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DER FORSCHUNG | DER LEHRE | DER BILDUNG
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Hopf algebras
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# Sheet 13

**Problem 1.** Let $\mathbf{0}$ denote the empty 1-manifold, $\mathbf{1}$ a fixed oriented circle and $\mathbf{n}$ the disjoint union $n$ circle $\mathbf{1}$. We consider $\mathsf{Cob}'(2)$ the full sub category of $\mathsf{Cob}(2)$ where objects are $\mathbf{0}, \mathbf{1}, \mathbf{2}, \dots$.

1. Prove that $\mathsf{Cob}'(2)$ and $\mathsf{Cob}(2)$ are equivalent as monoidal categories.

*Solution.* $\mathsf{Cob}'(2)$ *being a sub-category of* $\mathsf{Cob}(2)$, *we have an injection functor* $\mathcal{I}$. *Let us prove that* $\mathcal{I}$ *is fully faithful and essentially surjective.* $\mathsf{Cob}'(2)$ *being a full sub-category of* $\mathsf{Cob}(2)$, $\mathsf{I}$ *is fully faithful. Let* $\gamma$ *be an object of* $\mathsf{Cob}(2)$. *It is a collection of say* $k$ *circles. The cylinder-like cobordism between* $\gamma$ *and* $\mathbf{k} = \mathsf{I}(\mathbf{k})$ *is an isomorphism. Hence,* $\mathsf{I}$ *is essentially surjective. Hence it is an equivalence of category. Furthermore,* $\mathsf{I}$ *respect the tensor product. So that* $\mathsf{Cob}'(2)$ *and* $\mathsf{Cob}(2)$ *are equivalent as monoidal categories.* □

2. Remark that the category $\mathsf{Cob}'(2)$ is strict monoidal. We admit that the category $\mathsf{Cob}'(2)$ is generated as a strict monoidal category by the following morphisms: , , , , . What does mean *generated as a monoidal category*?

*Solution.* *Let* $(f_i)_{i\in I}$ *be a collection of morphism of a strict monoidal category* $\mathcal{C}$. *We say that* $(f_i)_{i\in I}$ *generated* $\mathcal{C}$, *if for every morphism* $g$ *of* $\mathcal{C}$, *there exist an integer* $k$, *a collection* $(i_j)_{j\in[1,k]}$ *of element of* $I$, *and two collections* $(c_j)_{j\in[1,k]}$ *and* $(c'_j)_{j\in[1,k]}$ *of objects of* $\mathcal{C}$ *such that:*

$$g = \prod_{j=1}^{k} \mathrm{id}_{c_j} \otimes f_{i_j} \otimes \mathrm{id}_{c'_j}.$$

*where the product is (of course) the composition. One should actually add to* $(f_I)_{i\in I}$ *the identity of the unit object in order to produce the identity of all object of* $\mathcal{C}$ □

3. We admit that the diffeomorphism type of a connected oriented surface is characterized by the number of component of its boundary and its genus. We admit as well that a connected cobordism from $\mathbf{m}$ to $\mathbf{n}$ has a unique representation of the form: How to interpret this surface whenever $m = 0$ or $n = 0$ ?

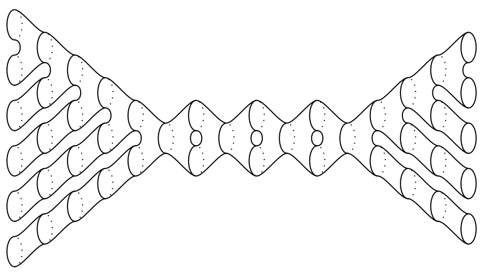


Figure 1: Example for $m = 6$, $n = 5$ and $g = 3$.

*Solution.* *If $m = 0$, one should think that the left tree part of the surface is replaced by . If $n = 0$, one should think that the right tree part of the surface is replaced by .* □

4. What does mean for a strict monoidal category to be presented by generators and relations?

*Solution.* *It means that there is a family of morphisms $(f_i)_{i\in I}$ and a family of couple $(r_j, r'_j)_{j\in J}$ where $r_j$ and $r'_j$ are finite words in the alphabet $\mathbb{A} := (\mathrm{id}_c \otimes f_i \otimes \mathrm{id}_{c'})_{c,c'\in obj(\mathcal{C}),i\in I}$ the interpretation of this words as morphism of $\mathcal{C}$ makes sense and with this interpration we have $r_j = r'_j$ for all $j$. Furthermore, we if two words $w_1$ and $w_2$ in the alphabet $\mathbb{A}$ make sense interpreted as morphism of $\mathcal{C}$ and are equal as morphisms, One can go from one to the other by a finite sequence of substitution of the type:*

$$\mathrm{id}_c \otimes r_i \otimes \mathrm{id}_{c'} \leftrightsquigarrow \mathrm{id}_c \otimes r'_i \otimes \mathrm{id}_{c'},$$

