METRIC MULTIVARIATE POISSON APPROXIMATION OF THE GENERALIZED MULTINOMIAL DISTRIBUTION

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The aim of this paper is to introduce the multivariate Charlier B expansion to the metric multivariate Poisson approximation of a generalized multinomial distribution considered by Barbour (1988) and Deheuvels and Pfeifer (1988a). Bounds for the total variation and the point metric are given.

1. Introduction. Let $X_1, \ldots, X_n$ ($n \in \mathbf{N} = \{1, 2, \ldots\}$) be independent random vectors of $\mathbf{R}^k$, $k \in \mathbf{N}$, where $X_j = (X_j(1), \ldots, X_j(k))$, $P(X_j = e_r) = p_{j,r} \in [0,1]$, and $P(X_j = 0) = 1 - \sum_{r=1}^{k} p_{j,r} \in [0,1]$, for $j \in \{1, \ldots, n\}$ and $r \in \{1, \ldots, k\}$. Here, e_r is the vector in $\mathbf{R}^k$ with 1 at position $r \in \{1, \ldots, k\}$ and 0 otherwise. For $l \in \mathbf{N}$ and $r \in \{1, \ldots, k\}$, let $\lambda_l = (\lambda_l(1), \ldots, \lambda_l(k))$, $\lambda_l(r) = \sum_{j=1}^{n} p_{j,r}^{l} > 0$, $\lambda = \lambda_1$, $\theta = (\theta(1), \ldots, \theta(k))$, $\theta(r) = \lambda_2(r)/\lambda(r)$. Set $S_n = \sum_{j=1}^{n} X_j$. Always, let $0^0 = 1$.

In this paper, the distribution P^{S_n} of S_n is approximated by the multivariate Poisson distribution $\mathcal{P}(t)$, for $t = (t_1, \ldots, t_k) \in (0, \infty)^k$, and related finite signed measures of higher order. Here

$$\mathcal{P}(t)(\{m\}) = \exp\left(-\sum_{r=1}^{k} t_r\right) \prod_{r=1}^{k} \frac{t_r^{m_r}}{m_r!} \qquad \text{for } m = (m_1, \ldots, m_k) \in \mathbf{Z}_+^k,$$

and $\mathbf{Z}_+$={0,1,2,...}. As measures of accuracy, the following metrics are considered:

$$d_\tau(P, Q) = \sup_{A \subseteq \mathbf{Z}_+^k} |P(A) - Q(A)| = \frac{1}{2} \sum_{m \in \mathbf{Z}_+^k} |P(\{m\}) - Q(\{m\})| \qquad \text{(total variation)}, \quad (1)$$

$$d_\pi(P, Q) = \sup_{m \in \mathbf{Z}_+^k} |P(\{m\}) - Q(\{m\})| \qquad \text{(point metric)}, \quad (2)$$

where P and Q are finite signed measures concentrated on $\mathbf{Z}_+^k$ satisfying $P(\mathbf{Z}_+^k) = Q(\mathbf{Z}_+^k)$.

Various authors treated this approximation problem. The most important papers, concerning the total variation, came from Barbour (1988) and Deheuvels and Pfeifer (1988a), using the Stein–Chen method and the semigroup method originally developed by Le Cam (1960), respectively.

Barbour (1988) showed that

$$d_\tau(P^{S_n}, \mathcal{P}(\lambda)) \le \sum_{j=1}^{n} \min\left\{\left[\frac{1}{2} + \log^+\left(2\sum_{r=1}^{k}\lambda(r)\right)\right] \sum_{r=1}^{k} \frac{p_{j,r}^2}{\lambda(r)}, \left(\sum_{r=1}^{k} p_{j,r}\right)^2\right\}, \quad (3)$$

where $\log^+(x) = \max\{0, \log(x)\}$ for $x \in \mathbf{R}$. Note that in the case $k = 1$, Barbour and Hall (1984) proved that $\frac{1}{32}\min\{\lambda_2(1), \theta(1)\} \le d_\tau(P^{S_n}, \mathcal{P}(\lambda)) \le (1 - e^{-\lambda(1)})\theta(1)$.

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Deheuvels and Pfeifer (1988a) developed some asymptotic formulae, which are too involved to state them here. It should be mentioned that their upper bounds for total variation [see their Theorem 2.2] are only useful for bounded $\sum_{j=1}^{n}(\sum_{r=1}^{k} p_{j,r})^2$. In the case $k = 1$, they used the univariate Charlier B expansion due to Charlier (1905) to prove [see Deheuvels et al. (1988b)]

$$d_\tau(P^{S_n}, \mathcal{P}(\lambda)) = \frac{1}{2}\lambda_2(1)e^{-\lambda(1)}\left(\frac{\lambda(1)^{a-1}}{a!}(a-\lambda(1)) + \frac{\lambda(1)^{b-1}}{b!}(\lambda(1)-b)\right) + R, \quad (4)$$

where $a = \lfloor\lambda(1) + 1/2 + \sqrt{\lambda(1)+1/4}\rfloor$, $b = \lfloor\lambda(1) + 1/2 - \sqrt{\lambda(1)+1/4}\rfloor$ and $|R| \leq \sqrt{2}\theta(1)^{3/2}/(1-\sqrt{2\theta(1)})$, for $\theta(1) < 1/2$ ($\lfloor x\rfloor \in \mathbf{Z}$ being defined by $x - 1 < \lfloor x\rfloor \leq x$, $x \in \mathbf{R}$).

