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RESEARCH STATEMENT: IN THE WORLD OF WEBS AND FOAMS

Introduction

My research is concerned with webs (graphs with some representation theoretical meaning), foams (cobordisms between webs) and categorification of knots invariants. These topics are related to many fields in mathematics such as low dimensional topology, differential geometry, geometric topology, representations of quantum groups, category theory and combinatorics.

In this document, I organized my research in four different projects. These projects are interconnected and all related by the facts that the underlying combinatorial objects are webs and/or foams.

A. **The colored $\mathfrak{sl}_N$-invariant.** The aim of this project (partly joint with Matthew Hogancamp) is to categorify the Reshetikhin-Turaev invariant associated with a framed link colored by an arbitrary finite dimensional $\mathfrak{sl}_N$-modules. Following the idea of Khovanov [Kho05] to categorify the colored Jones polynomial, we proceed in two steps: first we construct an explicit *tensor resolution* of every simple finite dimensional $\mathfrak{sl}_N$-modules, then we use an existent categorification of the $\mathfrak{sl}_N$-invariant.

B. **Signature invariant for webs.** The aim of this project (with Catherine Gille) is to define and study signature invariant for imbedded trivalent graphs. The idea is to use the 4-dimensional point of view introduced by Kauffman and Taylor [KT76]: If a graph w bounds a foam F, we construct a four manifold W_F whose signature (up to a correction depending on F) depends only on w. Whereas the construction of our invariant is pretty abstract, we want to explain how to compute the signature of an imbedded graph directly on a diagram.

C. **Categorification of spin networks.** The aim of this project (with Francesco Costantino) is to categorify the integral evaluation of spin networks defined in [Cos09]. The idea is to associate with every planar spin network Γ a finite complex $C(\Gamma)$ of graded $\mathbb{Z}$-module whose graded Euler characteristic is prescribed by the integral evaluation. Such a complex should be endowed with various algebraical structures. In particular, they should be DG-modules over the algebras defined by Khovanov in [Kho05].

D. **Kempe equivalence and dual canonical bases.** The aim of this project is to give a combinatorial recipe in order to compute the dual canonical bases of the spaces $I^\varepsilon := \mathrm{Hom}_{U_q(\mathfrak{sl}_3)}(\mathbb{C}(q), V^{\otimes\varepsilon})$. The computation of dual canonical bases is known to be a very difficult problem. In the case of I^ε Kuperberg [Kup96] gave a very nice combinatorial base thanks the so-called non-elliptic webs. This base is different but close to the dual canonical base. I want to use the Kempe equivalence of colorings of webs to compute the dual canonical base of I^ε.

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A. The colored $\mathfrak{sl}_N$-invariant

The $\mathfrak{sl}_N$-invariant (or Jones polynomial if $N = 2$) is a Laurent polynomial in q associated with an oriented link. This is defined (and computed) by interpreting a link diagram as a morphism in the category of modules over the quantum group $U_q(\mathfrak{sl}_N)$ (we assume here that q is generic): vertical strands are thought of as identity morphisms of the fundamental representation of $U_q(\mathfrak{sl}_N)$ (or its dual, depending on the orientation), cups ans caps are given by the duality structure and crossings are interpreted thanks to the braiding on $U_q(\mathfrak{sl}_N)\text{-}\mathsf{mod}$. Hence a link diagram D represents an endomorphism φ_D of $\mathbb{C}(q)$. The $\mathfrak{sl}_N$-invariant is equal to $\varphi_D(1)$.

The $\mathfrak{sl}_N$-invariant has been categorified in different (but related) ways at different level of generalities [KR08, MSV09, Wu14, QR14] (see [Kho00, Kho02, BN05] for the cases $N = 2, 3$). This means that there exists an $\mathfrak{sl}_N$-homology (the Khovanov homology if $N = 2$): with each link it associates a complex of graded $\mathbb{Z}$-modules (up to homotopy) whose graded Euler characteristic is equal to the $\mathfrak{sl}_N$-invariant of this link.

