UNIVERSITÄT HAMBURG

Sharp constants in the Poisson approximation

Bero Roos

Preprint–No. 2000-07 August 2000

FACHBEREICH MATHEMATIK

SCHWERPUNKT MATHEMATISCHE STATISTIK
UND STOCHASTISCHE PROZESSE

BUNDESSTR. 55, D – 20146 HAMBURG

SHARP CONSTANTS IN THE POISSON APPROXIMATION

Bero Roos

Schwerpunkt Math. Statistik und Stoch. Prozesse, Fachbereich Mathematik,
Universität Hamburg, Bundesstraße 55, D–20146 Hamburg, Germany.
E–mail: roos@math.uni-hamburg.de

Abstract. We present some new sharp bounds for several distances between the Poisson binomial distribution and the Poisson law with the same mean. It is shown that the constants involved cannot be reduced.

AMS 1991 subject classifications. Primary 60F05; secondary 60G50.

Key words and phrases. Poisson approximation, Poisson binomial distribution, probability metrics, sharp constants.

1 Introduction and main results

1.1 Motivation. This paper is devoted to the Poisson approximation of the Poisson binomial distribution with respect to several probability metrics, for which we give new sharp bounds. It will be shown that the constants involved cannot be reduced. Other publications concerning inequalities with small constants came, for example, from Le Cam (1960), Shorgin (1977), Serfling (1978), Barbour and Hall (1984), Gerber (1984), Deheuvels and Pfeifer (1986a, b, 1988), Deheuvels et al. (1988), Daley and Vere–Jones (1988, pp. 297–299), Rachev and Rüschendorf (1990), Witte (1990), Weba (1994), Xia (1997), and Roos (1995, 1999a, b). To some extent, the method of the present paper is a continuation of arguments in Roos (1999b), where we used Charlier's (1905) expansion [see also Schmidt (1933), Shorgin (1977), Deheuvels and Pfeifer (1988)].

1.2 Probability metrics. We proceed with the definition of the probability metrics, which are considered in this paper. The following notation is needed. Let Π be the set of all Poisson binomial distributions Q with mean $\lambda(Q)>0$. Hence $Q\in\Pi$ means that $n\in\mathbf{N}=\{1,2,\ldots\}$ and $p_1,\ldots,p_n\in[0,1]$ exist such that $\lambda(Q)=\sum_{j=1}^n p_j>0$ and $Q=*_{j=1}^n\mathcal{B}(1,p_j)$ is the convolution of the Bernoulli distributions $\mathcal{B}(1,p_1),\ldots,\mathcal{B}(1,p_n)$ with success probabilities $p_1,\ldots,p_n$; note that, letting $\lambda_2(Q)=\sum_{j=1}^n p_j^2$, the variance of Q is given by $\sigma^2(Q)=\lambda(Q)-\lambda_2(Q)$. Since $\sigma^2(Q)$ is a function of Q, this also applies to $\lambda_2(Q)$. Another reason for this is the easy observation that every $Q\in\Pi$ determines uniquely $n\in\mathbf{N}$ and $0<p_1\leq\ldots\leq p_n\leq 1$ such that $Q=*_{j=1}^n\mathcal{B}(1,p_j)$. Let $\mathbf{Z}_+=\{0,1,2,\ldots\}$ and $\mathbf{R}^{\mathbf{Z}_+}=\{f\,|\,f:\mathbf{Z}_+\longrightarrow\mathbf{R}\}$. For all $f\in\mathbf{R}^{\mathbf{Z}_+}$, set $f(m)=0$ if $m<0$ and let $\|f\|_q$, $(q\in\{1,\infty\})$ be the q–norm of f, i.e. $\|f\|_1=\sum_{m=0}^\infty|f(m)|$ and $\|f\|_\infty=\sup_{m\in\mathbf{Z}_+}|f(m)|$. We define the difference operator $\Delta:\mathbf{R}^{\mathbf{Z}_+}\longrightarrow\mathbf{R}^{\mathbf{Z}_+}$ by $(\Delta f)(m)=f(m-1)-f(m)$ for $f\in\mathbf{R}^{\mathbf{Z}_+}$ and $m\in\mathbf{Z}_+$. For the inverse $\Delta^{-1}:\mathbf{R}^{\mathbf{Z}_+}\longrightarrow\mathbf{R}^{\mathbf{Z}_+}$, we have $(\Delta^{-1}f)(m)=-\sum_{k=0}^m f(k)$. For $k\in\mathbf{N}$, let $\Delta^k=\Delta\circ\ldots\circ\Delta$ (resp. $\Delta^{-k}=(\Delta^{-1})\circ\ldots\circ(\Delta^{-1})$) be the k-th iterated composition of Δ (resp. Δ^{-1}) and denote by Δ^0 the identity mapping of $\mathbf{R}^{\mathbf{Z}_+}$ onto itself. In this paper, we consider the distance

$$d_q^{(i)}(Q)=\left\|\Delta^i(f_Q-\pi(\cdot,\lambda(Q)))\right\|_q,\qquad (q\in\{1,\infty\},\ i\in\{-2,-1,0\})$$

between $Q\in\Pi$ and the Poisson law $\mathcal{P}(\lambda(Q))$ with mean $\lambda(Q)$, where $f_Q\in\mathbf{R}^{\mathbf{Z}_+}$ and

$\pi(\cdot,\lambda(Q)) \in \mathbf{R}^{\mathbf{Z}_+}$ denote the counting densities of Q and $\mathcal{P}(\lambda(Q))$, i.e. $f_Q(m) = Q(\{m\})$ and $\pi(m,\lambda(Q)) = e^{-\lambda(Q)}(\lambda(Q))^m/m!$ for $m \in \mathbf{Z}_+$. In the case that no confusion occurs, we suppress the indicated argument Q, that is, we write λ for $\lambda(Q)$, λ_2 for $\lambda_2(Q)$, $d_q^{(i)}$ for $d_q^{(i)}(Q)$, and so on. Now we are led to the total variation distance $d_1^{(0)}/2$, the Kolmogorov metric $d_\infty^{(-1)}$, the Fortet–Mourier metric $d_1^{(-1)}$, the point metric $d_\infty^{(0)}$, and the stop–loss metrics $d_\infty^{(-2)}$ and $d_1^{(-2)}$. Several bounds for $d_q^{(i)}$ can be found in the publications cited above. For a treatment of the general theory of probability metrics, see, for example, Rachev (1991). In the following proposition, we present some basic relations for the distances $d_q^{(i)}$.

Proposition 1 *For each $Q \in \Pi$, we have*

$$\max\{d_\infty^{(0)},\, d_\infty^{(-1)}\} \le \frac{1}{2} d_1^{(0)} \le 2 d_\infty^{(-1)} \le d_1^{(-1)} = 2 d_\infty^{(-2)} \le 2 d_1^{(-2)} = \lambda_2. \tag{1}$$

Note that the inequality $\max\{d_\infty^{(0)},\, d_\infty^{(-1)}\} \le d_1^{(0)}/2$ is well-known and easy to prove. Further, $2d_\infty^{(-2)} \le 2d_1^{(-2)}$ is clear. In Section 2, one can find the proofs of the remaining equalities and inequalities. Observe that the non-trivial inequality $d_1^{(0)}/2 \le 2d_\infty^{(-1)}$ was shown in Daley and Vere–Jones (1988, p. 298) by the help of Newton's inequality (see Hardy, Littlewood, and Pólya (1952, pp. 104–105)). The idea behind the proof of this inequality can be used to show the surprising and non-trivial equalities $d_1^{(-1)} = 2d_\infty^{(-2)}$ and $2d_1^{(-2)} = \lambda_2$. It should be mentioned that the proof of Proposition 1 does not use a subadditivity property of $d_q^{(i)}$ (see also the remark after Proposition 2). Because of Proposition 1, it suffices to consider the distances $d_q^{(i)}$ only for $q \in \{1,\infty\}$ and $i \in \{-1, 0\}$.

