrices $\boldsymbol{K}$ and $\boldsymbol{M}$ in (**??**) and (**??**) we obtain the reduced matrices

$$\boldsymbol{K}_0 = \boldsymbol{K}_{mm} - \sum_{j=1}^{r} \boldsymbol{K}_{mmj} \quad , \quad \boldsymbol{M}_0 = \boldsymbol{M}_{mm} - \sum_{j=1}^{r} \boldsymbol{M}_{mmj},$$

where

$$\boldsymbol{K}_{mmj} := \boldsymbol{K}_{msj}\boldsymbol{K}_{ssj}^{-1}\boldsymbol{K}_{smj}$$

and

$$\boldsymbol{M}_{mmj} := \boldsymbol{K}_{msj}\boldsymbol{K}_{ssj}^{-1}\boldsymbol{M}_{smj} + \boldsymbol{M}_{msj}\boldsymbol{K}_{ssj}^{-1}\boldsymbol{K}_{smj} - \boldsymbol{K}_{msj}\boldsymbol{K}_{ssj}^{-1}\boldsymbol{M}_{ssj}\boldsymbol{K}_{ssj}^{-1}\boldsymbol{K}_{smj}.$$

Obviously the matrices $\boldsymbol{K}_{mmj}$ and $\boldsymbol{M}_{mmj}$ can be evaluated substructurewise (and hence completely in parallel) in the following way: Solve simultaneously the systems of linear equations

$$\boldsymbol{K}_{ssj}\boldsymbol{X}_j := \boldsymbol{K}_{smj}$$

where only those columns of $\boldsymbol{K}_{smj}$ have to be taken into account which belong to boundary variables of the j-th substructure and determine

$$\begin{aligned} \boldsymbol{K}_{mmj} &= \boldsymbol{K}_{msj}\boldsymbol{X}_j, \\ \boldsymbol{Y}_j &:= \boldsymbol{M}_{msj}\boldsymbol{X}_j, \\ \boldsymbol{M}_{mmj} &= \boldsymbol{Y}_j + \boldsymbol{Y}_j^t - \boldsymbol{X}_j^t\boldsymbol{M}_{ssj}\boldsymbol{X}_j. \end{aligned}$$

Because the matrices $\boldsymbol{K}_{ss}$ and $\boldsymbol{M}_{ss}$ are block diagonal the slave eigenvalue problem splits into r independent eigenvalue problems

$$\boldsymbol{K}_{ssj}\boldsymbol{\phi}_j = \omega \boldsymbol{M}_{ssj}\boldsymbol{\phi}_j, \quad j = 1, \dots, r. \tag{20}$$

Hence the modal matrix $\boldsymbol{\Phi}$ of the slave problem is block diagonal, $\boldsymbol{\Phi} = \text{diag}\,[\boldsymbol{\Phi}_j]$, and if the diagonal matrix $\boldsymbol{\Omega}_j$ contains in its diagonal the eigenvalues ω_{ji} of the j-th slave eigenvalue problem (**??**) then for $\bar{\boldsymbol{x}} \in \mathbb{R}^m$

$$\boldsymbol{S}^t\bar{\boldsymbol{x}} = \left\{ \begin{array}{c} \vdots \\ \boldsymbol{\Phi}_j^t \boldsymbol{M}_{smj}\bar{\boldsymbol{x}} - \boldsymbol{\Omega}_j^{-1}\boldsymbol{\Phi}_j^t \boldsymbol{K}_{smj}\bar{\boldsymbol{x}} \\ \vdots \end{array} \right\}. \tag{21}$$

Note that the slave eigenvalue problems (**??**) and those components of the vector $\boldsymbol{S}^t\bar{\boldsymbol{x}}$ which belong to different substructures again can be handled in parallel.

Since the substructures are usually much stiffer than the entire structure only very few substructure modes must be considered in the evaluation of the Rayleigh functional to achieve an eigenvalue approximation of good accuracy. Consequently we need not to solve the complete eigenvalue problems (**??**). Instead only a very small number of the smallest eigenvalues ω_{ji} and corresponding eigenvectors $\boldsymbol{\phi}_{ji}$ ($j = 1, \dots, r$, $i = 1, \dots, s_j$) have to be determined. With these we compute the improved eigenvalue approximation $\tilde{p}(\bar{\boldsymbol{x}})$ as the root of the shortened rational function (cf. Fig. 1)

$$\begin{aligned} \tilde{f}(\lambda, \bar{\boldsymbol{x}}) \;&:=\; -\bar{\boldsymbol{x}}^t\boldsymbol{K}_0\bar{\boldsymbol{x}} + \lambda\bar{\boldsymbol{x}}^t\boldsymbol{M}_0\bar{\boldsymbol{x}} + \sum_{j=1}^{r}\sum_{i=1}^{s_j}\frac{\lambda^2}{\omega_{ji}-\lambda}\left(\boldsymbol{\phi}_{ji}^t\boldsymbol{M}_{smj}\bar{\boldsymbol{x}} - \frac{1}{\omega_{ji}}\boldsymbol{\phi}_{ji}^t\boldsymbol{K}_{smj}\bar{\boldsymbol{x}}\right)^2 \\ &=:\; -\kappa_0 + \lambda\kappa_1 + \sum_{j=1}^{r}\sum_{i=1}^{s_j}\sigma_{ji}\frac{\lambda^2}{\omega_{ji}-\lambda}. \end{aligned} \tag{22}$$