Other publications, concerning the multivariate problem, are: Ahmad (1985), Arenbaev (1976), Barbour, Holst, and Janson (1992), Chen and M. Roos (1995), McDonald (1980), Sintes Blanc (1991), Wang (1986, 1989) and Witte (1993).

The univariate setting [i.e., the case $k = 1$] was investigated by many authors [for instance, see Barbour and Hall (1984), Barbour, Holst, and Janson (1992), Borovkov (1988), Chen (1975), Deheuvels and Pfeifer (1986a, b, 1988b), Kerstan (1964), Kruopis (1986), Le Cam (1960), Presman (1985), B. Roos (1996a, b), Serfling (1975), Shorgin (1977).]

Various metrics were considered in the univariate setting [for instance, the total variation, Kolmogorov metric, Fortet–Mourier metric, point metric, and l^p metric between distribution functions], while in the multivariate case, the total variation and the Kolmogorov metric [for this, see Sintes Blanc (1991)] were treated. Here, the point metric has not been considered before.

The aim of this paper is to introduce the multivariate Charlier B expansion to the given problem. The text refers to the works of Shorgin (1977), Deheuvels and Pfeifer (1988b), and B. Roos (1996a, b), concerning the univariate Poisson approximation with the help of the univariate Charlier B expansion. In Section 2, the multivariate Charlier B expansion of P^{S_n} is presented. Formulae and an estimate for the corresponding Charlier coefficients are derived. Section 3 is devoted to the results. Bounds for the distances between P^{S_n} and Poisson related signed measures in the total variation and point metric are given. As one of the most interesting results, it is proven that $d_\tau(P^{S_n}, \mathcal{P}(\lambda))$ is of order $\mathcal{O}([\sum_{r=1}^{k}\sqrt{\min\{\theta(r), \lambda_2(r)\}}]^2)$ [see Corollary 1].

Note that the Kolmogorov metric could also have been considered in this paper; but the resulting bounds would have been of the same order as for the total variation and are therefore omitted.

2. The multivariate Charlier B expansion. Let $\pi(m, x) = e^{-x}x^m/m!$ for $m \in \mathbf{Z}_+$, $x \in [0, \infty)$. For $f : \mathbf{Z}_+ \longrightarrow \mathbf{R}$, let $\Delta f : \mathbf{Z}_+ \longrightarrow \mathbf{R}$ be defined by $(\Delta f)(m) = f(m-1) - f(m)$, $m \in \mathbf{N}$ and $(\Delta f)(0) = -f(0)$. Further, let $\Delta^0 f = f$ and $\Delta^l f = \Delta(\Delta^{l-1} f)$ for

$l \in \mathbf{N}$. Finally, let $\Delta^l \pi(m, x) := \Delta^l(\pi(\cdot, x))(m)$ for $m, l \in \mathbf{Z}_+$, $x \in [0, \infty)$. It is easy to show that

$$\sum_{m=0}^{\infty} \Delta^l \pi(m, x) z^m = (z-1)^l \exp(x(z-1)), \tag{5}$$

for $l \in \mathbf{Z}_+$, $x \in [0, \infty)$, $z \in \mathbf{C}$, where $\mathbf{C}$ denotes the set of complex numbers. The following lemma gives the main argument of this paper.

Lemma 1 *For $m = (m_1, \ldots, m_k) \in \mathbf{Z}_+^k$, $t = (t_1, \ldots, t_k) \in (0, \infty)^k$,*

$$P(S_n = m) = \sum_{l_1=0}^{\infty} \cdots \sum_{l_k=0}^{\infty} a_l(t) \prod_{r=1}^{k} \Delta^{l_r} \pi(m_r, t_r), \tag{6}$$

where l stands for $(l_1, \ldots, l_k)$ and the coefficients $a_l(t)$ are defined by

$$\sum_{l_1=0}^{\infty} \cdots \sum_{l_k=0}^{\infty} a_l(t) z_1^{l_1} \ldots z_k^{l_k} = \exp\left(-\sum_{r=1}^{k} t_r z_r\right) \prod_{j=1}^{n} \left(1 + \sum_{r=1}^{k} p_{j,r} z_r\right). \tag{7}$$