*for $c$ and $c'$ objects of $\mathcal{C}$ so that this substitution make sense, and*

$$(\mathrm{id}_c \otimes f_i \otimes \mathrm{id}_{c'})(\mathrm{id}_d \otimes f_j \otimes \mathrm{id}_{d'}) \leftrightsquigarrow (\mathrm{id}_d \otimes f_i \otimes \mathrm{id}_{d'})(\mathrm{id}_c \otimes f_i \otimes \mathrm{id}_{c'})$$

*For any object $c, c', d$ and $d'$ so that this substitution makes sense and that the source and the target of $f_i$ and $f_j$ are "disjoint".*

□

5. Proves that $\mathsf{Cob}'(2)$ is presented by the relations given on figures 2 to 7.

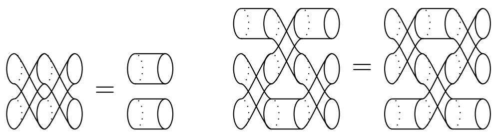


Figure 2: Braid-like relations.

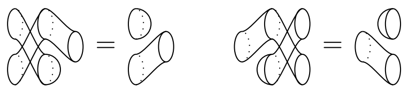


Figure 3: Twist and unit or counit.

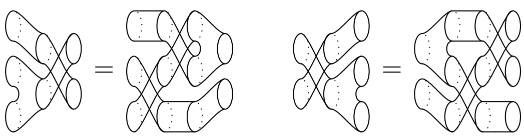


Figure 4: Twist and pair of pants.

*Solution.* *Let us denote $w_1$ and $w_2$ two words in the alphabet $\mathbb{A}$ representing the same morphism. First remark, that we can push all the twists to the left, we obtain then two words $w'_1 = v_1 t_1$ and $w'_2 = v_2 t_2$ where $t_1$ and $t_2$ are composition of twists and $v_1$ and $v_2$ do not contain any twist. The cobordism $v_1$ and $v_2$ represents a collection of disjoint surfaces presented without twist.*

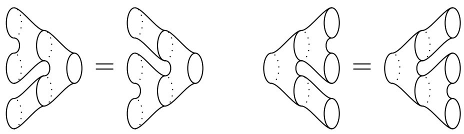


Figure 5: Associativity and coassociativity

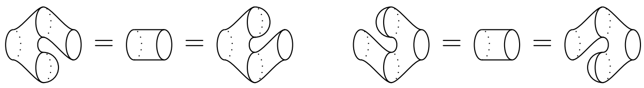


Figure 6: Unity and counity

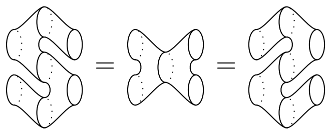
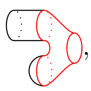
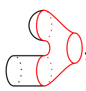
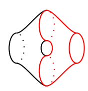


Figure 7: Frobenius-like relations

*Let us suppose for a moment that $v_1$ and $v_2$ are connected. We will prove that both $v_1$ and $v_2$ can be transform into the normal form given on figure 1. Hence we just need to consider $v_1$. Let $m$ be the number of input circles and $n$ be the number of output circles. we have then $\chi v_1 = 2 - 2g - m - n$ Let us denote by $a$ the number of , by $b$ the number of , by $c$ the number of and by $d$ the number of . From the additivity of the Euler characteristic, we get: $\chi v_1 = c + d - a - b = 2 - 2g - m - n$. The surface represented by $v_1$ has $m$ input circles and $n$ output circles, this comes as a composition of elementary pieces, so that we have: $n - m = d - c + b - a$. Hence we have:*

$$\begin{cases} a & = m + g - 1 + c \\ b & = n + g - 1 + d \end{cases}$$

*And let us "push" on the right as much as we can the . Three situations prevent us to push on the left:*

, , *and* 

*In the first two situations, we replace the local picture by a simple cylinder thanks to the relation of figure 6. This may happen at most $c$ times since there are $c$ . The last situation can happen at most $g$ times since the surface have genus $g$.*

*Remark now using the Frobenius-like condition and the associativity or the coassociavity that the handle operator commutes with and .*

*This means that we can "push" to the left all the way through at least $a - g - c = m - 1$ . On the other hand, starting from $m$ circle one cannot have more that $m - 1$ successive (one should treat the case $m = 0$ a little differently, saying that the cobordism have to start with a , we do not treat this case completely, since it is easy to adapt the general argument). Hence exactly $c$ disappear thanks to the unit relation and exactly $g$ are stuck in an handle operator. Doing the same reasoning for pushing the on the right we obtain that $v_1$ is equivalent to $u_1$ which consists of a tree of $m - 1$ followed by $g$ handle operators followed by a tree of $n - 1$ . Using the coassociativity and the associativity, we have that $v_1$ is equivalent to a normal form. This proves that one can go from $v_1$ to $v_2$ in a finite sequence of substitution.*

