Binomial Approximation to the Poisson Binomial Distribution: The Krawtchouk Expansion

Bero Roos

Institut für Mathematische Stochastik, Universität Hamburg, Bundesstraße 55, D–20146 Hamburg, Germany.
E–mail: roos@math.uni-hamburg.de

Abstract. The Poisson binomial distribution is approximated by a binomial distribution and also by finite signed measures, resulting from the corresponding Krawtchouk expansion. Bounds and asymptotic relations for the total variation distance and the point metric are given.

1 Introduction

1.1 The aim of this paper. We consider the sum S_n of $n \in \mathbf{N} = \{1, 2, \ldots\}$ independent Bernoulli random variables $X_1, \ldots, X_n$ with success probabilities

$$P(X_j = 1) = 1 - P(X_j = 0) = p_j \in [0, 1], \qquad j \in \{1, \ldots, n\}.$$

Since the distribution P^{S_n} of S_n has a complicated structure, it is often approximated by other distributions. To get higher accuracy, several authors deal with the corresponding asymptotic expansions. We can find publications on normal approximations and the Edgeworth expansion (see Uspensky [22, Chapter 7] for the binomial case; Makabe [12, §4]; Deheuvels et al. [5]; Mikhailov [13]; Volkova [23]) and Poisson approximations and expansions related to the Charlier polynomials (see Prohorov [14] for the binomial case; Le Cam [10]; Franken [7, §5]; Shorgin [18]; Deheuvels et al. [2], [3], [4]; Barbour et al. [1]; Roos [16], [17]).

In this paper, we consider the approximation of P^{S_n} by the binomial distribution $\mathcal{B}(n, p)$ with parameter n and arbitrary success probability p and also by finite signed measures, resulting from the corresponding Krawtchouk expansion of P^{S_n}. Here we have to deal with the Krawtchouk polynomials, being orthogonal with respect to the binomial counting density. We prove some bounds and asymptotic relations for the total variation distance d_τ and the point metric d_π between finite signed measures Q_1 and Q_2, which are concentrated on $\mathbf{Z}_+ = \{0, 1, \ldots\}$ and satisfy $Q_1(\mathbf{Z}_+) = Q_2(\mathbf{Z}_+)$:

$$\begin{aligned} d_\tau(Q_1, Q_2) &= \sup_{A \subseteq \mathbf{Z}_+} \Big| Q_1(A) - Q_2(A) \Big| = \frac{1}{2} \sum_{m=0}^{\infty} \Big| Q_1(\{m\}) - Q_2(\{m\}) \Big|, \\ d_\pi(Q_1, Q_2) &= \sup_{m \in \mathbf{Z}_+} \Big| Q_1(\{m\}) - Q_2(\{m\}) \Big|. \end{aligned}$$

The presented method is similar to that used by Shorgin [18] in the Poisson approximation. For refinements of Shorgin's method, see Deheuvels et al. [3], [4], and Roos [16], [17].

1.2 Some general notations. In what follows, let

$$\lambda = \sum_{j=1}^{n} p_j \in (0, n), \quad \overline{p} = \frac{\lambda}{n}, \quad \overline{q} = 1 - \overline{p}, \quad p \in [0, 1], \quad q = 1 - p,$$

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$$\gamma_k(p)=\sum_{j=1}^{n}(p-p_j)^k,\qquad \gamma_k=\gamma_k(\overline{p})\quad\text{for } k\in\mathbf{N},$$

and

$$\theta=\frac{\gamma_2}{n\overline{p}\,\overline{q}}=1-\frac{\operatorname{Var} S_n}{n\overline{p}\,\overline{q}}.$$

Remark. We have $\theta\leq\delta\min\{1,\,\delta[4\overline{p}\,\overline{q}]^{-1}\}$, where $\delta=p_{\max}-p_{\min}$, $p_{\max}=\max_{1\leq j\leq n}p_j$, and $p_{\min}=\min_{1\leq j\leq n}p_j$. The proof is given in Section 3.

Further, let

$$\begin{aligned} b(m,n,p) &= \Delta^0 b(m,n,p)=\mathcal{B}(n,p)(\{m\})\\ &= \begin{cases}\binom{n}{m}p^m q^{n-m} & \text{for } n,m\in\mathbf{Z}_+,\ m\leq n\\ 0 & \text{otherwise,}\end{cases}\end{aligned}$$

and

$$\Delta^j b(m,n,p)=\Delta^{j-1}b(m-1,n,p)-\Delta^{j-1}b(m,n,p)\qquad\text{for } j\in\mathbf{N}.$$

Here $\binom{x}{m}=\prod_{k=1}^{m}[(x-k+1)/k]$ for $m\in\mathbf{Z}_+$ and $x\in\mathbf{C}$. We write $\Delta^j b(\cdot,n,q)$ for the sequence $(\Delta^j b(m,n,q))_{m\in\mathbf{Z}_+}$ and set $\|f\|_\infty=\sup_{m\in\mathbf{Z}_+}|f(m)|$ and $\|f\|_1=\sum_{m=0}^{\infty}|f(m)|$ for $f:\mathbf{Z}_+\longrightarrow\mathbf{R}$. Always, let $0^0=1$. If not defined otherwise, i is the imaginary unit in $\mathbf{C}$. Let $\lfloor x\rfloor$ be the largest integer $\leq x\in\mathbf{R}$.

1.3 Known facts. Ehm [8, Theorem 1, Lemma 2] considered the approximation of P^{S_n} by $\mathcal{B}(n,\overline{p})$ and proved the following estimates by using the Stein–Chen method:

$$\begin{aligned}\frac{\theta}{124}\min\{1,\,n\overline{p}\,\overline{q}\} &\leq d_\tau(P^{S_n},\mathcal{B}(n,\overline{p}))=:d\leq\frac{1-\overline{p}^{n+1}-\overline{q}^{n+1}}{(n+1)\overline{p}\,\overline{q}}\,\gamma_2\\ &\leq \theta\min\{1,\,n\overline{p}\,\overline{q}\}.\end{aligned}\tag{1}$$

It follows that $\min\{\theta,\gamma_2\}$ and d are of the same order; further, d is small if and only if θ is small (see Ehm [8, Corollary 2]).

Using the Stein–Chen method, Barbour et al. [1, Theorem 9.E] and Soon [19, Corollary 1.3] treated the approximation of P^{S_n} by another binomial distribution $\mathcal{B}(\tilde{n},p)$, where the parameters $\tilde{n}\in\mathbf{N}$, $p\in(0,1)$ are chosen in such a way that both, P^{S_n} and $\mathcal{B}(\tilde{n},p)$, have the same mean and nearly the same variance. We only cite Soon's result: If $\lambda_r=\sum_{j=1}^{n}p_j^r$ for $r\in\mathbf{N}$, $\tilde{n}=\lfloor\lambda^2/\lambda_2+1/2\rfloor$, and $p=\lambda/\tilde{n}$, then

$$d_\tau(P^{S_n},\mathcal{B}(\tilde{n},p))\leq\frac{1-p^{\tilde{n}+1}-q^{\tilde{n}+1}}{(\tilde{n}+1)pq}\Big[2\Big(\lambda_3-\frac{\lambda_2^2}{\lambda}\Big)+\lambda\Big|p-\frac{\lambda_2}{\lambda}\Big|\Big].\tag{2}$$

Note that, if $\tilde{n}=\lambda^2/\lambda_2$, the variances of $\mathcal{B}(\tilde{n},p)$ and S_n coincide.

