Zur Lusternik-Schnirelmann-Kategorie Euklidischer Konfigurationsräume

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Abstract

We calculate the Lusternik-Schnirelmann-Category of the n-th ordered configuration spaces $\widetilde{C}^n(\mathbb{R}^m)$ of $\mathbb{R}^m$ and give bounds for the category of the corresponding unordered configuration spaces $C^n(\mathbb{R}^m)$. In many cases, e.g. if n is a power of 2, we determine $\mathrm{cat}(C^n(\mathbb{R}^m))$ precisely. We also obtain information on the sectional category of the fibrations $\widetilde{C}^n(\mathbb{R}^m) \to C^n(\mathbb{R}^m)$.

The sequel is the summary of [Rot05]. We give precise statements of the main results. Unless otherwise stated, we refer to [Rot05].

Theorem A
For all $m \geq 2$, i.e. as long as $\widetilde{C}^n(\mathbb{R}^m)$ is connected,

$$\mathrm{cat}(\widetilde{C}^n(\mathbb{R}^m)) = n - 1 \tag{1}$$

holds. The space $\widetilde{C}^n(\mathbb{R})$ consists of $n!$ contractible components, hence $\mathrm{cat}(\widetilde{C}^n(\mathbb{R})) = n! - 1$.

In the unordered case, we obtain

Theorem B
Let k_n be the number of 1's in the dyadic expansion of n. Then we have

$$(n - k_n) \cdot (m - 1) \leq \mathrm{cat}(C^n(\mathbb{R}^m)) \leq (n - 1) \cdot (m - 1). \tag{2}$$

If (i) n is a power of two or $n = 3$
or (ii) n is a prime and m is odd
or (iii) $m \leq 2$,
we have

$$\mathrm{cat}(C^n(\mathbb{R}^m)) = (n - 1) \cdot (m - 1). \tag{3}$$

If $n = p$ is a prime and m is even, then we also have

$$\mathrm{cat}(C^p(\mathbb{R}^m)) \geq (p - 1) \cdot (m - 2). \tag{4}$$

To derive theorem A for $m \geq 2$, we bound $\mathrm{cat}(\widetilde{C}^n(\mathbb{R}^m))$ from below by the integral cup-length (Folgerung 2.2) and from above by a standard argument (Satz 1.13), using that there is an $(m-2)$-connected $(n-1) \cdot (m-1)$-dimensional CW-model for $\widetilde{C}^n(\mathbb{R}^m)$. As the lower bound, this model is obtained through (co-)homological input due to Fred Cohen.

The lower bound for $\mathrm{cat}(C^n(\mathbb{R}^m))$ in (2) is in fact a lower bound for the sectional category of the covering $\widetilde{C}^n(\mathbb{R}^m) \to C^n(\mathbb{R}^m)$. It is obtained by finding a class of category weight $(n - k_n) \cdot (m-1)$ in the cohomology of $C^n(\mathbb{R}^m)$ with $\mathbb{Z}/2\mathbb{Z}$-coefficients. The existence of such a class follows from Vassiliev's observation, that the obvious morphism $H^*(\Sigma_n; \mathbb{Z}/2\mathbb{Z}) \to H^*(C^n(\mathbb{R}^m); \mathbb{Z}/2\mathbb{Z})$ is surjective (see [Vas92] or Satz 2.8 in [Rot05]) and the determination of the cohomological dimension $\mathrm{cohdim}_{\mathbb{Z}/2\mathbb{Z}}$ of the unordered configuration space $C^n(\mathbb{R}^m)$ (Satz 2.9). All this follows by examination of Vassiliev's CW-composition of the one-point-compactification $C^n(\mathbb{R}^m)_\infty$ of $C^n(\mathbb{R}^m)$. The sharpenings of this lower bound in theorem B in case when $n = p$ is a prime and m is odd is due to an analogous observation when we replace $\mathbb{Z}/2\mathbb{Z}$- by $\mathbb{Z}/p\mathbb{Z}$-coefficients. Here again, we use results due to Fred Cohen ([Coh73] and [CLM76]), which were proved anew by Erich Ossa [Oss96]. For $n = p = 3$ this sharpening can be generalized to even m using $\mathrm{cat}(C^3(\mathbb{R}^{m+1})) \leq \mathrm{cat}(C^3(\mathbb{R}^m)) + 2$ (Lemma 3.11).

The upper bound in (2) is also derived from Vassiliev's CW-model, when we apply the following proposition (see Satz 1.14), which we could not find in literature:

Proposition C
Let X be an n-dimensional CW-Complex such that $X - X^{(k-1)}$ is path-connected, where $X^{(r)}$ denotes the r-skeleton of X. Then we have

$$\mathrm{cat}(X - X^{(k-1)}) \leq n - k. \tag{5}$$

As a further improvement of theorem B, we prove that

$$\mathrm{cat}(C^n(\mathbb{R}^m)) \geq \mathrm{cat}(C^{n-1}(\mathbb{R}^m)).$$

This can be regarded as a corollary of the next proposition.

Proposition D
If $n = n_1 + \cdots + n_l$, then we have the inequality

$$\mathrm{cat}\left(C^n(\mathbb{R}^m)\right) \geq \mathrm{cat}\left(C^{n_1}(\mathbb{R}^m) \times \cdots \times C^{n_l}(\mathbb{R}^m)\right). \tag{6}$$

References

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