Universität Hamburg
DER FORSCHUNG | DER LEHRE | DER BILDUNG
Algebra and Number Theory
Mathematics department

PD Dr. Ralf Holtkamp
Prof. Dr. C. Schweigert
Hopf algebras
Winter term 2014/2015

Sheet 12

Problem 1. Let H be a finite dimensional Hopf algebra with an invertible antipode. We Set $E = \mathrm{End}(H)$.

1. Explain how to identify $H \otimes H^*$ and E (as vector spaces). Explain how to identify $E \otimes E$ with $\mathrm{End}(H \otimes H)$.

Solution. *Let (e_i) be a base of H and (e^i) the dual base of H^*. We consider the following pairs of isomorphisms:*

$$\begin{array}{ccc} E & \to & H \otimes H^* \\ f & \mapsto & \sum_i f(e_i) \otimes e^i \end{array} \quad \text{and} \quad \begin{array}{ccc} H \otimes H^* & \to & E \\ h \otimes \lambda & \mapsto & (x \mapsto h\lambda(x)) \end{array}$$

They are clearly mutually inverse one from the other and this shows how to identify E and $H \otimes H^*$. We consider the following pair of mutually inverse isomorphisms:

$$\begin{array}{ccc} E \otimes E & \to & End(H \otimes H) \\ f \otimes g & \mapsto & h \otimes h' \mapsto f(h) \otimes g(h') \end{array} \quad \text{and} \quad \begin{array}{ccc} End(H \otimes H) & \to & E \otimes E \\ e_i \otimes e_j \otimes e^k \otimes e^l & \mapsto & (e_i \otimes e^k) \otimes (e_j \otimes e^l) \end{array}$$

Where for the second one, we use the first identification. □

2. We consider E as an algebra for the convolution product (in particular the unit is $\eta_E = \eta \circ \epsilon$). Prove that the following formula defines a comultiplication on E:

$$\Delta_E(f)(x \otimes y) = \sum_{(x)} (1 \otimes x_{(1)}) \Delta(f(yx_{(2)}))(1 \otimes S(x_{(3)})).$$

What is the counite ϵ_E?

Solution. *We want to show that $(\Delta_E \otimes \mathrm{id}_E) \circ \Delta_E = (\mathrm{id}_E \otimes \Delta_E) \circ \Delta_E$. Let us compute:*

$$\begin{aligned}
&(\Delta_E \otimes \mathrm{id}_E) \circ \Delta_E(f)(x \otimes y \otimes z) \\
&= \sum_{(f)} \Delta_E(f_{(1)})(x \otimes y) \otimes f_{(2)}(z) \\
&= \sum_{(f),(x)} ((1 \otimes x_{(1)}) \Delta(f_{(1)}(yx_{(2)}))(1 \otimes S(x_{(3)}))) \otimes f_{(2)}(z) \\
&= \sum_{(f),(x)} ((1 \otimes x_{(1)}) \Delta(f(zy_{(2)}x_{(3)})_{(1)})(1 \otimes S(x_{(5)}))) \otimes y_{(1)}x_{(2)} f(zy_{(2)}x_{(3)})_{(2)} S(y_{(3)}x_{(4)}) \\
&= \sum_{(f(zy_{(2)}x_{(3)})),(x),(y)} ((1 \otimes x_{(1)}) \Delta(f(zy_{(2)}x_{(3)})_{(1)})(1 \otimes S(x_{(3)}))) \otimes y_{(1)}x_{(2)} f(zy_{(2)}x_{(3)})_{(2)} S(y_{(3)}x_{(5)}) \\
&= \sum_{(f(zy_{(2)}x_{(3)})),(x),(y)} f(zy_{(2)}x_{(3)})_{(1)} \otimes x_{(1)} f(zy_{(2)}x_{(3)})_{(2)} S(x_{(5)}) \otimes y_{(1)}x_{(2)} f(zy_{(2)}x_{(3)})_{(3)} S(y_{(3)}x_{(4)})
\end{aligned}$$

