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Bero Roos

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BUNDESSTR. 55, D – 20146 HAMBURG

KERSTAN'S METHOD IN THE MULTIVARIATE POISSON APPROXIMATION: AN EXPANSION IN THE EXPONENT

Bero Roos*

Abstract. The generalized multinomial distribution is approximated by finite signed measures, resulting from a Poisson type expansion in the exponent. In the univariate case, this expansion was first used by Kornya and Presman. We apply Kerstan's method and present a bound for the total variation distance with explicit constants.

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1. Introduction and results. Let S_n be the sum of independent Bernoulli random vectors $X_1, \ldots, X_n$ in $\mathbf{R}^k$ ($k, n \in \mathbf{N} = \{1, 2, \ldots\}$) with probabilities

$$\mathbf{P}\{X_j = e_r\} = p_{j,r} \in [0,1], \qquad \mathbf{P}\{X_j = 0\} = 1 - p_j \in [0,1]$$

for $j \in \{1, \ldots, n\}$ and $r \in \{1, \ldots, k\}$, where $p_j = \sum_{r=1}^k p_{j,r}$ and e_r denotes the vector in $\mathbf{R}^k$ with entry 1 at position r and 0 otherwise. We assume that $\lambda_r = \sum_{j=1}^n p_{j,r} > 0$ for all r.

In this paper, we are concerned with the approximation of the distribution P^{S_n} of S_n by the finite signed measures $\mathcal{P}_s$, ($s \in \mathbf{N}$) concentrated on $\mathbf{Z}_+^k$ with the generating function

$$\Psi_{\mathcal{P}_s}(z) = \sum_{l \in \mathbf{Z}_+^k} \mathcal{P}_s(\{l\}) z^l = \exp\Big(\sum_{m=1}^s G_m(z)\Big), \qquad (z = (z_1, \ldots, z_k) \in \mathbf{C}^k),$$

where $\mathbf{Z}_+ = \{0, 1, 2, \ldots\}$, $z^l = z_1^{l_1} \ldots z_k^{l_k}$ for $l = (l_1, \ldots, l_k) \in \mathbf{Z}_+^k$, and, for $m \in \{1, \ldots, s\}$ and $z \in \mathbf{C}^k$,

$$G_m(z) = \frac{(-1)^{m+1}}{m} \sum_{j=1}^n H_j(z)^m, \quad H_j(z) = \sum_{r=1}^k p_{j,r}(z_r - 1), \quad (j \in \{1, \ldots, n\}).$$

An approximation by $\mathcal{P}_s$ may be useful, since, for the probability generating function $\Psi_{S_n}(z)$ of P^{S_n}, the following relations hold:

$$\Psi_{S_n}(z) = \sum_{l \in \mathbf{Z}_+^k} \mathbf{P}\{S_n = l\}\, z^l = \prod_{j=1}^n (1 + H_j(z)) = \exp\Big(\sum_{m=1}^\infty G_m(z)\Big),$$

if $z \in \mathbf{C}^k$ with $\max_{1 \le r \le k} |z_r - 1| < 1$. Note that $\mathcal{P}_s(\mathbf{Z}_+^k) = \Psi_{\mathcal{P}_s}(1) = 1$.

In the univariate case $k = 1$, the first results concerning the above expansion are due to Kornya [9] and Presman [11]. Therefore we call this expansion

*Schwerpunkt Math. Statistik und Stoch. Prozesse, Fachbereich Mathematik, Universität Hamburg, Bundesstraße 55, D-20146 Hamburg, Germany.
E-mail: roos@math.uni-hamburg.de

a Kornya–Presman expansion. It should be mentioned that Presman considered the binomial case with $s = 2$. Further results came from Kruopis [10] and Barbour and Xia [2] (for $k = 1$ and $s = 2$), and Hipp [7] and Čekanavičius [3] (for $k = 1$ and arbitrary s). The multivariate Poisson case with $s = 1$ and arbitrary k was treated, for example, by Barbour [1], Deheuvels and Pfeifer [5], and Roos [13, 14]. For this case, see also the references in [13]. The method of this paper is originally due to Kerstan [8], who considered the case $k = s = 1$. Refinements of this method came from Daley and Vere-Jones [4, pp. 297–299], Witte [15], and Roos [12, Kapitel 8], [14].

