Improved Dynamic Substructuring on Distributed Memory MIMD-Computers

Kai Rothe and Heinrich Voss

Institute for Applied Mathematics, University of Hamburg, Federal Republic of Germany

ABSTRACT

For solving the eigenvalue problem of a structural system having a large number of degrees of freedom the authors introduced in [6] the improved condensation method using the Rayleigh functional of the exactly condensed nonlinear eigenvalue problem. In this paper we discuss the implementation of the method for substructured problems on a distributed memory MIMD Computer. Examples are presented which demonstrate the efficiency of the method at least if the number of substructures is not too large.

INTRODUCTION

In the dynamic analysis of structures using the finite element method very large eigenvalue problems arise, and condensation is often employed to reduce the number of degrees of freedom to manageable size. To this end Guyan [2] and Irons [3] introduced a method which is called static condensation nowadays and which is based upon eliminating a large number of coordinates (called slaves) and retaining a small portion of the unknown coordinates (called masters). However, it is well known that often only very few eigenvalues at the lower end of the spectrum are achieved with reasonable accuracy.

Several attempts have been made to increase the accuracy of the eigenvalues of the condensed problem. In [6] we took advantage of the structure of the exactly condensed eigenvalue problem $\boldsymbol{T}(\lambda)\boldsymbol{u} = \mathbf{o}$ which is nonlinear with respect to the parameter λ and the linearization of which is the static condensation. For $\boldsymbol{T}$ a Rayleigh functional p exists which has similar properties as the Rayleigh quotient for linear eigenvalue problems. Especially, if λ is an eigenvalue of $\boldsymbol{T}$ and $\boldsymbol{u}$ is a corresponding eigenvector, then $\boldsymbol{u}$ is a stationary point of p and $\lambda = p(\boldsymbol{u})$. Hence, a first order error of an eigenvector approximation $\bar{\boldsymbol{u}}$ to $\boldsymbol{u}$ yields a second order error of $p(\bar{\boldsymbol{u}})$ as an approximation to λ. Since the eigenvectors $\boldsymbol{u}^j$ of the statically condensed problem are usually good approximations to the master portions of the eigenvectors of the original problem the eigenvalue approximations from the condensed problem can be improved substantially if they are replaced by the values of the Rayleigh functional $p(\boldsymbol{u}^j)$.

The method is ideally suited for the application of substructuring methods in various ways. The condensation process can be performed substructurewise, and the data needed for the evaluation of the Rayleigh functional can be calculated substructurewise, too. Hence, most of the computations can be done in parallel, where a process is allocated to the master problem and to any substructure.

In this note we discuss the implementation of the improved condensation method for dynamic substructuring on a distributed memory computer. Exemplarily, we considere a finite element approximation of the free vibration problem of a rectangular clamped plate by Bogner-Fox-Schmidt elements. For coarse decompositions of the plate (4 and 6 substructures, respectively) and a moderate size of degrees of freedom (196 and 484, respectively) we report the performance of our algorithm (which is of very coarse granularity and hence only needs a negligible communication time) on a PARSYTECH transputer system. It turns out that the speedup and the efficiency of the algorithm is quite satisfactory as long as the number of substructures is not too large. For finer substructuring, however, the master problem has to be solved in parallel, too. This will be studied in a forthcoming paper.

THE CONDENSED EIGENVALUE PROBLEM

Free vibration analysis of structures results in the linear eigenvalue problem

$$\boldsymbol{K}\boldsymbol{x} = \lambda \boldsymbol{M}\boldsymbol{x}, \tag{1}$$

where the stiffness matrix $\boldsymbol{K} \in \mathbb{R}^{(n,n)}$ and the mass matrix $\boldsymbol{M} \in \mathbb{R}^{(n,n)}$ are real symmetric and positive definite, $\boldsymbol{x}$ is the vector of modal displacements, and λ is the square of the natural frequencies.

The size of the eigenvalue problem is reduced by the dynamic condensation method (cf. Leung [4], [5], Berkkan, Dokainish [1], Rothe, Voss [6]). To this end the degrees of freedom are partitioned into masters and slaves, and Equation (1) is rewritten in block form

$$\begin{bmatrix} \boldsymbol{K}_{mm} & \boldsymbol{K}_{ms} \\ \boldsymbol{K}_{sm} & \boldsymbol{K}_{ss} \end{bmatrix} \begin{Bmatrix} \boldsymbol{x}_m \\ \boldsymbol{x}_s \end{Bmatrix} = \lambda \begin{bmatrix} \boldsymbol{M}_{mm} & \boldsymbol{M}_{ms} \\ \boldsymbol{M}_{sm} & \boldsymbol{M}_{ss} \end{bmatrix} \begin{Bmatrix} \boldsymbol{x}_m \\ \boldsymbol{x}_s \end{Bmatrix} \tag{2}$$

where $\boldsymbol{x}_s \in \mathbb{R}^s$ denotes the vector of the slave variables to be eliminated and $\boldsymbol{x}_m \in \mathbb{R}^m$ the master variables to be retained.