$$
\begin{aligned}
&(\mathrm{id}_E \otimes \Delta_E) \circ \Delta_E(f)(x \otimes y \otimes z) \\
&\quad = \sum_{(f),(y)} f_{(1)}(x) \otimes ((1 \otimes y_{(1)})\Delta(f_{(2)}(zy_{(2)}))(1 \otimes S(y_{(3)}))) \\
&\quad = \sum_{(f(zy_{(2)}x_{(2)})),(y),(x)} f(zy_{(2)}x_{(2)})_{(1)} \otimes ((1 \otimes y_{(1)})\Delta(x_{(1)} f(zy_{(2)}x_{(2)})_{(2)} S(x_{(3)}))(1 \otimes S(y \\
&\quad = \sum_{(f(zy_{(2)}x_{(3)})),(y),(x)} f(zy_{(2)}x_{(2)})_{(1)} \otimes ((1 \otimes y_{(1)})(x_{(1)} \otimes x_{(2)}) f(zy_{(2)}x_{(3)})_{(2)} \otimes f(zy_{(2)}x \\
&\quad = \sum_{(f(zy_{(2)}x_{(3)})),(x),(y)} f(zy_{(2)}x_{(3)})_{(1)} \otimes x_{(1)} f(zy_{(2)}x_{(3)})_{(2)} S(x_{(5)}) \otimes y_{(1)} x_{(2)} f(zy_{(2)}x_{(3)})
\end{aligned}
$$

Hence we have: $(\Delta_E \otimes \mathrm{id}_E) \circ \Delta_E = (\mathrm{id}_E \otimes \Delta_E) \circ \Delta_E$. *I claim that* $f \mapsto \epsilon(f(1))$ *is a conunity. Indeed we have:*

$$
\begin{aligned}
&(\epsilon \otimes \mathrm{id}_H) \circ \Delta_E(f))(1 \otimes y) = (\epsilon(1) \otimes 1)(\epsilon \otimes \mathrm{id}_E) \circ \Delta(f(y1))(1 \otimes S(1)) = f(y) \\
&(\mathrm{id}_H \otimes \epsilon) \circ \Delta_E(f))(x \otimes 1) = \Delta_E(f)(x \otimes 1) = \sum_{(x)} (1 \otimes \epsilon(x_{(1)}))(\mathrm{id}_E \otimes \epsilon) \circ \Delta(f(x_{(2)}))(1 \otimes \epsilon(S(x_{(3)}))) \\
&\qquad = \epsilon(x_{(1)}) f(x_{(2)}) \epsilon(S(x_{(3)})) = f(x).
\end{aligned}
$$

This proves that $\epsilon_E : f \mapsto \epsilon(f(1))$ *is a counity for* Δ_E. □

3. Prove that E endowed with the Δ_E and the convolution (that we denote μ_E or $*$) is a bialgebra.

Solution. *We want to prove that* ϵ_E *and* Δ_E *are algebra morphisms. We start with* ϵ_E*:*

$$\epsilon_E(f * g) = \epsilon(\mu \circ (f \otimes g) \circ \Delta(1)) = \epsilon(f(1)g(1)) = \epsilon_E(f)\epsilon_E(g).$$

For Δ_E*, this is a little longer: we want to prove:* $\Delta_E(f * g) = \sum_{(f),(g)} f_{(1)} * g_{(1)} \otimes f_{(2)} * g_{(2)}$.

$$
\begin{aligned}
\Delta_E(f * g)(x \otimes y) &= \sum_{(x)} (1 \otimes x_{(1)})\Delta((f * g)(yx_{(2)}))(1 \otimes S(x_{(3)})) \\
&= \sum_{(x)} (1 \otimes x_{(1)})\Delta((f(y_{(1)}x_{(2)})g(y_{(2)}x_{(3)})))(1 \otimes S(x_{(4)}))
\end{aligned}
$$