In what follows, further notation is needed. Let

$$s \in \mathbf{N}, \qquad p_0 = \max_{1\le j\le n} p_j, \qquad \tilde{p}_0 = \sum_{r=1}^{k} \max_{1\le j\le n} p_{j,r},$$

$$\beta_s(x) = \sum_{j=1}^{n} \min\left\{x \sum_{r=1}^{k} \frac{p_{j,r}^2}{\lambda_r},\, p_j^2\right\}^{(s+1)/2}, \qquad (x \in [0,\infty)).$$

Observe that

$$p_0 \le \tilde{p}_0, \qquad \beta_s(x) \le \beta_1(x)^{(s+1)/2} \le (x\,\tilde{p}_0)^{(s+1)/2}. \tag{1}$$

For $y \in \mathbf{C}$, let

$$U_s(y) = \exp\left(\sum_{m=1}^{s} \frac{y^m}{m}\right), \qquad V_s(y) = [1-(1-y)U_s(y)]\frac{s+1}{y^{s+1}}.$$

Note that $(1-y)U_s(y)$ is the prime function used by Weierstrass (see Hille [6, p. 227]). For $x \in \mathbf{R}$, let $\lceil x\rceil$ denote the smallest integer $\ge x$. Let $A, B \in [0,1]$ with $A + B = 1$. However, to get reasonable inequalities, it must often be assumed that $A, B \in (0,1)$. In this paper, we frequently deal with power series of the type $W(z) = \sum_{l\in\mathbf{Z}_+^k} w_l z^l$, $(w_l \in \mathbf{R})$, which are absolutely convergent for all $z \in \mathbf{C}^k$. In particular, the order of summation may be chosen arbitrarily. We write $\|W(z)\| = \sum_{l\in\mathbf{Z}_+^k} |w_l|$ and use the easy fact that $\|W_1(z)W_2(z)\| \le \|W_1(z)\|\,\|W_2(z)\|$ for two power series $W_1(z)$ and $W_2(z)$.

In the following theorem, we give an upper bound for the total variation distance $d_s = 2^{-1}\|\Psi_{S_n}(z) - \Psi_{\mathcal{P}_s}(z)\|$ between P^{S_n} and $\mathcal{P}_s$.

Theorem 1 *Let*

$$c_1(s) = \begin{cases} (s+1)\,2^{-5/2}, & \textit{for odd } s,\\ (s+1)\,2^{1/[2(s+1)]-5/2}, & \textit{for even } s,\end{cases}$$

$$c_2(s,p_0) = \frac{e\,2^s\,\lceil s/2-1\rceil!}{\sqrt{2\pi}\,(s+1)}\,V_s(2p_0),$$

$$c_3(s,p_0) = \frac{e\,2^{s+1}}{s+1}\,V_s(2p_0), \qquad c_4(s,p_0) = 4e\sum_{m=2}^{s}\frac{(2p_0)^{m-2}}{m},$$

If $c_3(s,p_0)\,p_0^{s-1}\,\beta_1(2^{-3/2}B^{-1}) < 1$ *and* $c_4(s,p_0)\,\beta_1(2^{-3/2}A^{-1}) < 1$, *then*

$$d_s \le \frac{c_2(s,p_0)\,\beta_s(c_1(s)B^{-1})}{[1-c_3(s,p_0)\,p_0^{s-1}\,\beta_1(2^{-3/2}B^{-1})]^{\lceil s/2\rceil}[1-c_4(s,p_0)\,\beta_1(2^{-3/2}A^{-1})]}. \tag{2}$$

Remarks. (a) Considering the properties of the Weierstrass prime function (see Hille [6, p. 227]), we obtain $V_s(y) = \sum_{i=0}^{\infty} v_{s,i}\, y^i$, $(y \in \mathbf{C})$, where $v_{s,0} = 1$ and $v_{s,i} \geq 0$ for all $i \geq 1$. Therefore $V_s(x)$ is increasing in $x \in [0, \infty)$. Further, for these x, we have $V_s(x) \leq U_s(x)$, since this is equivalent to

$$1 \leq U_s(x)\Big(1 - x + \frac{x^{s+1}}{s+1}\Big),$$

which follows from the observation that the term on the right-hand side is increasing in $x \in [0, \infty)$. Note also that $U_s(x) \leq (1-x)^{-1}$ for $x \in [0, 1)$. The inequalities given here can be used to obtain upper bounds for the constants $c_2(s, p_0)$ and $c_3(s, p_0)$.