1.3 Further notation and facts. For $Q = *_{j=1}^n \mathcal{B}(1,p_j) \in \Pi$, $q \in \{1,\infty\}$, and $i \in \{-1,0\}$, let

$$\lambda_k(Q) = \sum_{j=1}^n p_j^k, \quad (k \in \mathbf{N}), \quad \theta(Q) = \frac{\lambda_2(Q)}{\lambda(Q)}, \quad \zeta_q^{(i)}(Q) = \frac{\theta(Q)}{\lambda(Q)^{(i+1)/2-1/(2q)}}. \tag{2}$$

Here and throughout the paper, we let $1/\infty = 0$. It follows from the above that, if q, i, and k are fixed, $\lambda_k(Q)$, $\theta(Q)$, and $\zeta_q^{(i)}(Q)$ are functions of Q. Note that $\lambda_1 = \lambda$ and $0 < \theta = 1 - \sigma^2/\lambda \le 1$. For $r \in [0,1]$, let $\Pi(r) = \{Q \in \Pi \,|\, \theta(Q) < r\}$. Further, for $k \in \mathbf{Z}_+$, set $\varphi_k(x) = (2\pi)^{-1/2}(d^k/dx^k)e^{-x^2/2}$, $(x \in \mathbf{R})$ and let $|\varphi_k|_q$ be the q–norm of φ_k, i.e. $|\varphi_k|_1 = \int_{\mathbf{R}} |\varphi_k(x)|\,dx$ and $|\varphi_k|_\infty = \sup_{x\in\mathbf{R}} |\varphi_k(x)|$. From (32) in Roos (1999b), we see that, for any specified $r \in (0,1)$, $q \in \{1,\infty\}$, and $i \in \{-1,0\}$, there exists an absolute constant $C = C_{r,q,i} > 0$ such that, for all $Q \in \Pi(r)$,

$$\left| \frac{d_q^{(i)}}{u_q^{(i)}\,\zeta_q^{(i)}} - 1 \right| \le C \min\left\{1, \frac{1}{\sqrt{\lambda}} + \theta\right\}, \qquad \text{where} \qquad u_q^{(i)} = \frac{|\varphi_{i+2}|_q}{2}. \tag{3}$$

Moreover, for either $q = \infty$, $i = -1$ or $q = 1$, $i = 0$, there exists an absolute constant $C > 0$ such that (3) holds for all $Q \in \Pi$. One should be aware that (3) holds even if we allow $q \in [1,\infty]$ and $i \in \{-2,-1,0,\ldots\}$, whereby we have to extend the definition of $d_q^{(i)}$, $\zeta_q^{(i)}$, and $u_q^{(i)}$ in an appropriate way. As follows from (3), for each choice of $q \in \{1,\infty\}$ and $i \in \{-1,0\}$, we have $d_q^{(i)} \sim u_q^{(i)}\,\zeta_q^{(i)}$ as $\theta \to 0$ and $\lambda \to \infty$. Here and elsewhere, we set $\nu_n \sim \eta_n$ when $\nu_n/\eta_n \to 1$. Note that the latter asymptotic relation was already obtained by Deheuvels and Pfeifer (1986a, b, 1988) and Roos (1995), but, in these papers, we can only find the following calculated values of $u_q^{(i)}$:

$$u_\infty^{(0)} = \frac{1}{2\sqrt{2\pi}}, \qquad u_\infty^{(-1)} = \frac{1}{2\sqrt{2\pi e}}, \qquad u_1^{(0)} = \frac{2}{\sqrt{2\pi e}}, \qquad u_1^{(-1)} = \frac{1}{\sqrt{2\pi}}. \tag{4}$$

1.4 Main results. In view of (3), set, for $q \in \{1, \infty\}$ and $i \in \{-1, 0\}$,

$$U_q^{(i)}(r) = \sup_{Q \in \Pi(r)} \frac{d_q^{(i)}(Q)}{\zeta_q^{(i)}(Q)}, \qquad (r \in (0,1)), \qquad U_q^{(i)}(0) = \lim_{r \downarrow 0} U_q^{(i)}(r).$$

Obviously, $U_q^{(i)}(r)$ is finite and nondecreasing as a function of $r \in (0, \infty)$, so that $U_q^{(i)}(0)$ is well-defined. It is noteworthy that (3) implies that

$$u_q^{(i)} = \lim_{r \downarrow 0,\, M \uparrow \infty} \left(\sup_{Q \in \Pi(r),\, \lambda(Q) \geq M} \frac{d_q^{(i)}(Q)}{\zeta_q^{(i)}(Q)} \right) \leq U_q^{(i)}(0). \tag{5}$$

The main purpose of the present paper is to evaluate $U_q^{(i)}(0)$. This will be achieved in Theorem 1 below. Introduce the constants

$$\tilde{U}_\infty^{(0)} = \frac{1}{2}\left(\frac{3}{2e}\right)^{3/2}, \qquad \tilde{U}_\infty^{(-1)} = \frac{1}{2e}, \qquad \tilde{U}_1^{(0)} = \frac{3}{2e}, \qquad \tilde{U}_1^{(-1)} = \frac{1}{\sqrt{2e}}. \tag{6}$$

Theorem 1 *For each choice $q \in \{1, \infty\}$ and $i \in \{-1, 0\}$, we have $U_q^{(i)}(0) = \tilde{U}_q^{(i)}$.*

A comparison between $\tilde{U}_q^{(i)}$ and $u_q^{(i)}$ (see (4) and (6)) shows that the difference $U_q^{(i)}(0) - u_q^{(i)}$ is positive and, in general, relatively small. Indeed, we have

$$\begin{aligned} \tilde{U}_\infty^{(0)} - u_\infty^{(0)} &= 0.0054\ldots, \qquad & \tilde{U}_\infty^{(-1)} - u_\infty^{(-1)} &= 0.0629\ldots, \\ \tilde{U}_1^{(0)} - u_1^{(0)} &= 0.0678\ldots, \qquad & \tilde{U}_1^{(-1)} - u_1^{(-1)} &= 0.0299\ldots. \end{aligned}$$

Theorem 1 can be deduced from the more precise Theorem 2, in which we give an explicit sharp upper bound for $d_q^{(i)}$ containing $\zeta_q^{(i)}$, θ, and the constant $\tilde{U}_q^{(i)}$. Let

$$A_\infty^{(0)} = \frac{3}{2}, \qquad A_\infty^{(-1)} = 1, \qquad A_1^{(0)} = 1, \qquad A_1^{(-1)} = \frac{1}{2}. \tag{7}$$

Recall that θ is defined in (2). Let, as usual, $1/0 = \infty$.

Theorem 2 (a) *For $Q \in \Pi$,*

$$d_\infty^{(0)} \leq \left(\tilde{U}_\infty^{(0)} + \frac{\sqrt{\theta}(6 - 4\sqrt{\theta})}{3(1 - \sqrt{\theta})^2} \right) \frac{\theta}{\sqrt{\lambda}}, \tag{8}$$

$$d_\infty^{(-1)} \leq \left(\tilde{U}_\infty^{(-1)} + \frac{6\sqrt{\theta}}{5(1 - \sqrt{\theta})} \right) \theta, \tag{9}$$

$$d_1^{(0)} \leq \left(\tilde{U}_1^{(0)} + \frac{7\sqrt{\theta}(3 - 2\sqrt{\theta})}{3(1 - \sqrt{\theta})^2} \right) \theta, \tag{10}$$

$$d_1^{(-1)} \leq \left(\tilde{U}_1^{(-1)} + \frac{8\sqrt{\theta}(2 - \sqrt{\theta})}{5(1 - \sqrt{\theta})^2} \right) \theta \sqrt{\lambda}. \tag{11}$$

(b) *The inequality for $d_q^{(i)}$ given in* (a) *can be replaced by "$\sim$", if $\theta \to 0$ and $\lambda \to A_q^{(i)}$.*

Remarks. (a) In the case $\theta = 1$, (8)–(11) reduce to the trivial statement $d_q^{(i)} \leq \infty$ for $q \in \{1, \infty\}$ and $i \in \{-1, 0\}$.

(b) For the better understanding of Theorem 2(b), we consider an example: If $q = 1$, $i = 0$, $\theta \to 0$, and $\lambda \to A_q^{(i)} = A_1^{(0)} = 1$, we have sharpness of (10), i.e.

$$d_1^{(0)} \sim \Big(\tilde{U}_1^{(0)} + \frac{7\sqrt{\theta}(3 - 2\sqrt{\theta})}{3(1-\sqrt{\theta})^2}\Big)\theta \sim \tilde{U}_1^{(0)}\,\theta = \frac{3}{2e}\,\theta.$$

If we assume that $\theta \to 0$ and $\lambda \to \infty$ instead of the conditions above, we obtain from (3) that $d_1^{(0)} \sim u_1^{(0)}\,\theta = 2\theta/\sqrt{2\pi e} < 3\theta/(2e)$; in the latter case, we do not have sharpness in (10).

