Multinomial and Krawtchouk approximations to the generalized multinomial distribution

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Abstract. The generalized multinomial distribution is approximated by multinomial distributions and also by finite signed measures resulting from the corresponding multivariate Krawtchouk expansion. Bounds for the total variation norm and the ℓ_∞–norm are presented. The method used is a multivariate extension of that in Roos [6], although additional complications occur.

1 Introduction

1.1 The problem. Let S_n be the sum of independent Bernoulli random vectors $X_1, \ldots, X_n$ in $\mathbf{R}^k$ with probabilities

$$P(X_j = e_r) = p_{j,r}, \qquad P(X_j = 0) = 1 - \sum_{r=1}^{k} p_{j,r} = p_{j,0},$$

for $j \in \{1, \ldots, n\}$ and $r \in \{1, \ldots, k\}$, where $e_r \in \mathbf{R}^k$ is the vector with entry 1 at position r and 0 otherwise.

In this paper, we consider the approximation of the generalized multinomial distribution P^{S_n} by multinomial distributions and also by finite signed measures resulting from the corresponding multivariate Krawtchouk expansion. Such approximations of P^{S_n} are useful, for example, in case of $X_1, \ldots, X_n$ being nearly identically distributed. The main task is to give some bounds for the approximation errors with respect to the total variation norm and the ℓ_∞–norm.

Using Stein's method, Loh [5, Theorem 5] has recently provided a somewhat complicated upper bound for the total variation distance between P^{S_n} and a multinomial distribution. In the univariate case $k = 1$, the problem reduces to the binomial approximation of the Poisson binomial distribution. It follows from Ehm's [3] results (see (1) below) that, in this case, Loh's [5, Corollary 3] bound has not the correct order.

The method of the present paper is a multivariate extension of that in Roos [6]. Further, some ideas of Shorgin [9] and Roos [7] are used. We have to deal with the multivariate generalizations of the Krawtchouk polynomials. A complication is that these are, contrary to the univariate case $k = 1$, no longer orthogonal with respect to the multinomial distribution [see (4),

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(5), and the remarks thereafter]. For the binomial approximation in the univariate case, our upper bound with respect to the total variation norm has the correct order [see Corollary 1 and formula (1)].

For Poisson approximations of P^{S_n} see, for example, Barbour [1], Deheuvels and Pfeifer [2], Roos [7, 8], and the references therein.

1.2 Further notations. We always assume that $m \in \mathbf{Z}_+$, $u, \tilde{u}, v, \tilde{v}, w \in \mathbf{Z}_+^k$, and $y, z \in \mathbf{C}^k$. We use the standard multi–index notation: For $z = (z_1, \dots, z_k)$ and $u = (u_1, \dots, u_k)$, we set $z^u = z_1^{u_1} \dots z_k^{u_k}$, $|u| = \sum_{r=1}^k u_r$, and $u! = u_1! \dots u_k!$. We always let $0^0 = 1$. We write $u \leq v$ in the case that $u_r \leq v_r$ for all r; if additionally at least one of these inequalities is strict, we write $u < v$. Similarly, $u \wedge v \in \mathbf{Z}_+^k$ is the vector with rth component $\min\{u_r, v_r\}$. All of our series and sums are carried out over subsets of $\mathbf{Z}_+^k$, unless otherwise specified. For example, the sign $\sum_{|u| \leq m}$ means the sum over all $u \in \mathbf{Z}_+^k$ with $|u| \leq m$.

Let $q = (q_1, \dots, q_k) \in (0,1)^k$ with $q_0 = 1 - \sum_{r=1}^k q_r \in (0,1)$, and, for $r \in \{0, \dots, k\}$,

$$\overline{p}_r = \frac{1}{n} \sum_{j=1}^n p_{j,r} \in (0,1), \qquad \overline{p} = (\overline{p}_1, \dots, \overline{p}_k), \qquad \gamma_m(r, q) = \sum_{j=1}^n (q_r - p_{j,r})^m.$$

Note that $\overline{p}_0 = 1 - \sum_{r=1}^k \overline{p}_r$ and that $\gamma_1(r, \overline{p}) = 0$ for all $r \in \{0, \dots, k\}$. The counting density $M(\cdot, m, q)$ of the multinomial distribution $\mathcal{M}(m, q)$ is given by

$$M(w, m, q) = \mathcal{M}(m, q)(\{w\}) = \begin{cases} \frac{m!}{w!\,(m-|w|)!}\, q^w\, q_0^{m-|w|} & \text{if } |w| \leq m, \\ 0 & \text{otherwise.} \end{cases}$$

In the case $k = 1$, $\mathcal{M}(m, q)$ is a binomial distribution, which is also denoted by $\mathcal{B}(m, q)$. For $f : \mathbf{Z}_+^k \longrightarrow \mathbf{R}$ and $r \in \{1, \dots, k\}$, let $\Delta_r f : \mathbf{Z}_+^k \longrightarrow \mathbf{R}$ with

$$(\Delta_r f)(w) = \begin{cases} f(w - e_r) - f(w) & \text{if } w_r \geq 1, \\ -f(w) & \text{if } w_r = 0. \end{cases}$$

Further, let $\Delta_r^0 f = f$ and $\Delta_r^m f = \Delta_r^{m-1}(\Delta_r f)$ for $m \in \mathbf{N}$. Clearly, $\Delta_r \Delta_s f = \Delta_s \Delta_r f$ for $r, s \in \{1, \dots, k\}$. Let $\Delta^u f = \Delta_1^{u_1} \dots \Delta_k^{u_k} f$. We set $\Delta^u M(w, m, q) = (\Delta^u M(\cdot, m, q))(w)$. Let $\|f\|_1 = \sum_{w \in \mathbf{Z}_+^k} |f(w)|$ and $\|f\|_\infty = \sup_{w \in \mathbf{Z}_+^k} |f(w)|$. For a finite signed measure Q concentrated on $\mathbf{Z}_+^k$ with counting density f_Q, let $\|Q\|_1 = \|f_Q\|_1$ and $\|Q\|_\infty = \|f_Q\|_\infty$. For $x \in \mathbf{C}$, let $\binom{x}{m} = \prod_{j=1}^m (x - j + 1)/j$.

1.3 Some known facts in the univariate case. In this subsection, let $k = 1$. Using the Stein–Chen method, Ehm [3, Theorem 1, Lemma 2] proved the following estimates:

$$\begin{aligned} \frac{\gamma_2(1, \overline{p})}{62} \min\left\{\frac{1}{n \overline{p}_1 \overline{p}_0}, 1\right\} &\leq \|P^{S_n} - \mathcal{B}(n, \overline{p}_1)\|_1 \\ &\leq 2\, \gamma_2(1, \overline{p}) \min\left\{\frac{1}{n \overline{p}_1 \overline{p}_0}, 1\right\}. \end{aligned} \qquad (1)$$

Hence the term on the right–hand side of (1) and $\|P^{S_n} - \mathcal{B}(n,\overline{p}_1)\|_1$ have the same order; further, this distance is small if and only if $\gamma_2(1,\overline{p})(n\overline{p}_1\overline{p}_0)^{-1}$ is small (see Ehm [3, Corollary 2]). A generalization of this fact for $k \in \mathbf{N}$ being fixed or bounded is shown in Corollary 2 of the present paper.