The colored $\mathfrak{sl}_N$-invariant (or colored Jones polynomial for $N = 2$) is a generalization of the $\mathfrak{sl}_N$-invariant: The strands of the link are now framed and colored by arbitrary finite dimensional representations of $U_q(\mathfrak{sl}_N)$.

The colored Jones polynomial has been categorified[1] by Khovanov [Kho05] using *tensor resolutions* of irreducible representations of $U_q(\mathfrak{sl}_2)$ and the (projective) functioriality of the $\mathfrak{sl}_2$-homology. In [Rob15b], I explain how to mimic this strategy in the case $N = 3$, using the functoriality of the $\mathfrak{sl}_3$-homology proved in [Cla09]. It turns out that the proofs on my paper, would be much simpler if the $\mathfrak{sl}_3$-homology would be known to be functorial for foam-like cobordisms.

Project A.1. *Prove functoriality of the $\mathfrak{sl}_3$-homology for foam-like cobordisms.*

The main difficulty for A.1 is to have a list of movie moves like the one given by Carter and Saito [CS93] in the case of (classical) link cobordism. A beginning of an answer is given Carter [Car] and by Queffelec in its PhD thesis [Que13].

In [Ras10], Rasmussen explained how to extract some geometrical information from the (filtered version of the) $\mathfrak{sl}_2$-homology: he defined the link invariant s_2 (called the Rasmussen invariant) which is a lower bound to the slice genus. This has been generalized for arbitrary N [Wu14, Lob09] and Lewark [Lew13] proved that these invariants are not always equal and conjectured [Lew14] that they are all linearly independent.

There is *a priori* no obstruction to define a similar invariant using the colored $\mathfrak{sl}_2$ and $\mathfrak{sl}_3$-homology. However some technical difficulties have to be handle and the geometrical interpretation of these invariants has to be changed or refined, since in the colored context, we are working with framed knots.

Project A.2. *Define colored $\mathfrak{sl}_2$ and $\mathfrak{sl}_3$ Rasmussen invariant and interpret them geometrically.*

I already mentioned the colored $\mathfrak{sl}_2$ and $\mathfrak{sl}_3$-homology. This naturally yields the following project:

Project A.3 (Joint with Matthew Hogancamp)**.** *Categorify the colored $\mathfrak{sl}_N$-invariant following the strategy of Khovanov* [Kho05].

[1]There is actually a different approach to this problematic, see for example [CK12, FSS12] and project C.

The essentially new[2] ingredient for this categorification is an explicit resolution C_λ of each finite dimensional simple $\mathfrak{sl}_N$-module V_λ in terms of modules of the form $\Lambda^{k_1}(\mathbb{C}^N)\otimes\dots\Lambda^{k_l}(\mathbb{C}^N)$. An interpretation of this resolution C_λ in terms of foams together with the $\mathfrak{sl}_N$-homology given by [QR14] should yield a categorification of the colored $\mathfrak{sl}_n$-invariant.

However this interpretation is not easy since the foams in [QR14] are very rigid. It would be very nice if one could give an alternative description of the $\mathfrak{sl}_N$-homology in terms of "flexible" foam. This amout to giving a way to evaluate closed foams and construct a (web, foam)-TQFT via a universal construction à la [BHMV95] like in [Kho04].

Project A.4. *Give an alternative evaluation of $\mathfrak{sl}_N$-foams and provide a fully foam version of the $\mathfrak{sl}_N$-homology.*

The complexes C_λ are naturally endowed with structures of DG-algebras and DG-coalgebras. This is compatible with the fact that they are conjecturally associated with the unknot colored by V_λ. Therefore, the $\mathfrak{sl}_N$-homology should produce complexes of DG-modules-comodules over the C_λ's.

Project A.5 (Joint with Matthew Hogancamp)**.** *Understand the representation theory of the DG-algebras C_λ's.*

B. Signature invariant for webs

Let L be a link and Σ a Seifert surface of K. The Seifert form ϕ on Σ is a bilinear form on $\mathbb{Q}$-module $H_1(\Sigma,\mathbb{Q})$ given by $\phi(a,b) = \mathrm{lk}(a^+,b^-)$. The signature $\sigma(L)$ of the symmetric bilinear form $\phi+\phi^t$ is called the *signature of* L and is a link invariant [Tro62, Mil68, Mur70].