The following corollary is a consequence of the Theorems 1 and 2. Further notation is needed. For $Q = *_{j=1}^{n} \mathcal{B}(1, p_j) \in \Pi$, set $p_0(Q) = \max_{1\leq j\leq n} p_j$. Let $\overline{\Pi}(r) = \{Q \in \Pi \,|\, p_0(Q) < r\}$, $(r \in [0,\, 1])$. Note that $\theta(Q) \leq p_0(Q)$, giving $\overline{\Pi}(r) \subseteq \Pi(r)$ for all $r \in [0, 1]$. For $q \in \{1, \infty\}$ and $i \in \{-1, 0\}$, let

$$\overline{U}_q^{(i)}(r) = \sup_{Q\in\overline{\Pi}(r)} \frac{d_q^{(i)}(Q)}{\zeta_q^{(i)}(Q)}, \qquad (r \in (0,1)), \qquad \overline{U}_q^{(i)}(0) = \lim_{r\downarrow 0} \overline{U}_q^{(i)}(r).$$

Corollary 1 *For all $q \in \{1, \infty\}$ and $i \in \{-1, 0\}$, we have $\overline{U}_q^{(i)}(0) = \tilde{U}_q^{(i)}$.*

1.5 A second problem of sharp constants. The problem of sharp constants given above has an easier counterpart, which we now discuss in greater detail. Let us have a look at (33) in Roos (1999b). From this, it follows that, for any choice of $M > 0$, $q \in \{1, \infty\}$, and $i \in \{-1, 0\}$, there exists an absolute constant $D = D_{M,q,i} > 0$ such that, for all $Q \in \Pi$ with $\lambda_2 = \lambda_2(Q) \leq M$,

$$\Big|\frac{d_q^{(i)}}{w_q^{(i)}\,\lambda_2} - 1\Big| \leq D\,\min\{1,\,\lambda\}, \qquad \text{where} \qquad w_q^{(i)} = \frac{1}{2}\,\|\Delta^{i+2}\mathbf{1}\|_q \tag{12}$$

and $\mathbf{1} \in \mathbf{R}^{\mathbf{Z}_+}$ is defined by setting $\mathbf{1}(0) = 1$ and $\mathbf{1}(n) = 0$ for $n \in \mathbf{N}$. Observe that, as (3), (12) also holds if we permit $q \in [1, \infty]$ and $i \in \{-2, -1, 0, \ldots\}$. Since $\lambda_2 \leq \lambda$, (12), in turn, implies that, for $q \in \{1, \infty\}$ and $i \in \{-1, 0\}$, $d_q^{(i)} \sim w_q^{(i)}\lambda_2$ as $\lambda \to 0$. Note that, in the case $(q, i) \in \{(\infty, -1),\, (1, 0),\, (1, -1)\}$, the latter asymptotic relation can be found in Deheuvels and Pfeifer (1986b). We have

$$w_\infty^{(0)} = 1, \qquad w_\infty^{(-1)} = \frac{1}{2}, \qquad w_1^{(0)} = 2, \qquad w_1^{(-1)} = 1. \tag{13}$$

In view of (12), we see that

$$\lim_{r\downarrow 0}\Big(\sup_{Q\in\Pi,\,\lambda(Q)<r} \frac{d_q^{(i)}(Q)}{\lambda_2(Q)}\Big) = w_q^{(i)}. \tag{14}$$

Equality (14) can be compared with the equalities given in (5), Theorem 1, and Corollary 1. The following proposition is an easy consequence of (1), (12), and (13) and gives an explicit sharp upper bound for $d_q^{(i)}$ containing λ_2 and the constant $w_q^{(i)}$.

Proposition 2 *For $Q \in \Pi$, $q \in \{1, \infty\}$, and $i \in \{-1, 0\}$, we have*

$$d_q^{(i)} \leq w_q^{(i)}\,\lambda_2. \tag{15}$$

If $\lambda \to 0$, all inequalities in (15) and in (1) can be replaced by "$\sim$".

Remark. In the case $(q, i) \in \{(\infty, -1), (1, 0), (1, -1)\}$, (15) is already known (see Le Cam (1960) for $(q, i) = (1, 0)$, Serfling (1978) for $(q, i) \in \{(1, 0), (\infty, -1)\}$ and Deheuvels et al. (1988) for $(q, i) = (1, -1)$). It should be mentioned that Serfling (1978, p. 573) pointed out that (15) with $(q, i) = (\infty, -1)$ is due to D. J. Daley. Note that in Serfling (1978, p. 570), the basic idea was to use the subadditivity property of $d_1^{(0)}$ and $d_\infty^{(-1)}$. This method can be applied to each $d_q^{(i)}$. Indeed all $d_q^{(i)}$ are subadditive, which can immediately be shown. Hence it can be used that, for $Q = *_{j=1}^n \mathcal{B}(1, p_j) \in \Pi$, $q \in \{1, \infty\}$, and $i \in \{-1, 0\}$, $d_q^{(i)}(Q) \leq \sum_{j=1}^n d_q^{(i)}(\mathcal{B}(1, p_j))$. Now it suffices to show the following elementary relations (see Serfling (1978, (4.1) and (5.2)) for the first two):

$$d_1^{(0)}(\mathcal{B}(1, p_j)) = 2p_j(1 - e^{-p_j}) \leq 2p_j^2, \qquad d_\infty^{(-1)}(\mathcal{B}(1, p_j)) = e^{-p_j} - 1 + p_j \leq \frac{p_j^2}{2},$$

$$d_1^{(-1)}(\mathcal{B}(1, p_j)) = 2(e^{-p_j} - 1 + p_j) \leq p_j^2, \qquad d_\infty^{(0)}(\mathcal{B}(1, p_j)) = p_j(1 - e^{-p_j}) \leq p_j^2.$$

Observe that, in the present paper, the proof of (1) and hence the proof of (15) does not need the subadditivity property of the distances $d_q^{(i)}$.

2 Remaining proofs

Proof of Proposition 1. Let $Q \in \Pi$. We show that (a) $2d_\infty^{(-1)} \leq d_1^{(-1)}$, (b) $d_1^{(0)}/2 \leq 2d_\infty^{(-1)}$, (c) $d_1^{(-1)} = 2d_\infty^{(-2)}$, and (d) $2d_1^{(-2)} = \lambda_2$. Let $\nu = [f_Q - \pi(\cdot, \lambda)] \in \mathbf{R}^{\mathbf{Z}_+}$. Since the mean of Q and $\mathcal{P}(\lambda)$ coincide, we have $\sum_{k=0}^\infty \Delta^{-1}\nu(k) = 0$, and therefore, letting $K_+ = \{k \in \mathbf{Z}_+ \,|\, \Delta^{-1}\nu(k) \geq 0\}$ and $K_- = \mathbf{Z}_+ \setminus K_+$, we obtain $\sum_{k \in K_+} \Delta^{-1}\nu(k) = -\sum_{k \in K_-} \Delta^{-1}\nu(k)$. Now we can show (a):

$$\begin{aligned} 2d_\infty^{(-1)} &= 2 \sup_{k \in \mathbf{Z}_+} |\Delta^{-1}\nu(k)| \leq 2 \sup_{K \subseteq \mathbf{Z}_+} \Big| \sum_{k \in K} \Delta^{-1}\nu(k) \Big| = 2 \sum_{k \in K_+} \Delta^{-1}\nu(k) \\ &= \sum_{k \in K_+} \Delta^{-1}\nu(k) - \sum_{k \in K_-} \Delta^{-1}\nu(k) = \sum_{k \in \mathbf{Z}_+} |\Delta^{-1}\nu(k)| = d_1^{(-1)}. \end{aligned}$$