*If $v_1$ and $v_2$ have several connected component, we use the argumentation for each connected component and we get the result.*

*We know have to deal with the twisted parts: for this we can use we have the braids-like relations (but note that there is no notion of over- and under- crossings). Using this relation we can write $t_1$ and $t_2$ such that all the twist involving the same connected component come first (we call $t_1'$ and $t_2'$ the corresponding words) and then all the twists involve two connected components (we call $t_1''$ and $t_2''$ the corresponding words). Supposing that our connected untwisted cobordisms are in the normal form, and thanks to the relation of figure 4, we can delete the words $t_1''$ and $t_2''$. Now $t_1''$ and $t_2''$ represent the same permutation (which is an invariant of the cobordism) and hence one can go from one to the other using the braid-like relations.*

□

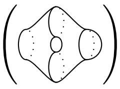
6. Let $\mathcal{F} : \mathsf{Cob}'(2) \to \mathsf{Vect}_{\mathbb{K}}$ be a TQFT. Prove that $\mathcal{F}(\mathbf{1})$ has a natural structure of commutative Frobenius algebra.

*Solution.* *The multiplication is given by* *. It is associative commutative and*  *is a unit thanks to the corresponding relations. The comultiplication and the counit are given by*  *and* *. And as the Frobenius like relation are satisfied, this proves that $\mathcal{F}(1)$ is a Frobenius algebra.* □

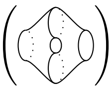
7. Prove on the other hand that if $A$ is a Frobenius algebra, this defines a $(1+1)$-TFT.

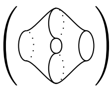
*Solution.* *This is clear: we send the multiplication, the comultiplication, the unit and the counit to the corresponding cobordisms. All the relation of Frobenius algebras are satisfied.* □

8. Prove that if

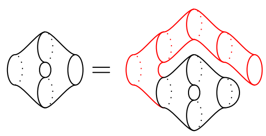
$\mathcal{F}$ (  ) $= 0,$

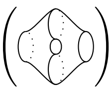
then

$\mathcal{F}$ (  ) $= 0.$

Prove that the same holds for (  )$^k$ for any integer $k$.

*Solution.* *We have:*



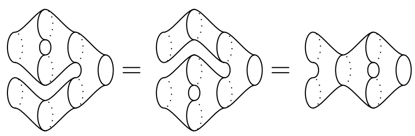
*A similar picture proves the case* (  )$^k$.

□

**Problem 2.** Let $A$ be a commutative Frobenius algebra and $F_A : \mathsf{Cob}(2) \to \mathrm{vect}_{\mathbb{K}}$ the TFT that sends the generators of $\mathsf{Cob}(2)$ to the corresponding structure morphisms of $A$. We denote the closed surface of genus $g$ by $\Sigma_g$.

1. Show that the $\mathbb{K}$-linear map $\omega : A \to A$ defined by $\mu\Delta$ is an $A$-bimodule homomorphism, where we consider $A$ as the bimodule with action given by left and right multiplication.

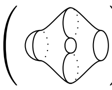
*Solution.* *This comes from the following equality at the level of cobordism:*



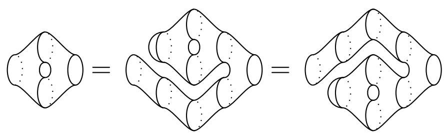
$\square$

2. Show that there is an element $w \in A$ such that $\omega(a) = wa = aw$ for all $a \in A$.

*Solution.* *If such an element exists, it is equal to* $\omega(1)$. *We set:*

$w = \mathcal{F}_A$ (  )

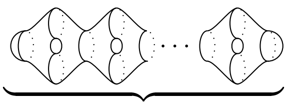
*The equality follow from:*



$\square$

3. Compute $F(\Sigma_g) \in \mathbb{K}$ for every $g \in \mathbb{N}$.

*Solution.* *We have:*

$\Sigma_g =$ 

 *appears $g$ times*

*Hence we have:*

$$\mathcal{F}_{\mathcal{A}}(\Sigma_g) = \epsilon(\omega^g(1)) = \epsilon(w^g).$$

$\square$

**Problem 3.** Let $G$ be a finite group (and hence endowed with the discrete topology) and $M$ be a manifold (with or without boundary). A *principal $G$-bundle* over $M$ is a smooth manifold $X$ with a right $G$-action (by smooth maps) and a map smooth map $\pi : X \to M$ such that:

For all point $m$ in $M$ there exists a neighborhood $U$ of $m$, such that $\phi_U : \pi^{-1}(U) \simeq U \times G$ and the following diagram commutes:

$$\begin{tikzcd} \pi^{-1}(U) \arrow[r, "\phi_U"] \arrow[d, "\pi"'] & U \times G \arrow[dl, "p_U"] \\ U & \end{tikzcd}$$

where $p_U$ is the projection on the first coordinate. We require as well that all the maps are compatible with the action of $G$ where $G$ acts by multiplication on $G$ and trivially on $U$.