For further results on the univariate case, see Roos [6] and the references therein.

2 Main Results

2.1 The Krawtchouk expansion of P^{S_n}.

Theorem 1 *For arbitrary q,*

$$P(S_n = w) = \sum_{|u|\leq n} a_u(q)\, \Delta^u M(w, n-|u|, q), \tag{2}$$

where the coefficients $a_u(q)$ are defined by the relation

$$\sum_{|u|\leq n} a_u(q)\, z^u = \prod_{j=1}^{n}\Big(1 + \sum_{r=1}^{k}(p_{j,r} - q_r)\, z_r\Big). \tag{3}$$

In this paper, we call the right–hand side of (2) the Krawtchouk expansion of P^{S_n} with parameter q; further, we call the coefficients $a_u(q)$ the Krawtchouk coefficients of P^{S_n} with parameter q. For the approximation of P^{S_n}, we use the finite signed measures $\mathcal{M}_t(n,q)$, $(t \in \{0,\dots,n\})$ concentrated on $\mathbf{Z}_+^k$ with counting density

$$\mathcal{M}_t(n,q)(\{w\}) = \sum_{|u|\leq t} a_u(q)\, \Delta^u M(w, n-|u|, q).$$

Remarks. 1. For $t \geq 1$, $\mathcal{M}_t(n,q)$ depends not only on n and q but also on the $p_{j,r}$.

2. For all t, we have $\mathcal{M}_t(n,q)(\mathbf{Z}_+^k) = 1$ and $\mathcal{M}_t(n,q)(\{w\}) = 0$ in the case $|w| > n$.

The Krawtchouk expansion of P^{S_n} and its coefficients are associated with two sets $\mathcal{K}_{m,q} = \{K_u(z,m,q) \,|\, |u| \leq m\}$ and $\tilde{\mathcal{K}}_{m,q} = \{\tilde{K}_u(z,m,q) \,|\, |u| \leq m\}$ of multivariate Krawtchouk polynomials with variable z, which are, for $|u| \leq m$, defined by

$$K_u(z,m,q) = \sum_{v\leq u} \binom{m - \sum_{r=1}^{k} z_r}{|u-v|} \frac{|u-v|!\,(-q)^{u-v}\, q_0^{|v|}}{(u-v)!} \prod_{r=1}^{k}\binom{z_r}{v_r} \tag{4}$$

and

$$\tilde{K}_u(z,m,q) = \sum_{v\leq u} \frac{(m-|v|)!\,(-q)^{u-v}}{(u-v)!\,(m-|u|)!} \prod_{r=1}^{k}\binom{z_r}{v_r}. \tag{5}$$

Remarks. 1. Apparently Griffiths [4] is the first author who treated a multivariate generalization of the Krawtchouk polynomials, though he did not give an explicit formula comparable with (4) or (5). For $k = 2$, another version of the Krawtchouk expansion and the two kinds of Krawtchouk polynomials above can also be found in Takeuchi and Takemura [11, Section 3], who considered various approximations of the counting density and the distribution function of the sum of not necessarily independent Bernoulli random vectors. The multivariate Krawtchouk polynomials for arbitrary k were given by Tratnik [12, formulas (3.3) and (3.4)] by means of the generalized Kampé de Fériet hypergeometric series. However, we use our notations, since they seem to be more convenient for our purposes. Note that in formula (3.4) of Tratnik [12] the "x_1" must be replaced with "$-x_1$".

2. In the case that $k = 1$ and $u \le m$, $K_u(z, m, q)$ coincides with $\tilde{K}_u(z, m, q)$ and also with the classical univariate Krawtchouk polynomial of uth degree and parameters m and q (see (22), (23), and, for example, Szegö [10, formula (2.82.2)]).

3. Suppose $m \ge 1$.

(a) The elements of $\mathcal{K}_{m,q}$ constitute a system of orthogonal polynomials with respect to the multinomial distribution $\mathcal{M}(m, q)$ if and only if $k = 1$ [see (26)].

(b) The elements of $\mathcal{K}_{m,q}$ and $\tilde{\mathcal{K}}_{m,q}$ constitute a biorthogonal system with respect to the multinomial distribution $\mathcal{M}(m, q)$ [see Tratnik [12, formula (3.9)] or (27)].

4. There is a formula for $\Delta^u M(w, n - |u|, q)$ (resp. for $a_u(q)$), containing the term $K_u(w, n, q)$ (resp. the terms $\tilde{K}_u(w, n, q)$ for $|w| \le n$) [see (21) and (7)].

In the following proposition, we provide some alternative formulae for the Krawtchouk coefficients of P^{S_n} with parameter q. Let $\mu_{(u)}$ be the uth factorial moment of S_n, defined by

$$\mu_{(u)} = \sum_{w \in \mathbf{Z}_+^k} P(S_n = w) \prod_{r=1}^{k} \left[\binom{w_r}{u_r} u_r! \right]. \tag{6}$$

Proposition 1 *If $1 \le |u| \le n$ and $R \in (0, \infty)^k$, then*

$$\begin{aligned} a_u(q) &= \sum_{|w| \le n} P(S_n = w)\, \tilde{K}_u(w, n, q) && (7) \\ &= -\frac{1}{|u|} \sum_{v < u} \frac{|u - v|!}{(u - v)!} \Big(\sum_{j=1}^{n} \prod_{r=1}^{k} (q_r - p_{j,r})^{u_r - v_r} \Big) a_v(q) && (8) \\ &= \sum_{v \le u} \frac{(n - |v|)!\, (-q)^{u-v}\, \mu_{(v)}}{(u - v)!\, (n - |u|)!\, v!} && (9) \\ &= \frac{1}{(2\pi)^k R^u} \int_0^{2\pi} \cdots \int_0^{2\pi} \exp\Big(-i \sum_{r=1}^{k} u_r x_r \Big) \end{aligned}$$

$$\times \prod_{j=1}^{n} \Big(1+\sum_{r=1}^{k}(p_{j,r}-q_r)\, R_r e^{ix_r}\Big)\, dx_1 \dots dx_k. \tag{10}$$

Remark. We have $a_0(q)=1$ and, for $r,s\in\{1,\dots,k\}$ with $r\neq s$,

$$\begin{aligned} a_{e_r}(q) &= -\gamma_1(r,q), & (11)\\ a_{e_r+e_s}(q) &= \gamma_1(r,q)\,\gamma_1(s,q)-\sum_{j=1}^{n}(q_r-p_{j,r})(q_s-p_{j,s}), & (12)\\ a_{2e_r}(q) &= \frac{1}{2}\Big(\gamma_1(r,q)^2-\gamma_2(r,q)\Big). & (13)\end{aligned}$$

Hence $\mathcal{M}_0(n,q)=\mathcal{M}(n,q)$ and

$$\mathcal{M}_1(n,q)(\{w\}) = M(w,n,q)\Big[1-\sum_{r=1}^{k}\frac{\gamma_1(r,q)}{nq_rq_0}\Big(q_r|w|+q_0w_r-nq_r\Big)\Big].$$

Observe that $\mathcal{M}_1(n,\overline{p})=\mathcal{M}(n,\overline{p})$. For $t\geq 2$, $\mathcal{M}_t(n,q)(\{w\})$ can be evaluated by using (4), (8), and (21).