Proof. Let $\varphi_{S_n} : \mathbf{C}^k \longrightarrow \mathbf{C}$ denote the probability generating function of S_n, defined by

$$\varphi_{S_n}(z) = \sum_{m_1=0}^{\infty} \cdots \sum_{m_k=0}^{\infty} P(S_n = (m_1, \ldots, m_k)) z_1^{m_1} \ldots z_k^{m_k} = \prod_{j=1}^{n} \left(1 + \sum_{r=1}^{k} p_{j,r}(z_r - 1)\right),$$

for $z = (z_1, \ldots, z_k) \in \mathbf{C}^k$. By the help of (7), this yields, in case of $|z_r| < 1$ for $r \in \{1, \ldots, k\}$,

$$\begin{aligned}
\varphi_{S_n}(z) &= \sum_{l_1=0}^{\infty} \cdots \sum_{l_k=0}^{\infty} a_l(t) \left[\prod_{r=1}^{k} (z_r - 1)^{l_r}\right] \exp\left(\sum_{r=1}^{k} t_r (z_r - 1)\right) \\
&= \sum_{l_1=0}^{\infty} \cdots \sum_{l_k=0}^{\infty} a_l(t) \prod_{r=1}^{k} \left[\sum_{m_r=0}^{\infty} \Delta^{l_r} \pi(m_r, t_r) z_r^{m_r}\right] \\
&= \sum_{m_1=0}^{\infty} \cdots \sum_{m_k=0}^{\infty} \left[\sum_{l_1=0}^{\infty} \cdots \sum_{l_k=0}^{\infty} a_l(t) \prod_{r=1}^{k} \Delta^{l_r} \pi(m_r, t_r)\right] z_1^{m_1} \ldots z_k^{m_k}.
\end{aligned}$$

Here, summations may be interchanged, because $|\Delta^l \pi(m, x)| \leq 2^l$ holds for all $l, m \in \mathbf{Z}_+$, $x \in [0, \infty)$, and therefore the iterated series converge absolutely. By comparing the power series, the assertion follows. ∎

By similar considerations as above, one shows that the iterated series (6) converge absolutely, so that, here, the order of summation is not relevant. The series (6) is called Charlier B expansion of P^{S_n}. The coefficients $a_l(t)$ are called Charlier coefficients of P^{S_n}. In order to derive upper bounds for the total variation and point metric, an inequality for the Charlier coefficients of P^{S_n} is needed.

In what follows, we use the notation

$$s = \sum_{r=1}^{k} l_r, \qquad \text{for } (l_1, \ldots, l_k) \in \mathbf{Z}_+^k$$

and

$$\eta(r, t_r) = 2\lambda_2(r) + (\lambda(r) - t_r)^2$$

for $r \in \{1, \ldots, k\}$, $(t_1 \ldots, t_k) \in (0, \infty)^k$.

Lemma 2 *Let* $l = (l_1, \ldots, l_k) \in \mathbf{Z}_+^k$, $s \geq 1$, $t = (t_1, \ldots, t_k) \in (0, \infty)^k$. *Further, let* $I_0(x) = \sum_{j=0}^{\infty} (x/2)^{2j}/(j!)^2$ *be the modified Bessel function of the first kind and order* 0 *and* $\beta(x) = I_0(x) e^{-x^2/4}$, $x \in \mathbf{R}$. *Then*

$$|a_l(t)| \leq \prod_{r=1}^{k} \left[\frac{(\eta(r, t_r) s e)^{l_r/2}}{2^{l_r/2} l_r^{l_r}} \beta\left(\frac{\sqrt{2}|\lambda(r) - t_r| l_r}{\sqrt{s\eta(r, t_r)}} \right) \right]. \tag{8}$$

Proof. Let $R_1, \ldots, R_k \in (0, \infty)$. By Cauchy's theorem,

$$a_l(t) = \frac{1}{(2\pi)^k R_1^{l_1} \ldots R_k^{l_k}} \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} h(R_1 e^{ix_1}, \ldots, R_k e^{ix_k}) \exp\left(-i \sum_{r=1}^{k} l_r x_r \right) dx_1 \ldots dx_k, \tag{9}$$

where $h(z) = \varphi_{S_n}(z_1 + 1, \ldots, z_k + 1) \exp(-\sum_{r=1}^{k} t_r z_r)$, $z = (z_1, \ldots, z_k) \in \mathbf{C}^k$. Because $\cos(x_1)\cos(x_2) + \sin(x_1)\sin(x_2) = \cos(x_1 - x_2)$, $x_1, x_2 \in \mathbf{R}$ and $1 + x \leq e^x$, $x \in \mathbf{R}$,

$$\begin{aligned} |h(R_1 e^{ix_1}, \ldots, R_k e^{ix_k})| &\leq \prod_{j=1}^{n} \left| 1 + \sum_{r=1}^{k} p_{j,r} R_r e^{ix_r} \right| \left| \exp\left(-\sum_{r=1}^{k} t_r R_r e^{ix_r} \right) \right| \\ &= \prod_{j=1}^{n} \left[1 + 2\sum_{r=1}^{k} p_{j,r} R_r \cos(x_r) + \sum_{r_1=1}^{k} \sum_{r_2=1}^{k} p_{j,r_1} p_{j,r_2} R_{r_1} R_{r_2} \cos(x_{r_1} - x_{r_2}) \right]^{1/2} \\ &\quad \times \exp\left(-\sum_{r=1}^{k} t_r R_r \cos(x_r) \right) \\ &\leq \exp\left(\sum_{r=1}^{k} (\lambda(r) - t_r) R_r \cos(x_r) + \frac{1}{2} \sum_{j=1}^{n} \left[\sum_{r=1}^{k} p_{j,r} R_r \right]^2 \right). \end{aligned}$$