Kauffman and Taylor [KT76] gave a beautiful 4-dimensional interpretation of the signature: The knot L is in $\mathbb{S}^3$ which is the boundary of B^4. We consider a properly imbedded oriented surface Σ in B^4 whose boundary is equal to L. If W_Σ is the double branched cover of B^4 along Σ, $\sigma(L)$ is equal to the signature of W_Σ as an oriented four manifold. It is even possible to consider non-orientable surfaces, in this case, a correction term depending on the normal Euler number of the surface appears (see [GL78]).

Together with Catherine Gille, we define[3] a signature invariant for colored trivalent graphs imbedded in $\mathbb{S}^3$. We consider an embedded trivalent graph Γ together with a coloring: Every edge has a color a, b or c, such that at every vertex all color are present. Let F be a colored foam properly imbedded in B^4 such that $\partial F = \Gamma$. Interpretating a, b and c as the non trivial elements of $\mathbb{Z}_2\oplus\mathbb{Z}_2$, one can construct W_F the $\mathbb{Z}_2\oplus\mathbb{Z}_2$-branched covering of B^4 along F. It turns out to be an orientable four manifold. We prove that $\sigma(\Gamma) = \sigma(W_F) + e(F)$ only depends on Γ, where $e(F)$ is analogue of the normal Euler number for foams (this should be compare with [GL79]).

Unlike the link case, we do not know yet how to compute the signature of a graph Γ in a combinatorial and systematical way. The main issue is that we need to compute some linking numbers in rational homology spheres and we do not understand yet how these can be read directly on a diagram of Γ. Once we will understand this, we will be able to make some computations.

Project B.1 (with Catherine Gille)**.** *Explain how to compute $\sigma(\Gamma)$ on the a diagram of Γ. Compute the signature of the simplest imbedded trivalent graphs. Relate the signatures of a given imbedded graph for different colorings.*

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[2]This should appear soon in a paper joint with Matthew Hogancamp.

[3]A paper is in preparation.

As we manage to define a signature invariant for imbedded colored trivalent graphs, it is natural to hope that related invariants still make sense for imbedded trivalent graphs.

Project B.2 (with Catherine Gille). *Define Tristram-Levine signature and Alexander modules for imbedded graphs.*

C. Spin networks

A spin network is a 3-regular framed graph with edges labeled by irreducible representations of $U_q(\mathfrak{sl}_2)$ imbedded in $\mathbb{S}^3$. One can evaluate the spin networks: The edges labeled by V_n are replaced by n parrallels strands and by a box representing the nth Jones-Wenzl projector. Around the vertices, the strands are connected one to another in the only possible crossing-less way[4]. Expending the sum defining the Jones-Wenzl projectors, we get a $\mathbb{Z}(q)$-linear combination of links. The evaluation of spin networks is the $\mathbb{Z}(q)$-linear combination of the Jones polynomial of these links.

The Jones-Wenzl projectors have been categorified by Frenkel, Stroppel and Sussan [FSS12] and by Krushkal and Cooper [CK12]. These construction can be extended to give a categorification of the evaluation of spin networks. The coefficients of the formular defining the Jones Wenzl projectors are not integrals. Hence, the both approaches of categorification involve chain complexes of infinite length and are therefore not very hard to manipulate and computations, even in small cases are very complicated.

Costantino [Cos09] gave a new normalization of the evaluation of spin networks. Whereas the classical evaluation is $\mathbb{Z}(q)$-valued, the new evaluation associates with a spin Network an element of $\mathbb{Z}[q, q^{-1}]$. The existence and the simplicity of the colored $\mathfrak{sl}_2$-homology leads to think that a simple categorification of the evaluation of spin networks exists.