(b) In the case $k = 1$ and $s = 2$, inequality (2) substantially coincides, up to constants, with the inequalities given by Presman [11, Assertion 1(a)] and Kruopis [10, Theorem 3], respectively. If k is arbitrary and $s = 1$, we can choose $A = 0$ and $B = 1$, since $c_4(1, p_0) = 0$; in this case, (2) is comparable with inequality (5) in Roos [14].

(c) There exist positive constants c_5 and $c_6(s)$, such that, if $\tilde{p}_0 \leq c_5$, the inequality $d_s \leq c_6(s)\, \beta_s(1)$ is valid for all s. Note that c_5 does not depend on s. Indeed for $s = 1$, such an inequality with $c_6(1) = 8.8$ was proved in Roos [14, formula (6)] without any restrictions on $\tilde{p}_0$. Further, letting $A = 0.745$ and using (1), (2), and Remark (a), it is easily shown that one can choose $c_5 = 1/4$. Here it suffices to verify that an $\epsilon \in (0, 1)$ exists such that, for all $s \in \{2, 3, \dots\}$ and $\tilde{p}_0 \leq 1/4$,

$$c_3(s, p_0)\, p_0^{s-1}\, \beta_1(2^{-3/2} B^{-1}) < 1 - \epsilon \quad \text{and} \quad c_4(s, p_0)\, \beta_1(2^{-3/2} A^{-1}) < 1 - \epsilon.$$

The special value for A was taken to obtain a large c_5. It should be mentioned that, in the general case $s \in \mathbf{N}$, it is not clear, whether the above condition $\tilde{p}_0 \leq c_5$ can be dropped.

For a successful approximation, we have to compute the values of $\mathcal{P}_s$. In the following proposition, we give a recursive formula for the counting density of $\mathcal{P}_s$. Observe that $\mathcal{P}_s(\{0\}) = \Psi_{\mathcal{P}_s}(0)$. For $l = (l_1, \dots, l_k) \in \mathbf{Z}_+^k$, we use the standard multi-index notation $|l| = l_1 + \dots + l_k$ and $l! = l_1! \dots l_k!$. Further, if additionally $t \in \mathbf{Z}_+^k$, we write $t \leq l$ in the case that $t_r \leq l_r$ for all r.

Proposition 1 *Let $l \in \mathbf{Z}_+^k$ with $|l| \geq 1$, $M_{l,s} = \{t \in \mathbf{Z}_+^k \mid 1 \leq |t| \leq s;\ t \leq l\}$, and*

$$b_t = \frac{(-1)^{|t|+1}\, |t|}{t!} \sum_{j=1}^{n} \Big(\sum_{m=0}^{s-|t|} \frac{(m + |t| - 1)!}{m!}\, p_j^m \Big) \prod_{r=1}^{k} p_{j,r}^{t_r}, \qquad (t \in M_{l,s}).$$

Then

$$\mathcal{P}_s(\{l\}) = \frac{1}{|l|} \sum_{t \in M_{l,s}} \mathcal{P}_s(\{l - t\})\, b_t.$$

2. Proofs. For the proof of Theorem 1, we need the following lemma.

Lemma 1 *Let $x \in (0,\infty)$ and $j \in \{1,\ldots,n\}$. Then*

$$\|V_s(-H_j(z))\| = V_s(2p_j), \tag{3}$$

$$\|H_j(z)^{s+1}\exp(xG_1(z))\| \leq \min\Big\{\frac{4\,c_1(s)}{x}\sum_{r=1}^{k}\frac{p_{j,r}^2}{\lambda_r},\ 4p_j^2\Big\}^{(s+1)/2}. \tag{4}$$

If $c_4(s,p_0)\,\beta_1(2^{-3/2}A^{-1})<1$, then

$$T_1(A) := \Big\|\exp\Big(A\,G_1(z)+\sum_{m=2}^{s}G_m(z)\Big)\Big\| \leq \frac{1}{1-c_4(s,p_0)\,\beta_1(2^{-3/2}A^{-1})}. \tag{5}$$