2.2 The bounds for the distances. We use the following notations: For $r\in\{1,\dots,k\}$, let

$$\eta(r,q)=2\gamma_2(r,q)+\gamma_1(r,q)^2,$$
$$\delta(r,q)=\eta(r,q)\min\Big\{\frac{1}{2nq_rq_0},\,\frac{2}{e}\Big\},\qquad \zeta(r,q)=\eta(r,q)\min\Big\{\frac{1}{2nq_rq_0},\,\frac{2e}{3}\Big\}.$$

Theorem 2 *Let $t\in\{0,\dots,n-1\}$. If $\sum_{r=1}^{k}\sqrt{\delta(r,q)}<e^{-1/2}$, then*

$$\|P^{S_n}-\mathcal{M}_t(n,q)\|_1 \leq \frac{e^{(t+1)/2}\big(\sum_{r=1}^{k}\sqrt{\delta(r,q)}\big)^{t+1}}{1-\sum_{r=1}^{k}\sqrt{e\delta(r,q)}}. \tag{14}$$

Remark. The proof of Theorem 2 will also show that

$$\|\mathcal{M}_t(n,q)\|_1 \leq \sum_{j=0}^{t}\Big(\sum_{r=1}^{k}\sqrt{e\delta(r,q)}\Big)^{j}. \tag{15}$$

Further, inequality (15) remains valid, if $t\geq 1$, $q=\overline{p}$, and the summand for $j=1$ in the first sum on the right–hand side is omitted.

The following corollary shows that it is possible to remove the singularity on the right–hand side of (14).

Corollary 1 *Let $t\in\{0,\dots,n-1\}$. Then*

$$\|P^{S_n}-\mathcal{M}_t(n,q)\|_1 \leq c_1(t)\, e^{(t+1)/2}\Big(\sum_{r=1}^{k}\sqrt{\delta(r,q)}\Big)^{t+1}, \tag{16}$$

where $c_1(t)=(1-x_t)^{-1}\leq 2t+3$ and x_t is the unique positive solution of the equation $x^{t+1}+x/2=1$. Relation (16) remains valid, if $t\geq 1$, $q=\overline{p}$, and $c_1(t)$ is replaced by

$$\tilde{c}_1(t)=\frac{1}{1-\tilde{x}_t}\leq\frac{2t-1}{t+1-\sqrt{t^2+2}},$$

where $\tilde{x}_t$ is the unique positive solution of the equation $\tilde{x}^{t+1}-\tilde{x}^2/2+\tilde{x}=1$.

Remarks. 1. We have $\tilde{c}_1(t) < c_1(t)$ for $t \geq 1$. The inequality given for $c_1(t)$ (resp. for $\tilde{c}_1(t)$) is an equality, if $t = 0$ (resp. if $t = 1$).

2. In view of (1), we see that

$$\|P^{S_n} - \mathcal{M}(n, \overline{p})\|_1 \geq \frac{1}{62} \max_{0 \leq r \leq k} \left(\gamma_2(r, \overline{p}) \min\left\{ \frac{1}{n\overline{p}_r(1 - \overline{p}_r)}, 1 \right\} \right). \tag{17}$$

Further, in the case $k = 1$, $t = 1$, and $q = \overline{p}$, the bound in (16) has the correct order; under these assumptions, the next corollary coincides with Corollary 2 of Ehm [3] (see Subsection 1.3 of the present paper).

In what follows, we write $S_n = (S_n(1), \ldots, S_n(k))$ and set $S_n(0) = n - \sum_{r=1}^k S_n(r)$.

Corollary 2 *In the case of fixed or bounded dimension k, the distance $\|P^{S_n} - \mathcal{M}(n, \overline{p})\|_1$ is small if and only if*

$$\sum_{r=0}^k \frac{\gamma_2(r, \overline{p})}{n\overline{p}_r(1 - \overline{p}_r)} = \sum_{r=0}^k \left[1 - \frac{\mathrm{Var} S_n(r)}{n\overline{p}_r(1 - \overline{p}_r)} \right]$$

is small.

The following proposition shows that sometimes $\|P^{S_n} - \mathcal{M}(n, \overline{p})\|_1$ can be estimated by using results for the univariate case.

Proposition 2 *If $p_{j,r} = b_j b'_r$ with $b_j, b'_r \in [0, \infty)$, for $j \in \{1, \ldots, n\}$, $r \in \{1, \ldots, k\}$, and if $T_n = \sum_{r=1}^k S_n(r)$, then*

$$\|P^{S_n} - \mathcal{M}(n, \overline{p})\|_1 = \left\| P^{T_n} - \mathcal{B}\left(n, \sum_{r=1}^k \overline{p}_r\right) \right\|_1.$$

Remarks. 1. Under the assumptions of Proposition 2, Ehm's [3] results [see (1)] lead to the inequalities

$$\frac{1}{62} \xi \leq \|P^{S_n} - \mathcal{M}(n, \overline{p})\|_1 \leq 2\xi,$$

where

$$\xi = \left[\sum_{j=1}^n \left(\sum_{r=1}^k \overline{p}_r - \sum_{r=1}^k p_{j,r} \right)^2 \right] \min\left\{ \frac{1}{n(\sum_{r=1}^k \overline{p}_r)\overline{p}_0}, 1 \right\}.$$

2. From Corollary 1 it follows that

$$\|P^{S_n} - \mathcal{M}(n, \overline{p})\|_1 \leq c\, k^\alpha \sum_{r=1}^k \gamma_2(r, \overline{p}) \min\left\{ \frac{1}{n\overline{p}_r\overline{p}_0}, 1 \right\} \tag{18}$$

with $c = 4/(2 - \sqrt{3})$ and $\alpha = 1$. But generally, (18) cannot hold with absolute constants $c \in (0, \infty)$ and $\alpha \in [0, 1)$. To verify this assertion, we assume that the conditions in Proposition 2 hold and let $n \in \{2, 3, \ldots\}$,

$k = n$, $b_j = (jn\sum_{m=1}^n 1/m)^{-1}$ for $j \in \{1,\dots,n\}$, and $b_1' = \dots = b_k' = 1$. Using the notations of Remark 1, we then have

$$\xi = \frac{\sum_{j=1}^n 1/j^2}{(\sum_{j=1}^n 1/j)^2} - \frac{1}{n},$$

whereas $c\,\xi/n^{1-\alpha}$ is the value of the right–hand side of (18).