Two principal $G$-bundle $(X, \pi)$ and $(X', \pi')$ are isomorphic if there exists a diffeomorphism $\psi : X \to X'$ which commutes with the $G$-action and such that $\pi' \circ \psi = \pi$.

1. Find two non-isomorphic $\mathbb{Z}/2\mathbb{Z}$-bundle of the circle $\mathbb{S}^1$. Find three non-isomorphic $\mathbb{Z}/3\mathbb{Z}$-bundle of the circle $\mathbb{S}^1$. Find 6 non-isomorphic $\mathbb{Z}/6\mathbb{Z}$-bundle of the circle $\mathbb{S}^1$.

*Solution.*

*Case $G = \mathbb{Z}/2\mathbb{Z}$:*

*Let us denote by $+1$ and $-1$ the two element of $\mathbb{Z}/2\mathbb{Z}$ and let us consider $\mathbb{S}^1 = z \in \mathbb{C}||z| = 1$. The two principal $G$-bundles we are looking for are:*

$$\begin{tikzcd} \mathbb{S}^1 \times \{+1,-1\} \arrow[d, "\pi_1:(z,a)\mapsto z"] \\ \mathbb{S}^1 \end{tikzcd} \quad \textit{and} \quad \begin{tikzcd} \mathbb{S}^1 \arrow[d, "\pi_2:z\mapsto z^2"] \\ \mathbb{S}^1 \end{tikzcd}$$

*where the action of $G$ is the multiplication on the second coordinate for the first bundle and the multiplication on by $+1$ and $-1$ interpreted as element of $\mathbb{C}$ for the second bundle. The two underlying covering being non-diffeomorphic we clear have two non-isomorphic principal $G$-bundles.*

*Case $G = \mathbb{Z}/3\mathbb{Z}$:*

*Let us denote by $g$ a fixed generator of $\mathbb{Z}/3\mathbb{Z}$. The three bundles we are looking for are:*

$$\begin{tikzcd} \mathbb{S}^1 \times \mathbb{Z}/3\mathbb{Z} \arrow[d, "\pi_1:(z,a)\mapsto z"] \\ \mathbb{S}^1 \end{tikzcd}, \quad \begin{tikzcd} \mathbb{S}^1 \arrow[d, "\pi_2:z\mapsto z^3"] \\ \mathbb{S}^1 \end{tikzcd} \quad \textit{and} \quad \begin{tikzcd} \mathbb{S}^1 \arrow[d, "\pi_3:z\mapsto z^3"] \\ \mathbb{S}^1 \end{tikzcd}$$

*where the actions of $G$ are:*

- *The multiplication on the second coordinate for the first bundle.*
- *$g$ acts by multiplication by $j = \exp\left(\frac{2i\pi}{3}\right)$ for the second bundle.*
- *$g$ acts by multiplication by $j^2$ for the second bundle.*

*These three principal $G$-bundles are clearly not isomorphic.*

*Case $G = \mathbb{Z}/6\mathbb{Z}$:*

*Let us denote by $g_2$ a fixed element of order 2 of $G$ and by $g_3$ a fixed element of order 3 of $G$. Then $g_6 = g_2 g_3$ is a generator of $G$. We denote by $G_2$ and $G_3$ the subgroups of $G$ generated by $g_2$ and $g_3$ respectively. The 6 principal $G$-bundles we are looking for are:*

$$\begin{tikzcd} \mathbb{S}^1 \times G \arrow[d, "\pi_1:(z,a)\mapsto z"] \\ \mathbb{S}^1 \end{tikzcd}, \quad \begin{tikzcd} \mathbb{S}^1 \times G_2 \arrow[d, "\pi_2:(z,a)\mapsto z^3"] \\ \mathbb{S}^1 \end{tikzcd}, \quad \begin{tikzcd} \mathbb{S}^1 \times G_2 \arrow[d, "\pi_3:(z,a)\mapsto z^3"] \\ \mathbb{S}^1 \end{tikzcd}, \quad \begin{tikzcd} \mathbb{S}^1 \times G_3 \arrow[d, "\pi_4:(z,a)\mapsto z^2"] \\ \mathbb{S}^1 \end{tikzcd}, \quad \begin{tikzcd} \mathbb{S}^1 \arrow[d, "\pi_5:(z,a)\mapsto z^6"] \\ \mathbb{S}^1 \end{tikzcd} \quad \textit{and} \quad \begin{tikzcd} \mathbb{S}^1 \arrow[d, "\pi_6:(z,a)\mapsto z^6"] \\ \mathbb{S}^1 \end{tikzcd}.$$