For a given eigenvector $\bar{\boldsymbol{x}}$ of the reduced problem the coefficients κ_0, κ_1 are positive and all σ_{ji} are non negative. Hence the function $\tilde{f}(\cdot, \bar{\boldsymbol{x}})$ is monotonically increasing and convex on J. Therefore the unique solution $\tilde{p}(\bar{\boldsymbol{x}})$ of $\tilde{f}(\lambda, \bar{\boldsymbol{x}}) = 0$ in the interval $(p(\bar{\boldsymbol{x}}), \bar{\lambda})$ can be computed with Newton's method with starting approximation $\bar{\lambda}$ which is globally and monotonically convergent (cf. Fig. 1).

In all examples that we considered we observed that we are able to reduce the relative errors of the eigenvalues $\bar{\lambda} \in (0\,,\,0.5\underline{\omega})$ of the condensed problem to less than 1% if we take into account the slave eigenvalues ω_{ji} satisfying

$$\omega_{ji} \;\; \leq \;\; 30\bar{\lambda}. \tag{23}$$

For eigenvalues $\lambda_j \notin (0.5\underline{\omega}\,,\,\underline{\omega})$ we could not achieve this accuracy not even via the exact Rayleigh functional. The reason is that the eigenvectors of the condensed problem approximate the master portions of the eigenvectors of the uncondensed problem the worse the closer the corresponding eigenvalue is to the minimal eigenvalue of the slave problem.

We approximate these eigenvalues with a combination of the dynamic condensation and an improvement via the Rayleigh functional. If we choose $J := (-\hat{\lambda}\,,\,\underline{\omega} - \hat{\lambda})$ then the exact condensation of the shifted problem (**??**) possesses the Rayleigh functional

$$p^*(\boldsymbol{x}) = p(\boldsymbol{x}) - \hat{\lambda}.$$

With a shift parameter $\hat{\lambda}$ close to $0.5\underline{\omega}$ we are able to compute approximations $\tilde{p}^*(\bar{\boldsymbol{x}}_j)$ of the eigenvalues of problem (??) in the interval $(0.5\underline{\omega}\,,\, 0.75\underline{\omega})$ with a relative error less than 1%, where $\bar{\boldsymbol{x}}_j$ is an eigenvector of the condensed problem of (??) and $\tilde{p}^*$ is the curtailment of p^* following the rule of thumb (??).

The eigenvalues in the interval $(0.75\underline{\omega}\,,\, 0.875\underline{\omega})$ are computed in the same way with a shift close to $0.75\underline{\omega}$ etc.

We have chosen the error bound 1% because this suffices for most applications. Other accuracy requirements lead to different partitions of the interval $(0\,,\,\underline{\omega})$ and to different thresholds for the slave eigenvalues to be considered in the curtailed Rayleigh functional.

Notice that we could have used a shortening of the original Rayleigh functional p to calculate approximations of the eigenvalues of (??) in $(0.5\underline{\omega}\,,\,\underline{\omega})$. With this however we would have needed much more slave modes to obtain the same accuracy.

4 The Computational Method

In this section we collect the algorithm for the computation of the eigenvalues of (??) which do not exceed a given number Λ. The computational steps are as follows:

(1) Compute the smallest eigenvalue ω_{j1} and corresponding eigenvector $\boldsymbol{\phi}_{j1}$ of the slave eigenvalue problem

$$\boldsymbol{K}_{ssj}\boldsymbol{\phi} = \omega \boldsymbol{M}_{ssj}\boldsymbol{\phi}\ , \quad j = 1, \ldots, r, \tag{24}$$

corresponding to the r substructures and set

$$\underline{\omega} := \min_{j=1,\ldots,r} \omega_{j1}.$$

If $\underline{\omega} \le \Lambda$ (or if $\underline{\omega}$ is close to Λ) then choose a finer substructuring.

(2) Compute the eigenvalues ω_{ji} which do not exceed $15\underline{\omega}$ and the corresponding eigenvectors $\boldsymbol{\phi}_{ji}$ of the slave eigenvalue problems (??). An easy calculation shows that following the rule (??) no slave eigenvalues greater than $15\underline{\omega}$ are needed.

(3) Set $\hat{\lambda} := 0$ and $k := 0$ (k counts the number of eigenvalues which are already determined).