Since $I_0(x) = \frac{1}{\pi} \int_0^{\pi} \exp(x \cos(y))\, dy$, $x \in \mathbf{R}$, this leads to

$$|a_l(t)| \leq \frac{1}{R_1^{l_1} \ldots R_k^{l_k}} \exp\left(\frac{1}{2} \sum_{j=1}^{n} \left[\sum_{r=1}^{k} p_{j,r} R_r \right]^2 + \frac{1}{4} \sum_{r=1}^{k} (\lambda(r) - t_r)^2 R_r^2 \right) \prod_{r=1}^{k} \beta((\lambda(r) - t_r) R_r).$$

Using Cauchy's inequality, the following estimate is valid:

$$\sum_{j=1}^{n} \left[\sum_{r=1}^{k} p_{j,r} R_r \right]^2 = \sum_{r_1=1}^{k} \sum_{r_2=1}^{k} \left(\sum_{j=1}^{n} p_{j,r_1} p_{j,r_2} \right) R_{r_1} R_{r_2} \leq \left(\sum_{r=1}^{k} \sqrt{\lambda_2(r)} R_r \right)^2.$$

Hence, $|a_l(t)| \leq g(R_1, \ldots, R_k) \prod_{r=1}^{k} \beta((\lambda(r) - t_r) R_r)$, where $g : (0, \infty)^k \longrightarrow \mathbf{R}$, defined by

$$g(R_1, \ldots, R_k) = \frac{1}{R_1^{l_1} \ldots R_k^{l_k}} \exp\left(\frac{1}{2} \left[\sum_{r=1}^{k} \sqrt{\frac{\eta(r, t_r)}{2}} R_r \right]^2 \right),$$

attains its minimum for $\tilde{R} = (\tilde{R}_1, \ldots, \tilde{R}_k)$, $\tilde{R}_r = \sqrt{2}l_r/\sqrt{s\eta(r,t_r)}$, $r \in \{1,\ldots,k\}$. Substituting $\tilde{R}$ for $(R_1, \ldots, R_k)$, the desired result is shown. ∎

Note that $0 < \beta(x) = \beta(|x|) \leq 1$, $x \in \mathbf{R}$. The next lemma gives a recursive formula for the Charlier coefficients of P^{S_n}. Lemmata 2 and 3 are multivariate generalizations of results of B. Roos (1996a, b) [for $k = 1$ and arbitrary t] and Shorgin (1977) [for $k = 1$ and $t = \lambda$].

Lemma 3 *Let $b \in \{1, \ldots, k\}$, $l = (l_1, \ldots, l_k) \in \mathbf{Z}_+^k$, $l_b \geq 1$, $t = (t_1, \ldots, t_k) \in (0, \infty)^k$. Then*

$$a_l(t) = -\frac{t_b}{l_b} a_{l-e_b}(t) - \frac{1}{l_b} \sum_{j \in A} \frac{l_b - j_b}{s - J} \binom{s - J}{l_1 - j_1, \ldots, l_k - j_k} (-1)^{s+J} a_j(t) \left[\sum_{i=1}^{n} \prod_{r=1}^{k} p_{i,r}^{l_r - j_r} \right], \tag{10}$$

where A denotes the set of all $j = (j_1, \ldots, j_k) \in \mathbf{Z}_+^k$ such that $0 \leq j_r \leq l_r$ for $r \in \{1, \ldots, k\} \setminus \{b\}$ and $0 \leq j_b \leq l_b - 1$, and further J is the sum $\sum_{r=1}^{k} j_r$.

Proof. It suffices to prove the assertion for $b = 1$. Let $z = (z_1, \ldots, z_k) \in (-1, 1)^k$, $g(z) = \exp(f(z))$, $f(z) = -\sum_{r=1}^{k} t_r z_r + \sum_{j=1}^{n} \ln(1 + \sum_{r=1}^{k} p_{j,r} z_r)$. Then

$$\begin{aligned} \frac{\partial^{l_1+\ldots+l_k}}{\partial z_1^{l_1} \ldots \partial z_k^{l_k}} g(z) &= \sum_{j_1=0}^{l_1-1} \binom{l_1 - 1}{j_1} \frac{\partial^{l_2+\ldots+l_k}}{\partial z_2^{l_2} \ldots \partial z_k^{l_k}} \left[\frac{\partial^{j_1}}{\partial z_1^{j_1}} g(z) \frac{\partial^{l_1-j_1}}{\partial z_1^{l_1-j_1}} f(z) \right] \\ &= \sum_{j_1=0}^{l_1-1} \sum_{j_2=0}^{l_2} \cdots \sum_{j_k=0}^{l_k} \binom{l_1 - 1}{j_1} \binom{l_2}{j_2} \cdots \binom{l_k}{j_k} \\ &\quad \times \frac{\partial^{j_1+\ldots+j_k}}{\partial z_1^{j_1} \ldots \partial z_k^{j_k}} g(z) \frac{\partial^{(l_1-j_1)+\ldots+(l_k-j_k)}}{\partial z_1^{l_1-j_1} \ldots \partial z_k^{l_k-j_k}} f(z). \end{aligned}$$