Proof. Let $V_s(y)=\sum_{i=0}^{\infty}v_{s,i}\,y^i$, $(y\in\mathbf{C})$ as in Remark (a) after Theorem 1. Using the polynomial theorem, we obtain

$$\begin{aligned}
V_s(-H_j(z)) &= \sum_{i=0}^{\infty}v_{s,i}\Big(p_j-\sum_{r=1}^{k}p_{j,r}z_r\Big)^i\\
&= \sum_{i=0}^{\infty}v_{s,i}\sum_{l\in\mathbf{Z}_+^k:\,|l|\leq i}\frac{i!\,p_j^{i-|l|}}{l!\,(i-|l|)!}\prod_{r=1}^{k}(-p_{j,r}z_r)^{l_r}\\
&= \sum_{l\in\mathbf{Z}_+^k}\Big[\sum_{i=|l|}^{\infty}\frac{v_{s,i}\,i!\,p_j^{i-|l|}}{l!\,(i-|l|)!}\prod_{r=1}^{k}p_{j,r}^{l_r}\Big](-z)^l,
\end{aligned}$$

leading to (3): $\|V(-H_j(z))\| = V_s(-H_j((-1,\ldots,-1))) = V_s(2p_j)$. Now we prove (4). For $s\in\{1,2\}$, this inequality was proved in Roos [14, formulas (19) and (26)]. For even s, we use

$$\begin{aligned}
\|H_j(z)^{s+1}\exp(xG_1(z))\| &\leq \Big\|H_j(z)\exp\Big(\frac{xG_1(z)}{s+1}\Big)\Big\|\\
&\quad\times\Big\|H_j(z)^2\exp\Big(\frac{2xG_1(z)}{s+1}\Big)\Big\|^{s/2}
\end{aligned}$$

and the fact that (4) also holds for $s=0$ with $c_1(0)=1/4$ (see Roos [14, formula (18)]). For odd s, the proof of (4) is similar. For the proof of (5), we may assume that $s\geq 2$. Using (4) and Stirling's formula, we obtain

$$\begin{aligned}
T_1(A) &= \Big\|\exp\Big(\sum_{m=2}^{s}\sum_{j=1}^{n}\frac{(-1)^{m+1}}{m}H_j(z)^m\Big)\exp(A\,G_1(z))\Big\|\\
&= \Big\|\exp(A\,G_1(z))+\sum_{i=1}^{\infty}\frac{1}{i!}\Big(\sum_{j=1}^{n}\sum_{m=2}^{s}\frac{(-1)^{m+1}}{m}H_j(z)^m\exp\Big(\frac{A}{i}\,G_1(z)\Big)\Big)^i\Big\|\\
&\leq 1+\sum_{i=1}^{\infty}\frac{1}{i!}\Big(\sum_{j=1}^{n}\sum_{m=2}^{s}\frac{\|H_j(z)\|^{m-2}}{m}\Big\|H_j(z)^2\exp\Big(\frac{A}{i}G_1(z)\Big)\Big\|\Big)^i\\
&\leq 1+\sum_{i=1}^{\infty}\frac{i^i}{i!}\Big[\frac{c_4(s,p_0)}{e}\beta_1(2^{-3/2}A^{-1})\Big]^i \leq \frac{1}{1-c_4(s,p_0)\,\beta_1(2^{-3/2}A^{-1})},
\end{aligned}$$

if $c_4(s,p_0)\,\beta_1(2^{-3/2}A^{-1})<1$. The lemma is proved. ∎

Proof of Theorem 1. Consider the following expansion of the difference of the generating functions of P^{S_n} and $\mathcal{P}_s$:

$$\begin{aligned}\Psi_{S_n}(z)-\Psi_{\mathcal{P}_s}(z) &= \Big[\prod_{j=1}^{n}\Big(1-\frac{(-H_j(z))^{s+1}}{s+1}V_s(-H_j(z))\Big)-1\Big]\Psi_{\mathcal{P}_s}(z)\\ &= \sum_{j=1}^{n}\ \sum_{1\le i(1)<\ldots<i(j)\le n}\ \prod_{\nu=1}^{j}\Big[\frac{(-1)^s V_s(-H_{i(\nu)}(z))}{s+1}H_{i(\nu)}(z)^{s+1}\\ &\qquad\times\exp\Big(\frac{B}{j}G_1(z)\Big)\Big]\exp\Big(A\,G_1(z)+\sum_{m=2}^{s}G_m(z)\Big).\end{aligned}$$

