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POISSON APPROXIMATION VIA THE CONVOLUTION WITH KORNYA–PRESMAN SIGNED MEASURES

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Abstract. We present an upper bound for the total variation distance between the generalized multinomial distribution and a finite signed measure, which is the convolution of two finite signed measures, one of which is of Kornya–Presman type. In the one-dimensional Poisson case, such a finite signed measure was first considered by Borovkov and Pfeifer [1]. We give asymptotic relations in the one-dimensional case, and, as an example, the i.i.d. record model is investigated. It turns out that here the approximation is of order $O(n^{-s}(\ln n)^{-(s+1)/2})$ for s being a fixed positive integer, whereas in the approximation with simple Kornya–Presman signed measures, we only have the rate $O((\ln n)^{-(s+1)/2})$.

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1. Introduction and main results. Let $X_1, X_2, X_3, \dots$ be a sequence of independent Bernoulli random vectors in $\mathbf{R}^k$ ($k \in \mathbf{N} = \{1, 2, \dots\}$) with probabilities

$$\mathbf{P}\{X_j = e_r\} = p_{j,r} \in [0,1], \qquad \mathbf{P}\{X_j = 0\} = 1 - p_j \in [0,1]$$

for $j \in \mathbf{N}$ and $r \in \{1, \dots, k\}$, where $p_j = \sum_{r=1}^k p_{j,r}$ and e_r denotes the vector in $\mathbf{R}^k$ with entry 1 at position r and 0 otherwise. Let $n \in \mathbf{N}$. We assume that $\lambda_r = \sum_{j=1}^n p_{j,r} > 0$ for all r and that $\sum_{j=1}^\infty p_j^2 < \infty$.

In Roos [7], we used Kerstan's method to prove a result concerning the approximation of the distribution P^{S_n} of the sum $S_n = \sum_{j=1}^n X_j$ by the finite signed measures $\mathcal{P}_s$, ($s \in \mathbf{N}$) of Kornya–Presman type (cf. [3], [4]), which are concentrated on $\mathbf{Z}_+^k$ and have the generating functions

$$\Psi_{\mathcal{P}_s}(z) = \sum_{l \in \mathbf{Z}_+^k} \mathcal{P}_s(\{l\}) z^l = \exp\Big(\sum_{m=1}^s G_m(z)\Big), \qquad (z = (z_1, \dots, z_k) \in \mathbf{C}^k),$$

where $\mathbf{Z}_+ = \{0, 1, 2, \dots\}$, $z^l = z_1^{l_1} \cdots z_k^{l_k}$ for $l = (l_1, \dots, l_k) \in \mathbf{Z}_+^k$, and, for $m \in \{1, \dots, s\}$ and $z \in \mathbf{C}^k$,

$$G_m(z) = \frac{(-1)^{m+1}}{m} \sum_{j=1}^n [H_j(z)]^m, \quad H_j(z) = \sum_{r=1}^k p_{j,r}(z_r - 1), \quad (j \in \mathbf{N}).$$

For relevant references concerning the Kornya–Presman expansion, see [7].

In the present paper, we give some results for the approximation of P^{S_n} by the convolution $\mu_s = \mathcal{P}_s * \nu_s$, ($s \in \mathbf{N}$), where ν_s is the finite signed measure with generating function

$$\Psi_{\nu_s}(z) = \prod_{j=1}^\infty W_s(-H_j(z)), \qquad (z \in \mathbf{C}^k),$$

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where

$$W_s(y) = (1-y)\,\exp\Big(\sum_{m=1}^{s}\frac{y^m}{m}\Big), \qquad (y \in \mathbf{C})$$

is the Weierstrass prime function. Since $\sum_{j=1}^{\infty} p_j^{s+1} < \infty$, $\Psi_{\nu_s}(z)$ is an entire function (see, e.g., [2, p. 169]). Therefore ν_s and, in turn, μ_s are finite signed measures. The idea behind the above expansion is that the generating function of S_n

$$\Psi_{S_n}(z) = \sum_{l\in\mathbf{Z}_+^k} \mathbf{P}\{S_n = l\} z^l = \exp\Big(\sum_{j=1}^{n}\sum_{m=1}^{\infty}\frac{(-1)^{m+1}}{m}[H_j(z)]^m\Big)$$

can be approximated by

$$\exp\Big(\sum_{m=1}^{s} G_m(z) + \sum_{j=1}^{\infty}\sum_{m=s+1}^{\infty}\frac{(-1)^{m+1}}{m}[H_j(z)]^m\Big) = \Psi_{\mu_s}(z), \tag{1}$$

if s or n is large. The main difference between the approximants μ_s and $\mathcal{P}_s$ lies in the fact that, in contrast to $\mathcal{P}_s$, μ_s contains the information about the whole sequence of distributions of the X_j, $(j \in \mathbf{N})$. This can be a disadvantage, but it turns out that the approximation by μ_s is much more precise than the one by $\mathcal{P}_s$. Note that, in the one-dimensional Poisson case $k = s = 1$, a result concerning the Kolmogorov metric has been given by Borovkov and Pfeifer [1] (see Section 2 below).