For (b)–(d), we proceed with an argument of Daley and Vere–Jones (1988, p. 298), who proved (b): There exist $k_0 = k_0(Q)$ and $k_1 = k_1(Q)$ in $\mathbf{Z}_+$ with $k_0 \leq k_1$ such that

$$\{k \in \mathbf{Z}_+ \,|\, \nu(k) \geq 0\} = \{k_0, \ldots, k_1\}. \tag{16}$$

Indeed, this can be shown by means of Newton's inequality (see Hardy, Littlewood, and Pólya (1952, pp. 104–105). From $\lambda > 0$ it easily follows that $\nu(0) < 0$ and hence $k_0 \geq 1$. We obtain the known result (b):

$$\frac{1}{2} d_1^{(0)} = \sum_{k=k_0}^{k_1} \nu(k) = \Delta^{-1}\nu(k_0 - 1) - \Delta^{-1}\nu(k_1) \leq 2\|\Delta^{-1}\nu\|_\infty = 2d_\infty^{(-1)}.$$

Further, in view of (16), we see that

(i) $\Delta^{-1}\nu(k) > 0$ for all $k \in \{0, \ldots, k_0 - 1\}$,

(ii) $\Delta^{-1}\nu(k+1) \leq \Delta^{-1}\nu(k) \;\Leftrightarrow\; k_0 - 1 \leq k \leq k_1 - 1$, and

(iii) $\Delta^{-1}\nu(k) < 0$ for all $k \in \{k_1, k_1 + 1, \ldots\}$.

Therefore $k_2 = k_2(Q) \in \{k_0 - 1, \ldots, k_1 - 1\}$ exists such that

$$K_+ = \{k \in \mathbf{Z}_+ \,|\, \Delta^{-1}\nu(k) \geq 0\} = \{0, \ldots, k_2\}. \tag{17}$$

Since $\sum_{k=0}^{\infty}\Delta^{-1}\nu(k)=0$, we have $-\Delta^{-2}\nu(k)=\sum_{j=0}^{k}\Delta^{-1}\nu(j)\geq 0$ for all $k\in\mathbf{Z}_+$. Hence, (17) leads to (c):

$$\begin{aligned} d_1^{(-1)} &= \sum_{k=0}^{\infty}|\Delta^{-1}\nu(k)| = \sum_{k=0}^{k_2}\Delta^{-1}\nu(k) - \sum_{k=k_2+1}^{\infty}\Delta^{-1}\nu(k) = 2\sum_{k=0}^{k_2}\Delta^{-1}\nu(k) \\ &= 2\sup_{j\in\mathbf{Z}_+}\left|\sum_{k=0}^{j}\Delta^{-1}\nu(k)\right| = 2\|\Delta^{-2}\nu\|_\infty = 2d_\infty^{(-2)}. \end{aligned}$$

Further, it is now an easy task to verify that (d) holds:

$$d_1^{(-2)} = \sum_{k=0}^{\infty}|\Delta^{-2}\nu(k)| = \sum_{k=0}^{\infty}\sum_{j=0}^{k}\Delta^{-1}\nu(j) = -\frac{1}{2}\sum_{m=0}^{\infty}m(m-1)\nu(m) = \frac{\lambda_2}{2}.$$

The proposition is proved. ∎

The proofs of Theorems 1 and 2 require sharp estimates of the norms $\|\Delta^{2+i}\pi(\cdot,t)\|_q$ for $q\in\{1,\infty\}$, $i\in\{-1,0\}$, and $t\in(0,\infty)$. The following three lemmas are necessary. Let us write $\Delta^k\pi(m,t)=(\Delta^k\pi(\cdot,t))(m)$ for $k,m\in\mathbf{Z}_+$ and $t\in(0,\infty)$.

Lemma 1 *Let $t\in(0,\infty)$ and $k,m\in\mathbf{Z}_+$. If $k+m\geq 1$, then*

$$\|\Delta^k\pi(\cdot,t)\|_\infty \leq 2^{k+1}\,\alpha_{k+2m}\left(\frac{k+2m}{4te}\right)^{(k+2m+1)/2} + \frac{2^k}{\pi}\sum_{j=1}^{m}\frac{\Gamma((k+2j-1)/2)}{(2t)^{(k+2j-1)/2}}, \tag{18}$$

where $\alpha_j=2^{-1}\sqrt{e}(1+\sqrt{\pi/(2j)})$, ($j\in\mathbf{N}$).

Proof. For $t\in(0,\infty)$ and $k\in\mathbf{Z}_+$, let $\gamma_k(t)=\int_0^{\pi/2}\exp(-2t\sin^2 x)\,\sin^k x\;dx$. First we will show that, for $t\in(0,\infty)$ and $k,m\in\mathbf{Z}_+$,

$$\gamma_k(t)\leq\gamma_{k+2m}(t)+\frac{1}{2}\sum_{j=1}^{m}\frac{\Gamma((k+2j-1)/2)}{(2t)^{(k+2j-1)/2}}\,. \tag{19}$$

For $m=0$, (19) is clear. Let us now consider the case $m=1$. Using the equality $\sin^2 x+\cos^2 x=1$, ($x\in\mathbf{R}$), we obtain, for $t\in(0,\infty)$ and $k\in\mathbf{Z}_+$,

$$\gamma_k(t)=\gamma_{k+2}(t)+\int_0^{\pi/2}\exp(-2t\sin^2 x)\,\sin^k x\,\cos^2 x\;dx.$$

Substituting $y=2t\sin^2 x$, the latter integral is equal to

$$\frac{1}{2(2t)^{(k+1)/2}}\int_0^{2t}e^{-y}\,y^{(k-1)/2}\sqrt{1-\frac{y}{2t}}\;dy\leq\frac{\Gamma((k+1)/2)}{2(2t)^{(k+1)/2}},$$

completing the proof of (19) for $m=1$. Now it is easy to show (19) by induction over m. To prove (18), we use (19) and Shorgin's (1977, proof of Lemma 6) inequalities

$$\|\Delta^k\pi(\cdot,t)\|_\infty\leq\frac{2^{k+1}}{\pi}\gamma_k(t),\qquad(t\in(0,\infty),\ k\in\mathbf{Z}_+), \tag{20}$$

$$\gamma_k(t)\leq\alpha_k\,\pi\left(\frac{k}{4te}\right)^{(k+1)/2},\qquad(t\in(0,\infty),\ k\in\mathbf{N}). \tag{21}$$

Note that (20) can be verified with complex analysis. Indeed, it is easy to see that

$$\sum_{l=0}^{\infty} \Delta^k \pi(l,t)\, z^l = \exp(t(z-1))(z-1)^k, \qquad (z \in \mathbf{C},\ t \in (0,\infty),\ k \in \mathbf{Z}_+)$$

and, by Cauchy's theorem, we are led to (20)

$$\begin{aligned} \|\Delta^k \pi(\cdot,t)\|_\infty &= \sup_{l \in \mathbf{Z}_+} |\Delta^k \pi(l,t)| = \sup_{l \in \mathbf{Z}_+} \frac{1}{2\pi} \left| \int_{-\pi}^{\pi} e^{-ilx} \exp(t(e^{ix}-1))(e^{ix}-1)^k\, dx \right| \\ &\le \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| \exp(t(e^{ix}-1)) \right| \left| e^{ix}-1 \right|^k dx = \frac{2^{k+1}}{\pi} \gamma_k(t). \end{aligned}$$

Here, i denotes the imaginary unit in $\mathbf{C}$. Now we obtain, for $t \in (0,\infty)$, $k,m \in \mathbf{Z}_+$ with $k+m \ge 1$, the following inequalities

$$\begin{aligned} \|\Delta^k \pi(\cdot,t)\|_\infty &\le \frac{2^{k+1}}{\pi} \gamma_k(t) \le \frac{2^{k+1}}{\pi} \Bigg(\gamma_{k+2m}(t) + \frac{1}{2} \sum_{j=1}^{m} \frac{\Gamma((k+2j-1)/2)}{(2t)^{(k+2j-1)/2}} \Bigg) \\ &\le \frac{2^{k+1}}{\pi} \Bigg(\alpha_{k+2m}\, \pi \left(\frac{k+2m}{4te} \right)^{(k+2m+1)/2} + \frac{1}{2} \sum_{j=1}^{m} \frac{\Gamma((k+2j-1)/2)}{(2t)^{(k+2j-1)/2}} \Bigg), \end{aligned}$$

giving the assertion. ■

For $t \in [0,\infty)$ and $m \in \mathbf{Z}_+$, let $L_m(t) = (m-t)\,\pi(m,t)$. Here and throughout the paper, we let $0^0 = 1$, leading to $L_m(0) = m(0^m/m!)e^{-0} = 0$ for all $m \in \mathbf{Z}_+$.