Hence,

$$\begin{aligned} a_l(t) &= \frac{1}{\prod_{r=1}^{k} (l_r!)} \left. \frac{\partial^{l_1+\ldots+l_k}}{\partial z_1^{l_1} \ldots \partial z_k^{l_k}} g(z) \right|_{z=0} \\ &= \frac{1}{l_1} \sum_{j_1=0}^{l_1-1} \sum_{j_2=0}^{l_2} \cdots \sum_{j_k=0}^{l_k} \frac{a_j(t)}{(l_1 - 1 - j_1)!(l_2 - j_2)! \ldots (l_k - j_k)!} \left. \frac{\partial^{(l_1-j_1)+\ldots+(l_k-j_k)}}{\partial z_1^{l_1-j_1} \ldots \partial z_k^{l_k-j_k}} f(z) \right|_{z=0}. \end{aligned}$$

Since

$$\left. \frac{\partial^{l_1+\ldots+l_k}}{\partial z_1^{l_1} \ldots \partial z_k^{l_k}} f(z) \right|_{z=0} = \begin{cases} \lambda(c) - t_c & \text{if } (l_1, \ldots, l_k) = e_c,\ c \in \{1, \ldots, k\} \\ \sum_{j=1}^{n} \left(\prod_{r=1}^{k} p_{j,r}^{l_r} \right) (s-1)!(-1)^{s-1} & \text{otherwise,} \end{cases}$$

for $(l_1, \ldots, l_k) \in \mathbf{Z}_+^k$, $s \geq 1$, the assertion follows immediately. ∎

It is clear that $a_0(t) = 1$ and, as the preceeding lemma shows, for $r, v \in \{1, \ldots, k\}$, $r \neq v$, $t = (t_1, \ldots, t_k) \in (0, \infty)^k$,

$$a_{e_r}(t) = \lambda(r) - t_r, \tag{11}$$

$$a_{2e_r}(t) = \frac{1}{2} \left((\lambda(r) - t_r)^2 - \lambda_2(r) \right), \tag{12}$$

$$a_{e_r+e_v}(t) = (\lambda(r) - t_r)(\lambda(v) - t_v) - \sum_{j=1}^{n} p_{j,r} p_{j,v}. \tag{13}$$

3. Main results. For the first main result of this paper, the next technical lemma is necessary.

Lemma 4 *Let* $k \in \mathbf{N}$, $l_1, \ldots, l_k \in \mathbf{Z}_+$. *Then*

$$\frac{s^s}{\prod_{r=1}^k l_r^{l_r}} \leq \left(\frac{s!}{\prod_{r=1}^k (l_r!)}\right)^2 = \binom{s}{l_1, \ldots, l_k}^2. \tag{14}$$

Proof. It suffices to assume that $l_1, \ldots, l_k \in \mathbf{N}$, $k \in \{2, 3, \ldots\}$. Then

$$\frac{s^s}{\prod_{r=1}^k l_r^{l_r}} = \prod_{r=1}^{k-1} \frac{u_{r+1}^{u_{r+1}}}{u_r^{u_r}\, l_{r+1}^{l_{r+1}}},$$

where $u_r = \sum_{m=1}^r l_m$ for $r \in \{1, \ldots, k\}$. For $a \in \mathbf{N}$, $b \in \{1, \ldots, a-1\}$,

$$\frac{a^a}{b^b (a-b)^{a-b}} = \left(\frac{a}{b}\right)^b \left(\frac{a}{a-b}\right)^{a-b} \leq \left[\prod_{m=1}^b \frac{a-m+1}{b-m+1}\right] \left[\prod_{m=1}^{a-b} \frac{a-m+1}{a-b-m+1}\right] = \binom{a}{b}^2.$$

The proof is easily completed. ■

For $f : \mathbf{Z}_+ \longrightarrow \mathbf{R}$, let $\|f\|_\infty = \sup_{m \in \mathbf{Z}_+} |f(m)|$ and $\|f\|_1 = \sum_{m=0}^\infty |f(m)|$.