(4) Repeat

(i) Set

$$\boldsymbol{L} := \boldsymbol{K} - \hat{\lambda}\boldsymbol{M}$$

and define the partition of $\boldsymbol{L}$ corresponding to that of $\boldsymbol{K}$ in eq. (??).

(ii) For $j = 1, \ldots, r$ solve for the matrix $\boldsymbol{X}_j$

$$\boldsymbol{L}_{ssj}\boldsymbol{X}_j = \boldsymbol{L}_{smj}$$

where only those columns of $\boldsymbol{L}_{smj}$ have to be taken into account which belong to boundary variables of the j-th substructure.

(iii) Perform the following matrix operations

$$\begin{aligned} \boldsymbol{L}_{mmj} &:= \boldsymbol{L}_{msj}\boldsymbol{X}_j \\ \boldsymbol{Y}_j &:= \boldsymbol{M}_{msj}\boldsymbol{X}_j \\ \boldsymbol{M}_{mmj} &:= \boldsymbol{Y}_j + \boldsymbol{Y}_j^t - \boldsymbol{X}_j^t \boldsymbol{M}_{ssj}\boldsymbol{X}_j \end{aligned}$$

and evaluate the condensed matrices

$$\boldsymbol{L}_0 := \boldsymbol{L}_{mm} - \sum_{j=1}^{r} \boldsymbol{L}_{mmj} \ , \quad \boldsymbol{M}_0 := \boldsymbol{M}_{mm} - \sum_{j=1}^{r} \boldsymbol{M}_{mmj}.$$

(iv) Determine the eigenvalues $\bar{\lambda}_{k+1} \leq \ldots \leq \bar{\lambda}_\ell$ and corresponding eigenvectors $\bar{\boldsymbol{x}}_j$ of the reduced problem

$$\boldsymbol{L}_0\boldsymbol{x} = \lambda \boldsymbol{M}_0\boldsymbol{x}$$

which do not exceed $\hat{\lambda} + 0.5(\underline{\omega} - \hat{\lambda})$.

(v) For $\mu = k+1, \ldots, \ell$ do

(α) Compute

$$\kappa_0 := \bar{\boldsymbol{x}}_\mu^t \boldsymbol{L}_0 \bar{\boldsymbol{x}}_\mu \ , \ \kappa_1 := \bar{\boldsymbol{x}}_\mu^t \boldsymbol{M}_0 \bar{\boldsymbol{x}}_\mu.$$

(β) For $j = 1, \ldots, r$ for $i = 1, \ldots, s_j$ calculate

$$\sigma_{ji} := \left(\boldsymbol{\phi}_{ji}^t \boldsymbol{M}_{smj} \bar{\boldsymbol{x}}_j - \frac{1}{\omega_{ji} - \hat{\lambda}} \boldsymbol{\phi}_{ji}^t \boldsymbol{L}_{smj} \bar{\boldsymbol{x}}_j \right)^2 ,$$

where s_j is chosen such that $\omega_{j,s_j} - \hat{\lambda} \leq 30(\bar{\lambda}_j - \hat{\lambda})$.

(γ) Determine the unique root $\tilde{p}^*(\bar{\boldsymbol{x}}_\mu) < \bar{\lambda}_\mu$ of

$$\tilde{f}(\lambda, \bar{\boldsymbol{x}}_\mu) = -\kappa_0 + \kappa_1\lambda + \sum_{j=1}^{r}\sum_{i=1}^{s_j} \sigma_{ji} \frac{\lambda^2}{\omega_{ji} - \hat{\lambda} - \lambda} = 0$$

(δ) Set $\lambda_\mu = \tilde{p}^*(\bar{\boldsymbol{x}}_\mu) + \hat{\lambda}$

(vi) Set $k := \ell$, $\hat{\lambda} := \hat{\lambda} + 0.5(\underline{\omega} - \hat{\lambda})$.

until $\hat{\lambda} > \Lambda$.

(5) For $\mu = 1, \ldots, k$

(i) set $\boldsymbol{x}_{m,\mu} := \bar{\boldsymbol{x}}_\mu$.

(ii) Compute (substructurewise) the slave portions of the eigenvectors

$$\boldsymbol{x}_{s,\mu} = -(\boldsymbol{K}_{ss} - \lambda_\mu \boldsymbol{M}_{ss})^{-1} (\boldsymbol{K}_{sm} - \lambda_\mu \boldsymbol{M}_{sm}) \boldsymbol{x}_{m,\mu}.$$

5 Numerical Results

5.1 Tapered Cantilever Beam

We consider the transversal vibrations of a tapered cantilever beam of length 1 with area of cross section $A_x := A_0(1-0.5x)^2$, $0 \le x \le 1$.

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Fig. 2. Substructuring of the tapered cantilever beam

The problem is described by the boundary eigenvalue problem

$$\begin{aligned} ((1-0.5x)^4 y'')'' &= \lambda(1-0.5x)^2 y \\ y(0) = y'(0) &= y''(1) = y'''(1) = 0, \end{aligned}$$

where $\lambda = \omega^2 \rho A_0/(EI_0)$, A_0 and I_0 are the area of the cross section and moment of inertia at $x=0$ respectively, ρ is the mass per unit volume, E is the modulus of elasticity and ω denotes the natural circular frequencies of the beam.