Lemma 2 *For $t \in [0,\infty)$ and $m \in \mathbf{N}$,*

$$-\frac{1}{e} \le L_0(t) \le 0, \qquad -0.31 \le L_1(t) \le 0.17, \qquad -0.3 \le L_2(t) \le 0.19, \tag{22}$$

$$-\exp\left(\frac{1}{2\sqrt{m}}\right) \le \sqrt{2\pi e}\, L_m(t) \exp\left(\frac{1}{12m+1}\right) \le \exp\left(-\frac{1}{4\sqrt{m+1/4}}\right). \tag{23}$$

In particular, for all $t \in [0,\infty)$ and $m \in \mathbf{Z}_+$, we have $|L_m(t)| \le 1/e$.

Proof. By easy calculus, we get, for $m \in \mathbf{Z}_+$,

$$\sup_{t \in [0,\infty)} L_m(t) = L_m(r_m^-) \ge 0, \qquad \inf_{t \in [0,\infty)} L_m(t) = L_m(r_m^+) < 0,$$

$$r_m^{\pm} = \left(\sqrt{m + \frac{1}{4}} \pm \frac{1}{2} \right)^2 = m + \frac{1}{2} \pm \sqrt{m + \frac{1}{4}},$$

leading to

$$L_m(r_m^+) \le L_m(t) \le L_m(r_m^-), \qquad (t \in [0,\infty),\ m \in \mathbf{Z}_+). \tag{24}$$

Considering $m \in \{0,1,2\}$, we obtain (22). For the proof of (23), let $m \in \mathbf{N}$ and $s_m^{\pm} = \mp(m - r_m^{\pm}) = \sqrt{r_m^{\pm}}$. Then $s_m^- \in (0, \sqrt{m})$, $s_m^+ \in (\sqrt{m}, \infty)$, $m = s_m^{\pm}(s_m^{\pm} \mp 1)$ and, using Stirling's formula [see Feller (1968, page 54)]

$$m! = \sqrt{2\pi}\, m^{m+1/2}\, e^{\vartheta(m)-m}, \qquad \frac{1}{12m+1} \le \vartheta(m) \le \frac{1}{12m}, \tag{25}$$

we get

$$\begin{aligned} L_m(r_m^\pm) &= (m - r_m^\pm)\, e^{-r_m^\pm}\, \frac{(r_m^\pm)^m}{m!} = \mp s_m^\pm\, e^{-r_m^\pm}\, \frac{(s_m^\pm)^{2m}}{\sqrt{2\pi}\, m^{m+1/2}\, e^{\vartheta(m)-m}} \\ &= \mp \frac{1}{\sqrt{2\pi}} \left(\frac{(s_m^\pm)^2}{m} \right)^{m+1/2} \exp(\mp s_m^\pm - \vartheta(m)). \end{aligned} \tag{26}$$

Using the inequality [see Mitrinović (1970, 3.6.18)]

$$\log(\nu) \le \max\left\{ \frac{2(\nu-1)}{\nu+1},\, \frac{\nu-1}{\sqrt{\nu}} \right\}, \qquad (\nu \in (0,\infty)),$$

it is easy to show that

$$\begin{aligned} \left(m+\frac12\right)\log\left(\frac{(s_m^-)^2}{m}\right) &\le \left(m+\frac12\right)\frac{2((s_m^-)^2 - m)}{(s_m^-)^2+m} = \frac{(2m+1)(1/2 - \sqrt{m+1/4})}{2m+1/2-\sqrt{m+1/4}} \\ = -\frac{2m+1}{2\sqrt{m+1/4}} &= -\sqrt{m+\frac14} - \frac{1}{4\sqrt{m+1/4}} = -s_m^- - \frac12 - \frac{1}{4\sqrt{m+1/4}} \end{aligned}$$

and

$$\begin{aligned} \left(m+\frac12\right)\log\left(\frac{(s_m^+)^2}{m}\right) &\le \left(m+\frac12\right)\frac{(s_m^+)^2 - m}{s_m^+\sqrt{m}} = \left(m+\frac12\right)\frac{s_m^+}{s_m^+\sqrt{m}} \\ &= \sqrt{m} + \frac{1}{2\sqrt{m}} \le \sqrt{m+\frac14} + \frac{1}{2\sqrt{m}} \le s_m^+ - \frac12 + \frac{1}{2\sqrt{m}}. \end{aligned}$$

If we combine these inequalities with (24), (25), and (26), we obtain, for $t \in [0,\infty)$ and $m \in \mathbf{N}$,

$$\begin{aligned} L_m(t) &\le L_m(r_m^-) = \frac{1}{\sqrt{2\pi}}\left(\frac{(s_m^-)^2}{m}\right)^{m+1/2} \exp(s_m^- - \vartheta(m)) \\ &\le \frac{1}{\sqrt{2\pi e}} \exp\left(- \frac{1}{4\sqrt{m+1/4}} - \frac{1}{12m+1} \right), \\ L_m(t) &\ge L_m(r_m^+) = -\frac{1}{\sqrt{2\pi}}\left(\frac{(s_m^+)^2}{m}\right)^{m+1/2} \exp(-s_m^+ - \vartheta(m)) \\ &\ge -\frac{1}{\sqrt{2\pi e}} \exp\left(\frac{1}{2\sqrt{m}} - \frac{1}{12m+1} \right). \end{aligned}$$

Therefore (23) is valid. Now, we prove the rest of the assertion. From (22) and (23) we easily obtain, for all $t \in [0,\infty)$ and $m \in \mathbf{Z}_+$, $L_m(t) \le 1/\sqrt{2\pi e} \le 1/e = 0.3678\ldots$. If $t \in [0,\infty)$ and $m \in \{0,1\}$, (22) yields $-L_m(t) \le 1/e$ and, in the case $m \in \{2,3,\ldots\}$, (23) leads to

$$-L_m(t) \le \frac{1}{\sqrt{2\pi e}} \exp\left(\frac{1}{2\sqrt{m}}\right) \le \frac{1}{\sqrt{2\pi e}} \exp\left(\frac{1}{2\sqrt{2}}\right) \le 0.35 \le \frac{1}{e}.$$

The lemma is proved. ■

For $x \in \mathbf{R}$, let $\lfloor x \rfloor, \lceil x \rceil \in \mathbf{Z}$ be defined by $x - 1 < \lfloor x \rfloor \le x \le \lceil x \rceil < x+1$.

Lemma 3 *For* $t \in (0,\infty)$,

$$\|\Delta\pi(\cdot,t)\|_1 \le \sqrt{\frac{2}{te}}, \tag{27}$$

$$\begin{aligned} \|\Delta\pi(\cdot,t)\|_\infty &\le \frac{1}{te}, & (28)\\ \|\Delta^2\pi(\cdot,t)\|_1 &\le \frac{3}{te}, & (29)\\ \|\Delta^2\pi(\cdot,t)\|_\infty &\le \left(\frac{3}{2te}\right)^{3/2}. & (30) \end{aligned}$$

In (27), (28), (29), (30) *equality holds, respectively, for* $t = 1/2$, $t = 1$, $t = 1$, $t = 3/2$.

Proof. Ad (27): This was shown by Deheuvels and Pfeifer (1988) [see also Deheuvels and Pfeifer (1986a, b)]. They also proved the identity

$$\|\Delta\pi(\cdot,t)\|_1 = 2e^{-t}\,\frac{t^{\lfloor t\rfloor}}{\lfloor t\rfloor!}, \qquad (t\in(0,\infty)). \tag{31}$$

Ad (28): Deheuvels and Pfeifer (1986b) proved that

$$\|\Delta\pi(\cdot,t)\|_\infty = \frac{1}{t}\max\{L_{b_+(t)}(t),\, -L_{b_-(t)}(t)\}, \qquad (t\in(0,\infty)), \tag{32}$$

where $b_\pm(t) = \lfloor t+1/2 \pm \sqrt{t+1/4}\rfloor$. Now, (28) follows from Lemma 2.