Theorem 1 *Let* $t = (t_1, \ldots, t_k) \in (0, \infty)^k$ *and*

$$\gamma(r, t_r) = \eta(r, t_r) \min\{1/(2t_r),\, e\}$$

for $r \in \{1, \ldots, k\}$. *If* $\sum_{r=1}^k \sqrt{2\gamma(r, t_r)} < 1$ *then*

$$d_\tau(P^{S_n}, \mathcal{P}(t)) \leq \sum_{r=1}^k |\lambda(r) - t_r| \min\{(2t_r e)^{-1/2}, 1\} + \frac{\left(\sum_{r=1}^k \sqrt{\gamma(r, t_r)}\right)^2}{1 - \sum_{r=1}^k \sqrt{2\gamma(r, t_r)}}. \tag{15}$$

If $Q(u, t)$ *denotes the finite signed measure concentrated on* $\mathbf{Z}_+^k$ *with counting density*

$$Q(u, t)(\{m\}) = \sum_{s=0}^u \sum_{l \in A_s} \left[a_l(t) \prod_{r=1}^k \Delta^{l_r} \pi(m_r, t_r)\right], \tag{16}$$

for $m = (m_1, \ldots, m_k) \in \mathbf{Z}_+^k$, *where* $u \in \mathbf{N}$ *and* $A_s = \{(l_1, \ldots, l_k) \in \mathbf{Z}_+^k \,|\, \sum_{r=1}^k l_r = s\}$ *for* $s \in \mathbf{Z}_+$, *then, in the case* $\sum_{r=1}^k \sqrt{2\gamma(r, t_r)} < 1$,

$$d_\tau(P^{S_n}, Q(u, t)) \leq 2^{(u-1)/2} \frac{\left(\sum_{r=1}^k \sqrt{\gamma(r, t_r)}\right)^{u+1}}{1 - \sum_{r=1}^k \sqrt{2\gamma(r, t_r)}}. \tag{17}$$

Proof. Let $T = \sum_{r=1}^k |\lambda(r) - t_r| \min\{(2t_r e)^{-1/2}, 1\}$. By the use of $\|\Delta^b \pi(\cdot, x)\|_1 \leq \min\{[2b/(xe)]^{b/2}, 2^b\}$, $x \in (0, \infty)$, $b \in \mathbf{Z}_+$ [see Deheuvels and Pfeifer (1988b)] in addition to (6), (8), (11) and (14), the inequality (15) is shown as follows. Assume that

$\sum_{r=1}^k \sqrt{2\gamma(r,t_r)} < 1$, then

$$\begin{aligned} d_\tau(P^{S_n}, \mathcal{P}(t)) &\le \frac{1}{2} \sum_{m \in \mathbf{Z}_+^k} \left| \sum_{l \in \mathbf{Z}_+^k \setminus A_0} a_l(t) \prod_{r=1}^k \Delta^{l_r} \pi(m_r, t_r) \right| \\ &\le \frac{1}{2} \sum_{l \in \mathbf{Z}_+^k \setminus A_0} |a_l(t)| \prod_{r=1}^k \|\Delta^{l_r}\pi(\cdot, t_r)\|_1 \\ &\le T + \frac{1}{2}\sum_{s=2}^\infty \sum_{l \in A_s} \prod_{r=1}^k \left[\frac{(\eta(r,t_r)se)^{l_r/2}}{2^{l_r/2} l_r^{l_r}} \min\left\{ \left(\frac{2l_r}{t_r e}\right)^{l_r/2}, 2^{l_r} \right\} \right] \\ &\le T + \frac{1}{2}\sum_{s=2}^\infty \sum_{l \in A_s} \binom{s}{l_1, \ldots, l_k} \prod_{r=1}^k (2\gamma(r,t_r))^{l_r/2} = T + \frac{\left(\sum_{r=1}^k \sqrt{\gamma(r,t_r)}\right)^2}{1 - \sum_{r=1}^k \sqrt{2\gamma(r,t_r)}}. \end{aligned}$$

The rest of the assertion is proven analogously. ∎

It is easy to verify that $Q(1,\lambda) = \mathcal{P}(\lambda)$ and, for $m = (m_1, \ldots, m_k) \in \mathbf{Z}_+^k$, $t = (t_1, \ldots, t_k) \in (0,\infty)^k$,

$$Q(1,t)(\{m\}) = \left[\prod_{r=1}^k \pi(m_r, t_r)\right] \left[1 + \sum_{r=1}^k \frac{1}{t_r}(\lambda(r) - t_r)(m_r - t_r)\right], \tag{18}$$

$$\begin{aligned} Q(2,t)(\{m\}) &= \left[\prod_{r=1}^k \pi(m_r, t_r)\right] \left[1 + \sum_{r=1}^k \frac{1}{t_r}(\lambda(r) - t_r)(m_r - t_r)\right. \\ &\quad + \frac{1}{2}\left(\sum_{r=1}^k \frac{1}{t_r}(\lambda(r) - t_r)(m_r - t_r)\right)^2 - \frac{1}{2}\sum_{j=1}^n \left(\sum_{r=1}^k \frac{p_{j,r}(m_r - t_r)}{t_r}\right)^2 \\ &\quad \left. - \frac{1}{2}\sum_{r=1}^k \frac{m_r}{t_r^2}[(\lambda(r) - t_r)^2 - \lambda_2(r)]\right]. \end{aligned} \tag{19}$$

As a consequence of Theorem 1,

$$d_\tau(P^{S_n}, \mathcal{P}(\lambda)) \le \frac{\left(\sum_{r=1}^k \sqrt{\gamma(r)}\right)^2}{1 - \sum_{r=1}^k \sqrt{2\gamma(r)}} \qquad \text{if } \sum_{r=1}^k \sqrt{2\gamma(r)} < 1, \tag{20}$$

where $\gamma(r) = \gamma(r, \lambda(r))$, $r \in \{1, \ldots, k\}$. The following corollary shows that the singularity in (20) can be removed.

