Universität Hamburg
DER FORSCHUNG | DER LEHRE | DER BILDUNG
Algebra and Number Theory
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Hopf algebras
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Sheet 8

Problem 1. Let H be a finite dimensional Hopf algebra. In this case we know that H^* is also a Hopf algebra. We consider the multiplication in H^* given by

$$\langle f \cdot g, h\rangle = \sum_{(h)} f(h_{(1)}) \cdot g(h_{(2)}) \quad f, g \in H^*, h \in H.$$

1. Show that the following defines a left resp. right action of the algebra H^* on the vector space H.

$$f \rightharpoonup h := \sum_{(h)} f(h_{(2)}) \cdot h_{(1)}, \qquad h \leftharpoonup f := \sum_{(h)} f(h_{(1)}) \cdot h_{(2)} \qquad f \in H^*, h \in H.$$

   Show that H is an H^*-bimodule with the above actions, i.e. $(f \rightharpoonup h) \leftharpoonup g = f \rightharpoonup (h \leftharpoonup g)$ for all $f, g \in H^*$ and $h \in H$.

2. Show that the following defines a left resp. right action of H on the vector space H^*

$$h \rightharpoonup f := (k \mapsto \langle f, kh\rangle), \qquad f \leftharpoonup h := (k \mapsto \langle f, hk\rangle) \qquad f \in H^*, h \in H.$$

   Show that H^* is an H-bimodule with the above actions, i.e. $(h \rightharpoonup f) \leftharpoonup k = h \rightharpoonup (f \leftharpoonup k)$ for all $f \in H^*$ and $h, k \in H$.

3. Show that H^* becomes a left H-module with

$$h.f := \sum_{(h)} h_{(1)} \rightharpoonup f \leftharpoonup S(h_{(2)}) \qquad f \in H^*, h \in H$$

   This action is called coadjoint (left) action of H on H^*.

4. How do the actions above look in the graphical notation introduced in the lecture?

Problem 2. Let H be a finite-dimensional Hopf algebra over a field $\mathbb{K}$. Assume there is a left integral $\lambda \in \mathcal{I}_\ell(H)$, such that $\epsilon(\lambda) = 1$. Further let M be a left H-module and $N \subset M$ a submodule.

1. Choose a $\mathbb{K}$-linear $\pi : M \to M$, with $\pi^2 = \pi$ and $\mathrm{im}\pi = N$. Show that

$$\Pi : M \to M, \quad m \mapsto \sum_{(\lambda)} \lambda_{(1)}.\pi(S(\lambda_{(2)}).m)$$

   is H-linear, $\Pi^2 = \Pi$ and $\mathrm{im}\Pi = N$.

2. Show that there is a complement for every H-submodule $N \subset M$, i.e. there exists an H-submodule P of M, such that $M = N \oplus P$.

Problem 3. Let $\mathbb{K}$ be a field of characteristic 2, and let $\mathfrak{g}$ be the following 2-dimensional Lie algebra over k: as a k-vector space, it is spanned by x and y, and $[x, y] = x$.

1. Show that $\mathfrak{g}$ can be endowed with a structure of restricted Lie algebra.

2. Recall the structure of restricted Lie algebra on $\mathcal{U}(\mathfrak{g}) = U(\mathfrak{g})/I$ where I is the ideal of $U(\mathfrak{g})$ generated by $a^{[2]} - a^2$, for all $a \in \mathfrak{g}$.

3. Give a basis of $\mathcal{U}(\mathfrak{g})$.

4. Recall the structure of Hopf Algebra on $\mathcal{U}(\mathfrak{g})$.

5. Compute the left and right integrals of H.

Problem 4. 1. Let H be a Hopf algebra, prove the following equality for all $f \in H^\star$ and all $x, y \in H$:

$$(f \rightharpoonup x)y = \sum_{(y)} (f \leftharpoonup y_{(2)}) \rightharpoonup (xy_{(1)})$$

2. Show that if J is a right ideal in H, then the right coideal (or equivalently the left rational H^*-module) generated by J is still a right ideal.

3. Show that it $K \subset H$ is a right ideal and a right coideal, then K is an H-Hopf module, and prove that $K = H$.

4. Prove that of an Hopf algebra H contains a non-zero finite dimensional right ideal, it is itself finite dimensional.

5. Prove that if H is a Hopf algebra and $\{0\} \neq J$ is a right coideal, then $JH = H$.