And:

$$
\begin{aligned}
&\sum_{(f),(g)} f_{(1)} * g_{(1)} \otimes f_{(2)} * g_{(2)}(x \otimes y) \\
&\quad = \sum_{(f),(g),(x),(y)} f_{(1)}(x_1) g_{(1)}(x_2) \otimes f_{(2)}(y_{(1)}) g_{(2)}(y_{(2)}) \\
&\quad = \sum_{(x)} f(y_{(1)}x_{(2)})_{(1)} g(y_{(2)}x_{(5)})_{(1)} \otimes x_{(1)} f(y_{(1)}x_{(2)})_{(2)} S(x_{(3)}) x_{(4)} g(y_{(2)}x_{(5)})_{(2)} S(x_{(6)}) \\
&\quad = \sum_{(x)} f(y_{(1)}x_{(2)})_{(1)} g(y_{(2)}x_{(5)})_{(1)} \otimes x_{(1)} f(y_{(1)}x_{(2)})_{(2)} \epsilon(x_{(3)}) g(y_{(2)}x_{(4)})_{(2)} S(x_{(5)}) \\
&\quad = \sum_{(x)} f(y_{(1)}x_{(2)})_{(1)} g(y_{(2)}x_{(5)})_{(1)} \otimes x_{(1)} f(y_{(1)}x_{(2)})_{(2)} g(y_{(2)}x_{(3)})_{(2)} S(x_{(4)}) \\
&\quad = \sum_{(x)} (1 \otimes x_{(1)})\Delta((f(y_{(1)}x_{(2)})g(y_{(2)}x_{(3)})))(1 \otimes S(x_{(4)}))
\end{aligned}
$$

Hence E *is a bialgebra.* □

4. Prove that the following formula define an antipode on E:

$$S_E(f)(x) = \sum_{(x)} S(x_{(1)})(S \circ f \circ S^{-1}(x_{(2)}))x_{(3)}.$$

Solution. We want to prove that:

$$\mu_E \circ (S_E \otimes \mathrm{id}_E) \circ \Delta_E = \eta_E \epsilon_E = \mu_E \circ (\mathrm{id}_E\ otimes S_E) \circ \Delta_E$$

. Let us compute:

$$\begin{aligned}
&(\mu_E \circ (S_E \otimes \mathrm{id}_E) \circ \Delta_E(f))(x) \\
&\quad = S_E(f_{(1)})(x_{(1)}) f_{(2)}(x_{(2)}) \\
&\quad = S(x_{(1)})(S \circ f_{(1)} \circ S^{-1}(x_{(2)}))x_{(3)} f_{(2)}(x_{(4)}) \\
&\quad = S(x_{(1)}) S(f_{(1)}(S^{-1}(x_{(2)})))x_{(3)} f_{(2)}(x_{(4)}) \\
&\quad = S(x_{(1)}) S((f(x_4 S^{-1}(x_{(2)})_{(2)}))_{(1)})x_{(3)} S^{-1}(x_{(2)})_{(1)}(f(x_4 S^{-1}(x_{(2)})_{(2)}))_{(2)} S(S^{-1}(x_{(2)})_{(3)}) \\
&\quad = S(x_{(1)}) S(f(x_6 S^{-1}(x_{(3)}))_{(1)})x_{(5)} S^{-1}(x_{(4)}) f(x_6 S^{-1}(x_{(3)}))_{(2)} S(S^{-1}(x_{(2)})) \\
&\quad = S(x_{(1)}) S(f(x_5 S^{-1}(x_{(3)}))_{(1)})\epsilon(x_{(4)}) f(x_5 S^{-1}(x_{(3)}))_{(2)} x_{(2)} \\
&\quad = S(x_{(1)}) S(f(x_5 S^{-1}(x_{(3)}))_{(1)}) f(x_5 S^{-1}(x_{(3)}))_{(2)} \epsilon(x_{(4)}) x_{(2)} \\
&\quad = S(x_{(1)}) \epsilon(f(x_5 S^{-1}(x_{(3)}))) \epsilon(x_{(4)}) x_{(2)} \\
&\quad = \epsilon(x_{(1)}) \epsilon(f(x_4 S^{-1}(x_{(2)}))) \epsilon(x_{(3)}) \\
&\quad = \epsilon(x_{(1)}) \epsilon(f(x_4 \epsilon(x_{(3)}) S^{-1}(x_{(2)}))) \\
&\quad = \epsilon(x_{(1)}) \epsilon(f(x_3 S^{-1}(x_{(2)}))) \\
&\quad = \epsilon(x_{(1)}) \epsilon(f(\epsilon(x_2))) \\
&\quad = \epsilon(x_{(1)}) \epsilon(x_{(2)}) \epsilon(f(1)) \\
&\quad = \epsilon(x) \epsilon(f(1)) \\
&\quad = \eta_E \circ \epsilon_E(f)
\end{aligned}$$