Other notes, dealing with binomial approximations, came from Le Cam [10, Section 4], Makabe [12, §3], Takeuchi et al. [21, Section 3], Jakševičius [9], and Loh [11, Corollary 3]. It should be mentioned that Takeuchi et al. considered the Krawtchouk expansion of the counting density and the distribution function of the sum of not necessarily independent Bernoulli random variables, whereas Jakševičius approximated the distribution of the sum of independent identically distributed lattice random variables by a binomial type law.

2 Main Results

2.1 The Krawtchouk expansion of P^{S_n}.

Theorem 1 *For $m \in \mathbf{Z}_+$ and arbitrary p,*

$$P(S_n = m) = \sum_{j=0}^{n} a_j(p)\, \Delta^j b(m, n-j, p), \tag{3}$$

where $a_0(p) = 1$ and

$$a_j(p) = \sum_{1 \le k(1) < \ldots < k(j) \le n} \prod_{r=1}^{j} (p_{k(r)} - p), \qquad j \in \{1, \ldots, n\}. \tag{4}$$

We call the right–hand side of (3) the Krawtchouk expansion of P^{S_n} with parameter p, and $a_0(p), \ldots, a_n(p)$ the corresponding Krawtchouk coefficients. For $s \in \{0, \ldots, n\}$, the finite signed measure $\mathcal{B}_s(n,p)$ concentrated on $\mathbf{Z}_+$, is defined by

$$\mathcal{B}_s(n,p)(\{m\}) = \sum_{j=0}^{s} a_j(p)\, \Delta^j b(m, n-j, p), \qquad m \in \mathbf{Z}_+. \tag{5}$$

Note that $\mathcal{B}_0(n,p) = \mathcal{B}(n,p)$ and that, for $s \ge 1$, $\mathcal{B}_s(n,p)$ depends not only on the indicated arguments but also on $p_1, \ldots, p_n$, though we omit these parameters in our notation. We derive our results for the total variation distance from the inequality

$$d_\tau(P^{S_n}, \mathcal{B}_s(n,p)) \le \frac{1}{2} \sum_{j=s+1}^{n} |a_j(p)|\, \|\Delta^j b(\cdot, n-j, p)\|_1.$$

For the point metric, we use a similar inequality. To get the bounds for the distances [see Theorem 2], we have to estimate the Krawtchouk coefficients of P^{S_n} and the norms $\|(\Delta^j b(\cdot, n-j, p))\|_t$ for $t \in \{1, \infty\}$ [see Lemmas 1, 2, and 4].

Remarks. 1. We have $\mathcal{B}_s(n,p)(\mathbf{Z}_+) = 1$, since $\sum_{m=0}^{\infty} \Delta^j b(m,n,p) = 0$ for $j \in \mathbf{N}$ and $n \in \mathbf{Z}_+$. Further, $\mathcal{B}_s(n,p)(\{m\}) = 0$ for $m \in \{n+1, n+2, \ldots\}$.

2. By induction over j, we get

$$\frac{d^j}{dp^j} b(m,n,p) = n^{[j]}\, \Delta^j b(m, n-j, p) \tag{6}$$

for $n, m \in \mathbf{Z}_+$, $j \in \{0, \ldots, n\}$, with $n^{[j]} = n!/(n-j)!$, and hence,

$$\mathcal{B}_s(n,p)(\{m\}) = \sum_{j=0}^{s} \frac{a_j(p)}{n^{[j]}} \frac{d^j}{dp^j} b(m,n,p), \qquad m \in \mathbf{Z}_+. \tag{7}$$

3. With our assumptions, relation (3.5) of Takeuchi et al. [21] is similar to (3).

In the following proposition, we give some alternative formulae for the Krawtchouk coefficients. We need the Krawtchouk polynomials $K_j(x,n,p) \in \mathbf{R}[x]$ defined by (see Szegö [20, (2.82.2)])

$$K_j(x,n,p) = \sum_{k=0}^{j} \binom{n-x}{j-k} \binom{x}{k} (-p)^{j-k} q^k, \qquad n, j \in \mathbf{Z}_+,\ x \in \mathbf{C}. \tag{8}$$

Proposition 1 *Let* $\mu_{(k)} = \sum_{m=k}^{n} m^{[k]} P(S_n = m)$, *for* $k \in \{0, \dots, n\}$, *be the* k*th factorial moment of* S_n. *Further, let* $\alpha \in (0, \infty)$ *be arbitrary. Then, for* $j \in \{1, \dots, n\}$,

$$\begin{aligned} a_j(p) &= \sum_{m=0}^{n} P(S_n = m)\, K_j(m, n, p) && (9)\\ &= -\frac{1}{j} \sum_{k=0}^{j-1} \Big[a_k(p)\, \gamma_{j-k}(p)\Big] && (10)\\ &= \sum_{k=0}^{j} \binom{n-k}{j-k} \frac{1}{k!} (-p)^{j-k}\, \mu_{(k)} && (11)\\ &= \frac{1}{2\pi\, \alpha^j} \int_0^{2\pi} e^{-ijx} \prod_{k=1}^{n} \Big[1 + (p_k - p)\alpha e^{ix}\Big]\, dx. && (12) \end{aligned}$$

Remarks. 1. It follows from (10) and (11) that $a_j(p)$ can be considered as a function of $(\gamma_1(p), \dots, \gamma_j(p))$ or of $(\mu_{(1)}, \dots, \mu_{(j)}, p)$.

2. Using (10), we get

$$a_1(p) = -\gamma_1(p), \qquad a_2(p) = \frac{1}{2}[\gamma_1(p)^2 - \gamma_2(p)], \tag{13}$$

$$a_3(p) = -\frac{1}{6}\gamma_1(p)^3 + \frac{1}{2}\gamma_1(p)\,\gamma_2(p) - \frac{1}{3}\gamma_3(p), \tag{14}$$

leading to $a_1(\overline{p}) = 0$, $a_2(\overline{p}) = -\gamma_2/2$, and $a_3(\overline{p}) = -\gamma_3/3$.

3. For $m \in \mathbf{Z}_+$, we have

$$\mathcal{B}_1(n,p)(\{m\}) = b(m,n,p)\Big[1 - \frac{\gamma_1(p)\,(m - np)}{npq}\Big],$$

and, if $n \in \{2, 3, \dots\}$,

$$\begin{aligned} \mathcal{B}_2(n,p)(\{m\}) &= b(m,n,p)\Big[1 - \frac{\gamma_1(p)\,(m-np)}{npq}\\ &+ \frac{\gamma_1(p)^2 - \gamma_2(p)}{2\,n(n-1)\,[pq]^2}\Big[m^2 - (1 + 2(n-1)p)m + n(n-1)p^2\Big]\Big]. \end{aligned}$$

Note that $\mathcal{B}(n, \overline{p}) = \mathcal{B}_1(n, \overline{p})$. For $3 \le s \le n$, $\mathcal{B}_s(n,p)$ can be evaluated by using (8), (10), and (39).