For the discretisation we employ finite elements with cubic hermite splines (beam elements). We divide the beam into 6 substructures of the same length and each substructure is subdivided into 5 elements of the same lenght.

Thus, problem (**??**) has dimension $n = 60$ and is condensed to dimension $m = 12$. The 6 substructure eigenproblems (**??**) are of dimension $s/r = 8$ and we get $\underline{\omega} \approx 190200$.

In Tab. 1 we denote by λ_j the eigenvalues of the 60×60 system (**??**), $\bar{\lambda}_j$, $\bar{\boldsymbol{x}}_j$ are the eigenpairs of the condensed and linearized 12×12 problem (**??**) and $\tilde{p}(\bar{\boldsymbol{x}}_j)$ are the improvements from the shortened Rayleigh functional, where the number $\tilde{s}$ of terms in the sum in (**??**) is determined by the rule of thumb (**??**).

With static condensation we obtain the eigenvalue approximations $\tilde{p}(\bar{\boldsymbol{x}}_1), \dots, \tilde{p}(\bar{\boldsymbol{x}}_6) \le 0.5\underline{\omega}$ with the predicted accuracy. For $\tilde{p}(\bar{\boldsymbol{x}}_7) > 0.5\underline{\omega}$ the relative error exceeds 1%.

For the shift parameter $\hat{\lambda} := 95100 \approx 0.5\underline{\omega}$ and dynamic condensation we obtain $\tilde{p}(\bar{\boldsymbol{x}}_7) < 0.75\underline{\omega}$ with the desired accuracy. For $\hat{\lambda} = 142650 \approx 0.75\underline{\omega}$ the relative error (0.11%) of $\tilde{p}(\bar{\boldsymbol{x}}_8)$ is already smaller than 1%, but because $\tilde{p}(\bar{\boldsymbol{x}}_8) > 0.875\underline{\omega}$ we compute it with the shift parameter $\hat{\lambda} = 166425 \approx 0.875\underline{\omega}$.

The eigenvalues $\lambda_9, \dots, \lambda_{12} > \underline{\omega}$ can not be approximated by the method discussed here.

j	λ_j	$\bar{\lambda}_j$	$\tilde{p}(\bar{\boldsymbol{x}}_j)$	$\frac{\bar{\lambda}_j-\lambda_j}{\lambda_j}, (\tilde{s}=0)$	$\frac{\tilde{p}(\bar{\boldsymbol{x}}_j)-\lambda_j}{\lambda_j},\ (\tilde{s})$	$\frac{p(\bar{\boldsymbol{x}}_j)-\lambda_j}{\lambda_j}, (\tilde{s}=48)$
shift $\hat{\lambda}=0$						
1	21.3920	21.3934	21.3934	$6.45\cdot10^{-5}$	$6.45\cdot10^{-5}$ (0)	$9.72\cdot10^{-12}$
2	382.110	382.447	382.447	$8.83\cdot10^{-4}$	$8.83\cdot10^{-4}$ (0)	$-4.96\cdot10^{-8}$
3	2359.93	2370.70	2370.70	$4.57\cdot10^{-3}$	$4.57\cdot10^{-3}$ (0)	$-8.52\cdot10^{-7}$
4	8429.82	8542.18	8502.60	$1.34\cdot10^{-2}$	$8.63\cdot10^{-3}$ (2)	$4.53\cdot10^{-6}$
5	22319.0	22882.2	22386.9	$2.52\cdot10^{-2}$	$3.04\cdot10^{-3}$ (6)	$-4.30\cdot10^{-5}$
6	48994.2	51968.2	49178.1	$6.07\cdot10^{-2}$	$3.75\cdot10^{-3}$ (7)	$-5.30\cdot10^{-4}$
7	94670.5	115510.	90865.5	$2.20\cdot10^{-1}$	$-4.02\cdot10^{-2}$ (10)	$-4.13\cdot10^{-2}$
8	166819.	224475.	undef.	$3.46\cdot10^{-1}$	undef.	undef.
9	274178.	419662.	undef.	$5.31\cdot10^{-1}$	undef.	undef.
shift $\hat{\lambda}=95100$						
7	94670.5	94671.0	94671.0	$5.50\cdot10^{-6}$	$5.50\cdot10^{-6}$ (0)	$-7.04\cdot10^{-11}$
8	166819.	179023.	163315.	$7.32\cdot10^{-2}$	$-2.10\cdot10^{-2}$ (9)	$-2.13\cdot10^{-2}$
9	274178.	343041.	undef.	$2.51\cdot10^{-1}$	undef.	undef.
shift $\hat{\lambda}=142650$						
8	166819.	168692.	166636.	$1.12\cdot10^{-2}$	$-1.10\cdot10^{-3}$ (6)	$-1.50\cdot10^{-3}$
9	274178.	317315.	undef.	$1.57\cdot10^{-1}$	undef.	undef.
shift $\hat{\lambda}=166425$						
8	166819.	166820.	166820.	$5.16\cdot10^{-6}$	$5.16\cdot10^{-6}$ (0)	$-8.22\cdot10^{-10}$
9	274178.	307294.	undef.	$1.20\cdot10^{-2}$	undef.	undef.

