Toward the colored $\mathfrak{sl}_N$ homology

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DER FORSCHUNG | DER LEHRE | DER BILDUNG

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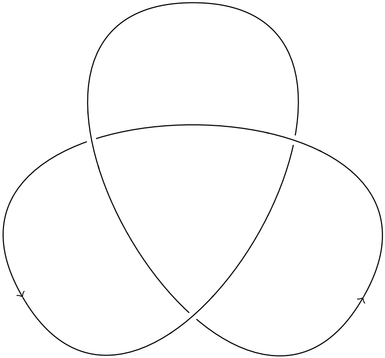
DMV

Jones polynomial



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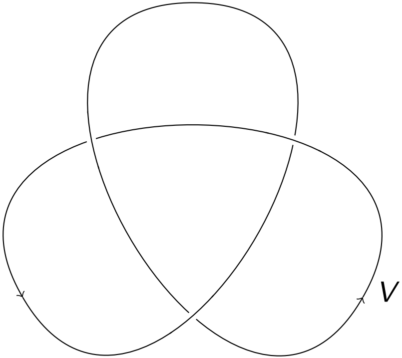
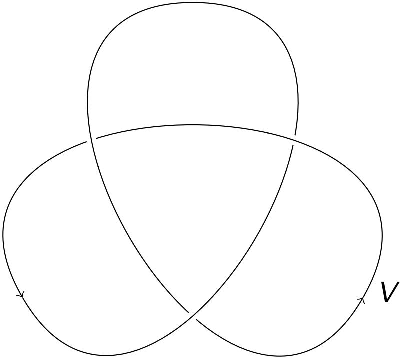


- V fundamental rep. of $U_q(\mathfrak{sl}_2)$,
- evaluations and coevaluations,
- braiding.

Jones polynomial



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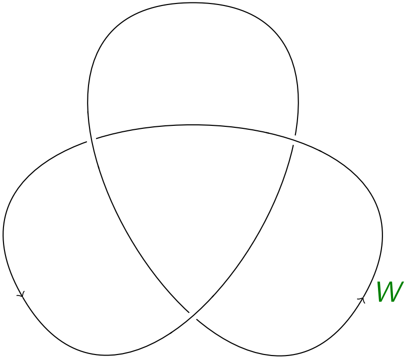
$$J(K) = -q^9 + q^5 + q^3 + q$$

Colored Jones polynomial



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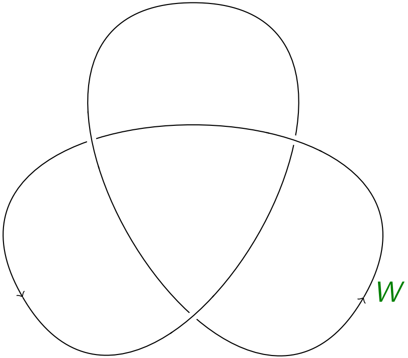

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- evaluations and coevaluations,
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$J(K, W)$ is an invariant of framed links.

Colored Jones polynomial

$$J(K, W_1 \oplus W_2) = J(K, W_1) + J(K, W_2).$$

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$$\boxed{J(K, V_n) = \sum_{k \in \mathbb{N}} (-1)^k \binom{n-k}{k} J(K_{//_{n-2k}}).}$$

Categorification

$$\begin{tikzcd}
\text{Links} \arrow[r, "\text{Khovanov homology, 2000}"] \arrow[dr, "\text{Jones polynomial}"'] & \text{Seq. of graded } \mathbb{Z}\text{-mod} \arrow[d, "\chi_q"] \\
& \text{Laurent pol.}
\end{tikzcd}$$

Categorification

$$\begin{tikzcd}[column sep=huge, row sep=huge]
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$$\text{Links} \xrightarrow{\text{Khovanov homology, 2000}} \begin{array}{l}\text{Seq. of}\\ \text{graded}\\ \mathbb{Z}\text{-mod}\end{array}$$

$$\boxed{J(K, V_n) = \sum_{k \in \mathbb{N}} (-1)^k \binom{n-k}{k} J(K_{II_{n-2k}}).}$$