Theorem 3 *Let* $t \in \{0,\dots,n-1\}$, $c_2 = \frac{1}{2\sqrt{3}}(1+\sqrt{\pi/2})$. *If* $\sum_{r=1}^k \sqrt{\zeta(r,q)} < 3^{-1/2}$, *then*

$$\begin{aligned}\|P^{S_n} - \mathcal{M}_t(n,q)\|_\infty \;\leq\; & 3^{(t+1)/2}\, c_2^k \left(\prod_{r=1}^k \min\left\{\frac{1}{nq_rq_0}, \frac{4e}{3}\right\}\right)^{1/2} \\ & \times \frac{(\sum_{r=1}^k \sqrt{\zeta(r,q)})^{t+1}}{1-\sum_{r=1}^k \sqrt{3\zeta(r,q)}}. \qquad (19)\end{aligned}$$

3 Proofs

Proof of Theorem 1. Inductively it is easy to show that

$$\sum_{w\in\mathbf{Z}_+^k} \Delta^u M(w,m,q)\, z^w = \Big(1+\sum_{r=1}^k q_r(z_r-1)\Big)^m \prod_{r=1}^k (z_r-1)^{u_r}. \qquad (20)$$

Hence, if $|z_r - 1| < 1$ for all $r \in \{1,\dots,k\}$, then

$$\begin{aligned}&\sum_{w\in\mathbf{Z}_+^k} P(S_n = w)\, z^w \\ &= \Big(1+\sum_{r=1}^k q_r(z_r-1)\Big)^n \prod_{j=1}^n \Big(1+\frac{\sum_{r=1}^k (p_{j,r}-q_r)(z_r-1)}{1+\sum_{r=1}^k q_r(z_r-1)}\Big) \\ &= \sum_{|u|\leq n} a_u(q) \Big(1+\sum_{r=1}^k q_r(z_r-1)\Big)^{n-|u|} \prod_{r=1}^k (z_r-1)^{u_r} \\ &= \sum_{w\in\mathbf{Z}_+^k} \Big(\sum_{|u|\leq n} a_u(q)\, \Delta^u M(w, n-|u|, q)\Big) z^w,\end{aligned}$$

giving the assertion. ■

The following two lemmas are devoted to some useful properties of the Krawtchouk polynomials.

Lemma 1 *If* $|w| \leq m$ *and* $|u| \leq m$, *then*

$$K_u(w,m,q)\, M(w,m,q) = \frac{m!\, q^u\, q_0^{|u|}}{u!\,(m-|u|)!}\, \Delta^u M(w, m-|u|, q). \qquad (21)$$

Further, if $|w| \le m$*, then*

$$\sum_{|u|\le m} K_u(w,m,q)\, z^u = \Big(1-\sum_{r=1}^k q_r z_r\Big)^{m-|w|} \prod_{r=1}^k (1+q_0 z_r)^{w_r} \tag{22}$$

and

$$\sum_{|u|\le m} \tilde{K}_u(w,m,q)\, z^u = \Big(1-\sum_{r=1}^k q_r z_r\Big)^{m-|w|} \prod_{r=1}^k \Big(z_r+1-\sum_{s=1}^k q_s z_s\Big)^{w_r}. \tag{23}$$

Proof. Using (4), it can be shown that, for $|u| \le m$,

$$\begin{aligned}
&\sum_{|w|\le m} K_u(w,m,q)\, M(w,m,q)\, z^w \\
&= \frac{m!\, q^u\, q_0^{|u|}}{u!\,(m-|u|)!} \Big(1+\sum_{r=1}^k q_r(z_r-1)\Big)^{m-|u|} \prod_{r=1}^k (z_r-1)^{u_r}.
\end{aligned}\tag{24}$$

Relation (21) is proved by using (20). The proof of (22) is similar to that of (24). We now verify (23). For $|w| \le m$, we have

$$\begin{aligned}
&\Big(1-\sum_{r=1}^k q_r z_r\Big)^{m-|w|} \prod_{r=1}^k \Big(z_r+1-\sum_{s=1}^k q_s z_s\Big)^{w_r} \\
&= \Big(1-\sum_{r=1}^k q_r z_r\Big)^{m-|w|} \prod_{r=1}^k \Big[\sum_{v_r=0}^{w_r} \binom{w_r}{v_r} z_r^{v_r} \Big(1-\sum_{s=1}^k q_s z_s\Big)^{w_r-v_r}\Big] \\
&= \sum_{v\le w} \Big(1-\sum_{r=1}^k q_r z_r\Big)^{m-|v|} \Big[\prod_{r=1}^k \binom{w_r}{v_r}\Big] z^v \\
&= \sum_{v\le w} \sum_{|\tilde{v}|\le m-|v|} \frac{(m-|v|)!\,(-q)^{\tilde{v}}}{\tilde{v}!\,(m-|v|-|\tilde{v}|)!} \Big[\prod_{r=1}^k \binom{w_r}{v_r}\Big] z^{v+\tilde{v}} \\
&= \sum_{|u|\le m} \sum_{v\le u} \frac{(m-|v|)!\,(-q)^{u-v}}{(u-v)!\,(m-|u|)!} \Big[\prod_{r=1}^k \binom{w_r}{v_r}\Big] z^u.
\end{aligned}$$

Here, the latter equality follows from the substitution $u = v + \tilde{v}$. The lemma is proved. ■

Lemma 2 *The relations*

$$\begin{aligned}
&\sum_{|u|\le m} \sum_{|v|\le m} \Big[\sum_{|w|\le m} M(w,m,q)\, K_u(w,m,q)\, K_v(w,m,q)\Big] y^u z^v \\
&= \Big[1+q_0^2 \sum_{r=1}^k q_r y_r z_r + q_0 \Big(\sum_{r=1}^k q_r y_r\Big)\Big(\sum_{r=1}^k q_r z_r\Big)\Big]^m &(25)\\
&= \sum_{|u|\le m} \sum_{|v|=|u|} \Big(\sum_{w\le u\wedge v} \frac{m!\,|u-w|!\, q^{u+v-w}\, q_0^{|w+u|}}{w!\,(m-|u|)!\,(u-w)!\,(v-w)!}\Big) y^u z^v, &(26)
\end{aligned}$$

are valid. Further, for $|u| \le m$ and $|v| \le m$,

$$\sum_{|w|\le m} M(w,m,q)\, K_u(w,m,q)\, \tilde{K}_v(w,m,q) = \frac{\delta_{u,v}\, m!\, q^u\, q_0^{|u|}}{u!\,(m-|u|)!}, \tag{27}$$

where $\delta_{u,v}$ is the Kronecker symbol.