Solving the second row of the matrix Equation (2) for $\boldsymbol{x}_s$

$$\boldsymbol{x}_s = -(\boldsymbol{K}_{ss} - \lambda \boldsymbol{M}_{ss})^{-1}(\boldsymbol{K}_{sm} - \lambda \boldsymbol{M}_{sm})\boldsymbol{x}_m, \tag{3}$$

and substituting it in the first row one obtains the dynamically condensed eigenvalue problem

$$\boldsymbol{T}(\lambda)\boldsymbol{x}_m = \mathbf{o}, \tag{4}$$

where

$$\boldsymbol{T}(\lambda) := -\boldsymbol{K}_{mm} + \lambda \boldsymbol{M}_{mm} + (\boldsymbol{K}_{ms} - \lambda \boldsymbol{M}_{ms})(\boldsymbol{K}_{ss} - \lambda \boldsymbol{M}_{ss})^{-1}(\boldsymbol{K}_{sm} - \lambda \boldsymbol{M}_{sm}).$$

The nonlinear matrix eigenvalue problem (4) of dimension m is equivalent to the original linear problem (1), if (1) and the slave eigenvalue problem do not have eigenvalues in common.

We take advantage of the modal properties of the slave problem to write $\boldsymbol{T}(\lambda)$ in a more convenient form. Let $\boldsymbol{\Phi} \in \mathbb{R}^{(n,n)}$ and $\boldsymbol{\Omega} := \operatorname{diag}[\omega_i] \in \mathbb{R}^{(n,n)}$ be the modal matrix and the spectral matrix of the slave eigenvalue problem

$$\boldsymbol{K}_{ss}\boldsymbol{\phi} = \omega \boldsymbol{M}_{ss}\boldsymbol{\phi}, \tag{5}$$

respectively, such that

$$\boldsymbol{\Phi}^t \boldsymbol{M}_{ss}\boldsymbol{\Phi} = \boldsymbol{I} \quad \text{and} \quad \boldsymbol{\Phi}^t \boldsymbol{K}_{ss}\boldsymbol{\Phi} = \boldsymbol{\Omega}. \tag{6}$$

Then $\boldsymbol{T}(\lambda)$ can be rewritten as (cf. Leung [4])

$$\boldsymbol{T}(\lambda) = -\boldsymbol{K}_0 + \lambda \boldsymbol{M}_0 + \boldsymbol{S}\boldsymbol{D}(\lambda)\boldsymbol{S}^t, \tag{7}$$

where

$$\begin{aligned}
\boldsymbol{K}_0 &:= \boldsymbol{K}_{mm} - \boldsymbol{K}_{ms}\boldsymbol{K}_{ss}^{-1}\boldsymbol{K}_{sm}, \\
\boldsymbol{M}_0 &:= \boldsymbol{M}_{mm} - \boldsymbol{K}_{ms}\boldsymbol{K}_{ss}^{-1}\boldsymbol{M}_{sm} - \boldsymbol{M}_{ms}\boldsymbol{K}_{ss}^{-1}\boldsymbol{K}_{sm} \\
&\qquad + \boldsymbol{K}_{ms}\boldsymbol{K}_{ss}^{-1}\boldsymbol{M}_{ss}\boldsymbol{K}_{ss}^{-1}\boldsymbol{K}_{sm}, \\
\boldsymbol{S} &:= \boldsymbol{M}_{ms}\boldsymbol{\Phi} - \boldsymbol{K}_{ms}\boldsymbol{\Phi}\boldsymbol{\Omega}^{-1}, \\
\boldsymbol{D}(\lambda) &:= \operatorname{diag}\left[\lambda^2/(\omega_i - \lambda)\right].
\end{aligned}$$

Neglecting the nonlinear part of (7) yields the statically condensed problem

$$(\boldsymbol{K}_0 - \lambda \boldsymbol{M}_0)\boldsymbol{x}_m = \mathbf{o} \tag{8}$$

which was introduced by Guyan [2] and Irons [3].

Usually in the literature approximations to some of the smallest eigenvalues of (1) and to the master portions $\boldsymbol{x}_m$ of the corresponding eigenvectors are obtained from the statically condensed problem (8), and the slave portions $\boldsymbol{x}_s$ are calculated by Equation (3). Observe however, that only very few eigenmodes are derived from (8) with sufficient accuracy.

In [6] the authors proposed the following improvement of the static condensation method using the Rayleigh functional of the nonlinear eigenvalue problem (4).

Let $\underline{\omega}$ be the smallest eigenvalue of the slave eigenvalue problem and $J := (0, \underline{\omega})$. Then, for any fixed vector $\boldsymbol{u} \in \mathbb{R}^m \setminus \{\mathbf{o}\}$ the real valued function

$$f(\cdot, \boldsymbol{u}) \,:\, J \to \mathbb{R} \quad , \quad \lambda \mapsto f(\lambda, \boldsymbol{u}) := \boldsymbol{u}^t \boldsymbol{T}(\lambda) \boldsymbol{u},$$

i.e.

$$f(\lambda, \boldsymbol{u}) = -\boldsymbol{u}^t \boldsymbol{K}_0 \boldsymbol{u} + \lambda \boldsymbol{u}^t \boldsymbol{M}_0 \boldsymbol{u} + \sum_{i=1}^{s} \frac{\lambda^2}{\omega_i - \lambda} \Big(\boldsymbol{\phi}_i^t \boldsymbol{M}_{sm} \boldsymbol{u} - \frac{1}{\omega_i} \boldsymbol{\phi}_i^t \boldsymbol{K}_{sm} \boldsymbol{u} \Big)^2 \quad (9)$$

is strictly monotonically increasing. Hence, the nonlinear equation $f(\lambda, \boldsymbol{u}) = 0$ has at most one solution in J, and therefore, it implicitly defines a functional

$$p \,:\, \mathbb{R}^m \supset D(p) \to J \quad , \quad f(p(\boldsymbol{u}), \boldsymbol{u}) = 0.$$

p is called the Rayleigh functional of the nonlinear eigenvalue problem (4), since for linear eigenvalue problems it reduces to the Rayleigh quotient. In accordance with the linear theory, it is possible to characterize the lower part of the spectrum of (4) as min-max-values of p (cf. Voss [7]).