Tab. 1. Exact, condensed and improved eigenvalues of the tapered cantilever beam with relative errors

5.2 Clamped Plate

The free vibration of a uniform thin elastic plate covering the region $\Omega := [0,2]\times[0,2]$ and clamped at the boundary $\partial\Omega$ is described by the boundary eigenvalue problem

$$\begin{aligned} \Delta^2 u &= \lambda u \quad \text{in } \Omega \\ u = \frac{\partial u}{\partial n} &= 0 \quad \text{on } \partial\Omega. \end{aligned}$$

We like to approximate the eigenvalues $\lambda = \omega^2 h\rho/D$ (ρ mass density of plate material, h plate thickness, D plate flexural rigidity, ω circular frequency of free vibration) of the problem. The region Ω is divided into $r = 9$ quadratic substructures of edge length $2/3$ (cf. Fig. 3).

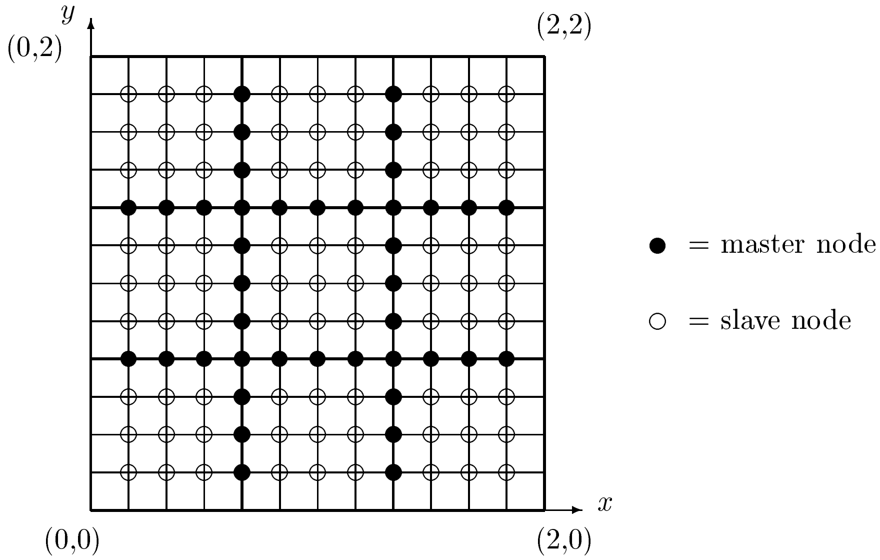

Fig. 3. Substructuring of the clamped plate

Each substructure is discretized by 16 quadratic Bogner-Fox-Schmidt elements (node variables u, u_x, u_y, u_{xy}). Thus problem (**??**) is of dimension $n = 484$ and after condensation we retain $m = 160$ master variables. Each substructure has $s/r = 36$ slave variables and we get $\underline{\omega} \approx 6581.9$.

The eigenvalues $\bar{\lambda}_1, \cdots, \bar{\lambda}_{10}$ of the statically condensed problem are contained in the interval $(0\,,\,0.5\underline{\omega})$ and the relative errors of the improved eigenvalue estimates $\tilde{p}(\bar{\boldsymbol{x}}_1), \cdots, \tilde{p}(\bar{\boldsymbol{x}}_{10})$ are less than 1%. For $\bar{\lambda}_{11} > 0.5\underline{\omega}$ this accuracy can not be achieved. Figs. 4 and 5 show the relative errors of $\tilde{p}(\bar{\boldsymbol{x}}_j)$ as functions of the number of terms which are taken into account in the definition of $\tilde{p}(\bar{\boldsymbol{x}}_j)$.

According to the dynamic condensation strategy described in Section 3 we obtain with shiftparameters $\hat{\lambda} = 0.5\underline{\omega} \approx 3290.95$, $\hat{\lambda} = 0.75\underline{\omega} \approx 4936.43$ and $\hat{\lambda} = 0.875\underline{\omega} \approx 5759.16$ respectively the improved eigenvalue approximations $\tilde{p}(\bar{\boldsymbol{x}}_{11}), \cdots, \tilde{p}(\bar{\boldsymbol{x}}_{13})$ of $\bar{\lambda}_{11}, \cdots, \bar{\lambda}_{13} < 0.75\underline{\omega}$ (cf. Fig. 6), $\tilde{p}(\bar{\boldsymbol{x}}_{14,15})$ of $\bar{\lambda}_{14,15} \in (0.75\underline{\omega}\,,\,0.875\underline{\omega})$ (cf. Fig. 7) and $\tilde{p}(\bar{\boldsymbol{x}}_{16}), \tilde{p}(\bar{\boldsymbol{x}}_{17})$ of $\bar{\lambda}_{16}, \bar{\lambda}_{17} \in (0.875\underline{\omega}\,,\,0.9375\underline{\omega})$ (cf. Fig. 8) respectively where the relative errors in all cases are smaller than 1%.