*where the $G$-action are given by:*

- *The multiplication on the second coordinate for the first bundle.*
- *For the second bundle, $g_2$ acts by multiplication on the second coordinate and $g_3$ acts by multiplication by $j = \exp\left(\frac{2i\pi}{3}\right)$ on the second coordinate.*
- *For the third bundle, $g_2$ acts by multiplication on the second coordinate and $g_3$ acts by multiplication by $j^2$ on the second coordinate.*
- *For the fourth bundle, $g_3$ acts by multiplication on the second coordinate and $g_2$ acts by multiplication by $-1$ on the second coordinate.*
- *For the fifth bundle, $g_3 g_2$ acts by multiplication by $\exp\left(\frac{2i\pi}{6}\right)$.*
- *For the sixth bundle, $g_3 g_2$ acts by multiplication by $\exp\left(\frac{2i\pi}{6}\right)$.*

*Note that there is a 1-1 correspondence between these principal $G$-bundles and the element of $G$: if $\mathbf{1}$ is a generator of $\pi_1(\mathbb{S}_1)$ the deck transformation corresponding to $\mathbf{1}$ is an element of $G$.* $\square$

2. Suppose that $M$ is connexe. Prove that there is a bijection:

$$\{\text{principal } G\text{-bundle of } M\}/\text{isomorphisms} \simeq \hom(\pi_1(M), G)/G.$$

where $G$ acts by conjugation. Let us recall that the fundamental group of a compact manifold is always finitely presented[1]

*Solution. First remark that to any based (i. e. we fix a base point of the base and the covering in a coherent way: the base point of the total space should correspond to the neutral element of $G$ and project on the base point of $M$) principal $G$-bundle, is associated a unique morphism $\phi : \pi_1(M) = \pi_1(M, *) \to G$: let $[\gamma]$ an element of the $\pi_1$, then $\phi([\gamma])$ is the unique element $g$ of $G$ such that $\widetilde{\gamma}(0) \cdot g = \widetilde{\gamma}(1)$ where $\widetilde{\gamma}$ is the unique based lift of $\gamma$ . Let us change the base point $*$ of $\widetilde{M}$ to $* \cdot h$ and denote $\widetilde{\gamma}'$ the new lift of $\gamma$, then we have $\widetilde{\gamma}' = \widetilde{\gamma} \cdot h$, hence $(\widetilde{'\gamma}(0) \cdot h) \cdot g = \widetilde{'\gamma}(1) \cdot h$.*

*This proves that we have a canonical map:*

$$\{\textit{principal } G\textit{-bundle of } M\}/\textit{isomorphisms} \to \hom(\pi_1(M), G)/G.$$

*Let $\phi$ be an element of $\hom(\pi_1(M), G)$. Then the kernel of $\phi$ only depend on $[\phi]$ and is a normal subgroup of $\pi_1(M)$. Hence there is a unique connected covering of $N$ associated to $\ker \phi$. This has a natural structure of principal $\operatorname{Im}\phi$ bundle. Using the isomorphism $\operatorname{Im}\phi \simeq g(\operatorname{Im}\phi)g^{-1}$ for $g \in G$ we have that $N$ is have a principal $g(\operatorname{Im}\phi)g^{-1}$-bundle structure. Let $g_1, \ldots g_k$ a set of element of $G$ representing $\operatorname{Im}\phi\backslash G$. We have the following partition of $G$:*

$$\bigcup_i (\mathrm{Im},\phi) g_i = G$$

*Let $\widetilde{M} = N \times \{g_1, \ldots, g_k\}$. It is endowed with a natural $G$-bundle structure, we denote it by $\widetilde{M}_\phi$. The conjugation $c_g$ by an element by $g$ gives us a an isomorphism between $M_\phi$ and $M_{c_h \circ \phi}$. This proves that we have a canonical map:*

$$\hom(\pi_1(M), G)/G \to \{\textit{principal } G\textit{-bundle of } M\}/\textit{isomorphisms}.$$

*One easily check that this two maps are inverses one from the other.* $\square$

[1] This follows from the fact that any compact manifold can be given a structure of finite CW-complex.