The following proposition shows that the first s moments of P^{S_n} and $\mathcal{B}_s(n,p)$ coincide.

Proposition 2 *For* $s \in \{0, \dots, n\}$, $k \in \{0, \dots, s\}$, *and* $\mu_{(k)}$ *as in Proposition 1,*

$$\sum_{m=k}^{n} m^{[k]}\, \mathcal{B}_s(n,p)(\{m\}) = \mu_{(k)}.$$

2.2 The bounds and asymptotic relations for the distances. In what follows, we use the notations

$$\eta(p) = 2\gamma_2(p) + \gamma_1(p)^2, \qquad \theta(p) = \frac{\eta(p)}{2npq} \qquad \text{for } p \in (0,1),$$

leading to $\eta(\overline{p}) = 2\gamma_2$ and $\theta(\overline{p}) = \theta$.

Theorem 2 *Let* $s \in \{0, \ldots, n\}$, $p \in (0, 1)$,

$$C_1(s) = \frac{\sqrt{e}(s+1)^{1/4}}{2}, \qquad C_2(s) = \frac{(2\pi)^{1/4} \exp(1/[24(s+1)]) 2^{(s-1)/2}}{(s+1)^{1/4}\sqrt{s!}},$$

$$C_3(s) = \frac{\sqrt{s+1}}{2}\Big(1 + \sqrt{\frac{\pi}{2(s+1)}}\Big).$$

Then

$$d_\tau(P^{S_n}, \mathcal{B}_s(n,p)) \leq C_1(s)\, \theta(p)^{(s+1)/2}\, \frac{1 - \frac{s}{s+1}\sqrt{\theta(p)}}{(1-\sqrt{\theta(p)})^2} \qquad \text{if } \theta(p) < 1, \tag{15}$$

$$d_\tau(P^{S_n}, \mathcal{B}_s(n,p)) \leq C_2(s)\, \eta(p)^{(s+1)/2} \Big(1 + \sqrt{2\eta(p)}\Big) \exp(2\eta(p)), \tag{16}$$

$$d_\pi(P^{S_n}, \mathcal{B}_s(n,p)) \leq C_3(s)\, \frac{\theta(p)^{(s+1)/2}(1 - \frac{s}{s+1}\sqrt{\theta(p)})}{\sqrt{npq}\,(1-\sqrt{\theta(p)})^2} \qquad \text{if } \theta(p) < 1. \tag{17}$$

Remarks. 1. For $p = \overline{p}$ and $s = 1$, we get estimates close to the upper bounds from (1).

2. Generally, an inequality $d_\pi(P^{S_n}, \mathcal{B}(n,\overline{p})) \leq c\,\theta[n\overline{p}\,\overline{q}]^{-1/2}$ with an absolute constant $c \in (0,\infty)$ cannot hold: If $k \in \mathbf{N}$, $n = 2k$, $p_1 = \ldots = p_k = 1$, and $p_{k+1} = \ldots = p_{2k} = 0$, then $\overline{p} = 1/2$,

$$d_\pi(P^{S_n}, \mathcal{B}(n,\overline{p})) \geq \Big|P(S_n = k) - b\Big(k, 2k, \frac{1}{2}\Big)\Big| = \Big[1 - \frac{(2k)!}{(k!)^2\, 2^{2k}}\Big] \overset{(k\to\infty)}{\longrightarrow} 1,$$

and $\theta[n\overline{p}\,\overline{q}]^{-1/2} = \sqrt{2/k} \overset{(k\to\infty)}{\longrightarrow} 0$.

Corollary 1 *Let* $n \in \{2, 3, \ldots\}$ *and*

$$x_\pm(n,p) = \Big\lfloor 1/2 + (n-1)p \pm \sqrt{1/4 + (n-1)pq}\Big\rfloor.$$

Then $d_\tau(P^{S_n}, \mathcal{B}(n,\overline{p})) = H + R$, *where*

$$H = \frac{|a_2(\overline{p})|}{2}\, \|\Delta^2 b(\cdot, n-2, \overline{p})\|_1 \tag{18}$$

$$= \frac{\gamma_2}{2\,(n-1)\overline{p}\,\overline{q}} \Big[\Big[x_+(n,\overline{p}) - (n-1)\overline{p}\Big]\, b(x_+(n,\overline{p}), n-1, \overline{p}) + \Big[(n-1)\overline{p} - x_-(n,\overline{p})\Big]\, b(x_-(n,\overline{p}), n-1, \overline{p})\Big] \tag{19}$$

and

$$|R| \leq d_\tau(P^{S_n}, \mathcal{B}_2(n,\overline{p}))$$

$$\leq |\gamma_3| \min\Big\{\frac{\sqrt{3}}{2\,[n\overline{p}\,\overline{q}]^{3/2}}, \frac{4}{3}\Big\} + \min\Big\{1.166\, \frac{\theta^2\,(1 - \frac{3}{4}\sqrt{\theta})}{(1-\sqrt{\theta})^2},\ 3.695\,\gamma_2^2\,(1 + 2\sqrt{\gamma_2})\exp(4\gamma_2)\Big\}. \tag{20}$$

Remark. We have $0 \leq x_-(n,p) \leq (n-1)p \leq x_+(n,p) \leq n-1$.

From the theory of orthogonal polynomials it follows that the zeros of the Krawtchouk polynomials $K_j(x,n,p)$ for $n \in \mathbf{N}$, $j \in \{1, \ldots, n\}$, and $p \in (0,1)$, are real, simple, and lie in the interval $(0,n)$. In what follows, we use this fact for $j = 3$.

Corollary 2 *Let $n \in \{2,3,\dots\}$ and $0 < x_1 < x_2 < x_3 < n+1$ be the zeros of $K_3(x, n+1, \overline{p}) \in \mathbf{R}[x]$. Then $d_\pi(P^{S_n}, \mathcal{B}(n,\overline{p})) = H' + R'$, where*

$$\begin{aligned} H' &= |a_2(\overline{p})|\,\|\Delta^2 b(\cdot, n-2, \overline{p})\|_\infty & (21)\\ &= \frac{\gamma_2}{n(n-1)[\overline{p}\,\overline{q}]^2} \max\Big\{|K_2(\lfloor x_i\rfloor, n, \overline{p})|\, b(\lfloor x_i\rfloor, n, \overline{p}) \,\Big|\, i \in \{1,2,3\}\Big\} & (22)\end{aligned}$$

and

$$\begin{aligned} |R'| &\le d_\pi(P^{S_n}, \mathcal{B}_2(n,\overline{p}))\\ &\le |\gamma_3| \min\Big\{\frac{2.398}{[n\overline{p}\,\overline{q}]^2}, 1\Big\}\\ &\quad + \min\Big\{\frac{1.627\,\theta^2\,(1-\frac{3}{4}\sqrt{\theta})}{\sqrt{n\overline{p}\,\overline{q}}\,(1-\sqrt{\theta})^2},\ 3.695\,\gamma_2^2\,(1+2\sqrt{\gamma_2})\exp(4\gamma_2)\Big\}. & (23)\end{aligned}$$

Now we present some asymptotic relations. Let us consider the following triangular scheme: We let n and $p_1,\dots,p_n$ depend on an additional parameter $k \in \mathbf{N}$ and assume that $k \to \infty$. Sometimes, we write $\theta^{(k)}$ for θ.