Good approximations of the eigenvalues $\lambda_{18}, \cdots, \lambda_{160} > \underline{\omega}$ can not be obtained by the method discussed here.

j	λ_j	λ_j $\hat{\lambda} = 0$	λ_j $\hat{\lambda} = 3290.95$	λ_j $\hat{\lambda} = 4936.43$	λ_j $\hat{\lambda} = 5759.16$
1	80.9389	81.3920			
2	336.751	342.225			
3	336.751	342.225			
4	732.194	751.618			
5	1083.07	1197.60			
6	1093.36	1210.76			
7	1703.04	1909.78			
8	1703.04	1909.78			
9	2776.73	3248.57			
10	2776.73	3248.57			
11	3029.33	3926.22	3046.78		
12	3672.98	4726.13	3676.07		
13	3702.85	4974.39	3728.78		
14	5500.06	6565.31	6152.36	5595.50	
15	5500.06	8785.50	6394.81	5595.50	
16	5996.21	8785.50	6394.81	6030.45	5999.21
17	6006.23	9082.03	7089.56	6317.31	6040.09
18	7284.89	10368.2	8163.47	7744.71	7582.50

Tab. 2. Exact and condensed eigenvalues of the clamped plate

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Fig. 4. Clamped plate: relative errors of $\tilde{p}(\bar{\boldsymbol{x}}_1), \cdots, \tilde{p}(\bar{\boldsymbol{x}}_{10})$ shift $\hat{\lambda} = 0$

bild5.eps14cm9cm

Fig. 5. Clamped plate: relative errors of $\tilde{p}(\bar{\boldsymbol{x}}_{11}), \cdots, \tilde{p}(\bar{\boldsymbol{x}}_{13})$ shift $\hat{\lambda} = 0$

bild6.eps14cm9cm

Fig. 6. Clamped plate: relative errors of $\tilde{p}(\bar{\boldsymbol{x}}_{11}), \cdots, \tilde{p}(\bar{\boldsymbol{x}}_{14})$ shift $\hat{\lambda} = 3290.95$

bild7.eps14cm9cm

Fig. 7. Clamped plate: relative errors of $\tilde{p}(\bar{\boldsymbol{x}}_{14,15}), \tilde{p}(\bar{\boldsymbol{x}}_{16})$ shift $\hat{\lambda} = 4936.43$

bild8.eps14cm9cm

Fig. 8. Clamped plate: relative errors of $\tilde{p}(\bar{\boldsymbol{x}}_{16}), \tilde{p}(\bar{\boldsymbol{x}}_{17})$ shift $\hat{\lambda} = 5759.16$

Most of the relative error polygons increase from a certain point on. The reason is that the approximations $\tilde{p}(\bar{\boldsymbol{x}}_j)$ fall below the exact eigenvalue (cf. (**??**)).

6 Concluding Remarks

A new method to substantially improve the eigenvalue estimates of condensed eigenvalue problems is presented. The approach takes advantage of the approximation properties of the Rayleigh functional of the exactly condensed problem. We demonstrate that the eigenvalues in the interval $(0\,,\,0.5\underline{\omega})$ (where $\underline{\omega}$ denotes the smallest eigenvalue of the slave eigenvalue problem) can be computed from the corresponding eigenvectors of the statically condensed problems such that the relative error is less than 1%. To compute the eigenvalues in the interval $(0.5\underline{\omega}\,,\,\underline{\omega})$ with the same accuracy a very small number of dynamically condensed problems has to be solved (in contrast to the dynamic condensation methods given by Leung and by Paz). The method is particularly efficient for dynamic substructure analysis because the condensed problems and the data needed in the Rayleigh functional can be calculated substructurewise. Moreover most of the computations can be performed fully in parallel.

Appendix

Proof of Theorem 2:
The inclusions (**??**) and (**??**) of the eigenvalues λ_j are direct consequences of the following error bounds which were shown in [**?**] by the second author

$$0 \;\leq\; \frac{\bar{\lambda}_j - \lambda_j}{\lambda_j} \;\leq\; \frac{\lambda_j}{\underline{\omega} - \lambda_j}, \quad j = 1, \cdots, \ell.$$

Theorem 1 yields

$$\lambda_1 \;=\; \min_{\boldsymbol{x} \in D(p)\setminus\{\mathbf{0}\}} p(\boldsymbol{x}) \;\leq\; p(\bar{\boldsymbol{x}}_1)$$

and the inequality

$$p(\bar{\boldsymbol{x}}_j) \;\leq\; \bar{\lambda}_j$$

follows from (**??**), (**??**) and (**??**) (cf. Fig. 1).

