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DER FORSCHUNG | DER LEHRE | DER BILDUNG
Algebra and Number Theory
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Hopf algebras
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Sheet 9

Problem 1. Let A and B be Hopf algebras. Consider the tensor categories $A-\mathsf{mod}$ and $B-\mathsf{mod}$ of finite dimensional left modules over A and B. A functor $F : A-\mathsf{mod} \to B-\mathsf{mod}$ is called exact, if for any short exact sequence

$$0 \to X \to Y \to Z \to 0$$

in $A-\mathsf{mod}$ the sequence

$$0 \to FX \to FY \to FZ \to 0$$

is exact in $B-\mathsf{mod}$.

Recall that an A-module P is called projective, if $\mathrm{Hom}_A(P, \bullet) : \mathcal{C} \to \mathsf{Vect}_{\mathbb{K}} = \mathbb{K}-\mathsf{mod}$ is an exact functor.

1. If P is projective, then $\bullet \otimes P$ is exact.

Solution. *The functor* $\_ \otimes P$ *has a left-adjoint* $\_ \otimes P^\vee$, *so* $\_ \otimes P$ *is left exact, and* $\_ \otimes P$ *has a right-adjoint* $\_ \otimes {}^\vee P$, *so* $\_ \otimes P$ *is right exact. Here we actually did not use that* P *is projective.* □

2. If P is projective, then $P^\vee$ is projective.

Solution. *Let* $0 \to X \to Y \to Z \to 0$ *be an exact sequence.*
The functor $\mathrm{Hom}(P^\vee, \_)$ *is isomorphic to the functor* $\mathrm{Hom}(\mathbb{K}, \_ \otimes P)$. *The natural isomorphism is given by:*

$$\begin{array}{rcl} Hom(P^\vee, M) & \simeq & \mathrm{Hom}(K, M \otimes P) \\ \phi & \mapsto & 1 \mapsto \sum \phi(e_i^*) \otimes e_i \\ f \mapsto f(p)m & \leftarrow\!\shortmid & m \otimes p \end{array}$$

From part 1. we know that

$$0 \to X \otimes P \to Y \otimes P \to Z \otimes P \to 0$$

is exact. Now by Lemma 3.2.11 of the script, the object $Z \otimes P$ *is projective since* P *is projective. Thus the above sequence splits which is equivalent to*

$$Y \otimes P \simeq (X \otimes P) \oplus (Z \otimes P). \tag{1}$$

If we apply $\mathrm{Hom}_H(\mathbb{K}, \_)$ *we get the sequence*

$$0 \to \mathrm{Hom}_H(\mathbb{K}, X \otimes P) \to \mathrm{Hom}_H(\mathbb{K}, Y \otimes P) \to \mathrm{Hom}_H(\mathbb{K}, Z \otimes P) \to 0 \tag{2}$$

Since the middle term is by (1) isomorphic to

$$\mathrm{Hom}_H(\mathbb{K}, Y \otimes P) \cong \mathrm{Hom}_H(\mathbb{K}, X \otimes P) \oplus \mathrm{Hom}_H(\mathbb{K}, Z \otimes P)$$

the sequence (2) is exact, thus $\mathrm{Hom}(P^\vee, \_)$ *is exact which means* $P^\vee$ *is projective.* □

Problem 2. Let A be an algebra over $\mathbb{K}$. An A-module M is called indecomposable, if $M = N \oplus N'$ implies that either N or N' is the zero module. An A-module M is called simple, if M and 0 are its only submodules.

Show that for a semi-simple algebra A every indecomposable module is simple.

Solution. *Let M be indecomposable. Assume that N is a submodule. Since A is semi-simple there is a complement P for N, i.e. $M = N \oplus P$. Since M is indecomposable we conclude $N = 0$ or $P = 0$. If $P = 0$ then $N = M$, so M and 0 are the only submodules of M thus M is simple.* □

Problem 3. We consider the following Hopf algebra H (called Sweedler's Hopf algebra): as an algebra it is given by the following quotient:

$$\mathbb{C}\langle C, X\rangle/(C^2 - 1, X^2, CX + XC)$$

where $\mathbb{C}\langle C, X\rangle$ is the algebra of non-commutative polynomials. The comultiplication is given by:

$$\Delta(C) = C \otimes C \quad \text{and} \quad \Delta(X) = C \otimes X + X \otimes 1.$$

1. Find a counity and an antipode and prove that H is indeed a Hopf algebra. Remark that H is neither commutative nor cocommutative.