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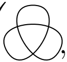
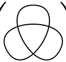
$J\Big($  $, V_3\Big) =$

$J\Big($  $\Big) - 2J\Big($  $\Big)$

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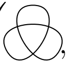
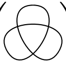
$$J\Big($$  $$,\, V_3\Big) =$$

$$J\Big($$  $$\Big) - 2J\Big($$  $$\Big)$$

$$V_3 \simeq \left(\begin{tikzcd} & V^{\otimes 3} \arrow[dl] \arrow[dr] & \\ V & \oplus & V \end{tikzcd} \right)$$

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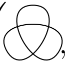
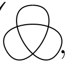
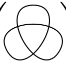
$J\Big($  $, V_3\Big) =$

$Kh\Big($  $, V_3\Big) :=$

$J\Big($  $\Big) - 2J\Big($  $\Big)$

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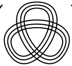
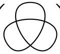
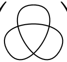
$Kh\Big($  $\Big)$

 $\times$  $\swarrow$ $\qquad$ $\searrow$  $\times$ 

$Kh\Big($  $\Big) \oplus Kh\Big($  $\Big)$

$\Bigg)$

1. Find a tensor resolution of simple modules.
2. Interpret this resolution in terms of cobordisms.
3. Use functoriality of our homology theory.

$\mathfrak{sl}_N$-homology

- Khovanov – Rozansky ('04) with matrix factorizations.
- Mackaay – Stošić – Vaz ('07) with foams and Kapustin-Li formula.
- Wu ('09) with matrix factorizations for all minuscule representations.

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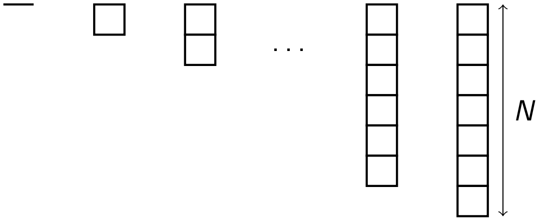
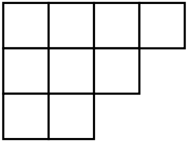
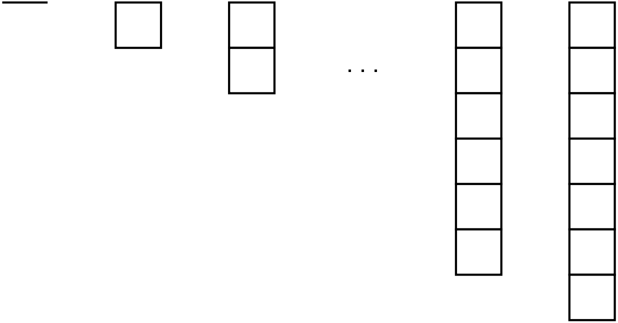
- Lauda – Queffelec – Rose ('12, '14) with ladders, foams and skew Howe duality. for all minuscule representations.



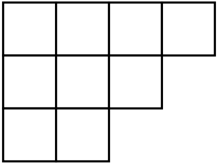
We want an explicit tensor resolution of



using



Hooks removal



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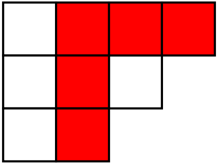

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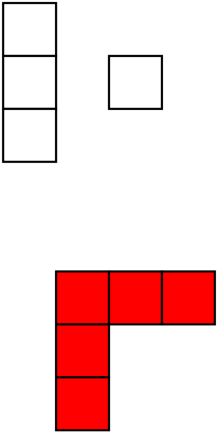

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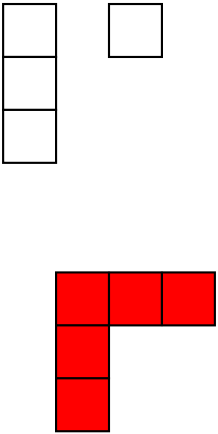