Corollary 1 *Let $\delta(r) = \min\{\theta(r), \lambda_2(r)\}$, $r \in \{1, \ldots, k\}$. Then*

$$\frac{1}{32}\max_{1 \le r \le k} \delta(r) \le d_\tau(P^{S_n}, \mathcal{P}(\lambda)) \le \frac{1}{2-\sqrt{3}}\left(\sum_{r=1}^k \sqrt{\gamma(r)}\right)^2 \le \frac{2ke}{2-\sqrt{3}}\sum_{r=1}^k \delta(r). \tag{21}$$

Proof. The first inequality is shown as follows. For $r \in \{1, \ldots, k\}$,

$$\begin{aligned} d_\tau(P^{S_n}, \mathcal{P}(\lambda)) &\ge \sup_{A \subseteq \mathbf{Z}_+} |P(S_n \in \mathbf{Z}_+^{r-1} \times A \times \mathbf{Z}_+^{k-r}) - \mathcal{P}(\lambda)(\mathbf{Z}_+^{r-1} \times A \times \mathbf{Z}_+^{k-r})| \\ &= d_\tau(\mathcal{PB}(n; p_{1,r}, \ldots, p_{n,r}), \mathcal{P}(\lambda(r))), \end{aligned}$$

where $\mathcal{PB}(n; p_{1,r}, \ldots, p_{n,r})$ is the Poisson binomial distribution with parameters $n \in \mathbf{N}$, $p_{1,r}, \ldots, p_{n,r} \in [0,1]$, i.e. the distribution of the sum of n independent random variables $Y_1, \ldots, Y_n$ with $P(Y_j = 1) = 1 - P(Y_j = 0) = p_{j,r}$, $j \in \{1, \ldots, n\}$. Hence, using a lower bound for the latter term obtaind by Barbour and Hall (1984),

$$d_\tau(P^{S_n}, \mathcal{P}(\lambda)) \geq \frac{1}{32} \max_{1 \leq r \leq k} \delta(r).$$

For the second inequality, it suffices to assume that $x := \sum_{r=1}^k \sqrt{2\gamma(r)} < 1$. But in this case, $d_\tau(P^{S_n}, \mathcal{P}(\lambda)) \leq \min\{1, f(x)\} \leq x^2/(2(2-\sqrt{3}))$, where $f(y) = y^2/(2(1-y))$, $y \in [0,1)$. By application of Cauchy's inequality, the third inequality is shown. ■

By Corollary 1 and

$$\frac{1}{k}\left(\sum_{r=1}^k \theta(r)\right)^2 \leq \sum_{r=1}^k \theta(r)^2 \leq \sum_{r=1}^k \delta(r) \leq \sum_{r=1}^k \theta(r),$$

it follows that, for fixed or bounded k, $d_\tau(P^{S_n}, \mathcal{P}(\lambda))$ tends to zero if and only if $\sum_{r=1}^k \theta(r)$ tends to zero.

In the general case, one can not remove the factor k from the right hand side of (21): Generally, an inequality of type $d_\tau(P^{S_n}, \mathcal{P}(\lambda)) \leq M k^\alpha \sum_{r=1}^k \delta(r)$ with absolute constants $M \in (0, \infty)$ and $\alpha \in [0,1)$ cannot hold. In order to verify this assertion, let, for example, $p_{j,r} = 1/(kn)$ for all $j \in \{1, \ldots, n\}$, $r \in \{1, \ldots, k\}$. Using an identity by Deheuvels and Pfeifer (1988a, Lemma 5.1) in addition to an asymptotic result by Deheuvels and Pfeifer (1986b), $d_\tau(P^{S_n}, \mathcal{P}(\lambda)) = d_\tau(\mathcal{B}(n, 1/n), \mathcal{P}(1)) \sim 3/(4en)$, $n \to \infty$. Under these assumptions, Corollary 1 leads to $d_\tau(P^{S_n}, \mathcal{P}(\lambda)) \leq 2e/((2-\sqrt{3})n)$.

Note that Corollary 1 shows that $d_\tau(P^{S_n}, \mathcal{P}(\lambda)) \leq c \sum_{r=1}^k \theta(r)$, with $c \leq (2ke)/(2 - \sqrt{3})$, whereas Barbour's result (3) implies $c \leq 1/2 + \log^+(2 \sum_{r=1}^k \lambda(r))$.

The next result concerns the point metric, for which the following lemma is needed.