From the definition of the Rayleigh functional for $j = 1, \cdots, \ell$ we get

$$f(p(\bar{\boldsymbol{x}}_j), \bar{\boldsymbol{x}}_j) = -\bar{\boldsymbol{x}}_j^t \boldsymbol{K}_0 \bar{\boldsymbol{x}}_j + p(\bar{\boldsymbol{x}}_j) \bar{\boldsymbol{x}}_j^t \boldsymbol{M}_0 \bar{\boldsymbol{x}}_j + \bar{\boldsymbol{x}}_j^t \boldsymbol{S} \boldsymbol{D}(p(\bar{\boldsymbol{x}}_j)) \boldsymbol{S}^t \bar{\boldsymbol{x}}_j = 0\ .$$

Hence

$$p(\bar{\boldsymbol{x}}_j) = \frac{\bar{\boldsymbol{x}}_j^t \boldsymbol{K}_0 \bar{\boldsymbol{x}}_j}{\bar{\boldsymbol{x}}_j^t \boldsymbol{M}_0 \bar{\boldsymbol{x}}_j} - \frac{\bar{\boldsymbol{x}}_j^t \boldsymbol{S} \boldsymbol{D}(p(\bar{\boldsymbol{x}}_j)) \boldsymbol{S}^t \bar{\boldsymbol{x}}_j}{\bar{\boldsymbol{x}}_j^t \boldsymbol{M}_0 \bar{\boldsymbol{x}}_j} = \bar{\lambda}_j - \frac{\bar{\boldsymbol{x}}_j^t \boldsymbol{S} \boldsymbol{D}(p(\bar{\boldsymbol{x}}_j)) \boldsymbol{S}^t \bar{\boldsymbol{x}}_j}{\bar{\boldsymbol{x}}_j^t \boldsymbol{M}_0 \bar{\boldsymbol{x}}_j} \tag{25}$$

On account of $\underline{\omega} := \min_{i=1,\ldots,s} \omega_i$ and $\boldsymbol{D}(p(\bar{\boldsymbol{x}}_j)) = \mathrm{diag}\ \left[p(\bar{\boldsymbol{x}}_j)^2/(\omega_i - p(\bar{\boldsymbol{x}}_j))\right]$ we get

$$\frac{\bar{\boldsymbol{x}}_j^t \boldsymbol{S} \boldsymbol{D}(p(\bar{\boldsymbol{x}}_j)) \boldsymbol{S}^t \bar{\boldsymbol{x}}_j}{\bar{\boldsymbol{x}}_j^t \boldsymbol{M}_0 \bar{\boldsymbol{x}}_j} \leq \frac{p(\bar{\boldsymbol{x}}_j)^2}{\underline{\omega} - p(\bar{\boldsymbol{x}}_j)} \cdot \frac{\bar{\boldsymbol{x}}_j^t \boldsymbol{S} \boldsymbol{S}^t \bar{\boldsymbol{x}}_j}{\bar{\boldsymbol{x}}_j^t \boldsymbol{M}_0 \bar{\boldsymbol{x}}_j}. \tag{26}$$

From the definition of $\boldsymbol{S}$ and (**??**) we obtain

$$\begin{aligned}
\boldsymbol{S}\boldsymbol{S}^t &= (\boldsymbol{M}_{ms}\boldsymbol{\Phi} - \boldsymbol{K}_{ms}\boldsymbol{\Phi}\boldsymbol{\Omega}^{-1})(\boldsymbol{\Phi}^t\boldsymbol{M}_{sm} - \boldsymbol{\Omega}^{-1}\boldsymbol{\Phi}^t\boldsymbol{K}_{sm}) \\
&= \boldsymbol{M}_{ms}\boldsymbol{\Phi}\boldsymbol{\Phi}^t\boldsymbol{M}_{sm} - \boldsymbol{M}_{ms}\boldsymbol{\Phi}\boldsymbol{\Omega}^{-1}\boldsymbol{\Phi}^t\boldsymbol{K}_{sm} \\
&\quad -\boldsymbol{K}_{ms}\boldsymbol{\Phi}\boldsymbol{\Omega}^{-1}\boldsymbol{\Phi}^t\boldsymbol{M}_{sm} + \boldsymbol{K}_{ms}\boldsymbol{\Phi}\boldsymbol{\Omega}^{-2}\boldsymbol{\Phi}^t\boldsymbol{K}_{sm}
\end{aligned}$$

$$
\begin{aligned}
&= \boldsymbol{M}_{ms}\boldsymbol{M}_{ss}{}^{-1}\boldsymbol{M}_{sm} - \boldsymbol{M}_{ms}\boldsymbol{K}_{ss}{}^{-1}\boldsymbol{K}_{sm} \\
&\quad -\boldsymbol{K}_{ms}\boldsymbol{K}_{ss}{}^{-1}\boldsymbol{M}_{sm} + \boldsymbol{K}_{ms}\boldsymbol{K}_{ss}{}^{-1}\boldsymbol{M}_{ss}\boldsymbol{K}_{ss}{}^{-1}\boldsymbol{K}_{sm} \\
&= \boldsymbol{M}_0 - (\underbrace{\boldsymbol{M}_{mm} - \boldsymbol{M}_{ms}\boldsymbol{M}_{ss}{}^{-1}\boldsymbol{M}_{sm}}_{=:\boldsymbol{M}_1}).
\end{aligned}
$$