In order to present the first result, we need further notation. Let

$$s \in \mathbf{N}, \qquad p_0 = \max_{1\le j\le n} p_j, \qquad \hat p_0 = \max_{j\ge n+1} p_j, \qquad \tilde p_0 = \sum_{r=1}^{k}\max_{1\le j\le n} p_{j,r},$$

$$\kappa_{s,j}(x) = \Big[\min\Big\{x\sum_{r=1}^{k}\frac{p_{j,r}^2}{\lambda_r},\ p_j^2\Big\}\Big]^{(s+1)/2}, \qquad (x\in[0,\infty),\ j\in\mathbf{N})$$

$$\beta_s(x) = \sum_{j=1}^{n}\kappa_{s,j}(x), \qquad \hat\beta_s(x) = \sum_{j=n+1}^{\infty}\kappa_{s,j}(x), \qquad (x\in[0,\infty)).$$

Observe that, for all $x \in [0,\infty)$, $\hat\beta_s(x) < \infty$. For $y \in \mathbf{C}$, let

$$V_s(y) = [1 - W_s(y)]\frac{s+1}{y^{s+1}}.$$

Some properties of the function V_s can be found in [7, Remark (a)]. For $x \in \mathbf{R}$, let $\lceil x\rceil$ denote the smallest integer $\ge x$. Let $A, B \in [0,1]$ with $A + B = 1$. In this paper, we frequently deal with power series of the type $Q(z) = \sum_{l\in\mathbf{Z}_+^k} q_l z^l$, $(q_l \in \mathbf{R})$, which are absolutely convergent for all $z \in \mathbf{C}^k$. In particular, the order of summation may be chosen arbitrarily. We write $\|Q(z)\| = \sum_{l\in\mathbf{Z}_+^k}|q_l|$ and use the easy fact that $\|Q_1(z)Q_2(z)\| \le \|Q_1(z)\|\,\|Q_2(z)\|$ for two power series $Q_1(z)$ and $Q_2(z)$. Let d_s (resp. d_s') be the total variation distance between P^{S_n} and $\mathcal{P}_s$ (resp. μ_s). For example, we have $d_s' = 2^{-1}\|\delta_s(z)\|$ with $\delta_s(z) = \Psi_{S_n}(z) - \Psi_{\mu_s}(z)$. In [7], we gave an upper bound for d_s. In the following theorem, we present our main result for d_s'.

Theorem 1 *Let*

$$\begin{aligned} c_1(s) &= \begin{cases} (s+1)\,2^{-5/2}, & \textit{for odd } s,\\ (s+1)\,2^{1/[2(s+1)]-5/2}, & \textit{for even } s, \end{cases}\\ c_2(s,p_0) &= \frac{e\,2^s\,\lceil s/2-1\rceil!}{\sqrt{2\pi}\,(s+1)}\,V_s(2p_0), \quad c_3(s,p_0)=\frac{e\,2^{s+1}}{s+1}\,V_s(2p_0). \end{aligned}$$

If $c_3(s,\hat p_0)\,\hat p_0^{s-1}\,\hat\beta_1(2^{-3/2}A^{-1})<1$ *and* $c_3(1,p_0)\,\beta_1(2^{-3/2}B^{-1})<1$, *then*

$$d_s' \le \frac{c_2(s,\hat p_0)\,\hat\beta_s(c_1(s)B^{-1})}{[1-c_3(s,\hat p_0)\,\hat p_0^{s-1}\,\hat\beta_1(2^{-3/2}B^{-1})]^{\lceil s/2\rceil}\,[1-c_3(1,p_0)\,\beta_1(2^{-3/2}A^{-1})]}. \tag{2}$$