Theorem 3 *Let us assume that $\gamma_2 \ne 0$ for sufficiently large k. Set*

$$v = \min\Big\{1,\ \frac{|\gamma_3|}{\gamma_2\sqrt{n\overline{p}\,\overline{q}}} + \frac{1}{n\overline{p}\,\overline{q}} + \theta\Big\}.$$

Then

$$d_\tau(P^{S_n}, \mathcal{B}(n,\overline{p})) = \frac{\theta\,[1+\mathcal{O}(v)]}{\sqrt{2\pi e}}, \qquad (24)$$

$$d_\pi(P^{S_n}, \mathcal{B}(n,\overline{p})) = \frac{\theta\,[1+\mathcal{O}(v)]}{2\sqrt{2\pi\, n\overline{p}\,\overline{q}}} \qquad \text{if } \limsup_{k\to\infty} \theta^{(k)} < 1. \qquad (25)$$

Remarks. 1. Since $|\gamma_3| \le \gamma_2$, the preceeding asymptotics remain valid if we replace v with $\min\{1, [n\overline{p}\,\overline{q}]^{-1/2} + \theta\}$.

2. The asymptotics of Theorem 3 have counterparts in the Poisson approximation: Prohorov [14, Theorem 2] gave an asymptotic formula for the total variation distance between a binomial and a Poisson distribution with the same means. As has been observed by Barbour et al. [1, p. 2], the statement of Prohorov's Theorem 2 is inaccurate. A correct version, in our notations, is: $d_\tau(\mathcal{B}(n,p), \mathcal{P}(np)) = (2\pi e)^{-1/2} p[1 + \mathcal{O}(\min\{1, [np]^{-1/2} + p\})]$, where $\mathcal{P}(np)$ denotes the Poisson distribution with mean np. A proof can be found in Roos [17]. In Prohorov's version, the "$+p$" is missing, which invalidates his result, for example, for $p = 1$, $n \to \infty$. Generalizations of this result for the Poisson approximation to the Poisson binomial distribution can be found in Deheuvels et al. [2], [3], and Roos [16], [17].

3 Proofs

Proof of the remark in Section 1.2. The first part of the asserted inequality follows from

$$\gamma_2 = \sum_{j=1}^n \Big(\frac{p_{\max}+p_{\min}}{2} - p_j\Big)(\overline{p} - p_j) \le \frac{\delta}{2}\sum_{j=1}^n |\overline{p} - p_j| \le \delta\, n\overline{p}\,\overline{q},$$

since, letting $A_+ = \{j \in \{1,\dots,n\} \,|\, p_j \ge \overline{p}\}$ and $A_- = \{1,\dots,n\} \setminus A_+$,

$$\frac{1}{2}\sum_{j=1}^n |\overline{p} - p_j| = n\overline{p}\,\overline{q} - \Big(\overline{p}\sum_{j\in A_+}(1-p_j) + \overline{q}\sum_{j\in A_-} p_j\Big) \le n\overline{p}\,\overline{q}.$$

By using $\gamma_2 \le \sum_{j=1}^{n}(\frac{p_{\max}+p_{\min}}{2} - p_j)^2$, the remaining part of the inequality is easily shown. ■

Proof of Theorem 1. We show the assertion by using generating functions and the equality

$$\sum_{m=0}^{n+j} \Delta^j b(m,n,p)\, z^m = [1+p(z-1)]^n\,(z-1)^j \tag{26}$$

for $j,n \in \mathbf{Z}_+$, $z \in \mathbf{C}$, which is easily proved by induction over j. For $z \in \mathbf{C}$, we have by independence

$$\begin{aligned}\sum_{m=0}^{\infty} P(S_n = m)\, z^m &= \prod_{j=1}^{n}\Big[[1+p(z-1)] + (p_j-p)(z-1)\Big]\\ &\overset{(4)}{=} \sum_{j=0}^{n} a_j(p)\,(z-1)^j\,[1+p(z-1)]^{n-j}\\ &\overset{(26)}{=} \sum_{m=0}^{n}\Big[\sum_{j=0}^{n} a_j(p)\,\Delta^j b(m,n-j,p)\Big]\, z^m.\end{aligned}$$

The proof is completed by comparing the power series. ■

Proof of Proposition 1. We use the generating functions

$$\sum_{j=0}^{n} a_j(p)\, z^j = \prod_{k=1}^{n}[1+(p_k-p)z], \qquad z \in \mathbf{C}, \tag{27}$$

and, for $n,m \in \mathbf{Z}_+$, $n \ge m$, $z \in \mathbf{C}$,

$$\sum_{j=0}^{n} K_j(m,n,p)\, z^j = [1+qz]^m\,(1-pz)^{n-m}. \tag{28}$$

Equality (27) is easy to prove, and for (28), see Szegö [20, (2.82.4)]. For $z \in \mathbf{C}$, $|z|<1$, we have

$$\begin{aligned}\sum_{j=0}^{n} a_j(p)\, z^j &\overset{(27)}{=} (1-pz)^n \prod_{k=1}^{n}\Big[1+p_k\Big(\frac{1+q\,z}{1-p\,z}-1\Big)\Big]\\ &= \sum_{m=0}^{n} P(S_n=m)\Big[\frac{1+q\,z}{1-pz}\Big]^m (1-pz)^n\\ &\overset{(28)}{=} \sum_{j=0}^{n}\Big[\sum_{m=0}^{n} P(S_n=m)\,K_j(m,n,p)\Big]\, z^j,\end{aligned}$$

$$\begin{aligned}\sum_{j=1}^{n} j\, a_j(p)\, z^j &\overset{(27)}{=} -\Big[\sum_{k=1}^{n}\frac{(p-p_k)z}{1-(p-p_k)z}\Big]\prod_{k=1}^{n}[1+(p_k-p)z]\\ &= -\Big[\sum_{j=1}^{\infty}\gamma_j(p)\, z^j\Big]\Big[\sum_{j=0}^{n} a_j(p)\, z^j\Big] = \sum_{j=1}^{n}\Big[-\sum_{k=0}^{j-1} a_k(p)\gamma_{j-k}(p)\Big] z^j,\end{aligned}$$

and, because of the binomial theorem and $\sum_{k=0}^{n}\frac{\mu_{(k)}}{k!}\, z^k = \prod_{j=1}^{n}(1+p_j z)$,

$$\begin{aligned}\sum_{j=0}^{n} a_j(p)\, z^j &\overset{(27)}{=} (1-p\,z)^n \prod_{j=1}^{n}\Big(1+\frac{p_j\,z}{1-pz}\Big)\\ &= \sum_{k=0}^{n}\frac{\mu_{(k)}}{k!}\, z^k\,(1-pz)^{n-k} = \sum_{j=0}^{n}\Big[\sum_{k=0}^{j}\binom{n-k}{j-k}\frac{(-p)^{j-k}\,\mu_{(k)}}{k!}\Big] z^j,\end{aligned}$$

from which (9), (10), and (11) follow. Equality (12) follows from (27) and Cauchy's theorem. ■