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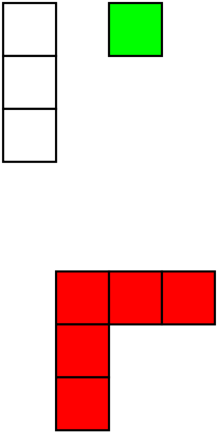

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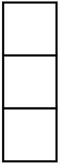

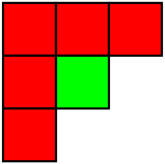

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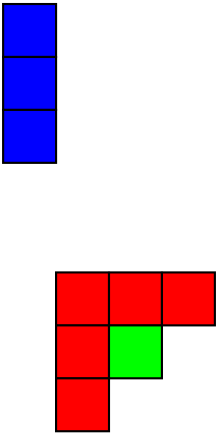

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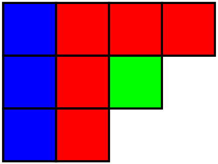

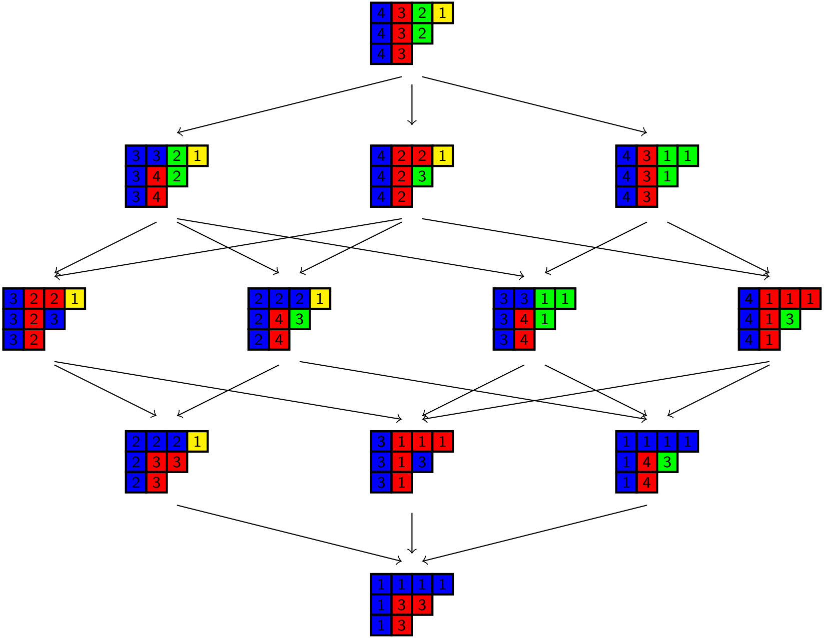
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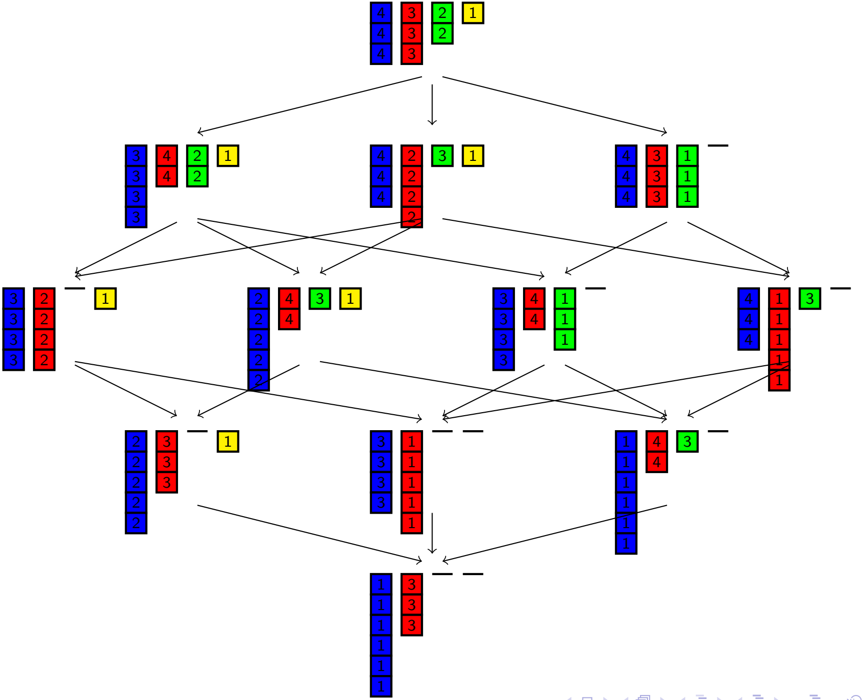
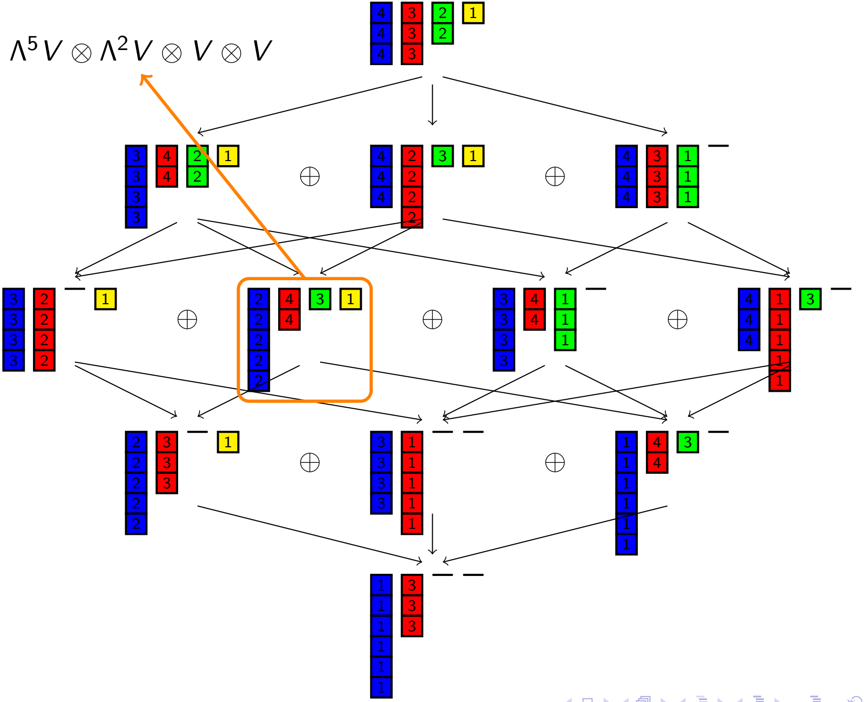


$$\Lambda^5 V \otimes \Lambda^2 V \otimes V \otimes V$$



- Maps are explicit and built with the elementary maps:

$$\begin{aligned} \Lambda^i V \otimes \Lambda^j V &\longrightarrow \Lambda^{i+1} V \otimes \Lambda^{j-1} V \\ x_1 \wedge \cdots \wedge x_i \otimes y_1 \wedge \cdots \wedge y_j &\longmapsto \\ &\sum_{k=1}^{j} (-1)^k x_1 \wedge \cdots \wedge x_i \wedge y_k \otimes y_1 \wedge \cdots \wedge \widehat{y_k} \wedge \cdots \wedge y_j. \end{aligned}$$

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- All squares commute. We use a signs assignment so that all squares anti-commute. This gives us a complex.

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Theorem (Hogancamp, R.)

The previous contruction yields a tensor resolution for every simple $\mathfrak{sl}_N$-module.

- The combinatorics does not depends on N.
- It hardly depends on the modules.
- The cobordism interpretation is clear.
- The tensor resolutions can be endowed with structures of:
  - DG-algebras,
  - DG-coalgebras.

Thank you!