Ad (29): Deheuvels and Pfeifer (1986a) proved that

$$\|\Delta^2\pi(\cdot,t)\|_1 = \frac{2}{t}\,[L_{b_+(t)}(t) - L_{b_-(t)}(t)], \qquad (t\in(0,\infty)), \tag{33}$$

where $b_\pm(t)$ is defined as in the proof of (28). If $b_-(t)\ge 5$, then $b_+(t)\ge 5$ and (29) immediately follows from Lemma 2:

$$\|\Delta^2\pi(\cdot,t)\|_1 \le \frac{2}{t\sqrt{2\pi e}}\left[1+\exp\left(\frac{1}{2\sqrt{5}}\right)\right] \le \frac{1.09}{t} \le \frac{3}{te} = \frac{1.10\ldots}{t}.$$

It remains to prove (29) in the case $0\le b_-(t)\le 4$. Easy calculations show that, for $t\in(0,\infty)$ and $m\in\mathbf{Z}_+$, $b_\pm(t) = m$ if and only if $t\in[m\mp\sqrt{m},\, m+1\mp\sqrt{m+1})$. Therefore, $0\le b_-(t)\le 4$ if and only if $(b_-(t), b_+(t))\in I$, where

$$I = \{\,(0,1),\,(0,2),\,(0,3),\,(1,4),\,(1,5),\,(2,5),\,(2,6),\,(2,7),\,(3,7),\,(3,8),\,(4,9),\,(4,10)\,\}.$$

Indeed, this follows from

$$\begin{aligned} b_-(t)=0 &\;\Leftrightarrow\; t\in[0,2-\sqrt{2})\cup[2-\sqrt{2},3-\sqrt{3})\cup[3-\sqrt{3},4-\sqrt{4})\\ &\;\Leftrightarrow\; b_-(t)=0 \text{ and } b_+(t)\in\{1,2,3\},\\ b_-(t)=1 &\;\Leftrightarrow\; t\in[4-\sqrt{4},5-\sqrt{5})\cup[5-\sqrt{5},2+\sqrt{2})\\ &\;\Leftrightarrow\; b_-(t)=1 \text{ and } b_+(t)\in\{4,5\},\\ b_-(t)=2 &\;\Leftrightarrow\; t\in[2+\sqrt{2},6-\sqrt{6})\cup[6-\sqrt{6},7-\sqrt{7})\cup[7-\sqrt{7},3+\sqrt{3})\\ &\;\Leftrightarrow\; b_-(t)=2 \text{ and } b_+(t)\in\{5,6,7\},\\ b_-(t)=3 &\;\Leftrightarrow\; t\in[3+\sqrt{3},8-\sqrt{8})\cup[8-\sqrt{8},9-\sqrt{9})\\ &\;\Leftrightarrow\; b_-(t)=3 \text{ and } b_+(t)\in\{7,8\},\\ b_-(t)=4 &\;\Leftrightarrow\; t\in[9-\sqrt{9},10-\sqrt{10})\cup[10-\sqrt{10},5+\sqrt{5})\\ &\;\Leftrightarrow\; b_-(t)=4 \text{ and } b_+(t)\in\{9,10\}. \end{aligned}$$

If $(b_-(t), b_+(t)) = (0,3)$, then $t \in [3-\sqrt{3}, 2)$ and

$$\|\Delta^2\pi(\cdot,t)\|_1 = \frac{2}{t}\,[L_3(t) - L_0(t)] = \frac{e^{-t}}{3}(-t^3+3t^2+6) \le \frac{3}{te},$$

which can directly be shown. The remaining 11 cases for $(b_-(t),\, b_+(t))$ can be treated using (24):

$$\|\Delta^2\pi(\cdot,t)\|_1 = \frac{2}{t}\,[L_{b_+(t)}(t) - L_{b_-(t)}(t)] \le \frac{2}{t}\,[L_{b_+(t)}(r^-_{b_+(t)}) - L_{b_-(t)}(r^+_{b_-(t)})] \le \frac{3}{te},$$

where $r^\pm_m$, $(m \in \mathbf{Z}_+)$ is defined as in the proof of Lemma 2. The latter inequality can immediately be verified by inserting the values of $(b_-(t), b_+(t)) \in I \setminus \{(0,3)\}$. Inequality (29) is proved.

Ad (30): If $t \ge 29$, then (30) follows from (18) with $m=3$ and $k=2$:

$$\|\Delta^2\pi(\cdot,t)\|_\infty \le \frac{1}{t^{3/2}}\left[4\sqrt{e}\left(1+\sqrt{\frac{\pi}{16}}\right)\left(\frac{2}{e}\right)^{9/2}\frac{1}{t^3} + \frac{4}{\pi}\sum_{j=1}^{3}\frac{\Gamma(j+1/2)}{2^{j+1/2}\,t^{j-1}}\right] \le \frac{0.40981}{t^{3/2}} \le \left(\frac{3}{2te}\right)^{3/2}.$$

It remains to show (30) in the case $t \in (0,\,29)$. As proved in Roos (1995, Lemma 2.3),

$$\|\Delta^2\pi(\cdot,t)\|_\infty = \max_{1\le j\le 3}|\Delta^2\pi(\lfloor x_j(t)\rfloor, t)|, \qquad (t \in (0,\infty)), \tag{34}$$

where $0 < x_1(t) < x_2(t) < x_3(t)$ are the real zeros of the 3rd Charlier polynomial $c_3(x,t) = x^3 - 3(t+1)x^2 + (3t^2+3t+2)x - t^3$. For $t \in (0,\infty)$, let

$$y_1(t) = t+1-\sqrt{3t+1}, \qquad y_2(t) = t+\frac{7}{6}-\sqrt{3t+\frac{11}{12}}, \qquad y_3(t) = t+\frac{2}{3},$$

$$y_4(t) = t+1, \qquad y_5(t) = t+1+\sqrt{3t+1}, \qquad y_6(t) = t+\frac{7}{6}+\sqrt{3t+\frac{11}{12}}.$$

Then it is easy to show that $y_j(t) < y_{j+1}(t)$ for all $1 \le j \le 5$ and

$$x_j(t) \in (y_{2j-1}(t),\, y_{2j}(t)), \qquad y_{2j}(t) - y_{2j-1}(t) \le 1/3, \qquad (1 \le j \le 3). \tag{35}$$

Indeed the first relation in (35) follows from the observation that

$$c_3(y_j(t),t) = -t < 0, \quad (j \in \{1,4,5\}) \quad \text{and} \quad c_3(y_j(t),t) = \frac{8}{27} > 0, \quad (j \in \{2,3,6\}).$$

This yields $\|\Delta^2\pi(\cdot,t)\|_\infty = \max_{1\le j\le 6}|\Delta^2\pi(\lfloor y_j(t)\rfloor, t)|$, $(t \in (0,\,\infty))$. For $t \in (0,\,29)$, we have $y_j(t) \le 39.6$ for all $1 \le j \le 6$ and therefore, by using some calculus,

$$t^{3/2}\|\Delta^2\pi(\cdot,t)\|_\infty = \max_{0\le m\le 39} t^{3/2}|\Delta^2\pi(m,t)| \le \max_{0\le m\le 39}\ \max_{s\in(0,\infty)} s^{3/2}|\Delta^2\pi(m,s)| \le \left(\frac{3}{2e}\right)^{3/2},$$

completing the proof of (30).