Proof of Proposition 2. For $j \in \{0, \dots, n\}$ and $k \in \{j, \dots, n\}$, we have

$$\sum_{m=k}^{n} \Delta^j b(m, n-j, p)\, m^{[k]} \overset{(26)}{=} \frac{d^k}{dz^k}\Big[(1+p(z-1))^{n-j}(z-1)^j\Big]\Big|_{z=1}$$
$$= \sum_{i=0}^{k}\binom{k}{i}\Big[\frac{d^{k-i}}{dz^{k-i}}(1+p(z-1))^{n-j}\frac{d^i}{dz^i}(z-1)^j\Big]\Big|_{z=1} = \frac{k!\,(n-j)!\,p^{k-j}}{(k-j)!\,(n-k)!}.$$

Similarly, $\sum_{m=k}^{n} \Delta^j b(m, n-j, p)\, m^{[k]} = 0$ for $k \in \{0, \dots, j-1\}$. Hence

$$\sum_{m=k}^{n} \mathcal{B}_s(n,p)(\{m\})\, m^{[k]} \overset{(5)}{=} \sum_{j=0}^{k} a_j(p)\,\frac{k!\,(n-j)!\,p^{k-j}}{(k-j)!\,(n-k)!} \overset{(11)}{=} \mu_{(k)}$$

for $s \in \{0, \dots, n\}$ and $k \in \{0, \dots, s\}$, giving the assertion. ■

To prove Theorem 2, we need the following four lemmas. Remember that $0^0 = 1$.

Lemma 1 *Let $I_0(x) = \sum_{m=0}^{\infty}(x/2)^{2m}/(m!)^2$ be the modified Bessel function of the first kind and order 0, $\beta(x) = e^{-x^2/4}I_0(x)$, $x \in \mathbf{R}$. Then, for $j \in \{1, \dots, n\}$ and $\gamma_2(p) \neq 0$,*

$$|a_j(p)| \leq \left[\frac{2\gamma_2(p) + \frac{(n-j)}{n}\gamma_1(p)^2}{2j}\right]^{j/2} \frac{n^{(n-j)/2}}{(n-j)^{(n-j)/2}} \times \beta\left(\sqrt{\frac{2j(n-j)\gamma_1(p)^2}{2n\gamma_2(p) + (n-j)\gamma_1(p)^2}}\right). \tag{29}$$

Proof. Let $\alpha, s \in (0, \infty)$ be arbitrary and $j \in \{1, \dots, n\}$. Using (12), the inequality $1 + x \leq e^x$, and the equality $I_0(x) = \frac{1}{2\pi}\int_0^{2\pi}\exp(x\cos y)\,dy$, $x \in \mathbf{R}$, we get

$$|a_j(p)| \leq \frac{1}{2\pi\alpha^j}\int_0^{2\pi}\prod_{k=1}^{n}\Big|1 + (p_k - p)\alpha e^{ix}\Big|\,dx$$
$$= \frac{1}{2\pi\alpha^j s^{n/2}}\int_0^{2\pi}\prod_{k=1}^{n}\Big[s + 2s(p_k - p)\alpha\cos x + s(p_k-p)^2\alpha^2\Big]^{1/2} dx$$
$$\leq \frac{\beta(s\gamma_1(p)\alpha)}{\alpha^j s^{n/2}}\exp\left(\frac{n(s-1)}{2} + \frac{\alpha^2}{4}\Big[2s\gamma_2(p) + s^2\gamma_1(p)^2\Big]\right).$$

Inequality (29) follows in the case $j \neq n$ by letting

$$\alpha = \left(\frac{2j}{2s\gamma_2(p) + s^2\gamma_1(p)^2}\right)^{1/2}, \qquad s = \frac{n-j}{n}.$$

If $j = n$, then (29) follows from the inequality between the arithmetic and geometric means:

$$|a_n(p)| \overset{(4)}{=} \left(\prod_{r=1}^{n}(p_r - p)^2\right)^{1/2} \leq \left(\frac{\gamma_2(p)}{n}\right)^{n/2}.$$

The lemma is proved. ■

Remark. The inequalities (29), $\beta(x) \le 1$ for $x \in \mathbf{R}$, and $(n-j)/n \le 1$ for $j \in \{1, \dots, n\}$ lead to the estimate

$$|a_j(p)| \le \Big[\frac{\eta(p)}{2j}\Big]^{j/2} \frac{n^{(n-j)/2}}{(n-j)^{(n-j)/2}}, \qquad j \in \{1, \dots, n\}, \tag{30}$$

which also holds in case of $\gamma_2(p) = 0$. We will use (30) in the proof of Theorem 2.

Lemma 2 *Let* $c_j = \frac{\sqrt{e}}{2}(1 + \sqrt{\pi/(2j)})$ *for* $j \in \mathbf{N}$, *and* $f_{j,n} : [0,1] \longrightarrow \mathbf{R}$ *with*

$$f_{j,n}(t) = \int_0^{\pi/2} \Big[1 - t \sin^2 x\Big]^{n/2} \sin^j x \, dx, \qquad j, n \in \mathbf{Z}_+, \, t \in [0,1].$$

Then

$$\|\Delta^j b(\cdot, n, p)\|_\infty \le \frac{2^{j+1}}{\pi} f_{j,n}(4pq), \qquad j, n \in \mathbf{Z}_+, \tag{31}$$

and, for $j \in \mathbf{N}$, $n \in \mathbf{Z}_+$, $t \in (0,1]$,

$$f_{j,n}(t) \le \frac{c_j \, \pi}{2^{j+1}} \Big(\frac{n}{n+j+1}\Big)^{n/2} \Big(\frac{4j}{(n+j+1)t}\Big)^{(j+1)/2}. \tag{32}$$