Lemma 5 *The following inequalities are valid:*

$$\|\Delta^l \pi(\cdot, x)\|_\infty \leq \min\left\{c[l/(xe)]^{(l+1)/2}, 2^l\right\}, \qquad l \in \mathbf{N},\ x \in (0, \infty), \tag{22}$$

where $c = \frac{\sqrt{e}}{2}(1 + \sqrt{\pi/2})$,

$$\|\Delta^0 \pi(\cdot, x)\|_\infty \leq \min\{(2xe)^{-1/2}, 1\}, \qquad x \in (0, \infty). \tag{23}$$

Proof. First, note that $\|\Delta^l \pi(\cdot, x)\|_\infty \leq \|\Delta^l \pi(\cdot, x)\|_1 \leq 2^l$, for $l \in \mathbf{Z}_+$, $x \in [0, \infty)$. The rest of (22) and (23) are results of Shorgin (1977) and Deheuvels and Pfeifer (1988b), respectively. ■

Note that a result by B. Roos (1996a, Satz 6.31 [being published in a subsequent paper]) asserts that (22) remains valid if c is replaced by $\frac{\sqrt{e}}{2}(1 + \sqrt{\pi/8})$.

Theorem 2 *Let* $t = (t_1, \dots, t_k) \in (0, \infty)^k$. *Let* c *be a constant satisfying inequality* (22) *and* $c_0 = (2e)^{-1/2} \max\{c, 1\}$. *Finally, let*

$$H = \sum_{r=1}^{k} |\lambda(r) - t_r| \min\{c/(t_r e), 2\} \prod_{\substack{v=1\\v\neq r}}^{k} \min\{(2t_v e)^{-1/2}, 1\}.$$

If $\sum_{r=1}^k \sqrt{2\gamma(r, t_r)} < 1$ *then*

$$d_\pi(P^{S_n}, \mathcal{P}(t)) \leq H + 2c_0^k \left[\prod_{r=1}^{k} \sqrt{\frac{2\gamma(r,t_r)}{\eta(r,t_r)}}\right] \frac{\left(\sum_{r=1}^k \sqrt{\gamma(r,t_r)}\right)^2}{1 - \sum_{r=1}^k \sqrt{2\gamma(r,t_r)}}. \tag{24}$$

If $Q(u, t)$ *denotes the finite signed measure with counting density* (16), *for* $u \in \mathbf{N}$, *then, under assumption of* $\sum_{r=1}^k \sqrt{2\gamma(r, t_r)} < 1$,

$$d_\pi(P^{S_n}, Q(u,t)) \leq 2^{(u+1)/2} c_0^k \left[\prod_{r=1}^{k} \sqrt{\frac{2\gamma(r,t_r)}{\eta(r,t_r)}}\right] \frac{\left(\sum_{r=1}^k \sqrt{\gamma(r,t_r)}\right)^{u+1}}{1 - \sum_{r=1}^k \sqrt{2\gamma(r,t_r)}}. \tag{25}$$

Proof. Let A_s be defined as in Theorem 1 and assume that $\sum_{r=1}^k \sqrt{2\gamma(r,t_r)} < 1$. Using (6), (11), (22), (23), (8), (14) and $b \leq 2^{b-1}$, $b \in \mathbf{N}$, the assertion follows as in the proof of Theorem 1:

$$\begin{aligned}
d_\pi(P^{S_n}, \mathcal{P}(t)) &\leq H + \sum_{s=2}^{\infty} \sum_{l \in A_s} \prod_{r=1}^{k} \left[\frac{(\eta(r,t_r)se)^{l_r/2}}{2^{l_r/2} l_r^{l_r}} \|\Delta^{l_r} \pi(\cdot, t_r)\|_\infty\right] \\
&\leq H + \sum_{s=2}^{\infty} \sum_{l \in A_s} \prod_{r=1}^{k} \left[\frac{s^{l_r/2}}{l_r^{l_r/2}} c_0 \sqrt{\frac{2\gamma(r,t_r)}{\eta(r,t_r)}} (2\gamma(r,t_r))^{l_r/2}\right] \\
&\leq H + 2c_0^k \left[\prod_{r=1}^{k} \sqrt{\frac{2\gamma(r,t_r)}{\eta(r,t_r)}}\right] \frac{\left(\sum_{r=1}^k \sqrt{\gamma(r,t_r)}\right)^2}{1 - \sum_{r=1}^k \sqrt{2\gamma(r,t_r)}}.
\end{aligned}$$

The rest of the assertion is proven analogously. ■

Theorem 2 yields

$$d_\pi(P^{S_n}, \mathcal{P}(\lambda)) \leq 2c_0^k \left[\prod_{r=1}^{k} \sqrt{\frac{\gamma(r)}{\lambda_2(r)}}\right] \frac{\left(\sum_{r=1}^k \sqrt{\gamma(r)}\right)^2}{1 - \sum_{r=1}^k \sqrt{2\gamma(r)}} \qquad \text{if } \sum_{r=1}^{k} \sqrt{2\gamma(r)} < 1, \tag{26}$$

where c_0 is defined as in Theorem 2. One can assume that $c_0 < 1$.

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