Remark 1 There exist positive constants C_1 and $C_2(s)$, such that, if n is sufficiently large and $\tilde p_0 \le C_1$, then, for all s, we have $d_s' \le C_2(s)\hat\beta_s(1)$. Note that C_1 is indeed independent of s. Letting $A=0.9$, we see that we can set $C_1=0.3$, since $\lim_{n\to\infty}\hat\beta_1(1)=0$ and an $\varepsilon>0$ exists, such that, for $\tilde p_0\le 0.3$ and A as above,

$$c_3(1,p_0)\beta_1(2^{-3/2}A^{-1}) \le \frac{e\,V_1(2\tilde p_0)}{\sqrt{2}\,A}\,\tilde p_0 \le 1-\varepsilon.$$

In Roos [7, Remark (c)], we have shown a similar assertion concerning the total variation distance d_s between P^{S_n} and $\mathcal{P}_s$: If $\tilde p_0 \le 1/4$, then $d_s \le C_3(s)\beta_s(1)$, where $C_3(s)$ is a constant depending only on s. In view of $\lim_{n\to\infty}\hat\beta_s(1)=0$, we see that, in the case of n being large, the approximation by μ_s is better than the one by $\mathcal{P}_s$.

We now give a recursive formula for the counting density of μ_s. Observe that $\mu_s(\{0\})=\Psi_{\mu_s}(0)$. For $l=(l_1,\dots,l_k)\in\mathbf{Z}_+^k$, we use the standard multi-index notation $|l|=l_1+\dots+l_k$ and $l!=l_1!\cdots l_k!$. Further, if additionally $t\in\mathbf{Z}_+^k$, we write $t\le l$ in the case that $t_r\le l_r$ for all r. For a set M, $\mathbf{1}_M(x)$ is set to be one if $x\in M$ and zero otherwise.

Proposition 1 *Let* $l\in\mathbf{Z}_+^k$ *with* $|l|\ge 1$, $I_l=\{t\in\mathbf{Z}_+^k\,|\,|t|\ge 1;\ t\le l\}$ *and, for* $t\in I_l$,

$$a_t = \frac{(-1)^{|t|+1}\,|t|!}{t!}\Big[\mathbf{1}_{\{1,\dots,s\}}(|t|)\sum_{j=1}^{n}\sum_{m=|t|-1}^{s-1} f_{j,m,t} + \sum_{j=1}^{\infty}\sum_{m=\max\{s,|t|-1\}}^{\infty} f_{j,m,t}\Big],$$

where

$$f_{j,m,t} = \binom{m}{|t|-1} p_j^{m-|t|+1}\prod_{r=1}^{k} p_{j,r}^{t_r}.$$

Then

$$\mu_s(\{l\}) = \frac{1}{|l|}\sum_{t\in I_l}\mu_s(\{l-t\})\,a_t.$$

The proof of this proposition will be omitted, since it can easily be done by using (1) and following the lines in the proof of Proposition 1 in [7].

2. The one-dimensional case. In this section, let always $k=1$. For $l\in\mathbf{N}$, let $\alpha_l=\sum_{j=1}^n p_j^l$ and $\hat\alpha_l=\sum_{j=n+1}^{\infty} p_j^l$. Instead of λ_1, we simply write λ. In the case $s=1$, Borovkov and Pfeifer [1, Theorem 2] derived the following asymptotic relation for the Kolmogorov distance

$$\rho_1=\sup_{x\in\mathbf{R}}|\mathbf{P}\{S_n\leq x\}-\mu_1((-\infty,x])|$$

between P^{S_n} and μ_1: If $\sum_{j=1}^{\infty}p_j=\infty$, then

$$\rho_1=\frac{\hat{\alpha}_2}{2\sqrt{2\pi e}\,\lambda}(1+o(1)),\qquad(n\to\infty).\tag{3}$$

In the following theorem, we show a similar but somewhat more precise asymptotic relation for the total variation distance in the general case $s\in\mathbf{N}$.

Theorem 2 *Let*

$$\tau_s=\int_{\mathbf{R}}|\varphi_{s+1}(x)|\,dx,\qquad\varphi_{s+1}(x)=\frac{1}{\sqrt{2\pi}}\frac{d^{s+1}}{dx^{s+1}}e^{-x^2/2},\qquad(x\in\mathbf{R}).$$

If $\sum_{j=1}^{\infty}p_j=\infty$, *then*

$$d_s'=\frac{\tau_s\,\hat{\alpha}_{s+1}}{2(s+1)\lambda^{(s+1)/2}}\Big(1+O\Big(\frac{1}{\sqrt{\lambda}}\Big)\Big),\qquad(n\to\infty).\tag{4}$$