Proof. Equality (25) is easily shown by using (22). A further computation of the right–hand side of (25) leads to

$$\begin{aligned}
&\sum_{j=0}^{m} \binom{m}{j} \Big(1 + q_0^2 \sum_{r=1}^{k} q_r y_r z_r\Big)^j \Big[q_0 \Big(\sum_{r=1}^{k} q_r y_r\Big)\Big(\sum_{r=1}^{k} q_r z_r\Big)\Big]^{m-j} \\
&= \sum_{j=0}^{m} \sum_{|w|\le j} \sum_{|\tilde{u}|=m-j} \sum_{|\tilde{v}|=m-j} \frac{m!\,(m-j)!\, q^{w+\tilde{u}+\tilde{v}}\, q_0^{2|w|+m-j}}{w!\,(j-|w|)!\,\tilde{u}!\,\tilde{v}!}\, y^{w+\tilde{u}}\, z^{w+\tilde{v}} \\
&= \sum_{|\tilde{u}|\le m} \sum_{|\tilde{v}|=|\tilde{u}|} \sum_{|w|\le m-|\tilde{u}|} \frac{m!\,|\tilde{u}|!\, q^{w+\tilde{u}+\tilde{v}}\, q_0^{2|w|+|\tilde{u}|}\, y^{w+\tilde{u}}\, z^{w+\tilde{v}}}{w!\,(m-|\tilde{u}|-|w|)!\,\tilde{u}!\,\tilde{v}!}.
\end{aligned}$$

Here, the latter equality follows from the substitution $j = m - |\tilde{u}|$. Substituting $\tilde{u} = u - w$ and $\tilde{v} = v - w$ and observing that, here, $|v| = |\tilde{v}| + |w| = |\tilde{u}| + |w| = |u|$, (26) is shown. Relation (27) is due to Tratnik [12, formula (3.9)] and can easily be verified by using (22) and (23). The proof is completed. ∎

Remark. For $k = 2$, similar versions of (21), (22), and (27) can be found in Takeuchi and Takemura [11, Section 3].

Proof of Proposition 1. Let $|z_r| < 1/2$ for all $r \in \{1, \ldots, k\}$. By (3) and (23), we get

$$\begin{aligned}
&\sum_{|u|\le n} a_u(q)\, z^u \\
&= \Big(1 - \sum_{r=1}^{k} q_r z_r\Big)^n \prod_{j=1}^{n} \Big(1 + \sum_{r=1}^{k} p_{j,r} \Big[\frac{z_r + 1 - \sum_{s=1}^{k} q_s z_s}{1 - \sum_{s=1}^{k} q_s z_s} - 1\Big]\Big) \\
&= \sum_{|w|\le n} P(S_n = w) \Big(1 - \sum_{r=1}^{k} q_r z_r\Big)^{n-|w|} \prod_{r=1}^{k} \Big(z_r + 1 - \sum_{s=1}^{k} q_s z_s\Big)^{w_r} \\
&= \sum_{|u|\le n} \Big(\sum_{|w|\le n} P(S_n = w)\, \tilde{K}_u(w,n,q)\Big) z^u,
\end{aligned}$$

giving (7). Using (3) we obtain (8):

$$\begin{aligned}
&\sum_{1\le|u|\le n} |u|\, a_u(q)\, z^u = \sum_{r=1}^{k} z_r \frac{\partial}{\partial z_r} \sum_{|u|\le n} a_u(q)\, z^u \\
&= \sum_{r=1}^{k} z_r \Big(\sum_{j=1}^{n} \frac{p_{j,r} - q_r}{1 + \sum_{s=1}^{k} (p_{j,s} - q_s) z_s}\Big) \prod_{j=1}^{n} \Big(1 + \sum_{s=1}^{k} (p_{j,s} - q_s) z_s\Big)
\end{aligned}$$

$$
\begin{aligned}
&= -\sum_{j=1}^{n}\sum_{m=1}^{\infty}\Big(\sum_{r=1}^{k}(q_r-p_{j,r})z_r\Big)^m\sum_{|u|\le n}a_u(q)\,z^u\\
&= -\Big[\sum_{|u|\ge 1}\frac{|u|!}{u!}\Big(\sum_{j=1}^{n}\prod_{r=1}^{k}(q_r-p_{j,r})^{u_r}\Big)z^u\Big]\Big[\sum_{|u|\le n}a_u(q)\,z^u\Big]\\
&= -\sum_{1\le|u|\le n}\Big[\sum_{v<u}\frac{|u-v|!}{(u-v)!}\Big(\sum_{j=1}^{n}\prod_{r=1}^{k}(q_r-p_{j,r})^{u_r-v_r}\Big)a_v(q)\Big]z^u.
\end{aligned}
$$

Using the equality $\sum_{|u|\le n}\mu_{(u)}z^u/u! = \prod_{j=1}^{n}(1+\sum_{r=1}^{k}p_{j,r}z_r)$ and (3), we get

$$
\begin{aligned}
\sum_{|u|\le n}a_u(q)\,z^u &= \Big(1-\sum_{r=1}^{k}q_rz_r\Big)^n\prod_{j=1}^{n}\Big(1+\frac{\sum_{r=1}^{k}p_{j,r}z_r}{1-\sum_{r=1}^{k}q_rz_r}\Big)\\
&= \sum_{|v|\le n}\frac{\mu_{(v)}}{v!}z^v\Big(1-\sum_{r=1}^{k}q_rz_r\Big)^{n-|v|}\\
&= \sum_{|v|\le n}\sum_{|w|\le n-|v|}\frac{(n-|v|)!\,(-q)^w\,\mu_{(v)}}{w!\,(n-|v|-|w|)!\,v!}z^{v+w}\\
&= \sum_{|u|\le n}\Big(\sum_{v\le u}\frac{(n-|v|)!\,(-q)^{u-v}\,\mu_{(v)}}{(u-v)!\,(n-|u|)!\,v!}\Big)z^u,
\end{aligned}
$$

giving (9). Equality (10) follows from (3) and Cauchy's theorem. The proposition is proved. ∎

For the proof of Theorem 2, we need the following three lemmas.

Lemma 3 *Let $I_0(x)=\sum_{m=0}^{\infty}(x/2)^{2m}/(m!)^2$ be the modified Bessel function of the first kind and order 0, $\beta(x)=I_0(x)e^{-x^2/4}$, $(x\in\mathbf{R})$. If $1\le|u|\le n$ and $\gamma_2(r,q)>0$ for all $r\in\{1,\dots,k\}$, then*

$$
\begin{aligned}
|a_u(q)| \;\le\; & \frac{n^{(n-|u|)/2}\,|u|^{|u|/2}}{(n-|u|)^{(n-|u|)/2}\,u^u}\prod_{r=1}^{k}\Big[\Big(\gamma_2(r,q)+\frac{n-|u|}{2n}\gamma_1(r,q)^2\Big)^{u_r/2}\\
&\times\beta\Big(\sqrt{\frac{2u_r^2\,(n-|u|)\,\gamma_1(r,q)^2}{|u|\,(2n\,\gamma_2(r,q)+(n-|u|)\,\gamma_1(r,q)^2)}}\Big)\Big].
\end{aligned}
$$