We take advantage of the approximation properties of the Rayleigh functional which are similar to those of the Rayleigh quotient in the linear case. Since the eigenvectors of $\boldsymbol{T}$ are stationary points of p, a first order error of an eigenvector approximation $\bar{\boldsymbol{u}}$ yields a second order error of the value of the Rayleigh functional $p(\bar{\boldsymbol{u}})$ for the corresponding eigenvalue (if $\bar{\boldsymbol{u}} \in D(p)$).

Let $\bar{\lambda} \in J$ be an eigenvalue of the statically condensed problem (8) and let $\bar{\boldsymbol{u}}$ be a corresponding eigenvector. Then

$$f(0, \bar{\boldsymbol{u}}) = -\bar{\boldsymbol{u}}^t \boldsymbol{K}_0 \bar{\boldsymbol{u}} < 0 \le \bar{\boldsymbol{u}}^t \boldsymbol{S} \boldsymbol{D}(\bar{\lambda}) \boldsymbol{S}^t \bar{\boldsymbol{u}} = f(\bar{\lambda}, \bar{\boldsymbol{u}}). \quad (10)$$

Hence, if $\bar{\boldsymbol{u}}^j$ is an eigenvector corresponding to the j-th eigenvalue $\bar{\lambda}_j$ of the statically condensed problem, then $p(\bar{\boldsymbol{u}}^j)$ is defined and can be expected to be a much better approximation to λ_j than $\bar{\lambda}_j$ (at least, if $\bar{\boldsymbol{u}}^j$ is a good approximation to the master portion of the eigenvector corresponding to λ_j).

DYNAMIC SUBSTRUCTURING

The evaluation of the Rayleigh functional $p(\bar{\boldsymbol{u}})$ for a given vector $\bar{\boldsymbol{u}} \in \mathbb{R}^m$ seems to be very costly since the complete modal information of the slave eigenvalue problem (the dimension of which usually is still very large) is needed . In the free vibration analysis of structures, however, the slave variables can be chosen such that the matrices $\boldsymbol{K}_{ss}$ and $\boldsymbol{M}_{ss}$ are block diagonal, and the slave eigenvalue problem decouples to a number of independent problems of manageable size.

Suppose that r substructures are considered and that they connect to each other through the master variables on the interfaces only. If the slave variables are numbered appropriately, then the stiffness matrix is given by

$$\boldsymbol{K} = \begin{bmatrix} \boldsymbol{K}_{mm} & \boldsymbol{K}_{ms1} & \boldsymbol{K}_{ms2} & \dots & \boldsymbol{K}_{msr} \\ \boldsymbol{K}_{sm1} & \boldsymbol{K}_{ss1} & \boldsymbol{O} & \dots & \boldsymbol{O} \\ \boldsymbol{K}_{sm2} & \boldsymbol{O} & \boldsymbol{K}_{ss2} & \dots & \boldsymbol{O} \\ \dots & \dots & \dots & \dots & \dots \\ \boldsymbol{K}_{smr} & \boldsymbol{O} & \boldsymbol{O} & \dots & \boldsymbol{K}_{ssr} \end{bmatrix},$$

and the mass matrix $\boldsymbol{M}$ has the same block form.

Here, the coupling of the interior degrees of freedom of the j-th substructure and the master variables is given by the matrices $\boldsymbol{K}_{smj} = \boldsymbol{K}^t_{msj}$ and $\boldsymbol{M}_{smj} = \boldsymbol{M}^t_{msj}$, respectively. Observe that only those columns of $\boldsymbol{K}_{smj}$ and $\boldsymbol{M}_{smj}$ which correspond to master variables on the boundary of the j-th substructure can be different from the nullvector.

Taking advantage of the blockstructure of $\boldsymbol{K}$ and $\boldsymbol{M}$ the reduced matrices $\boldsymbol{K}_0$ and $\boldsymbol{M}_0$ can be calculated substructurewise, and hence, completely in parallel. Obviously,

$$\boldsymbol{K}_0 = \boldsymbol{K}_{mm} - \sum_{j=1}^{r} \boldsymbol{K}_{mmj} := \boldsymbol{K}_{mm} - \sum_{j=1}^{r} \boldsymbol{K}_{msj}\boldsymbol{K}^{-1}_{ssj}\boldsymbol{K}_{smj}$$

and

$$\boldsymbol{M}_0 = \boldsymbol{M}_{mm} - \sum_{j=1}^{r} \boldsymbol{M}_{mmj},$$

where

$$\begin{aligned} \boldsymbol{M}_{mmj} \quad := \quad & \boldsymbol{K}_{msj}\boldsymbol{K}^{-1}_{ssj}\boldsymbol{M}_{smj} + \boldsymbol{M}_{msj}\boldsymbol{K}^{-1}_{ssj}\boldsymbol{K}_{smj} \\ & -\boldsymbol{K}_{msj}\boldsymbol{K}^{-1}_{ssj}\boldsymbol{M}_{ssj}\boldsymbol{K}^{-1}_{ssj}\boldsymbol{K}_{smj}. \end{aligned}$$