$$\begin{aligned}
&(\mu_E \circ (\mathrm{id}_E \otimes S_E) \circ \Delta_E(f))(x) \\
&\quad = f_{(1)}(x_{(1)}) S_E(f_{(2)})(x_{(2)}) \\
&\quad = f_{(1)}(x_{(1)}) S(x_{(2)})(S \circ f_{(2)} \circ S^{-1}(x_{(3)}))x_{(4)} \\
&\quad = f(S^{-1}(x_{(5)})x_{(2)})_{(1)} S(x_{(4)}) S(x_{(1)} f(S^{-1}(x_{(5)})x_{(2)})_{(2)}) S(x_{(3)}) x_{(6)} \\
&\quad = f(S^{-1}(x_{(5)})x_{(2)})_{(1)} S(x_{(1)} f(S^{-1}(x_{(5)})x_{(2)})_{(2)} S(x_{(3)}) x_{(4)}) x_{(6)} \\
&\quad = \epsilon(x_{(3)}) f(S^{-1}(x_{(4)})x_{(2)})_{(1)} S(f(S^{-1}(x_{(4)})x_{(2)})_{(2)}) S(x_{(1)}) x_{(5)} \\
&\quad = \epsilon(x_{(3)}) \epsilon(f(S^{-1}(x_{(4)})x_{(2)})) S(x_{(1)}) x_{(5)} \\
&\quad = \epsilon(f(S^{-1}(x_{(3)})x_{(2)})) S(x_{(1)}) x_{(4)} \\
&\quad = \epsilon(f(\epsilon(x_{(2)}))) S(x_{(1)}) x_{(3)} \\
&\quad = \epsilon(f(1)) S(x_{(1)}) \epsilon(x_{(2)}) x_{(3)} \\
&\quad = \epsilon(f(1)) \epsilon(x) \\
&\quad = \eta_E \circ \epsilon_E(f)
\end{aligned}$$

□

5. Prove that the maps $p_H : E \to H$ and $p_{H^*} : E \to H^{*\mathrm{cop}}$ defined by:

$$p_H(f) = f(1) \qquad \text{and} \qquad p_{H^*}(f) = \epsilon \circ f$$

are morphisms of Hopf algebras.

Solution. *We just need to show that p_H and p_{H^*} are morphisms of algebra and coalgebra. Let us compute the for multiplication:*

$$\begin{aligned} p_H(fg) &= fg(1) = f(1)g(1) = p_H(f)p_H(g) \\ p_H(\eta \circ \epsilon) &= \eta \circ \epsilon(1) = 1 \\ p_{H^*}(fg)(x) &= \epsilon \circ fg(x) \\ &= \epsilon(f(x_1)g(x_2)) \\ &= \epsilon(f(x_1))\epsilon(g(x_2)) \\ &= (p_{H^*}(f)p_{H^*}(g))(x) \\ p_{H^*}(\eta \circ \epsilon) &= \epsilon(1)\epsilon \\ &= \epsilon. \end{aligned}$$

For the comulitplication:

$$\begin{aligned} (p_H \otimes p_H)(\Delta_E(f)) &= \Delta_E(f)(1 \otimes 1) \\ &= (1 \otimes 1)\Delta(f(1))(1 \otimes 1) \\ &= \Delta(f(1)) = \Delta(p_H(f))\epsilon \circ p_H(f) & = \epsilon(f(1)) = \epsilon_E(f) \\ (p_{H^*} \otimes p_{H^*})(\Delta_E(f))(x \otimes y) &= (\epsilon \otimes \epsilon)\Delta_E(f)(x \otimes y) \\ &= (\epsilon \otimes \epsilon)(1 \otimes x_{(1)})\Delta(f(yx_{(2)}))(1 \otimes S(x_3)) \\ &= \epsilon(x_{(1)})\epsilon(f(yx_{(2)}))\epsilon(x_3) \\ &= \epsilon(f(yx)) \\ &= \Delta_{H^*}^{cop}(p_{H^*}(f))(x \otimes y) \end{aligned}$$

$\epsilon_{H^*} \circ p_{H^*}(f) = (p_{H^*}(f))(1) = \epsilon(f(1)) = \epsilon_E(f).$