The indicated sharpness of the inequalities (27)–(30) is easily verified by using (31)–(34). ■

Remark. The formulas (31)–(34) for the norms $\|\Delta^k\pi(\cdot,t)\|_q$, $(q \in \{1,\infty\},\ k \in \{1,2\})$ given in the proof of Lemma 3 can also be obtained from a more general result (see Roos (1999b, Corollaries 1, 2)). For a precise statement, we need some preparations. Let

$$c_k(x,t) = \sum_{j=0}^{k}\binom{k}{j}\binom{x}{j} j!\,(-t)^{k-j}, \qquad (t, x \in \mathbf{R},\ k \in \mathbf{Z}_+)$$

denote the Charlier polynomial of degree k. Observe that

$$\Delta^k \pi(m,t) = \frac{1}{t^k}\,\pi(m,t)\,c_k(m,t), \qquad (t \in (0,\infty),\ k, m \in \mathbf{Z}_+), \tag{36}$$

(see, for example, Roos (1999b, formula (6))). From the theory of orthogonal polynomials it is known that the zeros of the Charlier polynomials $c_k(x,t)$, ($k \in \mathbf{N}$, $t \in (0,\infty)$), are real, simple, and located in the interval $(0,\infty)$. The result is now as follows: If $k \in \mathbf{Z}_+$, $t \in (0,\infty)$, and $0 < x_1(t) < \ldots < x_{k+1}(t)$ denote the zeros of $c_{k+1}(x,t)$, then

$$\|\Delta^k \pi(\cdot,t)\|_\infty = \max_{1\le j\le k+1} |\Delta^k \pi(\lfloor x_j(t)\rfloor, t)|, \tag{37}$$

$$\|\Delta^{k+1}\pi(\cdot,t)\|_1 = 2\Big|\sum_{j=1}^{k+1}(-1)^j \Delta^k \pi(\lfloor x_j(t)\rfloor,t)\Big| = 2\sum_{j=1}^{k+1}|\Delta^k\pi(\lfloor x_j(t)\rfloor,t)|. \tag{38}$$

Note that the identities (37) and (38) explain the connection between the formulas for $\|\Delta\pi(\cdot,t)\|_\infty$ and $\|\Delta^2\pi(\cdot,t)\|_1$ given in the proof of Lemma 3 (see (32) and (33)).

Proof of Theorem 2. Let $q \in \{1,\infty\}$ and $i \in \{-1,0\}$. We may assume that $Q \in \Pi(1)$. Using Theorem 2 in Roos (1999b) with the parameters $t = \lambda$, $k = 0$, $s = 2$, and $j = 1$, we obtain $d_q^{(i)} = H + R$, where

$$H = \frac{\lambda_2}{2}\,\|\Delta^{i+2}\pi(\cdot,\,\lambda)\|_q, \qquad |R| \le \frac{1+\sqrt{\pi/2}}{2^{1-1/q}\,(i+3)^{\lceil a+1/q\rceil - a - 1/q}} \sum_{j=i+3}^{\infty} \frac{\theta^{(j-i)/2}}{\lambda^a}\, j^{\lceil a+1/q\rceil},$$

and $a = a_q^{(i)} = (i+1)/2 - 1/(2q)$. Applying Lemma 3, we are led to (8)–(11). Now we prove the sharpness of the inequalities. From (33) in Roos (1999b) it follows that, for any choice of $M > 0$, $q \in \{1,\infty\}$, and $i \in \{-1,0\}$, there exists an absolute constant $B = B_{M,q,i} > 0$ such that, for all $Q \in \Pi$ with $\lambda_2 = \lambda_2(Q) \le M$,

$$\left|\frac{d_q^{(i)}}{\lambda_2\|\Delta^{2+i}\pi(\cdot,A_q^{(i)})\|_q/2} - 1\right| \le B\left(\frac{\lambda_3}{\lambda_2} + \lambda_2 + |\lambda - A_q^{(i)}|\right) \le B(1+\sqrt{M})(\sqrt{\lambda_2} + |\lambda - A_q^{(i)}|),$$

since $\lambda_3 \le \lambda_2^{3/2}$. Therefore, if $\lambda \to A_q^{(i)}$ and $\theta \to 0$, we also have $\lambda_2 \to 0$, and letting a as above,

$$d_q^{(i)} \sim \frac{\lambda_2}{2}\|\Delta^{2+i}\pi(\cdot,A_q^{(i)})\|_q \sim \frac{1}{2}(A_q^{(i)})^{a+1}\|\Delta^{2+i}\pi(\cdot,A_q^{(i)})\|_q\frac{\theta}{\lambda^a} = \tilde{U}_q^{(i)}\,\zeta_q^{(i)}.$$

For the latter equality, we used (6), (7), and Lemma 3. Hence inequalities (8)–(11) are sharp. The proof is completed. ■

Proof of Theorem 1. We show $U_q^{(i)}(0) = \tilde{U}_q^{(i)}$ by proving "$\le$" and "$\ge$" separately. In view of (8)–(11), define, for $r \in (0,1)$,

$$h_\infty^{(0)}(r) = \frac{\sqrt{r}(6-4\sqrt{r})}{3(1-\sqrt{r})^2}, \qquad h_\infty^{(-1)}(r) = \frac{6\sqrt{r}}{5(1-\sqrt{r})},$$

$$h_1^{(0)}(r) = \frac{7\sqrt{r}(3-2\sqrt{r})}{3(1-\sqrt{r})^2}, \qquad h_1^{(-1)}(r) = \frac{8\sqrt{r}(2-\sqrt{r})}{5(1-\sqrt{r})^2}.$$

Now we obtain, for $q \in \{1,\infty\}$ and $i \in \{-1,0\}$,

$$U_q^{(i)}(0) = \lim_{r\downarrow 0}\left(\sup_{Q\in\Pi(r)} \frac{d_q^{(i)}(Q)}{\zeta_q^{(i)}(Q)}\right) \le \lim_{r\downarrow 0}\left(\sup_{Q\in\Pi(r)}\left[\tilde{U}_q^{(i)} + h_q^{(i)}(\theta(Q))\right]\right) \le \tilde{U}_q^{(i)}.$$

It remains to show "$\geq$". For $n \in \{3,4,\ldots\}$, $q \in \{1,\infty\}$, and $i \in \{-1,0\}$, let $Q_{n,q,i} = \mathcal{B}(n, A_q^{(i)}/n)$ be the binomial distribution with parameters n and $A_q^{(i)}/n$. Here $\theta(Q_{n,q,i}) = A_q^{(i)}/n \leq 3/(2n) \leq 1/2$, i.e. $Q_{n,q,i} \in \Pi(2A_q^{(i)}/n)$. Now Theorem 2(b) tells us that $d_q^{(i)}(Q_{n,q,i}) \sim \tilde{U}_q^{(i)} \zeta_q^{(i)}(Q_{n,q,i})$ as $n \to \infty$. Hence

$$U_q^{(i)}(0) = \lim_{r\downarrow 0}\Big(\sup_{Q\in\Pi(r)} \frac{d_q^{(i)}(Q)}{\zeta_q^{(i)}(Q)}\Big) = \lim_{n\uparrow\infty}\Big(\sup_{Q\in\Pi(2A_q^{(i)}/n)} \frac{d_q^{(i)}(Q)}{\zeta_q^{(i)}(Q)}\Big) \geq \lim_{n\uparrow\infty} \frac{d_q^{(i)}(Q_{n,q,i})}{\zeta_q^{(i)}(Q_{n,q,i})} = \tilde{U}_q^{(i)}. \tag{39}$$

The theorem is proved. ∎

Proof of Corollary 1. As in the proof of Theorem 1, the equality asserted is shown by proving "$\leq$" and "$\geq$" separately. Since $\overline{\Pi}(r) \subseteq \Pi(r)$ for all $r \in [0,1]$, "$\leq$" follows from Theorem 1:

$$\overline{U}_q^{(i)}(0) = \lim_{r\downarrow 0}\Big(\sup_{Q\in\overline{\Pi}(r)} \frac{d_q^{(i)}(Q)}{\zeta_q^{(i)}(Q)}\Big) \leq \lim_{r\downarrow 0}\Big(\sup_{Q\in\Pi(r)} \frac{d_q^{(i)}(Q)}{\zeta_q^{(i)}(Q)}\Big) = U_q^{(i)}(0) = \tilde{U}_q^{(i)}.$$

The proof of "$\geq$" was substantially done in the proof of Theorem 1: In (39) one has to consider $\overline{\Pi}$ instead of Π. Further, note that $p_0(Q_{n,q,i}) = A_q^{(i)}/n$. ∎

