Universität Hamburg
DER FORSCHUNG | DER LEHRE | DER BILDUNG
Algebra and Number Theory
Mathematics department

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Hopf algebras
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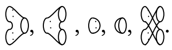
Sheet 13

Problem 1. Let $\mathbf{0}$ denote the empty 1-manifold, $\mathbf{1}$ a fixed oriented circle and $\mathbf{n}$ the disjoint union n circle $\mathbf{1}$. We consider $\mathsf{Cob}'(2)$ the full sub category of $\mathsf{Cob}(2)$ where objects are $\mathbf{0}, \mathbf{1}, \mathbf{2}, \ldots$.

1. Prove that $\mathsf{Cob}'(2)$ and $\mathsf{Cob}(2)$ are equivalent as monoidal categories.
2. Remark that the category $\mathsf{Cob}'(2)$ is strict monoidal. We admit that the category $\mathsf{Cob}'(2)$ is generated as a strict monoidal category by the following morphisms: . What does mean *generated as a monoidal category*?
3. We admit that the diffeomorphism type of a connected oriented surface is characterized by the number of component of its boundary and its genus. We admit as well that a connected cobordism from $\mathbf{m}$ to $\mathbf{n}$ has a unique representation of the form: How to interpret this surface whenever $m = 0$ or $n = 0$?

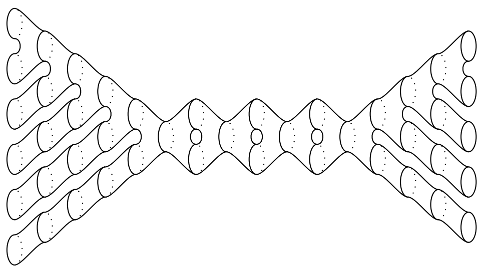


Figure 1: Example for $m = 6$, $n = 5$ and $g = 3$.

4. What does mean for a strict monoidal category to be presented by generators and relations?
5. Proves that $\mathsf{Cob}'(2)$ is presented by the relations given on figures 2 to 7.

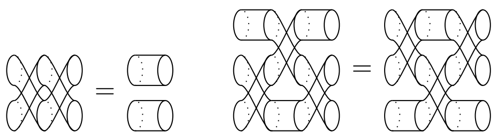


Figure 2: Braid-like relations.

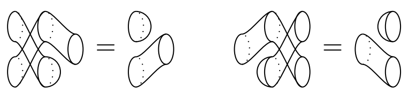


Figure 3: Twist and unit or counit.

6. Let $\mathcal{F} : \mathsf{Cob}'(2) \to \mathsf{Vect}_{\mathbb{K}}$ be a TQFT. Prove that $\mathcal{F}(\mathbf{1})$ has a natural structure of Frobenius algebra.
7. Prove on the other hand that if A is a Frobenius algebra, this defines a $(1 + 1)$-TFT.

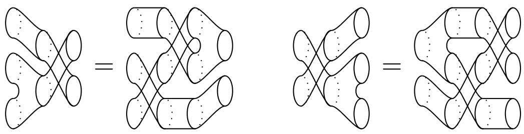


Figure 4: Twist and pair of pants.

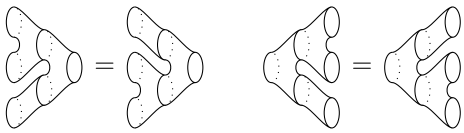


Figure 5: Associativity and coassociativity

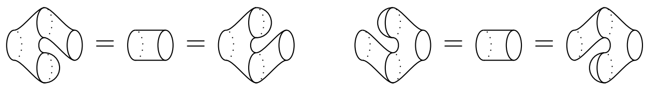


Figure 6: Unity and counity

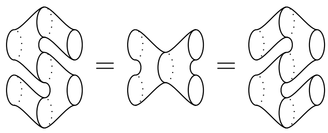
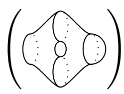
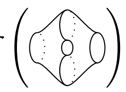


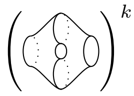
Figure 7: Frobenius-like relations

8. Prove that if

$\mathcal{F}$  $= 0,$

then

$\mathcal{F}$  $= 0.$

Prove that the same holds for  ^{k} for any integer k.

Problem 2. Let A be a commutative Frobenius algebra and $F_A : \mathsf{Cob}(2) \to \mathrm{vect}_{\mathbb{K}}$ the TFT that sends the generators of $\mathsf{Cob}(2)$ to the corresponding structure morphisms of A. We denote the closed surface of genus g by Σ_g.

1. Show that the $\mathbb{K}$-linear map $\omega : A \to A$ defined by $\mu\Delta$ is an A-bimodule homomorphism, where we consider A as the bimodule with action given by left and right multiplication.

2. Show that there is an element $w \in A$ such that $\omega(a) = wa = aw$ for all $a \in A$.

3. Compute $F(\Sigma_g) \in \mathbb{K}$ for every $g \in \mathbb{N}$.

Problem 3. Let G be a finite group (and hence endowed with the discrete topology) and M be a manifold (with or without boundary). A *principal G-bundle* over M is a smooth manifold X with a right G-action (by smooth maps) and a map smooth map $\pi : X \to M$ such that:

For all point m in M there exists a neighborhood U of m, such that $\phi_U : \pi^{-1}(U) \simeq U \times G$ and the following diagram commutes:

$$\begin{tikzcd} \pi^{-1}(U) \arrow[r, "\phi_U"] \arrow[d, "\pi"'] & U \times G \arrow[dl, "p_U"] \\ U & \end{tikzcd}$$

where p_U is the projection on the first coordinate. We require as well that all the maps are compatible with the action of G where G acts by multiplication on G and trivially on U.

Two principal G-bundle (X, π) and (X', π') are isomorphic if there exists a diffeomorphism $\psi : X \to X'$ which commutes with the G-action and such that $\pi' \circ \psi = \pi$.

1. Find two non-isomorphic $\mathbb{Z}/2\mathbb{Z}$-bundle of the circle $\mathbb{S}^1$. Find three non-isomorphic $\mathbb{Z}/3\mathbb{Z}$-bundle of the circle $\mathbb{S}^1$. Find 6 non-isomorphic $\mathbb{Z}/6\mathbb{Z}$-bundle of the circle $\mathbb{S}^1$.

2. Suppose that M is connexe. Prove that there is a bijection:

$$\{\text{principal } G\text{-bundle of } M\}/\text{isomorphisms} \simeq \hom(\pi_1(M), G)/G.$$

where G acts by conjugation. Let us recall that the fundamental group of a compact manifold is always finitely presented[1]

3. We consider the category $\mathsf{Cob}(n+1)$. If M is a closed n-manifold, we set:

$$\mathcal{F}(M) = \mathbb{K}\{\text{principal } G\text{-bundle of } M\}/\text{isomorphisms}$$

and if $W : M_1 \to M_2$ is a cobordism between M_1 and M_2 two n-manifolds, we set:

$$\mathcal{F}(W)([X_1]) = \sum_{[X_2] \in \{\text{principal } G\text{-bundle of } M_2\}/\sim} \quad \sum_{\substack{[X] \in \{\text{principal } G\text{-bundle of } W_2\}/\sim \\ \text{with } [X] \text{ induces } [X_1] \text{ on } M_1 \\ \text{and } [X] \text{ induces } [X_2] \text{ on } M_2}} \frac{[X_2]}{|\mathrm{Aut}(X)|}$$

Prove that this defines a functor. Is it monoidal?

[1]This follows from the fact that any compact manifold can be given a structure of finite CW-complex.