□

6. Prove that the map $p_H \otimes p_{H^*} \circ \Delta_E$ is the identification of the first question.

Solution. *The ismorphism in the direction $H \otimes H^* \to E$ is easy to write down so let us it to evaluate $p_H \otimes p_{H^*} \circ \Delta_E(f)$. Let $x \in H$ we want to prove:*

$$(p_H \otimes p_{H^*} \circ \Delta_E(f))(x) = f(x)$$

$$\begin{aligned} p_H \otimes p_{H^*} \circ \Delta_E(f)(x) &= p_H(f_{(1)}) \otimes p_{H^*}(f_{(2)})(x) \\ &= f_1(1)\epsilon(f_{(2)}(x)) \\ &= (\mathrm{id} \otimes \epsilon)((1 \otimes 1)\Delta(f(x))(1 \otimes S(1))) \\ &= f(x) \end{aligned}$$

□

7. We define a linear form on $E \otimes E$ via the following formula:

$$r(f \otimes g) = \epsilon(f \circ g)(1)).$$

Prove that r admits an inverse $\overline{r}$ for the convolution product (on $E \otimes E$).

Solution. We look for an application $\overline{r} : E \otimes E \to \mathbb{K}$ such that:

$$\overline{r} * r(f \otimes g) = r * \overline{r}(f \otimes g) = \epsilon_{E \otimes E}(f \otimes g) = \epsilon_E(f)\epsilon_E(g) = \epsilon(f(1))\epsilon(g(1)) = \epsilon(f(1)g(1))$$

Let us recall that if a and b are two linear form on $E \otimes E$, then $a * b$ is given by the following formula:

$$(a * b)(f \otimes g) = a(f_{(1)} \otimes g_{(1)})b(f_{(2)} \otimes g_{(2)})$$

We claim that $\overline{r}(f \otimes g) = f(S^{-1}(g(1))$. Let us compute:

$$\begin{aligned}
\overline{r} * r(f \otimes g) &= \overline{r}(f_{(1)} \otimes g_{(1)})r(f_{(2)} \otimes g_{(2)}) \\
&= \epsilon(f_{(1)}(S^{-1}(g_{(1)}(1))))\epsilon(f_{(2)}(g_{(2)}(1))) \\
&= \epsilon(f_{(1)}(S^{-1}(g(1)_{(1)})))\epsilon(f_{(2)}(g(1))_{(2)}) \\
&= (\epsilon \otimes \epsilon)((1 \otimes S^{-1}(g(1)_{(1)})_{(1)})\Delta(f(g(1)_2 S^{-1}(g(1)_{(1)})_{(2)}))(1 \otimes S(S^{-1}(g(1)_{(1)})_{(3)}))) \\
&= \epsilon(S(g(1)_{(3)}))\epsilon(f(g(1)_4 S(g(1)_{(2)})))\epsilon((g(1)_{(1)})) \\
&= \epsilon(g(1)_{(3)})\epsilon(f(g(1)_4 S^{-1}(g(1)_2)))\epsilon(g(1)_{(1)})) \\
&= \epsilon(f(g(1)_4\epsilon(g(1)_{(3)})S^{-1}(g(1)_2\epsilon(g(1)_{(1)})))) \\
&= \epsilon(f(g(1)_4\epsilon(g(1)_{(3)})S^{-1}(g(1)_2\epsilon(g(1)_{(1)})))) \\
&= \epsilon(f(g(1)_2 S^{-1}(g(1)_1))) \\
&= \epsilon(f(\epsilon(g(1)))) \\
&= \epsilon(f(1))\epsilon(g(1))
\end{aligned}$$

The other direction:

$$\begin{aligned}
r * \overline{r}(f \otimes g) &= r(f_{(1)} \otimes g_{(1)})\overline{r}(f_{(2)} \otimes g_{(2)}) \\
&= \epsilon(f_{(1)}(g(1)_{(1)}))\epsilon(f_{(2)}(S^{-1}(g(1)_{(2)}))) \\
&= (\epsilon \otimes \epsilon)((1 \otimes g(1)_{(1)})\Delta(f(S^{-1}(g(1)_4)g(1)_{(2)}))(1 \otimes S(g(1)_{(3)}))) \\
&= \epsilon(g(1)_{(1)})\epsilon(f(S^{-1}(g(1)_4)g(1)_{(2)}))\epsilon(S(g(1)_{(3)})) \\
&= \epsilon(f(S^{-1}(g(1)_4)\epsilon(g(1)_{(1)})g(1)_{(2)}))\epsilon(g(1)_{(3)}) \\
&= \epsilon(f(S^{-1}(g(1)_4)\epsilon(g(1)_{(1)})g(1)_{(2)}\epsilon(g(1)_{(3)}))) \\
&= \epsilon(f(S^{-1}(g(1)_2)g(1)_{(1)})) \\
&= \epsilon(f(\epsilon(g(1)))) \\
&= \epsilon(f(1))\epsilon(g(1))
\end{aligned}$$

□

8. Prove that the following equality holds: $r * \mu_E = \mu_E^{\mathrm{op}} * r$ where $*$ denote the convolution on $E \otimes E$.

Solution. Let us compute:

$$\begin{aligned}
r * \mu_E(f \otimes g)(x) &= \epsilon(f_{(1)} \circ g_{(1)}(1))(f_{(2)}g_{(2)})(x) \\
&= \epsilon(f_{(1)} \circ g_{(1)}(1))f_{(2)}(x_{(1)})g_{(2)}(x_{(2)}) \\
&= \epsilon(f(g(x_{(2)})_{(1)}))f_{(2)}(x_{(1)})(g(x_{(2)}))_{(2)} \\
&= \epsilon(f(x_{(1)}g(x_{(2)})_{(2)})_{(1)})g(x_{(2)})_{(1)}f(x_{(1)}g(x_{(2)})_{(2)})_{(2)}S(g(x_{(2)})_{(3)})(g(x_{(2)}))_{(4)} \\
&= \epsilon(f(x_{(1)}g(x_{(2)})_{(2)})_{(1)})g(x_{(2)})_{(1)}f(x_{(1)}g(x_{(2)})_{(2)})_{(2)}S(g(x_{(2)})_{(3)})(g(x_{(2)}))_{(4)} \\
&= \epsilon(f(x_{(1)}g(x_{(2)})_{(2)})_{(1)})g(x_{(2)})_{(1)}f(x_{(1)}g(x_{(2)})_{(2)})_{(2)}\epsilon(g(x_{(2)})_{(3)})) \\
&= \epsilon(f(x_{(1)}g(x_{(2)})_{(2)})_{(1)})g(x_{(2)})_{(1)}f(x_{(1)}g(x_{(2)})_{(2)})_{(2)} \\
&= \epsilon(f(x_{(1)}g(x_{(2)})_{(2)}))g(x_{(2)})_{(1)}.
\end{aligned}$$

And:

$$\begin{aligned}
\mu_E^{op} * r(f \otimes g)(x) &= g_{(1)}f_{(1)}(x)\epsilon(f_{(2)} \circ g_{(2)}(1)) \\
&= g_{(1)}(x_{(1)})f_{(1)}(x_{(2)})\epsilon(f_{(2)}(g_{(2)}(1))) \\
&= g(x_{(2)})_{(1)}f_{(1)}(x_{(4)})\epsilon(f_{(2)}(x_{(1)}g(x_{(2)})_{(2)}S(x_{(3)}))) \\
&= g(x_{(2)})_{(1)}f(x_{(1)}g(x_{(2)})_{(2)}S(x_{(3)})x_{(5)})_{(1)}\epsilon(x_{(4)}f(x_{(2)}g(x_{(2)})_{(2)}S(x_{(3)})x_{(5)})_{(2)}S(x_{(6)})) \\
&= g(x_{(2)})_{(1)}f(x_{(1)}g(x_{(2)})_{(2)}S(x_{(3)})x_{(5)})\epsilon(x_{(4)})\epsilon(S(x_{(6)})) \\
&= g(x_{(2)})_{(1)}f(x_{(1)}g(x_{(2)})_{(2)}S(x_{(3)})x_{(4)})) \\
&= g(x_{(2)})_{(1)}f(x_{(1)}g(x_{(2)})_{(2)}\epsilon(x_{(3)})) \\
&= \epsilon(f(x_{(1)}g(x_{(2)})_{(2)}))g(x_{(2)})_{(1)}. \\
&= r * \mu_E(f \otimes g)(x)
\end{aligned}$$

□

9. Prove that the following relations hold:

$$r \circ (\mu_E \otimes \mathrm{id}_E) = r_{13} * r_{23} r \circ (\mathrm{id}_E \otimes \mu_E) = r_{13} * r_{12}$$

where the notation r_{ik} means that r is applied to the ith and kth tensors and ϵ is applied everywhere else.