Proof. From (10), it follows that, for $1\le|u|\le n$ and arbitrary $R\in(0,\infty)^k$,

$$
|a_u(q)|\le\frac{1}{(2\pi)^k\,R^u}\int_0^{2\pi}\cdots\int_0^{2\pi}\Big|h(R_1e^{ix_1},\dots,R_ke^{ix_k})\Big|\,dx_1\dots dx_k,
$$

where $h(z)=\prod_{j=1}^{n}[1+\sum_{r=1}^{k}(p_{j,r}-q_r)z_r]$. Let $\alpha\in(0,\infty)$ be arbitrary. Using $\cos b_1\,\cos b_2+\sin b_1\,\sin b_2=\cos(b_1-b_2)\in[-1,1]$, $1+b_1\le e^{b_1}$ for $b_1,b_2\in\mathbf{R}$, and Cauchy's inequality, we obtain

$|h(R_1e^{ix_1},\dots,R_ke^{ix_k})|$

$$= \frac{1}{\alpha^{n/2}} \prod_{j=1}^{n} \Big[\alpha + 2\alpha \sum_{r=1}^{k} (p_{j,r} - q_r) R_r \cos x_r$$

$$+ \alpha \sum_{r=1}^{k} \sum_{s=1}^{k} (p_{j,r} - q_r)(p_{j,s} - q_s) R_r R_s \cos(x_r - x_s)\Big]^{1/2}$$

$$\leq \frac{1}{\alpha^{n/2}} \exp\Big(\frac{n(\alpha-1)}{2} - \alpha \sum_{r=1}^{k} \gamma_1(r,q) R_r \cos x_r + \frac{\alpha}{2}\Big(\sum_{r=1}^{k} \sqrt{\gamma_2(r,q)} R_r\Big)^2\Big).$$

Since

$$\sum_{r=1}^{k} b_r \leq \Big(\sum_{r=1}^{k} \sqrt{b_r}\Big)^2, \qquad \Big(\sum_{r=1}^{k} \sqrt{b_r}\Big)^2 + \Big(\sum_{r=1}^{k} \sqrt{b'_r}\Big)^2 \leq \Big(\sum_{r=1}^{k} \sqrt{b_r + b'_r}\Big)^2$$

for $b, b' \in [0,\infty)^k$, and $I_0(x) = \frac{1}{2\pi} \int_0^{2\pi} \exp(x \cos \nu)\, d\nu$ for $x \in \mathbf{R}$, we get

$$|a_u(q)| \leq \frac{\prod_{r=1}^{k} \beta(\alpha\gamma_1(r,q) R_r)}{\alpha^{n/2}\, R^u} \times \exp\Big(\frac{n(\alpha-1)}{2} + \frac{1}{4}\Big(\sum_{r=1}^{k} R_r \sqrt{2\alpha\gamma_2(r,q) + \alpha^2 \gamma_1(r,q)^2}\Big)^2\Big).$$

Finally, for $r \in \{1,\dots,k\}$ and $\epsilon > 0$, set

$$\alpha = \frac{n - |u| + \epsilon}{n}, \qquad R_r = \sqrt{\frac{2u_r^2 + \epsilon}{|u|\,(2\alpha\,\gamma_2(r,q) + \alpha^2\,\gamma_1(r,q)^2)}}$$

and let ϵ tend to zero. The proof is completed. ∎

Remarks. 1. We have $0 \leq \beta(x) = \beta(|x|) \leq 1$ for $x \in \mathbf{R}$.

2. From Lemma 3, we get the inequality

$$|a_u(q)| \leq \frac{n^{(n-|u|)/2}\, |u|^{|u|/2}}{(n-|u|)^{(n-|u|)/2}\, 2^{|u|/2}\, u^u} \prod_{r=1}^{k} \eta(r,q)^{u_r/2} \tag{28}$$

for $1 \leq |u| \leq n$, being also valid in the case of $\gamma_2(r,q) = 0$ for some $r \in \{1,\dots,k\}$.

Lemma 4 *The following inequality is valid:*

$$\frac{|u|^{|u|}}{u^u} \leq \exp\Big(\sum_{r=2}^{k} u_r\Big) \frac{|u|!}{u!}. \tag{29}$$

Proof. We may assume that $k \in \{2,3,\dots\}$ and $u \in \mathbf{N}^k$. For $s \in \{1,\dots,k\}$, let $v_s = \sum_{r=1}^{s} u_r$. Using the inequalities

$$\frac{b_1^{b_1}}{b_2^{b_2}\,(b_1-b_2)^{b_1-b_2}} \leq e\sqrt{b_2}\binom{b_1}{b_2}, \qquad b_1 \in \mathbf{N}, \quad b_2 \in \{1,\dots,b_1\}$$

(see Roos [6, Lemma 3]) and $1+x \le e^x$, $x \in \mathbf{R}$, we get

$$\begin{aligned}\frac{|u|^{|u|}}{u^u} &= \prod_{r=1}^{k-1} \frac{v_{r+1}^{v_{r+1}}}{v_r^{v_r}\, u_{r+1}^{u_{r+1}}} \le e^{k-1}\sqrt{u_2\ldots u_k}\,\frac{|u|!}{u!}\\ &\le \exp\Big(\frac{1}{2}\sum_{r=2}^{k}(u_r+1)\Big)\frac{|u|!}{u!} \le \exp\Big(\sum_{r=2}^{k} u_r\Big)\frac{|u|!}{u!}.\end{aligned}$$

The lemma is proved. ∎

Lemma 5 *The following inequalities hold:*

$$\|\Delta^u M(\cdot,m,q)\|_1 \le \Big(\frac{u!\,m!}{(m+|u|)!\,q^u\,q_0^{|u|}}\Big)^{1/2} \le \Big(\frac{e^{|u|}\,u^u\,m^m}{(m+|u|)^{m+|u|}\,q^u\,q_0^{|u|}}\Big)^{1/2}. \tag{30}$$

Proof. Using (21), Cauchy's inequality, and (26), we obtain

$$\begin{aligned}&\Big(\|\Delta^u M(\cdot,m,q)\|_1\,\frac{(m+|u|)!\,q^u\,q_0^{|u|}}{u!\,m!}\Big)^2\\ &\le \sum_{|w|\le m+|u|} M(w,m+|u|,q)\Big[K_u(w,m+|u|,q)\Big]^2\\ &= \frac{(m+|u|)!\,q^u\,q_0^{2|u|}}{u!\,m!}\sum_{w\le u}\Big[\prod_{r=1}^{k}\binom{u_r}{w_r}\Big]\frac{|w|!}{w!}\,q^w\,q_0^{-|w|}\\ &\le \frac{(m+|u|)!\,q^u\,q_0^{2|u|}}{u!\,m!}\sum_{w\le u}\Big[\prod_{r=1}^{k}\binom{u_r}{w_r}\Big]\sum_{|v|=|w|}\frac{|w|!}{v!}\,q^v\,q_0^{-|v|}\\ &= \frac{(m+|u|)!\,q^u\,q_0^{|u|}}{u!\,m!},\end{aligned}$$

