Universität Hamburg
DER FORSCHUNG | DER LEHRE | DER BILDUNG
Algebra and Number Theory
Mathematics department

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Hopf algebras
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Sheet 7

Problem 1 (Adjoint functors).

1. If A is an algebra, we denote by $A^\times$ the set of invertible element in A. Show that this fits in a functor setting, and find a left adjoint functor of $\bullet^\times$.
2. If $\mathfrak{g}$ is a Lie algebra, we denote by $U(\mathfrak{g})$ the enveloping algebra of $\mathfrak{g}$. Show that this fits in a functor settings, and find a right adjoint functor of $U(\bullet)$.
3. If C is a coalgerba, we denote by $G(C)$ the set of group like element of C. Show that this fits in a functor settings, and find a left adjoint functor of $G(\bullet)$.
4. If R is a commutative ring without zero divisors, we denote by $\mathfrak{F}(R)$ the field of fractions of R. Show that this fits in a functor settings, and find a right adjoint functors of $\mathfrak{F}(\bullet)$.

Problem 2 (The Hopf algebra $U(\mathfrak{sl}_2)$). We consider the Lie algebra $\mathfrak{sl}_2 = \mathfrak{sl}_2(\mathbb{C})$. As a vector space, it consists of all 2×2 matrices with complex coefficient and which have trace equal to 0. The Lie bracket is given by the commutator of the classical matrix product. Choose a base of $\mathfrak{sl}_2$:

1. Prove that $\mathfrak{sl}_2$ is isomorphic to the Lie algebra generated by E, F and H subjected to the relations:
$$[H,E] = -[E,H] = 2E, \quad [H,F] = -[F,H] = -2F \quad \text{and} \quad [E,F] = -[F,E] = H.$$
2. Recall the definition of $U(\mathfrak{sl}_2)$, compute Δ, ϵ and S on the generators.
3. Prove that $\mathfrak{sl}_2$ has no non-trivial ideal[1], that is: there is no non-trivial subspace $\mathfrak{i}$ such that $[\mathfrak{i}, \mathfrak{sl}_2] \subseteq \mathfrak{i}$. (Consider an element X in such a subspace and compute $[H,[H,X]]$, then discuss according to the different possible cases).
4. A representation of $\mathfrak{sl}_2$ is *irreducible* if it contains no non-trivial sub-representation of $\mathfrak{g}$. Let V be a finite dimensional irreducible representation of $\mathfrak{sl}_2$. Let v be an element of $V \setminus \{0\}$ such that there exists a complex number λ such that $H \cdot v = \lambda v$ (we say that v is an *weight vector*). Prove that if $E \cdot v \neq 0$, it is as well a weight vector.
5. Prove that there exists an *highest weight vector* in V, that is a weight vector such that $E \cdot v = 0$.
6. Let v be an highest weight vector in V. Prove that $V = < F^n \cdot v | n \in \mathbb{N} >$.
7. Describe all the finite dimensional representation of $\mathfrak{g}$.

Problem 3. Let H be a Hopf algebra of dimension n ($< \infty$).

1. Suppose first that as a $\mathbb{K}$-algebra, H is isomorphic to $\mathbb{K} \times \mathbb{K} \times \cdots \times \mathbb{K}$, prove that $G(H^*)$ has order n.
2. Deduce that H is isomorphic as a Hopf algebra to $(\mathbb{K}G)^*$ for some group G (the dual of the group algebra of G).
3. Suppose now that H is isomorphic as a Hopf algebra to $(\mathbb{K}G)^*$, for some finite group G, prove that H is isomorphic to $\mathbb{K} \times \mathbb{K} \times \cdots \times \mathbb{K}$.

[1] This property is the *simplicity* of $\mathfrak{sl}_2$.

Problem 4. We define $\mathcal{O}(M_n(\mathbb{K}))$ as the commutative algebra $\mathbb{K}[X_{i,j} \mid 1 \leq i,j \leq n]$ of polynomials in n^2 indeterminates $\{X_{i,j}\}_{1\leq i,j\leq n}$ together with the maps Δ and ϵ defined by

$$\Delta(X_{i,j}) := \sum_{k=1}^{n} X_{i,k} \otimes X_{k,j} \qquad \text{and} \qquad \epsilon(X_{i,j}) := \delta_{i,j}.$$

1. Show that $\mathcal{O}(M_n(\mathbb{K}))$ is a bialgebra.
2. Consider the $(n \times n)$-matrix $X = (X_{i,j})_{1\leq i,j\leq n}$ with entries in $\mathbb{K}[X_{i,j}]$. Show that $g := \det X \in \mathcal{O}(M_n(\mathbb{K}))$ is group-like, i.e. $\Delta(g) = g \otimes g$.
3. Show that $\mathcal{O}(M_n(\mathbb{K}))$ is not a Hopf algebra. (Hint: Is $\det X$ multiplicatively invertible?)
4. Let I be the two-sided ideal of $\mathcal{O}(M_2(\mathbb{K}))$ generated by $\det X - 1$. Show that $\mathcal{O}(M_2(\mathbb{K}))/I$ is a Hopf algebra where the antipode is given by $S(X_{i,j} + I) = (X^{-1})_{i,j} + I$. Here X^{-1} is the matrix $\begin{pmatrix} X_{2,2} & -X_{1,2} \\ -X_{2,1} & X_{1,1} \end{pmatrix}$. How can one generalize this for larger n?