Thus, if the matrix $\boldsymbol{X}_j$ is the solution of

$$\boldsymbol{K}_{ssj}\boldsymbol{X}_j := \boldsymbol{K}_{smj}$$

where only those columns of $\boldsymbol{K}_{smj}$ have to be taken into account which belong to master variables on the boundary of the j-th substructure, then

$$\begin{aligned}\boldsymbol{K}_{mmj} &= \boldsymbol{K}_{msj}\boldsymbol{X}_j,\\ \boldsymbol{M}_{mmj} &= \boldsymbol{Y}_j + \boldsymbol{Y}_j^t - \boldsymbol{X}_j^t\boldsymbol{M}_{ssj}\boldsymbol{X}_j, \quad \boldsymbol{Y}_j := \boldsymbol{M}_{msj}\boldsymbol{X}_j.\end{aligned}$$

Moreover, since the matrices $\boldsymbol{K}_{ss}$ and $\boldsymbol{M}_{ss}$ are block diagonal the slave eigenvalue problem splits into r independent eigenvalue problems

$$\boldsymbol{K}_{ssj}\boldsymbol{\phi}_j = \omega\boldsymbol{M}_{ssj}\boldsymbol{\phi}_j, \quad j = 1, \dots, r. \tag{11}$$

Therefore, the modal matrix $\boldsymbol{\Phi} = \text{diag}\,[\boldsymbol{\Phi}_j]$ and the spectral matrix $\boldsymbol{\Omega} = \text{diag}\,[\boldsymbol{\Omega}_j]$ of the slave problem are block diagonal, and for $\bar{\boldsymbol{u}} \in \mathrm{I\!R}^m$

$$\boldsymbol{S}^t\bar{\boldsymbol{u}} = \left\{\begin{array}{c}\vdots\\ \boldsymbol{\Phi}_j^t\boldsymbol{M}_{smj}\bar{\boldsymbol{u}} - \boldsymbol{\Omega}_j^{-1}\boldsymbol{\Phi}_j^t\boldsymbol{K}_{smj}\bar{\boldsymbol{u}}\\ \vdots\end{array}\right\}. \tag{12}$$

Hence, the slave eigenvalue problems (11) and those components of the vector $\boldsymbol{S}^t\bar{\boldsymbol{u}}$ which belong to different substructures again can be handled in parallel.

A considerable saving of work can be made by the observation that usually the substructures are much stiffer than the entire structure. Hence, only very few substructure modes have to be considered in the evaluation of the Rayleigh functional to achieve an eigenvalue approximation of good accuracy. Consequently we need not to solve the complete eigenvalue problems (11). Instead only a very small number of the smallest eigenvalues ω_{ji} and corresponding eigenvectors $\boldsymbol{\phi}_{ji}$ ($j = 1, \dots, r$, $i = 1, \dots, s_j$) have to be determined. With these we compute the improved eigenvalue approximation $\tilde{p}(\bar{\boldsymbol{u}})$ as the root of the shortened rational function

$$\begin{aligned}\tilde{f}(\lambda, \bar{\boldsymbol{u}}) &:= -\bar{\boldsymbol{u}}^t\boldsymbol{K}_0\bar{\boldsymbol{u}} + \lambda\bar{\boldsymbol{u}}^t\boldsymbol{M}_0\bar{\boldsymbol{u}}\\ &\qquad + \sum_{j=1}^{r}\sum_{i=1}^{s_j}\frac{\lambda^2}{\omega_{ji}-\lambda}\Big(\boldsymbol{\phi}_{ji}^t\boldsymbol{M}_{smj}\bar{\boldsymbol{u}} - \frac{1}{\omega_{ji}}\boldsymbol{\phi}_{ji}^t\boldsymbol{K}_{smj}\bar{\boldsymbol{u}}\Big)^2\\ &=: -\kappa_0 + \lambda\kappa_1 + \sum_{j=1}^{r}\sum_{i=1}^{s_j}\sigma_{ji}\frac{\lambda^2}{\omega_{ji}-\lambda}.\end{aligned} \tag{13}$$

For a given eigenvector $\bar{\boldsymbol{u}}$ of the reduced problem the coefficients κ_0, κ_1 are positive and all σ_{ji} are non negative. Thus, the function $\tilde{f}(\cdot, \bar{\boldsymbol{u}})$ is monotonically increasing and convex on J, and the unique solution $\tilde{p}(\bar{\boldsymbol{u}})$ of $\tilde{f}(\lambda, \bar{\boldsymbol{u}}) = 0$ in the interval $(p(\bar{\boldsymbol{u}})\,, \bar{\lambda})$ can be computed with Newton's method with starting approximation $\bar{\lambda}$ which is globally and monotonically convergent.

In all examples that we considered we observed that we are able to reduce the relative errors of the eigenvalues $\bar\lambda \in (0\,,\,0.5\underline{\omega})$ of the condensed problem to less than 1% if we took into account the slave eigenvalues ω_{ji} satisfying

$$\omega_{ji} \quad < \quad 15\underline{\omega}\ . \tag{14}$$

For an eigenvalue $\bar\lambda \in (0.5\underline{\omega}\,,\,\underline{\omega})$ usually the approximation of the master portion of the eigenvector of (1) by the eigenvector of the reduced problem corresponding to $\bar\lambda$ is not good enough to obtain an error of about 1%, not even if one uses the exact Rayleigh functional. In [6] we discussed a shift strategy to cope with this interval.