$\boldsymbol{M}_0$ and $\boldsymbol{M}_1$ are positive definite as projections of the positive definite matrix $\boldsymbol{M}$, thus

$$
0 \;\le\; \frac{\bar{\boldsymbol{x}}_j^t \boldsymbol{S}\boldsymbol{S}^t \bar{\boldsymbol{x}}_j}{\bar{\boldsymbol{x}}_j^t \boldsymbol{M}_0 \bar{\boldsymbol{x}}_j} = \frac{\bar{\boldsymbol{x}}_j^t (\boldsymbol{M}_0 - \boldsymbol{M}_1) \bar{\boldsymbol{x}}_j}{\bar{\boldsymbol{x}}_j^t \boldsymbol{M}_0 \bar{\boldsymbol{x}}_j} \;\le\; 1, \tag{27}
$$

and from (**??**), (**??**) and (**??**) we finally obtain

$$
\bar{\lambda}_j - \frac{p(\bar{\boldsymbol{x}}_j)^2}{\underline{\omega} - p(\bar{\boldsymbol{x}}_j)} \le p(\bar{\boldsymbol{x}}_j),
$$

i.e.

$$
\bar{\lambda}_j \frac{1}{1 + \bar{\lambda}_j/\underline{\omega}} \le p(\bar{\boldsymbol{x}}_j).
$$

References

[1] **B. Irons**
Structural Eigenvalue Problems: Elimination of Unwanted Variables, AIAA J. 3 (1965) 961 - 962

[2] **R. J. Guyan**
Reduction of Stiffness and Mass Matrices, AIAA J. 3 (1965) 380

[3] **M. Geradin**
Error Bounds for Eigenvalue Analysis by Elimination of Variables, J. Sound Vibration 19 (1971) 111 - 132

[4] **D. L. Thomas**
Errors in Natural Frequency Calculations Using Eigenvalue Economization, Internat. J. Numer. Meth. Engrg. 18 (1982) 1521 - 1527

[5] **H. Voss**
An Error Bound for Eigenvalue Analysis by Nodal Condensation, in: J. Albrecht, L. Collatz and W. Velte, eds., Numerical Treatment of Eigenvalue Problems 3, Internat. Series Numer. Math. 69 (Birkhäuser, Stuttgart, 1983) 205 - 214

[6] **C. P. Johnson, R. R. Craig Jr., A. Yargicoglu and R. Rajatabhothi**
Quadratic Reduction for the Eigenproblem, Internat. J. Numer. Meth. Engrg. 15 (1980) 911 - 923

[7] **Y. T. Leung**
An Accurate Method of Dynamic Condensation in Structural Analysis, Internat. J. Numer. Meth. Engrg. 12 (1978) 1705 - 1715

[8] **Y. T. Leung**
An Accurate Method of Dynamic Substructuring with Simplified Computation, Internat. J. Numer. Meth. Engrg. 14 (1979) 1241 - 1256

[9] **M. Paz**
Dynamic Condensation, AIAA J. 22 (1984) 724 - 727

[10] **H. R. Schwarz**
Methode der Finiten Elemente (Teubner, Stuttgart, 3.Aufl. 1992)

[11] **L. E. Suarez and M. P. Singh**
Dynamic Condensation Method for Structural Eigenvalue Problems, AIAA J. 30 (1992) 1046 - 1054

[12] **P. Ruge**
Dynamische Kondensation in rein modaler Darstellung, Z. Angew. Math. Mech. 67 (1987) 336 - 337

[13] **K. Berkkan and M. A. Dokainish**
Accuracy of Condensed Eigenvalue Solution, AIAA J. 28 (1990) 1324 - 1325

[14] **K. Rothe**
Least Squares Element Method for Boundary Eigenvalue Problems, Internat. J. Numer. Meth. Engrg. 33 (1992) 2129 - 2143

[15] **H. Voss and B. Werner**
A Minimax Principle for Nonlinear Eigenvalue Problems with Applications to Nonoverdamped Systems, Math. Meth. Appl. Sci. 4 (1982) 415 - 424

[16] **R. D. Henshell, J. H. Ong**
Automatic Masters for Eigenvalue Economisation, Earthquake Engrg. Struct. Dyn. 3 (1975) 375 - 383