Solution. The left-hand sides are easy to undersand:

$$\begin{aligned}
r \circ (\mu_E \otimes \mathrm{id}_E)(f \otimes g \otimes h) &= r(fg \otimes h) = \epsilon(fg(h(1))) = \epsilon(f(h(1)_{(1)}))\epsilon(g(h(1)_{(2)})) \\
r \circ (\mathrm{id}_E \otimes \mu_E)(f \otimes g \otimes h) &= r(f \otimes gh) = \epsilon(f(gh(1))) = \epsilon(f(g(1)h(1)))
\end{aligned}$$

Let us compute the right hand-sides:

$$\begin{aligned}
(r_{13} * r_{23})(f \otimes g \otimes h) &= r_{13}(f_{(1)} \otimes g_{(1)} \otimes h_{(1)})r_{23}(f_{(2)} \otimes g_{(2)} \otimes h_{(2)}) \\
&= \epsilon(f_{(1)}(h_{(1)}(1)))\epsilon_E(g_{(1)})\epsilon(g_{(2)}(h_{(2)}(1)))\epsilon_E(f_{(2)}) \\
&= \epsilon(f_{(1)}(h_{(1)}(1)))\epsilon(g_{(1)}(1))\epsilon(g_{(2)}(h_{(2)}(1)))\epsilon(f_{(2)}(1)) \\
&= \epsilon(f_{(1)}(h(1)_{(1)}))\epsilon(g(h(1)_{(2)})_{(1)})\epsilon(g(h(1)_{(2)})_{(2)})\epsilon(f_{(2)}(1)) \\
&= \epsilon(f(h(1)_{(2)})_{(1)})\epsilon(g(h(1)_{(4)})_{(1)})\epsilon(g(h(1)_{(4)})_{(2)})\epsilon(h(1)_{(1)}f(h(1)_{(2)})_{(2)}S(h(1)_{(3)})) \\
&= \epsilon(h(1)_{(1)})\epsilon(f(h(1)_{(2)})_{(1)})\epsilon(f(h(1)_{(2)})_{(2)})\epsilon(S(h(1)_{(3)}))\epsilon(g(h(1)_{(4)})_{(1)})\epsilon(g(h(1)_{(4)})_{(2)}) \\
&= \epsilon(h(1)_{(1)})\epsilon(f(h(1)_{(2)}))\epsilon(h(1)_{(3)})\epsilon(g(h(1)_{(4)})) \\
&= \epsilon(f(h(1)_{(1)}))\epsilon(g(h(1)_{(2)})) \\
&= r \circ (\mu_E \otimes \mathrm{id}_E)(f \otimes g \otimes h).
\end{aligned}$$

And:

$$\begin{aligned}
(r_{13} * r_{12})(f \otimes g \otimes h) &= r_{13}(f_{(1)} \otimes g_{(1)} \otimes h_{(1)}) r_{12}(f_{(2)} \otimes g_{(2)} \otimes h_{(2)}) \\
&= \epsilon(f_{(1)}(h_{(1)}(1)))\epsilon_E(g_{(1)})\epsilon(f_{(2)}(g_{(2)}(1)))\epsilon_E(h_{(2)}) \\
&= \epsilon(f_{(1)}(h(1)_{(1)}))\epsilon(g(1)_{(1)})\epsilon(f_{(2)}(g(1)_{(2)}))\epsilon(h(1)_{(2)}) \\
&= \epsilon(f(g(1)_{(2)}h(1)_{(2)})_{(1)})\epsilon(g(1)_{(1)})\epsilon(h(1)_{(1)}f(g(1)_{(2)}h(1)_{(2)})_{(2)}S(h(1)_{(3)}))\epsilon(h(1)_{(4)}) \\
&= \epsilon(f(g(1)_{(2)}h(1)_{(2)})_{(1)})\epsilon(g(1)_{(1)})\epsilon(h(1)_{(1)})\epsilon(f(g(1)_{(2)}h(1)_{(2)})_{(2)})\epsilon(h(1)_{(3)}))\epsilon(h(1)_{(4)}) \\
&= \epsilon(f(g(1)_{(2)}h(1)_{(2)}))\epsilon(g(1)_{(1)})\epsilon(h(1)_{(1)})\epsilon(h(1)_{(3)}))\epsilon(h(1)_{(4)}) \\
&= \epsilon(f(g(1)h(1)_{(1)}))\epsilon(h(1)_{(2)})) \\
&= \epsilon(f(g(1)h(1))).
\end{aligned}$$