SUBSTRUCTURING AND PARALLEL PROCESSES

For r substructures we create $r+1$ processes, one named 'master' and one for each substructure $(S_1,\cdots,S_r)$ which work in parallel as shown in Table 1.

step	master	$S_j \qquad j=1,\cdots,r$
initialisation		
1	compute $\boldsymbol{K}_{mm}, \boldsymbol{M}_{mm}$	compute $\boldsymbol{K}_{ssj}, \boldsymbol{K}_{smj}, \boldsymbol{M}_{ssj}, \boldsymbol{M}_{smj}$ solve $\boldsymbol{K}_{ssj}\boldsymbol{X}_j = \boldsymbol{K}_{smj}$ for $\boldsymbol{X}_j$ $\boldsymbol{K}_{mmj} := \boldsymbol{K}_{msj}\boldsymbol{X}_j$ $\boldsymbol{Y}_j := \boldsymbol{M}_{msj}\boldsymbol{X}_j$ $\boldsymbol{M}_{mmj} := \boldsymbol{Y}_j + \boldsymbol{Y}_j^{\,t} - \boldsymbol{X}_j^{\,t}\boldsymbol{M}_{ssj}\boldsymbol{X}_j$
communication		
2	solve $\boldsymbol{K}_0\bar{\boldsymbol{u}}_\mu = \bar\lambda_\mu \boldsymbol{M}_0\bar{\boldsymbol{u}}_\mu$ $\mu=1,\cdots,l$	compute $\omega_{j1}\le\ldots\le\omega_{j,s_j}$ and $\boldsymbol{\phi}_{ji}$ from $\boldsymbol{K}_{ssj}\boldsymbol{\phi}_{ji} = \omega_{ji}\boldsymbol{M}_{ssj}\boldsymbol{\phi}_{ji}$ such that $\omega_{j,s_j} \le 30\omega_{j1} < \omega_{j,s_j+1}$
communication		
3	compute $\bar{\boldsymbol{u}}_\mu^t\boldsymbol{K}_0\bar{\boldsymbol{u}}_\mu, \bar{\boldsymbol{u}}_\mu^t\boldsymbol{M}_0\bar{\boldsymbol{u}}_\mu$ $\mu=1,\cdots,l$	compute $\sqrt{\sigma_{ji}} = \boldsymbol{\phi}_{ji}^t\boldsymbol{M}_{smj}\bar{\boldsymbol{u}}_\mu - \boldsymbol{\phi}_{ji}^t\boldsymbol{K}_{smj}\bar{\boldsymbol{u}}_\mu/\omega_{ji}$ $\mu=1,\cdots,l,\ i=1,\ldots,s_j$
communication		
4	compute $\tilde p(\bar{\boldsymbol{u}}_\mu)$ from $\tilde f(\lambda,\bar{\boldsymbol{u}}_\mu)=0$ $\mu=1,\cdots,l$	idle
communication		
5	idle	compute $(\boldsymbol{K}_{ssj} - \tilde p(\bar{\boldsymbol{u}}_\mu)\boldsymbol{M}_{ssj})(\boldsymbol{x}_{s,\mu})_j =$ $-(\boldsymbol{K}_{smj} - \tilde p(\bar{\boldsymbol{u}}_\mu)\boldsymbol{M}_{smj})\bar{\boldsymbol{u}}_\mu \qquad \mu=1,\cdots,l$

Table 1. Steps of parallel computation

For the necessary communication these processes are connected among each other so that each S_i has a direct or indirect connection to the master. In the following examples we show some possible connection topologies which depend on the substructuring of the underlying domain (cf. Figure 2 and Figure 5).

We anticipate here that the idle time of $S_1, \cdots, S_r$ at step 4 does not have a significant influence on the load balancing because step 4 consumes only a negligible small part at the total computing time (cf. Figure 3 and Figure 6).

NUMERICAL RESULTS

We tested the improved dynamic substructuring method on a distributed memory PARSYTEC transputer system equipped with T800 INMOS transputers (25MHz, 4 MB RAM) under the distributed operating system 'helios'. Since each processor has a multiprocessing capability we were able to execute more than one process from Table 1 on every processor which turned out to be extremly important for a good load balancing of the system in the examples below (cf. Figure 3 and Figure 6).

We do not discuss the mapping of the process topology to the processor network. The operating system 'helios' has the option to do this mapping automatically, although this is not always the fastest solution. In our examples we tested different mappings, and we report the results of the best (fastest) mapping. This was possible because we considered examples with a small number of processes. However, for problems with a larger number of processes this point requires further investigations.

We considere the free vibration problem of a uniform thin clamped plate covering the rectangular region $\Omega := [0,4] \times [0,3]$ which is described by the eigenvalue problem

$$\begin{aligned} \Delta^2 u &= \lambda u \quad \text{in } \Omega \\ u = \frac{\partial u}{\partial n} &= 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{15}$$

We discretized the problem by rectangular Bogner-Fox-Schmidt elements (node variables u, u_x, u_y, u_{xy}) and considered different decompositions of Ω into substructures.

The eigenproblems in step 2 of Table 1 were solved using the EISPACK-library. The general eigenproblem was transformed by Cholesky decomposition to a special eigenvalue problem, this one was reduced to tridiagonal form, the eigenvalues of which were determined by bisection. The eigenvectors of the tridiagonal system were computed by inverse iteration, and then were backtransformed.

EXAMPLE 1: 4 SUBSTRUCTURES

First we consider a discretization of (15) by 64 equidistant rectangular Bogner-Fox-Schmidt elements. Thus the uncondensed problem (1) is of dimension $n = 196$.