Project C.1 (With Francesco Costantino). *Categorify the integral version of the evaluation of spin networks.*

The idea is to use the categorification of the colored Jones polynomial as a guideline: With each spin network Γ should be associated a complex C_Γ of graded $\mathbb{Z}$-module. Conjecturally, these complexes have many nice algebraical structures: If Γ contains an edge colored by n, C_Γ should be a DG-module-comodule over the algebra C_n described in [Kho05] (the evaluation actually suggests that C_Γ should be a free C_n-module). If Γ is θ-shape (or more generally if it has an axe of symmetry), its associated complex should be a DG-algebra-coalgebra. All these structures should guide us in our construction.

One can define spin networks for other quantum groups. The nice normalization of the evaluation defined by Costantino in the case of $U_q(\mathfrak{sl}_2)$ might have analogue for these other groups.

Project C.2. *Define integral evaluations of $U_q(\mathfrak{sl}_N)$-spin networks. Eventually categorify these new evaluations.*

D. Kempe equivalence of colorings and dual canonical bases

In my thesis [Rob13b, chapter 5], I proved that all 3-edge-colorings of a planar trivalent bipartite graph (that is $\mathfrak{sl}_3$-webs) are Kempe equivalent. This means that one can transform any coloring into any other by a finite sequence of local moves. This was actually already known [Moh07], but my proof is very different and show that the local moves are well-behaved with

[4] A triangular inequality condition on the labels of adjacent vertices ensures that such a connection is possible.

respect to the algebraical interpretation of webs. I used this fact to compute some partial traces. In [CKM14] Cautis, Kamnitzer and Morrison defined a notion of colorings for $\mathfrak{sl}_N$-webs (see [Rob15a] as well). It is possible to define a notion of Kempe equivalence for these colorings.

Project D.1. *Prove that for every planar $\mathfrak{sl}_N$-web, all colorings are Kempe equivalent.*

Kuperberg [Kup96][5] gave a beautiful base of $I^\varepsilon := \mathrm{Hom}_{U_q(\mathfrak{sl}_3)}(\mathbb{C}(q), V^{\otimes \varepsilon})$ where $V^{\otimes \varepsilon}$ is a tensor product of V^+ and V^-, the fundamental representation of $U_q(\mathfrak{sl}_3)$ and its dual. This base is parametrized by non-elliptic ε-webs[6]. It has been hoped, for a little time, that these web-bases would be dual-canonical[7]. However, Khovanov and Kuperberg [KK99] proved that it was not the case.

If w is a non-elliptic ε-web, the colorings of w gives the coefficients of the element of the base associated to w in the tensor base of $V^{\otimes \varepsilon}$. The web-bases are not dual canonical because some non-elliptic webs admits too many colorings. The notion of Kempe equivalence permits to reduce the number of colorings, which seems to be precisely the right things to do.

Project D.2. *Relate the web-base and the dual canonical base of I^ε using Kempe equivalence.*

Categorification and more specifically Khovanov – Kuperberg algebras K^ε [MPT14, Rob13a] provided another hope to compute the dual-canonical base, since the indecomposable K^ε-modules decategorify onto the dual-canonical base. This approach even permits to show the dual canonical base is uni-triangular with respect to the web-base [MPT14, Rob15d]. However, the indecomposable K^ε-modules are very difficult to compute. In [Rob15c], I developed some tools called *red graphs* in order to decompose K^ε-modules, but so far I did not manage to give complete lists of indecomposable K^ε-modules.

Project D.3. *Describe the indecomposable K^ε-modules using red graphs.*

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[5] This is where the word *web* has been introduced.

[6] An ε-web is a $\mathfrak{sl}_3$-web imbedded in the half plane, with boundary equal to the sequence of signs ε. It is non-elliptic when it contains no circle, no digon and no square.

[7] It is difficult to explain shortly what the dual-canonical bases are, but for us this is bases $\mathcal{B}^\varepsilon$ which behave well under a certain bilinear form $(\cdot|\cdot)$: $(b_i|b_j)$ is in $q\mathbb{N}[q]$ when $i \neq j$ and $(b_i|b_i) = 1 + P(q)$ with $P(q) \in q\mathbb{N}[q]$.

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