By using the polynomial theorem, we obtain $2d_s\le T_1(A)\,T_2(B)$, where $T_1(A)$ is defined as in Lemma 1 and

$$\begin{aligned}T_2(B) &:= \sum_{j=1}^{n}\frac{1}{j!}\Big[\sum_{i=1}^{n}\frac{\|V_s(-H_i(z))\|}{s+1}\Big\|H_i(z)^{s+1}\exp\Big(\frac{B}{j}G_1(z)\Big)\Big\|\Big]^j\\ &\le \sum_{j=1}^{n}\Big(\frac{1}{j!}\Big[\frac{V_s(2p_0)}{s+1}(2p_0)^{s-1}\sum_{i=1}^{n}\Big\|H_i(z)^2\exp\Big(\frac{B}{j}G_1(z)\Big)\Big\|\Big]^{j-1}\\ &\qquad\times\frac{V_s(2p_0)}{s+1}\sum_{i=1}^{n}\Big\|H_i(z)^{s+1}\exp\Big(\frac{B}{j}G_1(z)\Big)\Big\|\Big)\\ &\le \frac{2\,c_2(s,p_0)\,\beta_s(c_1(s)B^{-1})}{\lceil s/2-1\rceil!}\sum_{j=1}^{n}j^{s/2-1}\Big[c_3(s,p_0)\,p_0^{s-1}\beta_1(2^{-3/2}B^{-1})\Big]^{j-1}.\end{aligned}$$

For the latter inequality, we used (4) and Stirling's formula. In view of (5) and the relations

$$\sum_{j=1}^{\infty}j^{\nu}x^{j-1}\le\frac{d^{\nu}}{dx^{\nu}}\sum_{j=1}^{\infty}x^{j+\nu-1}=\frac{d^{\nu}}{dx^{\nu}}\sum_{j=0}^{\infty}x^{j}=\frac{\nu!}{(1-x)^{\nu+1}}$$

for $x\in[0,1)$ and $\nu\in\mathbf{Z}_+$, we see that (2) is valid. The proof is completed.■

Proof of Proposition 1. Using the binomial and the polynomial theorem we obtain

$$\begin{aligned}&[\Psi_{\mathcal{P}_s}(z)]^{-1}\sum_{l\in\mathbf{Z}_+^k}|l|\,\mathcal{P}_s(\{l\})z^l=[\Psi_{\mathcal{P}_s}(z)]^{-1}\sum_{r=1}^{k}z_r\frac{\partial}{\partial z_r}\Psi_{\mathcal{P}_s}(z)\\ &= \sum_{r=1}^{k}z_r\sum_{m=1}^{s}\frac{(-1)^{m+1}}{m}\sum_{j=1}^{n}\frac{\partial}{\partial z_r}[H_j(z)^m]\\ &= \sum_{j=1}^{n}\Big(\sum_{r=1}^{k}p_{j,r}z_r\Big)\sum_{m=0}^{s-1}\Big(p_j-\sum_{r=1}^{k}p_{j,r}z_r\Big)^m\\ &= -\sum_{j=1}^{n}\sum_{m=0}^{s-1}\sum_{i=0}^{m}\binom{m}{i}p_j^{m-i}\Big(-\sum_{r=1}^{k}p_{j,r}z_r\Big)^{i+1}\\ &= -\sum_{j=1}^{n}\sum_{i=0}^{s-1}\sum_{l\in\mathbf{Z}_+^k:\,|l|=i+1}\ \sum_{m=i}^{s-1}\binom{m}{i}\frac{|l|!}{l!}p_j^{m-i}\Big(\prod_{r=1}^{k}(-p_{j,r})^{l_r}\Big)z^l\end{aligned}$$

$$= \sum_{l\in\mathbf{Z}_+^k:\,1\le|l|\le s} \Big[\frac{(-1)^{|l|+1}\,|l|!}{l!} \sum_{j=1}^{n} \Big(\sum_{m=|l|-1}^{s-1} \binom{m}{|l|-1} p_j^{m-|l|+1}\Big) \prod_{r=1}^{k} p_{j,r}^{l_r}\Big] z^l$$

$$= \sum_{l\in\mathbf{Z}_+^k:\,1\le|l|\le s} b_l\, z^l.$$

Therefore

$$\sum_{l\in\mathbf{Z}_+^k} |l|\,\mathcal{P}_s(\{l\}) z^l = \Psi_{\mathcal{P}_s}(z) \sum_{l\in\mathbf{Z}_+^k:\,1\le|l|\le s} b_l\, z^l = \sum_{l\in\mathbf{Z}_+^k:\,|l|\ge 1}\ \sum_{t\in M_{l,s}} \mathcal{P}_s(\{l-t\})\, b_t\, z^l.$$

Comparing the power series, the assertion is shown. ∎

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