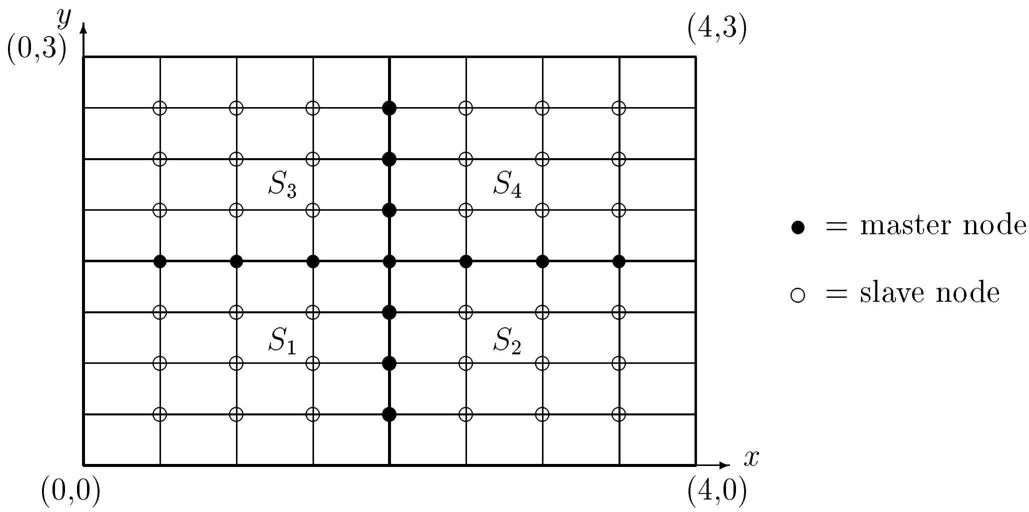

Figure 1. Discretized clamped plate (4 substructures)

We divided Ω into $r = 4$ substructures of edge length 2 in x-direction and 3/2 in y-direction (cf. Figure 1). Thus, the master problem is of dimension $m = 52$, and the slave problem splits into 4 independent problems each of dimension $s/r = 36$. The minimal slave eigenvalue is $\underline{\omega} \approx 163.36$.

Three eigenvalues $\bar{\lambda}_1$, $\bar{\lambda}_2$ and $\bar{\lambda}_3$ of the statically condensed problem are contained in the interval $(0\,,\,0.5\underline{\omega})$. The relative errors of $\bar{\lambda}_j$, $j = 1, 2, 3$, up to 21% are reduced to less than 1% for the improved eigenvalue approximates $\tilde{p}(\bar{\boldsymbol{u}}^1)$, $\tilde{p}(\bar{\boldsymbol{u}}^2)$ and $\tilde{p}(\bar{\boldsymbol{u}}^3)$ (cf. Table 2).

j	λ_j	$\bar{\lambda}_j$	$(\bar{\lambda}_j - \lambda_j)/\lambda_j$	$\tilde{p}(\bar{\boldsymbol{u}}^j)$	$(\tilde{p}(\bar{\boldsymbol{u}}^j) - \lambda_j)/\lambda_j$
1	10.1669	10.3742	$2.0 \cdot 10^{-2}$	10.1761	$9.1 \cdot 10^{-4}$
2	28.6623	31.3936	$9.5 \cdot 10^{-2}$	28.6874	$8.8 \cdot 10^{-4}$
3	56.3857	67.9643	$2.1 \cdot 10^{-1}$	56.6577	$4.8 \cdot 10^{-3}$
4	79.6953	91.1612	$1.4 \cdot 10^{-1}$	78.7445	$-1.2 \cdot 10^{-2}$
5	90.3928	175.7582	$9.4 \cdot 10^{-1}$	undef.	undef.

Table 2. Eigenvalues (4 substructures)

For parallel computation we define the five processes 'master' and $S_1, \cdots, S_4$ (cf. Figure 1) which work off the five steps of Table 1. The connection topology of them is shown in Figure 2.

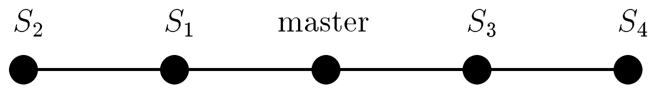

Figure 2. Connection topology (4 substructures)

At the end of step 1 $\boldsymbol{K}_0$ and $\boldsymbol{M}_0$ have to be updated. To circumvent a bottleneck at the master process we send the matrices $\boldsymbol{K}_{mm2}$ and $\boldsymbol{M}_{mm2}$ to process S_1 (resp. $\boldsymbol{K}_{mm4}$ and $\boldsymbol{M}_{mm4}$ to S_3) and perform an intermediate summation

processors	time [sec]	speedup	efficiency [%]
1	21.152	1.00	100.0
2	12.486	1.69	84.7
3	8.577	2.47	82.2
4	8.411	2.51	62.9
5	6.948	3.04	60.9

Table 3. Speedup and efficiency (4 substructures)

Table 3 contains the total CPU time, the speed up and the efficiency if the five processes are distributed to 1 through 5 processors.

Three processors seem to give a balanced compromise between minimal total time and optimal employment of the hardware resources in use measured by the efficiency. For this case the optimal distribution of the processes to the processors as well as the load diagram is shown in Figure 3, where a thin horizontal line indicates that the corresponding processor is busy.