3. We consider the category $\mathsf{Cob}(n+1)$. If $M$ is a closed $n$-manifold, we set:

$$\mathcal{F}(M) = \mathbb{K}\{\text{principal } G\text{-bundle of } M\}/\text{isomorphisms}$$

and if $W : M_1 \to M_2$ is a cobordism between $M_1$ and $M_2$ two $n$-manifolds and $X_1$ and $X_2$ two principal $G$-bundle over $M_1$ and $M_2$ respectively. We set

$$C(X_1, W, X_2) = \{(\phi_1, Y, \phi_2) | Y \text{ is a principal } G\text{-bundle of } W,\ \phi_i\text{'s are isomorphisms from } Y_{|M_i} \text{ to } X_i\}$$

Two elements of $(\phi_1, Y, \phi_2)$ and $(\phi_1', Y', \phi_2')$ of $C(X_1, W, X_2)$ are equivalent if there exists an isomorphism of $G$-bundle $Y \to Y'$ such that: $\phi_i = \phi_i' \circ \alpha_{|M_i}$ (if moreover $(\phi_1, Y, \phi_2) = (\phi_1', Y', \phi_2')$, the set of such morphism is denoted by $\mathrm{Aut}((\phi_1, Y, \phi_2))$). We denote by $\simeq$ this equivalence relation. We set:

$$\mathcal{F}(W)([X_1], [X_2]) = \sum_{[\phi_1, Y, \phi_2] \in C(X_1, W, X_2)/\simeq} \frac{1}{|\mathrm{Aut}(X_2)||\mathrm{Aut}((\phi_1, Y, \phi_2))|}$$

where $\mathcal{F}(W)([X_1], [X_2])$ is the coefficient of $[X_2]$ in the image of $[X_1]$ by $\mathcal{F}(W)$. Prove that this defines a functor. Is it monoidal?

*Solution. Let us first prove that the cylinder $M \times [0,1]$ on $M$ is sent on the identity: A principal $G$-bundle over $M \otimes [0,1]$ is nothing but a (principal $G$-bundle over $M$)$\times[0,1]$. Hence the set $C(X_1, M \times [0,1], X_2)$ is empty if $X_1 \not\simeq X_2$ and a set of representants of equivalence classes of $C(X_1, M \times [0,1], X_1)$ is given by $(\phi_1, X_1 \times [0,1], \phi \circ \phi_1)_{\phi \in \mathrm{Aut}(X_1)}$. Thus, we have:*

$$\mathcal{F}(W)([X_1], [X_2]) = \begin{cases} 0 & \textit{if } [X_1] \neq [X_2], \\ 1 & \textit{else.} \end{cases}$$

*Let $W_1 : M_1 \to M_2$ and $W_2 : M_2 \to M_3$ and let $X_1$ and $X_3$ be principal $G$-bundle over $M_1$ and $M_3$ respectively. We want to compare $\mathcal{F}(W_2) \circ \mathcal{F}(W_1)([X_1], [X_3])$ and $\mathcal{F}(W_2 \circ W_1)([X_1], [X_3])$.*

*Before starting the computations let us make a few remarks which will be useful in the sequence.*