Proof. Using (26) and Cauchy's theorem, we get

$$\Delta^j b(m, n, p) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-ixm} \Big[1 + p(e^{ix} - 1)\Big]^n (e^{ix} - 1)^j \, dx \tag{33}$$

for $j, m, n \in \mathbf{Z}_+$. By (33), (31) is easily shown. Observe that $4pq \in [0,1]$. To prove (32), Shorgin's [18, proof of his Lemma 6] inequality

$$\int_0^{\pi/2} \exp(-2s \sin^2 x) \sin^j x \, dx \le c_j \, \pi \Big(\frac{j}{4se}\Big)^{(j+1)/2} \tag{34}$$

for $j \in \mathbf{N}$, $s \in (0, \infty)$, and the estimate $x \le e^{x-1}$ for $x \in \mathbf{R}$ are used: For $t \in (0,1]$, $j, n \in \mathbf{N}$, and arbitrary $y \in (0, \infty)$,

$$\begin{aligned} f_{j,n}(t) &= \frac{1}{y^{n/2}} \int_0^{\pi/2} \Big(y - y\, t \sin^2 x\Big)^{n/2} \sin^j x \, dx \\ &\le \frac{e^{(y-1)\, n/2}}{y^{n/2}} \int_0^{\pi/2} \exp\Big(-2\Big(\frac{y\, n\, t}{4}\Big) \sin^2 x\Big) \sin^j x \, dx \\ &\le c_j \, \pi \, g(y) e^{-n/2} \Big(\frac{j}{n\, t\, e}\Big)^{(j+1)/2}, \end{aligned}$$

where $g : (0, \infty) \longrightarrow \mathbf{R}$, $g(y) = e^{ny/2} y^{-(n+j+1)/2}$. The function g attains its minimum at $y_0 = (n+j+1)/n$. Substituting $y = y_0$, (32) is shown for $n \ne 0$. In the case $n = 0$, $j \in \mathbf{N}$, and $t \in (0,1]$, we have

$$f_{j,0}(t) = \int_0^{\pi/2} \sin^j x \, dx \le \int_0^{\pi/2} \sin x \, dx = 1 \le c_j \, \pi \Big(\frac{j}{(j+1)t}\Big)^{(j+1)/2},$$

since $[j/(j+1)]^{(j+1)/2} \ge 1/2$ and $c_j \ge \sqrt{e}/2$ for $j \in \mathbf{N}$. This shows the validity of (32) also in this case and completes the proof. ∎

Remark. Combining the estimates (31) and (32), we get:

$$\|\Delta^j b(\cdot, n, p)\|_\infty \le c_j \Big(\frac{n}{n+j+1}\Big)^{n/2} \Big(\frac{j}{(n+j+1)pq}\Big)^{(j+1)/2} \tag{35}$$

for $j \in \mathbf{N}$, $n \in \mathbf{Z}_+$, $p \in (0,1)$, and c_j as in Lemma 2.

The following lemma is needed for Lemma 4.

Lemma 3 *For $n \in \mathbf{N}$ and $j \in \{1, \dots, n\}$,*

$$\frac{n^n}{j^j\,(n-j)^{n-j}} \le e\,\sqrt{j}\binom{n}{j}. \tag{36}$$

Proof. It suffices to prove (36) for $n \in \{2, 3, \dots\}$ and $j \in \{1, \dots, n-1\}$. For these values, let

$$T(n,j) = \frac{n^n\, j!\,(n-j)!}{j^j\,(n-j)^{n-j}\, n!}\sqrt{\frac{n}{j\,(n-j)}}.$$

In what follows, it is shown that $T(n,j) \le [n/(n-1)]^{n-1/2}$, from which inequality (36) can be derived. Let $f(x) = [(x+1)/x]^{x+1/2}$, $x \in (0,\infty)$. Since $f(x)$ is decreasing and

$$\frac{T(n,j)}{T(n,j+1)} = \frac{f(j)}{f(n-j-1)}, \qquad n \in \{3,4,\dots\},\ j \in \{1,\dots,n-2\},$$

we have $T(n,j) \le T(n,j+1)$ if and only if $n-1 \le 2j$. Hence

$$T(n,j) \le \max\{T(n,1), T(n,n-1)\} = \Big(\frac{n}{n-1}\Big)^{n-1/2},$$

giving the assertion. ■

Lemma 4 *For $j \in \mathbf{N}$, $n \in \mathbf{Z}_+$, and $p \in (0,1)$,*

$$\|\Delta^j b(\cdot,n,p)\|_1 \le \left[\binom{n+j}{j}[pq]^j\right]^{-1/2} \tag{37}$$

$$\le \sqrt{e}\,j^{1/4}\Big(\frac{n}{n+j}\Big)^{n/2}\Big(\frac{j}{(n+j)pq}\Big)^{j/2}. \tag{38}$$

Proof. We need the following relations:

$$K_j(m,n+j,p)\,b(m,n+j,p) = \binom{n+j}{j}[pq]^j\,\Delta^j b(m,n,p), \tag{39}$$

$$\sum_{m=0}^{n} b(m,n,p)\,K_j(m,n,p)\,K_r(m,n,p) = \delta_{j,r}\binom{n}{j}[pq]^j \tag{40}$$

for $m,n,j,r \in \mathbf{Z}_+$ and $p \in [0,1]$. Here $\delta_{j,r}$ is the Kronecker symbol. Identity (39) follows from (26) and

$$\sum_{m=0}^{n+j} K_j(m,n+j,p)\,b(m,n+j,p)\,z^m \overset{(8)}{=} \binom{n+j}{j}[pq]^j(1+p(z-1))^n(z-1)^j.$$

For (40), see Szegö [20, (2.82.6)]. The inequalities (37) and (38) follow from (39), (40), (36), and Cauchy's inequality:

$$\|\Delta^j b(\cdot,n,p)\|_1 = \left[\binom{n+j}{j}[pq]^j\right]^{-1}\sum_{m=0}^{n+j} b(m,n+j,p)\,|K_j(m,n+j,p)|$$

$$\le \left[\binom{n+j}{j}[pq]^j\right]^{-1/2} \le e^{1/2}\,j^{1/4}\Big(\frac{n}{n+j}\Big)^{n/2}\Big(\frac{j}{(n+j)\,p\,q}\Big)^{j/2}.$$

The lemma is proved. ■

Proof of Theorem 2. It suffices to prove the assertion in the case $s \le n-1$. Using (3), (30), (38), and the inequality $j^{1/4} \le j(s+1)^{-3/4}$ for $j \ge s+1$, we obtain in the case $\theta(p) < 1$,

$$d_\tau(P^{S_n}, \mathcal{B}_s(n,p)) \le \frac{1}{2} \sum_{j=s+1}^{n} |a_j(p)| \, \|\Delta^j b(\cdot, n-j, p)\|_1$$
$$\le \frac{\sqrt{e}}{2} \sum_{j=s+1}^{n} \theta(p)^{j/2} j^{1/4} \le C_1(s)\, \theta(p)^{(s+1)/2} \frac{1 - \frac{s}{s+1}\sqrt{\theta(p)}}{(1-\sqrt{\theta(p)})^2},$$

giving (15). To prove (16), we use (3), (30), the inequality

$$\|\Delta^j b(\cdot, n, p)\|_1 \le 2^j, \qquad j, n \in \mathbf{Z}_+, \; p \in [0,1], \tag{41}$$