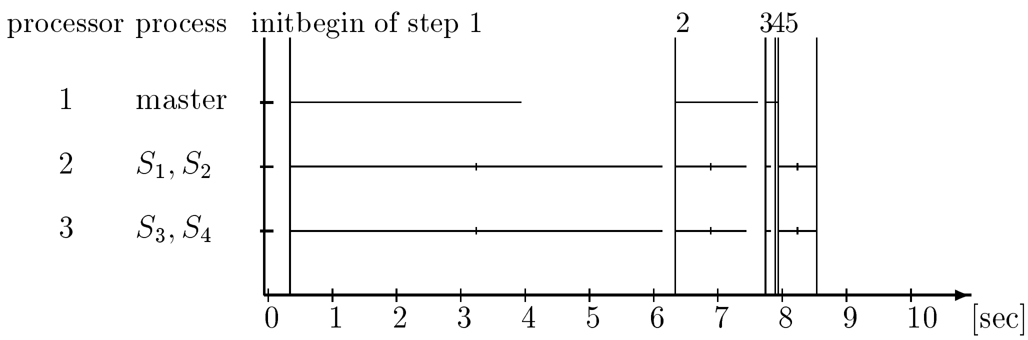

Figure 3. Loading diagram for 3 processors (4 substructures)

The gaps at the end of the steps 1, 2 and 3 in the diagrams of processor 2 and 3 indicate the communication time. Since the improved dynamic substructuring is of very coarse granularity the costs of communication are nearly negligible. Especially, this means that the system dependend initialisation and data transfer rates are of subordinate importance, and the same holds true for the process topology as long as we consider a small number of processes.

Finally, for comparison we solved the original eigenvalue problem directly by the EISPACK-routines mentioned above. To obtain the smallest three eigenvalues a CPU time of 70.2 sec was necessary. Thus, even if the improved condensation is executed on a single processor only 30.1% of the CPU time is needed, for 3 processors the time is reduced to 12.2%.

EXAMPLE 2: 6 SUBSTRUCTURES

The region Ω is discretized equidistantly by 144 rectangular Bogner-Fox-Schmidt elements yielding an uncondensed problem (1) of dimension $n = 484$.

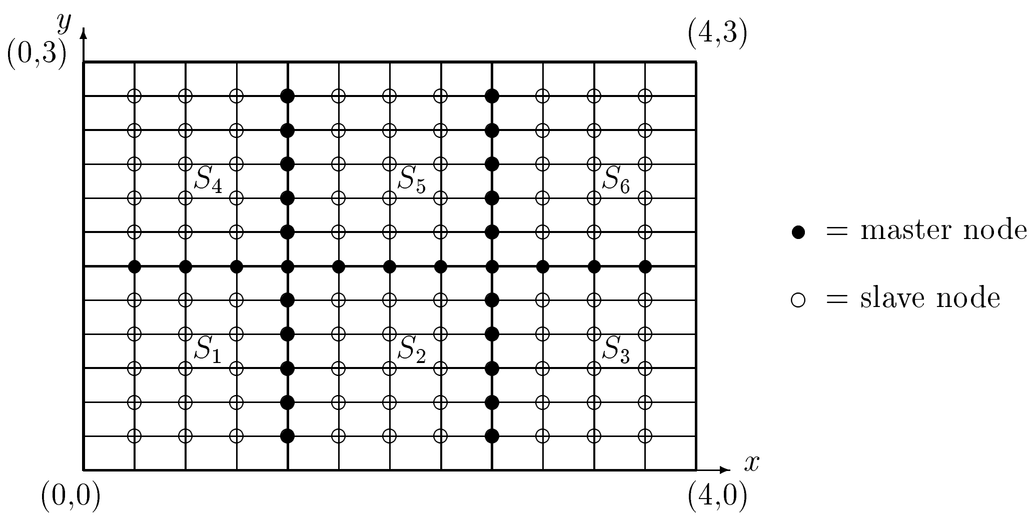

Figure 4. Discretized clamped plate (6 substructures)

We divided Ω into $r = 6$ identical substructures of edge length 4/3 in x-direction and 3/2 in y-direction (cf. Figure 4). Thus, the master problem is of dimension $m = 124$, and the slave problem splits into 6 independent eigenproblems of dimension $s/r = 60$. The minimal slave eigenvalue is $\underline{\omega} \approx 331.52$.

Five eigenvalues $\bar{\lambda}_1, \cdots, \bar{\lambda}_5$ of the statically condensed problem are contained in the interval $(0\,,\,0.5\underline{\omega})$ and the relative errors are reduced from up to 13% to less than 1% by the use of the curtailed Rayleigh functional $\tilde{p}$ (cf. Table 4).

j	λ_j	$\bar{\lambda}_j$	$(\bar{\lambda}_j - \lambda_j)/\lambda_j$	$\tilde{p}(\bar{\boldsymbol{u}}^j)$	$(\tilde{p}(\bar{\boldsymbol{u}}^j) - \lambda_j)/\lambda_j$
1	10.1601	10.2878	$1.3 \cdot 10^{-2}$	10.1727	$1.2 \cdot 10^{-3}$
2	28.6409	29.3124	$2.3 \cdot 10^{-2}$	28.6538	$4.5 \cdot 10^{-4}$
3	56.3358	62.7372	$1.1 \cdot 10^{-1}$	56.4001	$1.1 \cdot 10^{-3}$
4	79.4394	90.1488	$1.3 \cdot 10^{-1}$	79.5550	$1.5 \cdot 10^{-3}$
5	90.2518	101.7402	$1.3 \cdot 10^{-1}$	90.5234	$3.0 \cdot 10^{-3}$
6	165.7031	220.9188	$3.3 \cdot 10^{-1}$	190.5394	$1.5 \cdot 10^{-1}$
7	190.1096	238.9103	$2.6 \cdot 10^{-1}$	192.6623	$1.3 \cdot 10^{-2}$
8	198.1060	291.4454	$4.7 \cdot 10^{-1}$	165.2930	$-1.7 \cdot 10^{-1}$
9	256.1018	320.5638	$2.5 \cdot 10^{-1}$	231.4208	$-9.6 \cdot 10^{-2}$
10	309.8084	528.7235	$7.1 \cdot 10^{-1}$	undef.	undef.