giving the first inequality of (30). The second inequality follows from (29). ∎

Proof of Theorem 2. Let $I=\{r\in\{1,\ldots,k\}\,|\,q_r\le e/(4nq_0)\}$ and $I^c=\{1,\ldots,k\}\setminus I$. For $u\in\mathbf{Z}_+^k$, let $v(u)=\sum_{r\in I} u_r\,e_r$. To simplify notations, we write v for $v(u)$. Using (30) and the inequality $1+x\le e^x$, $x\in\mathbf{R}$, we get, for $|u|\le n$,

$$\begin{aligned}\|\Delta^u M(\cdot,n-|u|,q)\|_1 &\le 2^{|v|}\,\|\Delta^{u-v}M(\cdot,n-|u|,q)\|_1\\ &\le 2^{|v|}\Big(\frac{e^{|u-v|}\,(u-v)^{u-v}\,(n-|u|)^{n-|u|}}{(n-|v|)^{n-|v|}\,q^{u-v}\,q_0^{|u-v|}}\Big)^{1/2}\end{aligned}$$

and

$$\frac{n^{n-|v|}\,(u-v)^{u-v}}{(n-|v|)^{n-|v|}\,u^u}\le 1.$$

Using

$$\frac{|u|^{|u|}}{u^u}\le\Big(\frac{|u|!}{u!}\Big)^2 \tag{31}$$

(see Roos [7, Lemma 4]), (2), and (28), we obtain

$$\begin{aligned}
\|P^{S_n} - \mathcal{M}_t(n,q)\|_1 &\le \sum_{j=t+1}^{n} \sum_{|u|=j} |a_u(q)| \, \|\Delta^u M(\cdot, n-|u|, q)\|_1 \\
&\le \sum_{j=t+1}^{n} \sum_{|u|=j} \sqrt{\frac{|u|^{|u|} n^{n-|v|} (u-v)^{u-v}}{u^u \, (n-|v|)^{n-|v|} \, u^u}} \prod_{r\in I} (2\eta(r,q))^{u_r/2} \prod_{r\in I^c} \Big(\frac{e\eta(r,q)}{2nq_r q_0}\Big)^{u_r/2} \\
&\le \frac{e^{(t+1)/2} \big(\sum_{r=1}^{k} \sqrt{\delta(r,q)}\big)^{t+1}}{1-\sum_{r=1}^{k} \sqrt{e\delta(r,q)}},
\end{aligned}$$

giving the assertion. ■

Proof of Corollary 1. The assertions can easily be shown by using (14), the inequality $\|P^{S_n} - \mathcal{M}_t(n,q)\|_1 \le 1 + \|\mathcal{M}_t(n,q)\|_1$, the remark after Theorem 2, and Bernoulli's inequality. ■

Proof of Corollary 2. For $r \in \{0,\dots,k\}$, $A_+ = \{j \in \{1,\dots,n\} \,|\, p_{j,r} \ge \overline{p}_r\}$, and $A_- = \{1,\dots,n\} \setminus A_+$, we have

$$\begin{aligned}
\gamma_2(r,\overline{p}) &\le \Big(\sum_{j\in A_+} (p_{j,r} - \overline{p}_r)\Big)^2 + \Big(\sum_{j\in A_-} (\overline{p}_r - p_{j,r})\Big)^2 \\
&= \frac{1}{2}\Big(\sum_{j=1}^{n} |p_{j,r} - \overline{p}_r|\Big)^2 \le 2[n\overline{p}_r(1-\overline{p}_r)]^2
\end{aligned}$$

and $\gamma_2(r,\overline{p}) \le n\overline{p}_r(1-\overline{p}_r)$ (see Roos [6, proof of the remark in Section 1.2]). This leads to

$$\gamma_2(r,\overline{p}) \min\Big\{\frac{1}{n\overline{p}_r(1-\overline{p}_r)}, 1\Big\} \ge \frac{1}{2}\Big(\frac{\gamma_2(r,\overline{p})}{n\overline{p}_r(1-\overline{p}_r)}\Big)^2.$$

Using (17), we get

$$\|P^{S_n} - \mathcal{M}(n,\overline{p})\|_1 \ge \frac{1}{124(k+1)^2}\Big(\sum_{r=0}^{k} \frac{\gamma_2(r,\overline{p})}{n\overline{p}_r(1-\overline{p}_r)}\Big)^2.$$

For a similar upper bound, we may suppose that $\overline{p}_0 \ge 1/(k+1)$, which can be achieved by interchanging of components. Hence, by Corollary 1,

$$\|P^{S_n} - \mathcal{M}(n,\overline{p})\|_1 \le \frac{ek(k+1)}{2-\sqrt{3}} \sum_{r=0}^{k} \frac{\gamma_2(r,\overline{p})}{n\overline{p}_r(1-\overline{p}_r)},$$

completing the proof. ■

Proof of Proposition 2. For $r \in \{1,\dots,k\}$, let

$$q_r = \frac{\overline{p}_r}{\sum_{s=1}^{k} \overline{p}_s} = \frac{p_{j,r}}{\sum_{s=1}^{k} p_{j,s}}, \qquad (j \in \{1,\dots,n\}).$$

Then

$$\sum_{w\in\mathbf{Z}_+^k} P(S_n=w)\,z^w = \prod_{j=1}^n \Big[1+\Big(\sum_{r=1}^k p_{j,r}\Big)\Big(\sum_{r=1}^k q_r z_r - 1\Big)\Big]$$
$$= \sum_{j=0}^n P(T_n=j)\Big(\sum_{r=1}^k q_r z_r\Big)^j = \sum_{|w|\le n}\frac{|w|!}{w!}\,q^w\,P(T_n=|w|)\,z^w$$

and

$$\sum_{w\in\mathbf{Z}_+^k}\mathcal{M}(n,\overline{p})(\{w\})\,z^w = \sum_{|w|\le n}\frac{|w|!}{w!}\,q^w\,\mathcal{B}\Big(n,\sum_{r=1}^k \overline{p}_r\Big)(\{|w|\})\,z^w.$$

The proof is easily completed. ■

For the proof of Theorem 3, we need the following two lemmas.

Lemma 6 *Let* $c_3=\frac{\sqrt{e}}{2}(1+\sqrt{\pi/2})$ *and* $\alpha\in(0,\infty)$. *Then*

$$\int_0^{\pi/2}\exp(-2\alpha\sin^2 x)\,\sin^m x\,dx \le \min\Big\{\frac{\pi}{2},\,c_3\,\pi\Big(\frac{\max\{m,1/3\}}{4\alpha e}\Big)^{(m+1)/2}\Big\}. \tag{32}$$

Proof. Shorgin [9, proof of Lemma 6] has shown that, for $m\in\mathbf{N}$,

$$\int_0^{\pi/2}\exp(-2\alpha\,\sin^2 x)\,\sin^m x\,dx \le \frac{\pi\sqrt{e}}{2}\Big(1+\sqrt{\frac{\pi}{2m}}\Big)\Big(\frac{m}{4\alpha e}\Big)^{(m+1)/2}.$$