3 Concluding remark

In Roos (1998, Lemma 5, Theorem 2) an inequality

$$\|\Delta^k \pi(\cdot,t)\|_\infty \leq \alpha\Big(\frac{k}{te}\Big)^{(k+1)/2}, \qquad (t \in (0,\infty),\ k \in \mathbf{N})$$

was used, where $\alpha \in (0,\infty)$ is an absolute constant independent of k and t. From (20) and (21) it follows, that we can choose $\alpha = \alpha_1$. In Roos (1998), it was also claimed without a proof that α can be replaced with $\alpha_4 = 2^{-1}\sqrt{e}(1+\sqrt{\pi/8})$. Now this assertion can be checked. In the case $k \neq 3$, we argue with (20), (21), and the inequalities in Lemma 3. In particular, we use the fact that α_k is decreasing in k. Let us now consider the case $k = 3$. For t large enough, we obtain the assertion from Lemma 1 with $m = 1$: Indeed, for $t \geq 2.009$,

$$t^2\|\Delta^3\pi(\cdot,t)\|_\infty \leq \frac{2}{\pi} + \frac{125(1+\sqrt{\pi/10})}{8e^{5/2}\,t} \leq 1.6329 \leq \alpha_4\Big(\frac{3}{e}\Big)^2 = 1.633\ldots.$$

For $t \in (0, 2.009)$, we must work harder: Using Corollary 1 in Roos (1999b) (see (37) of the present paper), we have $\|\Delta^3\pi(\cdot,t)\|_\infty = \max_{1\leq j\leq 4}|\Delta^3\pi(\lfloor x_j(t)\rfloor, t)|$ for $t \in (0,\infty)$, where $0 < x_1(t) < \ldots < x_4(t)$ are the zeros of the 4th Charlier polynomial

$$c_4(x,t) = x^4 - (4t+6)x^3 + (6t^2+12t+11)x^2 - (4t^3+6t^2+8t+6)x + t^4.$$

Now let

$$z_1(t) = t+1-\sqrt{(3+\sqrt{6})t+1}, \qquad z_2(t) = t+2-\sqrt{(3+\sqrt{6})t+1},$$

$$z_3(t) = t+\frac{1}{2}-\sqrt{(3-\sqrt{6})t+\frac{1}{4}}, \qquad z_4(t) = t+\frac{3}{2}-\sqrt{(3-\sqrt{6})t+\frac{1}{4}},$$

$$z_5(t) = t + \frac{1}{2} + \sqrt{(3-\sqrt{6})t + \frac{1}{4}}, \qquad z_6(t) = t + \frac{3}{2} + \sqrt{(3-\sqrt{6})t + \frac{1}{4}},$$
$$z_7(t) = t + \frac{3}{2} + \sqrt{(3+\sqrt{6})t + \frac{1}{4}}, \qquad z_8(t) = t + \frac{5}{2} + \sqrt{(3+\sqrt{6})t + \frac{1}{4}},$$
$$\tilde{z}_j(t) = z_j(t) \quad \text{for } j \in \{1,4,5,6,7,8\},$$
$$\tilde{z}_2(t) = \min\{z_2(t), z_3(t)\}, \quad \tilde{z}_3(t) = \max\{z_2(t), z_3(t)\}.$$

Using some straightforward calculus, one can show that $\tilde{z}_j(t) \le \tilde{z}_{j+1}(t)$ for all $1 \le j \le 7$ and

$$x_j(t) \in (\tilde{z}_{2j-1}(t), \tilde{z}_{2j}(t)), \qquad 0 < \tilde{z}_{2j}(t) - \tilde{z}_{2j-1}(t) \le 1, \qquad (1 \le j \le 4). \tag{40}$$

The first relation in (40) follows from the fact that

$$c_4(z_j(t), t) > 0, \quad (j \in \{1,4,5,8\}) \qquad \text{and} \qquad c_4(z_j(t), t) < 0, \quad (j \in \{2,3,6,7\}).$$

Hence

$$\|\Delta^3 \pi(\cdot, t)\|_\infty = \max_{1 \le j \le 8} |\Delta^3 \pi(\lfloor z_j(t) \rfloor, t)| = \max_{0 \le j \le \lfloor z_8(t) \rfloor} |\Delta^3 \pi(j, t)|, \quad (t \in (0, \infty)). \tag{41}$$

By using (36) and (41), it is easy to show that $\|\Delta^3\pi(\cdot,t)\|_\infty = e^{-t}(3-t)$ for $t \in (0,2]$, from which the assertion follows for $t \in (0,2]$. The remaining case $t \in (2, 2.009)$ can be treated by using (41) directly.

Acknowledgement

This paper is based on parts of the authors Ph.D. thesis written at the University of Oldenburg in 1996 under the supervision of Professor Dietmar Pfeifer, to whom the author is grateful for his guidance and a number of helpful discussions. The author would like to thank the referee for his valuable and stimulating comments, which led to an improvement of the paper.

References

[1] Barbour, A.D. and P. Hall (1984), On the rate of Poisson convergence, *Math. Proc. Camb. Philos. Soc.* **95**, 473–480.

[2] Charlier, C.V.L. (1905), Die zweite Form des Fehlergesetzes, *Ark. Mat. Astr. Fys.* **2,15**, 1–8.

[3] Daley, D.J. and D. Vere–Jones (1988), An Introduction to the Theory of Point Processes, New York: Springer Verlag.

[4] Deheuvels, P., A. Karr, D. Pfeifer and R. Serfling (1988), Poisson approximations in selected metrics by coupling and semigroup methods with applications, *J. Stat. Plann. Inference* **20**, 1–22.

[5] Deheuvels, P. and D. Pfeifer (1986a), A semigroup approach to Poisson approximation, *Ann. Probab.* **14**, 663–676.

[6] Deheuvels, P. and D. Pfeifer (1986b), Operator semigroups and Poisson convergence in selected metrics. *Semigroup Forum* **34**, 203–224. Errata. *Semigroup Forum* **35**, 251.

[7] Deheuvels, P. and D. Pfeifer (1988), On a relationship between Uspensky's theorem and Poisson approximations, *Ann. Inst. Statist. Math.* **40**, 671–681.

[8] Feller, W. (1968), An Introduction to Probability Theory and Its Applications I, New York: John Wiley.

[9] Gerber, H.U. (1984), Error bounds for the compound Poisson approximation, *Insur. Math. Econ.* **3**, 191–194.

[10] Hardy, G.H., J.E. Littlewood and G. Pólya (1952), Inequalities, Cambridge: Cambridge University Press.

[11] Le Cam, L. (1960), An approximation theorem for the Poisson binomial distribution, *Pac. J. Math.* **10**, 1181–1197.

[12] Mitrinović, D.S. (1970), Analytic Inequalities, Berlin: Springer Verlag.

[13] Rachev, S.T. (1991), Probability Metrics and the Stability of Stochastic Models, Chichester: John Wiley.

[14] Rachev, S.T. and L. Rüschendorf (1990), Approximation of sums by compound Poisson distributions with respect to stop–loss distances, *Adv. Appl. Probab.* **22**, 350–374.

[15] Roos, B. (1995), A semigroup approach to Poisson approximation with respect to the point metric, *Statist. Probab. Lett.* **24**, 305–314.

[16] Roos, B. (1998), Metric multivariate Poisson approximation of the generalized multinomial distribution, *Theory Probab. Appl.* **43**, 306–315.

[17] Roos, B. (1999a), On the rate of multivariate Poisson convergence, *J. Multivariate Anal.* **69**, 120–134.

[18] Roos, B. (1999b), Asymptotics and sharp bounds in the Poisson approximation to the Poisson binomial distribution, *Bernoulli* **5**, 1021–1034.

[19] Schmidt, E. (1933), Über die Charlier–Jordansche Entwicklung einer willkürlichen Funktion nach der Poissonschen Funktion und ihren Ableitungen, *Z. Angew. Math. Mech.* **13**, 139–142.

[20] Serfling, R.J. (1978), Some elementary results on Poisson approximation in a sequence of Bernoulli trials, *SIAM Rev.* **20**, 567–579.

[21] Shorgin, S.Y. (1977), Approximation of a generalized binomial distribution, *Theory Probab. Appl.* **22**, 846–850.

[22] Weba, M. (1994), Zur Genauigkeit der Approximation einer Gesamtschadenverteilung durch eine zusammengesetzte Poisson–Verteilung, *Bl. Dtsch. Ges. Versicherungsmath.* **21**, 313–317.

[23] Witte, H.–J. (1990), A unification of some approaches to Poisson approximation, *J. Appl. Probab.* **27**, 611–621.

[24] Xia, A. (1997), On using the first difference in the Stein–Chen method, *Ann. Appl. Probab.* **7**, 899–916.