Table 4. Eigenvalues (6 substructures)

For parallel computation we consider seven processes: 'master' and $S_1, \cdots, S_6$ (cf. Figure 4) which work off the five steps of Table 1. The connection topology of them is shown in Figure 5.

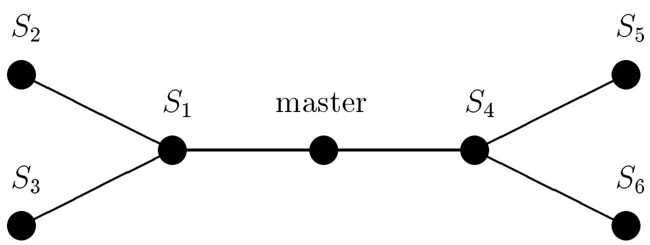

Figure 5. Connection topology (6 substructures)

At the end of step 1 $\boldsymbol{K}_0$ and $\boldsymbol{M}_0$ are updated using an intermediate summation of the submatrices from S_2 and S_3 by S_1 and from S_5 and S_6 by S_4, respectively, to avoid a bottleneck at the master processor.

processors	time [sec]	speedup	efficiency [%]
1	88.72	1.00	100.0
2	50.59	1.75	87.7
3	41.97	2.11	70.5
4	35.85	2.47	61.9
5	33.11	2.68	53.6
6	32.89	2.70	45.0
7	29.59	3.00	42.8

Table 5. Speedup and efficiency (6 substructures)

A balanced relation between a good speedup and a reasonable efficiency is attained for 4 processors, for which the load balancing is shown in Figure 6. As in our first example we recognize that the costs of communication are comparatively small (they consume only ??.?% of the total execution time).

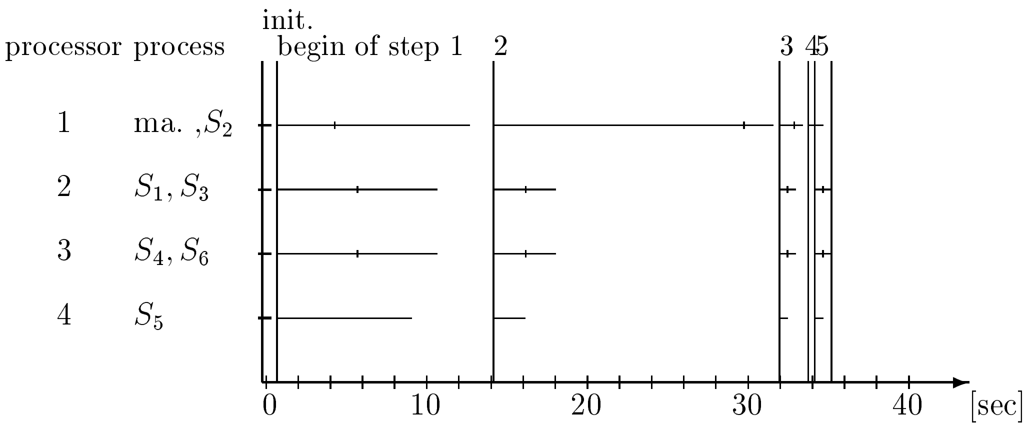

Figure 6. Loading diagram for 4 processors (6 substructures)

The direct computation of the eigenvalues $\lambda_1, \ldots, \lambda_5$ of the original eigenvalue problem of dimension $n = 484$ by the EISPACK-routines as mentioned above took 910.94 sec, where the main part is spent for the reduction to tridiagonal form. Thus, even if the improved substructuring is executed on a single processor only 9.7% of the CPU time is needed, for 4 processors the time is reduced to 3.9%.

Although the speedup and the efficiency are quite satisfactory, this example already demonstrates that the parallelization of the improved substructuring with a coarse granularity as described here is of limited value. The efficiency of the total program is influenced by two conflicting facts. Since the dimension of the master problem is significantly larger than the dimensions of the slave eigenvalue problems corresponding to the substructures, the solution of the master eigenvalue problem dominates the computing time in step 2 although each of the processors 2, 3 and 4 run two slave processes. The processors 2,3 and 4 are idle for ??.?% of step 2. On the other hand, in step 1 the matrices $\boldsymbol{K}_{mmj}$ and $\boldsymbol{M}_{mmj}$ have to be determined by the processors 2, 3 and 4 and the master processor 1 is idle for ??.?% of the execution time.

For real life problems the reduction of the efficiency by this conflict will be more pronounced. To obtain the relevant part of the spectrum of the original system (1) with sufficient accuracy one has to choose a fine substructuring (Observe, that only those eigenvalue approximations $\bar{\lambda}_j$ from the condensed problem such that $\bar{\lambda}_j < 0.5\underline{\omega}$ can be improved substantially by the Rayleigh functional). Therefore, the dimension of the master problem will be large compared to the dimensions of the substructure slave eigenvalue problems, and, if the load of the processors in step 1 is sufficiently ballanced, then the solution of the statically condensed problem will dominate the total solution process. Thus, to derive an efficient parallel eigensolver on the basis of the improved substructuring method, we have to solve the condensed eigenvalue problem in parallel, too. This will be studied in a forthcoming paper.

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