Similarly, under the same assumptions, the approximation by $\mathcal{P}_s$ *yields the expansion*

$$d_s=\frac{\tau_s\,\alpha_{s+1}}{2(s+1)\lambda^{(s+1)/2}}\Big(1+O\Big(\frac{1}{\sqrt{\lambda}}\Big)\Big),\qquad(n\to\infty).\tag{5}$$

Note that simple calculations yield

$$\begin{aligned}\tau_1&=\frac{4}{\sqrt{2\pi e}}\approx0.968,\qquad\tau_2=\frac{2(4+e^{3/2})}{\sqrt{2\pi}e^{3/2}}\approx1.510,\\ \tau_3&=\frac{4\sqrt{6}[\sqrt{3-\sqrt{6}}\exp(\sqrt{6}/2)+\sqrt{3+\sqrt{6}}\exp(-\sqrt{6}/2)]}{\sqrt{2\pi}e^{3/2}}\approx2.801.\end{aligned}$$

Example 1 In what follows, we consider the i.i.d. record model (cf. [1]). Let $Z_1,Z_2,\dots$ be independent and identically distributed random variables with a continuous cumulative distribution function. We say that Z_j, $(j\in\mathbf{N})$ is a record of this sequence when $Z_j>\max\{Z_1,\dots,Z_{j-1}\}$, where we set $\max\emptyset=-\infty$. The corresponding record indicators X_j', $(j\in\mathbf{N})$ are defined by $X_j'=1$ when Z_j is a record and $X_j'=0$ otherwise. Rényi [5] has shown that, here, the X_j' form a sequence of independent Bernoulli random variables with $\mathbf{P}\{X_j'=1\}=1/j$, $(j\in\mathbf{N})$. Since $X_1'=1$ is a constant, we omit it and set $X_j=X_{j+1}'$, $(j\in\mathbf{N})$. Now $p_j=1/(j+1)$, $(j\in\mathbf{N})$. As $n\to\infty$, we have $\hat{\alpha}_{s+1}=s^{-1}n^{-s}+O(n^{-(s+1)})$ and $\lambda=\ln n+\gamma+O(n^{-1})$, where $\gamma=0.5772\dots$ is Euler's constant. In fact, the first asymptotic relation can easily be deduced from the simple equality

$$\frac{1}{(n+1)^s}=s\sum_{j=n+1}^{\infty}\frac{1}{j(j+1)^s}+\sum_{m=0}^{s-2}\binom{s}{m}\sum_{j=n+1}^{\infty}\frac{1}{j^{s-m}(j+1)^s}.$$

Therefore, by Theorem 2,

$$d'_s = \frac{\tau_s\,[1+O((\ln n)^{-1/2})]}{2s(s+1)\,n^s\,(\ln n)^{(s+1)/2}}, \qquad d_s = \frac{\tau_s\,(\zeta(s+1)-1)\,[1+O((\ln n)^{-1/2})]}{2(s+1)\,(\ln n)^{(s+1)/2}},$$

where ζ is the Riemann zeta function. Here, we see that the approximation error d'_s is considerably smaller than d_s. The difference in the order is the factor n^{-s}. It should be mentioned, that an asymptotic relation for the Kolmogorov distance between P^{S_n} and $\mathcal{P}_1$ can be found (without proof) in [1, formula (4)]. A proof can easily be obtained from the more general relation (32) in Roos [6]. Observe that Borovkov and Pfeifer [1, Theorem 1] derived another finite signed measure with the help of the gamma function and gave an explicit bound of order $O(n^{-2})$ for the Kolmogorov distance to P^{S_n}.

3. Proofs. For the proof of Theorem 1, we need the following two lemmas.

Lemma 1 *(cf. [7]) Let $x \in (0,\infty)$ and $j \in \mathbf{N}$. Then*

$$\begin{aligned} \|V_s(-H_j(z))\| &= V_s(2p_j), \\ \|[H_j(z)]^{s+1}\exp(xG_1(z))\| &\le \left[\min\left\{\frac{4\,c_1(s)}{x}\sum_{r=1}^{k}\frac{p_{j,r}^2}{\lambda_r},\ 4p_j^2\right\}\right]^{(s+1)/2}. \end{aligned}$$

Lemma 2 *We have*

$$K_1(B) := \left\|\Psi_{S_n}(z)\,\exp(-BG_1(z))\right\| \le \frac{1}{1-c_3(1,p_0)\,\beta_1(2^{-3/2}A^{-1})}.$$