Stirling's formula (see Feller [6, p. 54])

$$j! = \sqrt{2\pi}\, j^{j+1/2} \exp\left(\vartheta_j - j\right), \qquad \vartheta_j \in \left[\frac{1}{12j+1}, \frac{1}{12j}\right], \; j \in \mathbf{N},$$

and the inequalities $1 + x \le e^x$ for $x \in \mathbf{R}$, and

$$\sum_{j=m}^{\infty} \frac{x^j}{\sqrt{j!}} \le \frac{x^m}{\sqrt{m!}} \sum_{j=0}^{\infty} \frac{x^j}{\sqrt{j!}} \binom{j}{\lfloor j/2 \rfloor}^{1/2} \le \frac{x^m}{\sqrt{m!}} (1+x)\exp(x^2)$$

for $x \in [0, \infty)$, $m \in \mathbf{Z}_+$. We get

$$\begin{aligned} d_\tau(P^{S_n}, \mathcal{B}_s(n,p)) &\le \frac{1}{2} \sum_{j=s+1}^{n} |a_j(p)|\, 2^j \le \frac{1}{2} \sum_{j=s+1}^{\infty} \left(\frac{2\,\eta(p)\, e}{j}\right)^{j/2} \\ &\le \frac{(2\pi)^{1/4} \exp(1/[24(s+1)])}{2} \sum_{j=s+1}^{\infty} \frac{(2\,\eta(p))^{j/2}}{\sqrt{j!}} j^{1/4} \\ &\le C_2(s)\, \eta(p)^{(s+1)/2} \left(1 + \sqrt{2\eta(p)}\right) \exp(2\eta(p)). \end{aligned}$$

The proof of (17) is similar. Here we use (3), (30), (35), and the inequalities $1 + x \le e^x$ for $x \in \mathbf{R}$ and $j^{1/2} \le j(s+1)^{-1/2}$ for $j \ge s+1$. ■

Proof of Corollary 1. Only (19) and (20) require a proof. We have

$$d_\tau(P^{S_n}, \mathcal{B}_2(n, \overline{p})) \le \frac{1}{2} |a_3(\overline{p})| \, \|\Delta^3 b(\cdot, n-3, \overline{p})\|_1 + d_\tau(P^{S_n}, \mathcal{B}_3(n, \overline{p}))$$

if $n \ge 3$, and $R = 0$ in case of $n = 2$. Inequality (20) can be shown by using Theorem 2, (41), $a_3(\overline{p}) = -\gamma_3/3$ and, in case of $n \ge 3$,

$$\|\Delta^3 b(\cdot, n-3, \overline{p})\|_1 \overset{(37)}{\le} \left[\binom{n}{3} [\overline{p}\,\overline{q}]^3\right]^{-1/2} \le \left(\frac{3}{n\overline{p}\,\overline{q}}\right)^{3/2}.$$

Identity (19) follows from $a_2(\overline{p}) = -\gamma_2/2$ and the next lemma. ■

Lemma 5 *Let* $n \in \{2, 3, \dots\}$, $p \in (0,1)$, *and* $x_\pm(n,p)$ *as in Corollary 1. Then*

$$\begin{aligned} \|\Delta^2 b(\cdot, n-2, p)\|_1 &= \frac{2}{(n-1)pq} \Big[[x_+(n,p) - (n-1)p]\, b(x_+(n,p), n-1, p) \\ &\quad + [(n-1)p - x_-(n,p)]\, b(x_-(n,p), n-1, p) \Big]. \end{aligned} \tag{42}$$

Proof. Let $v_m = \Delta^2 b(m, n-2, p)$ for $m \in \mathbf{Z}_+$. Observe that $x_\pm(n,p)$ are the integer parts of the zeros of $K_2(x,n,p) = \frac{1}{2}[x^2-(1+2(n-1)p)x+n(n-1)p^2] \in \mathbf{R}[x]$, and hence,

$$\begin{aligned}
\|(v_m)_{m\in\mathbf{Z}_+}\|_1 &\overset{(39)}{=} \binom{n}{2}^{-1}[pq]^{-2}\sum_{m=0}^{n}|K_2(m,n,p)|\,b(m,n,p)\\
&= \sum_{m=0}^{x_-(n,p)} v_m - \sum_{m=x_-(n,p)+1}^{x_+(n,p)} v_m + \sum_{m=x_+(n,p)+1}^{\infty} v_m\\
&= 2\Big[\Delta^1 b(x_+(n,p), n-2, p) - \Delta^1 b(x_-(n,p), n-2, p)\Big].
\end{aligned}$$

By using (39), the proof is completed. ∎

Corollary 2 is easily proved by using the following lemma.

Lemma 6 *Let $n \in \{2,3,\ldots\}$, $p \in (0,1)$, and $0 < x_1 < x_2 < x_3 < n+1$ be the zeros of $K_3(x, n+1, p) \in \mathbf{R}[x]$. Then*

$$\begin{aligned}
\|\Delta^2 b(\cdot, n-2, p)\|_\infty &= \frac{2}{n(n-1)[pq]^2}\\
&\quad\times \max\Big\{|K_2(\lfloor x_i\rfloor, n, p)|\,b(\lfloor x_i\rfloor, n, p)\,\Big|\, i \in \{1,2,3\}\Big\}. \qquad (43)
\end{aligned}$$

Proof. For $m \in \{0,\ldots,n-1\}$ and v_m as in the proof of Lemma 5,

$$\begin{aligned}
v_m \le v_{m+1} \;&\Leftrightarrow\; \Delta^3(m+1, n-2, p) \le 0 \overset{(39)}{\Leftrightarrow} K_3(m+1, n+1, p) \le 0\\
&\Leftrightarrow\; [0 \le m+1 \le x_1 \quad\text{or}\quad x_2 \le m+1 \le x_3]
\end{aligned}$$

and

$$v_m \ge v_{m+1} \;\Leftrightarrow\; [x_1 \le m+1 \le x_2 \;\text{ or }\; x_3 \le m+1 \le n].$$

Hence, letting $x_0 = 0$ and $x_4 = n$,

$$\sup_{m\in\mathbf{Z}_+} v_m = \max_{i\in\{1,3\}} v_{\lfloor x_i\rfloor}, \qquad \inf_{m\in\mathbf{Z}_+} v_m = \min_{i\in\{0,2,4\}} v_{\lfloor x_i\rfloor}.$$

This yields $\sup_{m\in\mathbf{Z}_+}|v_m| = \max_{i\in\{0,\ldots,4\}}|v_{\lfloor x_i\rfloor}|$. Since $0 \le v_0 \le v_{\lfloor x_1\rfloor}$ and $0 \le v_n \le v_{\lfloor x_3\rfloor}$, the assertion follows from (39). ∎

For the proof of Theorem 3, we need the following two lemmas.

Lemma 7 *Let $p \in (0,1)$, $n \in \mathbf{N}$, and $m \in \{0,\ldots,n\}$. Further, let $x = (m-np)[npq]^{-1/2}$, satisfying $|x| \le A$, where $A \in (0,\infty)$ is an absolute constant. Then*

$$b(m,n,p) = \frac{e^{-x^2/2}}{\sqrt{2\pi npq}}\Big[1 + \frac{(x^3-3x)(q-p)}{6\sqrt{npq}} + \mathcal{O}\Big(\frac{1}{npq}\Big)\Big],$$

where the constant intervening in $\mathcal{O}([npq]^{-1})$ depends on A only.