- *Let $X_2$ be a principal $G$-bundle over $M_2$. Then $\mathrm{Aut}(X_2)$ acts on: $C(X_1, W_1, X_2)$ and $C(X_2, W_2, X_3)$ via:*
$$\beta \cdot (\phi_1, Y_1, \phi_2) = (\phi_1, Y_1, \beta \circ \phi_2) \quad \textit{and} \quad \beta \cdot (\phi_2', Y_2, \phi_3) = (\beta \circ \phi_2', Y_2, \beta \circ \phi_3)$$
*This action is clearly compatible with the equivalence relations and leads to actions on $C(X_1, W_1, X_2)/\simeq$ and $C(X_2, W_2, X_3)/\simeq$. We have $\beta \cdot (\phi_1, Y_1, \phi_2) \simeq (\phi_1, Y_1, \phi_2)$ if and only if there exists an automorphism $\alpha_1$ of the $G$-bundle $Y_1$ such that $\phi_1 = \phi_1 \circ (\alpha_1)_{|M_1}$ and $\phi_2 = \beta \circ \phi_2 \circ (\alpha_1)_{|M_2}$. Similarly We have $\beta \cdot (\phi_1, Y_2, \phi_3) \simeq (\phi_2', Y_2, \phi_3)$ if and only if there exists an automorphism $\alpha_2$ of the $G$-bundle $Y_2$ such that $\phi_3 = \phi_3 \circ (\alpha_2)_{|M_3}$ and $\phi_2' = \beta \circ \phi_2' \circ (\alpha_2)_{|M_2}$.*
- *Let $X_2$ be a principal $G$-bundle over $M_2$ and $(\phi_1, Y_1, \phi_2)$ and $(\phi_2', Y_2, \phi_3)$ elements of $C(X_1, M_1, X_2)$ and $C(X_2, M_2, X_3)$, then $(\phi_1, Y_1 \cup_{(\phi_2')^{-1} \circ \phi_2} Y_2, \phi_3)$ is an element of $C(X_1, M_2 \circ M_1, X_3)$. Moreover, every element of $C(X_1, M_2 \circ M_1, X_3)$ are obtain via this construction ($X_2$ should run on a set of representant of isomoprhism classes of principal $G$-bundle over $M_2$.)*
- *As we just said, we have a natural injection $\iota$ from $C(X_1, W_1, X_2) \times C(X_2, W_2, X_3)$ in $C(X_1, W_2 \circ W_1, X_3)$. The (diagonal) action of $\mathrm{Aut}(X_2)$ on $C(X_1, W_1, X_2) \times C(X_2, W_2, X_3)$ let $\iota^{-1}((\phi_1, Y_1 \cup_{(\phi_2')^{-1} \circ \phi_2} Y_2, \phi_3))$ stable since $(\beta \circ \phi_2')^{-1} \circ \beta\phi_2 = (\phi_2')^{-1} \circ \phi_2$ and is transitive on this set.*
- *Let $(\phi_1, Y, \phi_3)$ be an element of $C(X_1, W_2 \circ W_1, X_3)$ and $\alpha$ is an element of $\mathrm{Aut}((\phi_1, Y, \phi_3))$. The restriction of $\alpha$ to $M_2$ is an automorphism of the $Y_{|M_2}$. If $X_2$ is a principal $G$ bundle over $M_2$ and we have an identification of $Y_{|M_2}$ with $X_2$, this yields a morphism:*

$$\mathrm{Aut}((\phi_1, Y, \phi_3)) \to \mathrm{Aut}(X_2)$$

- *With the canonical notation, we have the following exact sequence:*

$$1 \to \mathrm{Aut}((\phi_1, Y_1, \phi_2)) \times \mathrm{Aut}((\phi_2', Y_2, \phi_3)) \to \mathrm{Aut}((\phi_1, Y_1 \cup_{(\phi_2')^{-1}\circ\phi_2} Y_2, \phi_3)) \to \mathrm{Aut}(X_2)$$

*The first arrow is well defined, since the automorphisms of $(\phi_1, Y_1, \phi_2))$ and $(\phi_2', Y_2, \phi_3)$ are the identity on $(Y_1)_{M_2}$ and $(Y_2)_{M_2}$. The second arrow comes from the previous remark and the identification provided by $\phi_2$. The exactness is traight forward, since the first arrow is clearly injective and because an automorphism of $(\phi_1, Y_1 \cup_{(\phi_2')^{-1}\circ\phi_2} Y_2, \phi_3)$ induces the identity on $X_2$ precisely when it decomposes as automorphisms of $(\phi_1, Y_1, \phi_2))$ and $(\phi_2', Y_2, \phi_3)$ glued together along $M_2$ (where they are trivial by hypothesis). Note that the last arrow need not to be surjective. Note that the image of the last arrow are precisely the automorphism are given by $\phi_2 \alpha_{|M_2} \phi_2^{-1}$ where $\alpha$ is an automorphism of $(\phi_1, Y_1 \cup_{(\phi_2')^{-1}\circ\phi_2} Y_2, \phi_3)$. Hence this is precisely the stabilizer of $([\phi_1, Y_1, \phi_2], [\phi_2', Y_2, \phi_3])$ for the (diagonal) action of $\mathrm{Aut}(X_2)$ we consider in the first remark. This proves that:*

$$|\mathrm{Aut}((\phi_1, Y_1 \cup_{(\phi_2')^{-1}\circ\phi_2} Y_2, \phi_3))| = |\mathrm{Aut}((\phi_1, Y_1, \phi_2))||\mathrm{Aut}((\phi_1, Y_1, \phi_2))||\mathrm{Stab}_{\mathrm{Aut}(X_2)}([\phi_1, Y_1, \phi_2], [\phi_2', Y_2, \phi_3])|$$