   Solution. *First note that Δ is indeed a morphism of algebra and that the multiplication is a morphism of co-algebra. The counity is straightforward: we have to set $\epsilon(1) = \epsilon(C) = 1$ because they are group-like. From this follows that $\epsilon(X) = 0$ and there for $\epsilon(CX) = 0$. We have to set $S(1) = 1$ and $S(C) = C^{-1} = C$, the expression of $\Delta(X)$ impose $S(X) = -CX = XC$. This leads to $S(CX) = X$. The only thing to check is that the antipode does what it should on CX. We have:*

   $$m \circ (S \otimes \mathrm{id}) \circ \Delta(CX) = m(S(1) \otimes CX + S(CX) \otimes C) = CX + XC = 0 \quad \textit{and}$$
   $$m \circ (\mathrm{id} \otimes S) \circ \Delta(CX) = m(1 \otimes S(CX) + CX \otimes S(C)) = X + CXC = X - X.$$

   □

2. Find all (up to isomorphism) simple H-modules.

   Solution. *First of all remark, that if a module is 1 dimensional, then it is simple. We will prove that (up to isomorphism) there are exactly two simple modules, and both of them are 1-dimensional. Let M be a simple H-module. The action of C on M is diagonalisable (because the minimal polynomial of C has simple roots). We may right $M = M_{+1} \oplus M_{-1}$ where the indices indicates the diagonal action of C. Suppose M_{+1} is not trivial. Let m in M_{+1} a non zero element. If $X \cdot m = 0$, then $\mathbb{K}m$ is a sub-module of M and hence is equal to M. If $X \cdot m \neq 0$ then $X \cdot m$ belongs to M_{-1} and the $\mathbb{K}(X \cdot m)$ is a sub-module of M. This is absurd. The same argumentation works when M_{-1} is trivial. We have shown that there are exactly two simple H-modules. On both of them X acts trivially, while C acts as $\pm\mathrm{id}$. We denote them by V_{+1} and V_{-1}* □

3. Prove that the tensor product of two simple modules is simple.

   Solution. *This is a direct consequence of what we said before. We have:*

   $$V_{+1} \otimes V_{+1} \simeq V_{-1} \otimes V_{-1} \simeq V_{+1} \qquad \textit{and} \qquad V_{-1} \otimes V_{+1} \simeq V_{+1} \otimes V_{-1} \simeq V_{-1}$$

   □

4. Find all (up to isomorphism) projective indecomposable H-modules.

Solution. *We will show that there are exactly two projective indecomposable H-modules and both of them have dimension 2. First of all, remark that a projective indecomposable module must be a sub-module of H itself. Let us show that the two simple modules we found before are not projective:*

Let $\pi : H \to V_{+1}$ be the linear map given by $\pi(1) = \pi(C) \neq 0$ and $\pi(CX) = \pi(X) = 0$. This is a surjective H-module map, but it has no section. Hence V_{+1} is not projctive.

Let $\pi : H \to V_{-1}$ be the linear map given by $\pi(1) = \pi(C) = 0$ and $\pi(CX) = -\pi(X) \neq 0$. This is a surjective H-module map, but it has no section. Hence V_{+1} is not projctive.

So there is now two options: either H is itself indecomposable or it splits into two indecomposable projective modules but they might be isomorphic. Let us show that we have:

$$H \simeq P_{+1} \oplus P_{-1}$$

with P_{+1} and P_{-1} non-isomorphic.

Let $P_{+1} = \langle 1 + C, X + XC \rangle$ and $P_{-1} = \langle 1 - C, X - XC \rangle$. It is easy to show that they are 2-dimensional H-modules. To show that there are non-isomorphic, one should realize that the action of X eigenspace of C are different.

□

5. Prove that the tensor product of any two projective indecomposable H-modules is a direct sum of 2 projective indecomposable H-modules.

Solution. *This is clear since the tensor product of two projective modules is projective. More precisely, we have:*

$$P_{-1} \otimes P_{-1} \simeq P_{-1} \otimes P_{+1} \simeq P_{+1} \otimes P_{-1} \simeq P_{+1} \otimes P_{+1} \simeq P_{-1} \oplus P_{+1}.$$

□

Problem 4. Let H be a finite dimensional Hopf algebra. We suppose that S as an odd order (ie the smallest positiv n such that $S^n = \mathrm{id}_H$ is odd).

1. Prove that H is commutative.

Solution. *Let $n = 2k + 1$ be the rank of S, then we have for all x and y in H:*

$$xy = S^{2k+1}(xy) = S(S^{2k}(x)S^{2k}(y)) = S(S^{2k}(y))S(S^{2k}(x)) = S^{2k+1}(y)S^{2k+1}(x) = yx.$$

Hence H is commutative. □

2. Prove that H is cocommutative.

Solution. *Let us recall that $\Delta \circ S = \tau \circ (S \otimes S) \circ \Delta = (S \otimes S) \circ \tau \circ \Delta$. Hence we have:*

$$\Delta = \Delta \circ S^{2k+1} = \tau^{2k+1} \circ (S^{2k+1} \otimes S^{2k+1}) \circ \Delta = \tau \circ \Delta.$$

This show that H is cocommutative. □

3. Prove that $S = \mathrm{id}$

Solution. From question 1 we deduce thanks to 2.5.9 in the script, that the rank of n *is smaller than* 2*, so that it is equal to 1.* $\square$

4. Give an example of such a Hopf algebra.

Solution. We need to have $S = \mathrm{id}$*. Hence if we consider the Hopf algebra* $\mathbb{K}G$ *with* G *a group satisfying* $g^{-1} = g$*. This implies that* $G = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \cdots \times \mathbb{Z}/2\mathbb{Z}$*.* $\square$