Using the inequality $2x/\pi\le\sin x$ for $x\in[0,\pi/2]$, we get, in the case $m=0$,

$$\int_0^{\pi/2}\exp(-2\alpha\,\sin^2 x)\,dx \le \int_0^{\pi/2}\exp\Big(-\frac{8}{\pi^2}\,\alpha\,x^2\Big)\,dx \le c_3\,\pi\,\Big(\frac{1/3}{4\alpha e}\Big)^{1/2}.$$

This gives one part of (32). The remaining part is obvious. The lemma is proved. ■

Lemma 7 *Let* c_3 *be as in Lemma 6,* $I\subseteq\{1,\dots,k\}$, *and* $I^c=\{1,\dots,k\}\setminus I$. *If* $u\ne 0$, *then*

$$\begin{aligned}\|\Delta^u M(\cdot,m,q)\|_\infty \;&\le\; \Big(\frac{m}{m+|u|}\Big)^{m/2}\Big(\prod_{r\in I}(4e)^{u_r/2}\Big)\\ &\quad\times\prod_{r\in I^c}\Big[\frac{c_3}{\sqrt{e}}\Big(\frac{\max\{u_r,\,1/3\}}{(m+|u|)q_r q_0}\Big)^{(u_r+1)/2}\Big].\end{aligned}\tag{33}$$

Proof. By (20) and Cauchy's theorem, we have

$$\|\Delta^u M(\cdot,m,q)\|_\infty \le \frac{1}{(2\pi)^k}\int_{-\pi}^{\pi}\cdots\int_{-\pi}^{\pi}H(x)\,dx_1\dots dx_k,$$

where, for $x \in [-\pi, \pi]^k$,

$$\begin{aligned} H(x) &= \Big|1 + \sum_{r=1}^{k} q_r (e^{ix_r} - 1)\Big|^m \prod_{r=1}^{k} |e^{ix_r} - 1|^{u_r} \\ &= \Big(q_0^2 + 2q_0 \sum_{r=1}^{k} q_r \cos x_r + \sum_{r=1}^{k}\sum_{s=1}^{k} q_r q_s \cos(x_r - x_s)\Big)^{m/2} \\ &\quad \times 2^{|u|} \prod_{r=1}^{k} |\sin(x_r/2)|^{u_r} \\ &\le 2^{|u|}\Big(1 - 4q_0 \sum_{r=1}^{k} q_r \sin^2(x_r/2)\Big)^{m/2} \prod_{r=1}^{k} |\sin(x_r/2)|^{u_r}. \end{aligned}$$

Here, we used the relations $1 - \cos b_1 = 2\sin^2(b_1/2)$ and $\cos b_1 \cos b_2 + \sin b_1 \sin b_2 = \cos(b_1 - b_2) \le 1$ for $b_1, b_2 \in \mathbf{R}$. Hence

$$\|\Delta^u M(\cdot, m, q)\|_\infty \le \frac{2^{|u|+k}}{\pi^k} f_{u,m}(q),$$

where, for $\epsilon \in [0, \infty)$,

$$f_{u,\epsilon}(q) = \int_0^{\pi/2} \dots \int_0^{\pi/2} \Big(1 - 4q_0 \sum_{r=1}^{k} q_r \sin^2 x_r\Big)^{\epsilon/2} \prod_{r=1}^{k} \sin^{u_r} x_r \; dx_1 \dots dx_k.$$

For $\epsilon \in (0, \infty)$ and arbitrary $\alpha \in (0, \infty)$,

$$\begin{aligned} f_{u,\epsilon}(q) &\le \frac{e^{\epsilon(\alpha-1)/2}}{\alpha^{\epsilon/2}} \prod_{r=1}^{k} \Big(\int_0^{\pi/2} \exp\big(-2\alpha\epsilon q_r q_0 \sin^2 x_r\big) \sin^{u_r} x_r \; dx_r\Big) \\ &\le \pi^k\, g(\alpha)\, 2^{-\#I} \prod_{r \in I^c} \Big[c_3 \Big(\frac{\max\{u_r, 1/3\}}{4\epsilon q_r q_0 e}\Big)^{(u_r+1)/2}\Big], \end{aligned}$$

where $g(\alpha) = \alpha^{-(\epsilon+\nu)/2} e^{\epsilon(\alpha-1)/2}$, $\nu = \sum_{r \in I^c}(u_r + 1)$, and $\#I$ denotes the number of elements of I. Here, we used the inequality $1 + b \le e^b$ for $b \in \mathbf{R}$ and (32). Now, $g(\alpha)$ attains its minimum at $\alpha_0 = (\epsilon + \nu)/\epsilon$. Substituting $\alpha = \alpha_0$, we obtain

$$f_{u,\epsilon}(q) \le \pi^k \frac{\epsilon^{\epsilon/2} (\epsilon + |u|)^{\nu/2}}{(\epsilon + \nu)^{(\epsilon+\nu)/2}} 2^{-\#I} \prod_{r \in I^c} \Big[c_3 \Big(\frac{\max\{u_r, 1/3\}}{4(\epsilon + |u|) q_r q_0}\Big)^{(u_r+1)/2}\Big].$$

Application of the inequality $1 + b \le e^b$ leads to the desired result in the case $m \ne 0$. For $m = 0$, the assertion is proved by letting $\epsilon \downarrow 0$ in the inequalities above. ∎

Proof of Theorem 3. Let c_3 be as in Lemma 6, $I = \{r \in \{1, \dots, k\} \,|\, q_r \le 3/(4enq_0)\}$, and $I^c = \{1, \dots, k\} \setminus I$. Using (28), (33), (31), the inequality

$$\frac{(\max\{m, 1/3\})^{m+1}}{m^m} \le 3^{m-1},$$

and

$$\prod_{r\in I^c}\frac{1}{nq_rq_0}=\Big(\frac{3}{4e}\Big)^{\#I}\prod_{r=1}^{k}\min\Big\{\frac{1}{nq_rq_0},\frac{4e}{3}\Big\},$$

we obtain (19):

$$\begin{aligned}\|P^{S_n}-\mathcal{M}_t(n,q)\|_\infty &\leq \sum_{j=t+1}^{n}\sum_{|u|=j}|a_u(q)|\,\|\Delta^u M(\cdot,n-|u|,q)\|_\infty\\ \leq\ & \sum_{j=t+1}^{n}\sum_{|u|=j}\frac{|u|!}{u!}\Big(\prod_{r\in I}(2e\eta(r,q))^{u_r/2}\Big)\prod_{r\in I^c}\Big(\frac{c_3}{\sqrt{3enq_rq_0}}\Big(\frac{3\eta(r,q)}{2nq_rq_0}\Big)^{u_r/2}\Big)\\ \leq\ & 3^{(t+1)/2}\,c_2^k\Big(\prod_{r=1}^{k}\min\Big\{\frac{1}{nq_rq_0},\frac{4e}{3}\Big\}\Big)^{1/2}\frac{(\sum_{r=1}^{k}\sqrt{\zeta(r,q)})^{t+1}}{1-\sum_{r=1}^{k}\sqrt{3\zeta(r,q)}},\end{aligned}$$

completing the proof. ■

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