*We are now ready to compute:*

$$\begin{aligned}
&\mathcal{F}(W_2) \circ \mathcal{F}(W_1)([X_1], [X_3]) \\
&= \sum_{[X_2] \text{eq. cl. of ppal } G\text{-bundle over } M_2} \sum_{\substack{[\phi_1, Y_1, \phi_2] \in C(X_1, W_1, X_2)/\simeq \\ [\phi_1', Y_2, \phi_2] \in C(X_2, W_2, X_3)/\simeq}} \frac{1}{|\mathrm{Aut}((\phi_1, Y_1, \phi_2))||\mathrm{Aut}((\phi_1, Y_1, \phi_2))||\mathrm{Aut}(X_2)||\mathrm{Aut}(X_3)|} \\
&= \sum_{[X_2]} \sum_{\substack{[\phi_1, Y, \phi_3] \in C(X_1, W_2 \circ W_1, X_3)/\simeq \\ \text{such that } Y_{|M_2} \simeq X_2}} \sum_{\substack{[\phi_1, Y_1, \phi_2] \in C(X_1, W_1, X_2)/\simeq \\ [\phi_1', Y_2, \phi_2] \in C(X_2, W_2, X_3)/\simeq \\ \text{such that } [\phi_1, Y_1 \cup_{(\phi_2')^{-1}\circ\phi_2} Y_2, \phi_3)] \simeq [\phi_1, Y, \phi_3]}} \frac{1}{|\mathrm{Aut}((\phi_1, Y_1, \phi_2))||\mathrm{Aut}((\phi_1, Y_1, \phi_2))||\mathrm{Aut}(X_2)||\mathrm{Aut}(X_3)|} \\
&= \sum_{[X_2]} \sum_{\substack{[\phi_1, Y, \phi_3] \in C(X_1, W_2 \circ W_1, X_3)/\simeq \\ \text{such that } Y_{|M_2} \simeq X_2}} \sum_{\substack{[\phi_1, Y_1, \phi_2] \in C(X_1, W_1, X_2)/\simeq \\ [\phi_1', Y_2, \phi_2] \in C(X_2, W_2, X_3)/\simeq \\ \text{such that } [\phi_1, Y_1 \cup_{(\phi_2')^{-1}\circ\phi_2} Y_2, \phi_3)] \simeq [\phi_1, Y, \phi_3]}} \frac{|\mathrm{Stab}_{\mathrm{Aut}(X_2)}([\phi_1, Y_1, \phi_2], [\phi_2', Y_2, \phi_3])|}{|\mathrm{Aut}((\phi_1, Y, \phi_3))||\mathrm{Aut}(X_2)||\mathrm{Aut}(X_3)|} \\
&= \sum_{[X_2]} \sum_{\substack{[\phi_1, Y, \phi_3] \in C(X_1, W_2 \circ W_1, X_3)/\simeq \\ \text{such that } Y_{|M_2} \simeq X_2}} \sum_{\substack{[\phi_1, Y_1, \phi_2] \in C(X_1, W_1, X_2)/\simeq \\ [\phi_1', Y_2, \phi_2] \in C(X_2, W_2, X_3)/\simeq \\ \text{such that } [\phi_1, Y_1 \cup_{(\phi_2')^{-1}\circ\phi_2} Y_2, \phi_3)] \simeq [\phi_1, Y, \phi_3]}} \frac{1}{|\mathrm{Aut}((\phi_1, Y, \phi_3))||\iota^{-1}(\phi_1, Y, \phi_3)||\mathrm{Aut}(X_3)|} \\
&= \sum_{[X_2]} \sum_{\substack{[\phi_1, Y, \phi_3] \in C(X_1, W_2 \circ W_1, X_3)/\simeq \\ \text{such that } Y_{|M_2} \simeq X_2}} \frac{|\iota^{-1}(\phi_1, Y, \phi_3)|}{|\mathrm{Aut}((\phi_1, Y, \phi_3))||\iota^{-1}(\phi_1, Y, \phi_3)||\mathrm{Aut}(X_3)|} \\
&= \sum_{[X_2]} \sum_{\substack{[\phi_1, Y, \phi_3] \in C(X_1, W_2 \circ W_1, X_3)/\simeq \\ \text{such that } Y_{|M_2} \simeq X_2}} \sum_{\substack{[\phi_1, Y_1, \phi_2] \in C(X_1, W_1, X_2)/\simeq \\ [\phi_1', Y_2, \phi_2] \in C(X_2, W_2, X_3)/\simeq \\ \text{such that } [\phi_1, Y_1 \cup_{(\phi_2')^{-1}\circ\phi_2} Y_2, \phi_3)] \simeq [\phi_1, Y, \phi_3]}} |\frac{1}{|\mathrm{Aut}((\phi_1, Y, \phi_3))||\mathrm{Aut}(X_3)|} \\
&= \sum_{[\phi_1, Y, \phi_3] \in C(X_1, W_2 \circ W_1, X_3)/\simeq} \frac{1}{|\mathrm{Aut}(X_3)||\mathrm{Aut}((\phi_1, Y, \phi_3))|} \\
&= \mathcal{F}(W_2 \circ W_3)([X_1], [X_3]).
\end{aligned}$$

*The functor is clearly monoidal.*

$\square$