Proof. Using the expansion

$$\begin{aligned} \Psi_{S_n}(z)\,\exp(-BG_1(z)) &= \prod_{j=1}^{n}\left[1-\frac{[H_j(z)]^2}{2}V_1(-H_j(z))\right]\exp(A\,G_1(z)) \\ &= \sum_{j=1}^{n}\ \sum_{1\le i(1)<\cdots<i(j)\le n}\ \prod_{a=1}^{j}\left[\frac{-V_1(-H_{i(a)}(z))}{2}[H_{i(a)}(z)]^2\exp\left(\frac{A}{j}\,G_1(z)\right)\right] \\ &\quad + \exp(A\,G_1(z)) \end{aligned}$$

and the polynomial theorem, we are led to

$$K_1(B) \le 1+\sum_{j=1}^{\infty}\frac{1}{j!}\left[\frac{1}{2}\sum_{i=1}^{n}V_1(2p_i)\left\|[H_i(z)]^2\exp\left(\frac{A}{j}G_1(z)\right)\right\|\right]^j.$$

Application of Lemma 1 and Stirling's formula yields the assertion. ■

Proof of Theorem 1. We have

$$\begin{aligned} \delta_s(z) &= \lim_{\eta\to\infty}\left[1-\prod_{j=1}^{\eta}\left[1-\frac{[-H_{j+n}(z)]^{s+1}}{s+1}V_s(-H_{j+n}(z))\right]\right]\Psi_{S_n}(z) \\ &= \lim_{\eta\to\infty}\left[-\sum_{j=1}^{\eta}\ \sum_{1\le i(1)<\cdots<i(j)\le\eta}\ \prod_{a=1}^{j}\left[\frac{(-1)^s V_s(-H_{i(a)+n}(z))}{s+1}\right.\right. \\ &\qquad \left.\left.\times\,[H_{i(a)+n}(z)]^{s+1}\exp\left(\frac{B}{j}G_1(z)\right)\right]\right]\Psi_{S_n}(z)\exp(-B\,G_1(z)). \quad (6) \end{aligned}$$

By using the polynomial theorem, we obtain $2d_s \le K_1(B)\,K_2(B)$, where $K_1(B)$ is defined as in Lemma 2 and

$$K_2(B) \;:=\; \sum_{j=1}^{\infty}\frac{1}{j!}\Big[\sum_{i=n+1}^{\infty}\frac{V_s(2p_i)}{s+1}\,\Big\|[H_i(z)]^{s+1}\exp\Big(\frac{B}{j}G_1(z)\Big)\Big\|\Big]^j.$$

Proceeding as in the proof of Theorem 1 in Roos [7], the proof is easily completed. ■

Proof of Theorem 2. We have

$$\Big|2\,d_s' - \frac{\hat{\alpha}_{s+1}}{s+1}\|(z-1)^{s+1}\Psi_{S_n}(z)\|\Big| \le Y_1 + Y_2,$$

where, using the properties of V_s (see [7, Remark (a)]),

$$\begin{aligned} Y_1 &= \Big\|\sum_{i=n+1}^{\infty}\frac{p_i^{s+2}}{s+1}\,\frac{[V_s(-p_i(z-1))-1]}{p_i(z-1)}(z-1)^{s+2}\Psi_{S_n}(z)\Big\| \\ &= O(\hat{\alpha}_{s+2}\|(z-1)^{s+2}\Psi_{S_n}(z)\|), \end{aligned}$$

and Y_2 denotes the norm of the sum of terms over $j = 2, 3, \dots$ in the expansion (6), i.e. the summand for $j = 1$ has to be left out. As in the proof of Theorem 1, one can now show that $Y_2 = O(\hat{\alpha}_{s+1}^2/\lambda^{s+1})$. In view of formula (32) in Roos [6], we see that, for $l \in \mathbf{N}$,

$$\|(z-1)^l\Psi_{S_n}(z)\| = \|(z-1)^l\exp(\lambda(z-1))\| + O\Big(\frac{\alpha_2}{\lambda^{(l+2)/2}}\Big),$$

whereas Proposition 4 in Roos [6] gives

$$\|(z-1)^l\exp(\lambda(z-1))\| = \frac{\tau_{l-1}}{\lambda^{l/2}} + O\Big(\frac{1}{\lambda^{(l+1)/2}}\Big).$$

Combining the above estimates, we obtain (4). The proof of (5) is similar and therefore omitted. ■

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