Proof. See Uspensky [22, p. 135, problem 7] or, in case of fixed p, Rényi [15, p. 151, Theorem 1] . ∎

Lemma 8 *For $p \in (0,1)$ and $n \in \{2,3,\ldots\}$,*

$$\|\Delta^2 b(\cdot, n-2, p)\|_1 = \frac{4}{\sqrt{2\pi e}\,npq}\Big[1 + \mathcal{O}\Big(\frac{1}{npq}\Big)\Big], \qquad (44)$$

$$\|\Delta^2 b(\cdot, n-2, p)\|_\infty = \frac{1}{\sqrt{2\pi}\,[npq]^{3/2}}\Big[1 + \mathcal{O}\Big(\frac{1}{npq}\Big)\Big]. \qquad (45)$$

Proof. Since $\|\Delta^2 b(\cdot, n-2, p)\|_1 \le 4$, we may assume $npq \to \infty$. First, we prove (44) by using Lemma 5. Let $n \in \{2, 3, \ldots\}$, $p \in (0,1)$, $x_\pm(n,p)$ as in Corollary 1, and $y_\pm(n,p) = (x_\pm(n,p) - (n-1)p)[(n-1)pq]^{-1/2}$. For simplicity, we omit the indicated arguments for $x_\pm$ and $y_\pm$. Using Lemma 7 and the relations

$$y_\pm \mp 1 = \mathcal{O}\Big(\frac{1}{\sqrt{npq}}\Big), \qquad e^{-y_\pm^2/2} = \frac{2 \mp y_\pm}{\sqrt{e}} + \mathcal{O}\Big(\frac{1}{npq}\Big),$$

$$\frac{1}{\sqrt{(n-1)pq}} = \frac{1}{\sqrt{npq}} + \mathcal{O}\Big(\frac{1}{[npq]^{3/2}}\Big),$$

we get

$$b(x_\pm, n-1, p) = \frac{2 \mp y_\pm}{\sqrt{2\pi e\, npq}} + \frac{(y_\pm^3 - 3y_\pm)(q-p)}{6\sqrt{2\pi e}\, npq} + \mathcal{O}\Big(\frac{1}{[npq]^{3/2}}\Big).$$

Since

$$x_\pm - (n-1)p = y_\pm\Big(\sqrt{npq} + \mathcal{O}\Big(\frac{1}{\sqrt{npq}}\Big)\Big), \qquad |y_+| - |y_-| = \mathcal{O}\Big(\frac{1}{\sqrt{npq}}\Big),$$

it follows that

$$\begin{aligned}
&[x_+ - (n-1)p]\, b(x_+, n-1, p) + [(n-1)p - x_-]\, b(x_-, n-1, p) \\
&= \frac{2}{\sqrt{2\pi e}} + \frac{[(y_+^4 - y_-^4) - 3(y_+^2 - y_-^2)](q-p)}{6\sqrt{2\pi e}\, npq} + \mathcal{O}\Big(\frac{1}{npq}\Big) \\
&= \frac{2}{\sqrt{2\pi e}} + \mathcal{O}\Big(\frac{1}{npq}\Big),
\end{aligned}$$

giving relation (44). Now we prove (45). For $j \in \mathbf{Z}_+$, $n \in \mathbf{N}$, $t \in (0,1]$, and $f_{j,n}(t)$ as in Lemma 2, we have

$$f_{j,n}(t) \le f_{j+2,n}(t) + \frac{1}{2}\Gamma\Big(\frac{j+1}{2}\Big)\Big(\frac{2}{nt}\Big)^{(j+1)/2}, \tag{46}$$

since, by using $\sin^2 x + \cos^2 x = 1$, $x \in \mathbf{R}$, we get $f_{j,n}(t) = f_{j+2,n}(t) + I_{j,n}(t)$, where

$$\begin{aligned}
I_{j,n}(t) &= \int_0^{\pi/2} \Big[1 - t\sin^2 x\Big]^{n/2} \sin^j x\, \cos^2 x\, dx \\
&= \int_0^t \frac{[1-y]^{n/2}\, y^{(j-1)/2}}{2t^{(j+1)/2}} \sqrt{1 - \frac{y}{t}}\, dy \le \frac{1}{2t^{(j+1)/2}} \int_0^t e^{-ny/2}\, y^{(j-1)/2}\, dy \\
&\le \frac{1}{2}\Gamma\Big(\frac{j+1}{2}\Big)\Big(\frac{2}{nt}\Big)^{(j+1)/2}.
\end{aligned}$$

With $j = 2$, $n \ge 3$, and $p \in (0,1)$, it easily follows from (31), (46), and (32) that, for an absolute constant $M \in (0, \infty)$,

$$\|\Delta^2 b(\cdot, n-2, p)\|_\infty \le \frac{1}{\sqrt{2\pi}\,[npq]^{3/2}} + \frac{M}{[npq]^{5/2}}.$$

To prove (45), we show a similar lower bound. We have

$$\|\Delta^2 b(\cdot, n-2, p)\|_\infty \ge |\Delta^2 b(\lfloor np \rfloor, n-2, p)| \overset{(39)}{=} T_1\, T_2\, T_3,$$

where, letting $r = np - \lfloor np \rfloor$,

$$\begin{aligned} T_1 &= \frac{2}{n(n-1)[pq]^2} = \frac{2}{[npq]^2} + \mathcal{O}\Big(\frac{1}{[npq]^3}\Big), \\ T_2 &= |K_2(\lfloor np \rfloor, n, p)| = \frac{1}{2}|npq + r(p-q-r)| = \frac{npq}{2} + \mathcal{O}(1), \\ T_3 &= b(\lfloor np \rfloor, n, p) = \frac{1}{\sqrt{2\pi npq}} + \mathcal{O}\Big(\frac{1}{[npq]^{3/2}}\Big). \end{aligned}$$

For the latter bound, Lemma 7 has been used. Hence

$$T_1\, T_2\, T_3 = \frac{1}{\sqrt{2\pi}\,[npq]^{3/2}} + \mathcal{O}\Big(\frac{1}{[npq]^{5/2}}\Big),$$

leading to

$$\|\Delta^2 b(\cdot, n-2, p)\|_\infty \geq \frac{1}{\sqrt{2\pi}\,[npq]^{3/2}} - \frac{M'}{[npq]^{5/2}}$$

for an absolute constant $M' \in (0, \infty)$ and sufficiently large npq. The lemma is proved. ∎

Proof of Theorem 3. In the case $\limsup_{k\to\infty} \theta^{(k)} < 1$, the relations (24) and (25) are easily shown by using the Corollaries 1, 2, Lemma 8, and the inequalities (15), (17) for $s = 1, \overline{p} = p$. Since $d_\tau(P^{S_n}, \mathcal{B}(n, \overline{p})) \leq 1$, the above condition for (24) can be dropped. ∎

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