□

10. Prove that the dual of E is naturally isomorphic to $D(H)$.

Problem 2. Let A and B be Hopf algebras over a field $\mathbb{K}$. A Hopf-pairing is a linear map $\sigma : A \otimes B \to \mathbb{K}$ such that for all $a, a' \in A$ and $b, b' \in B$

$$\sigma(aa' \otimes b) = \sigma(a \otimes b_{(2)}) \cdot \sigma(a' \otimes b_{(1)}) \qquad \sigma(1 \otimes b) = \epsilon(b)$$
$$\sigma(a \otimes bb') = \sigma(a_{(1)} \otimes b) \cdot \sigma(a_{(2)} \otimes b') \qquad \sigma(a \otimes 1) = \epsilon(a)$$

1. Prove that $A \otimes B$ becomes an associative, unital algebra with unit $1_A \otimes 1_B$, if we set

$$(a \otimes b)(a' \otimes b') := \sigma(a'_{(1)}\ S(b_{(1)})) \cdot \sigma(a'_{(3)} \otimes b_{(3)}) aa'_{(2)} \otimes b_{(2)}b'.$$

Solution. Let us proves that the product is associative: Let $a, a', a'' \in A$ and $b, b', b'' \in B$. We compute $((a \otimes b)(a' \otimes b'))(a'' \otimes b'')$. Applying the definition we get

$$\begin{aligned}
&\sigma(a'_{(1)} \otimes S(b_{(1)})) \cdot \sigma(a'_{(3)} \otimes b_{(3)}) \cdot \sigma(a''_{(1)} \otimes S((b_{(2)}b')_{(1)})) \cdot \sigma(a''_{(3)} \otimes (b_{(2)}b')_{(3)}) \\
&\cdot aa'_{(2)}a''_{(2)} \otimes (b_{(2)}b')_{(2)}b'' \\
=&\sigma(a'_{(1)} \otimes S(b_{(1)})) \cdot \sigma(a'_{(3)} \otimes b_{(5)}) \cdot \sigma(a''_{(1)} \otimes S(b'_{(1)})S(b_{(2)})) \cdot \sigma(a''_{(3)} \otimes b_{(4)}b'_{(3)}) \\
&\cdot aa'_{(2)}a''_{(2)} \otimes b_{(3)}b'_{(2)}b'' \quad \text{[}\Delta\text{ alg. hom. + coass. + } S \text{ anti-alg. hom.]} \\
=&\sigma(a'_{(1)} \otimes S(b_{(1)})) \cdot \sigma(a'_{(3)} \otimes b_{(5)}) \cdot \sigma((a''_{(1)})_{(1)} \otimes S(b'_{(1)})) \cdot \sigma((a''_{(1)})_{(2)} \otimes S(b_{(2)})) \\
&\cdot \sigma((a''_{(3)})_{(1)} \otimes b_{(4)}) \cdot \sigma((a''_{(3)})_{(2)} \otimes b'_{(3)}) \cdot aa'_{(2)}a''_{(2)} \otimes b_{(3)}b'_{(2)}b'' \quad \text{[}\sigma\text{ Hopf pairing]} \\
=&\sigma(a'_{(1)} \otimes S(b_{(1)})) \cdot \sigma(a'_{(3)} \otimes b_{(5)}) \cdot \sigma(a''_{(1)} \otimes S(b'_{(1)})) \cdot \sigma(a''_{(2)} \otimes S(b_{(2)})) \\
&\cdot \sigma(a''_{(4)} \otimes b_{(4)}) \cdot \sigma(a''_{(5)} \otimes b'_{(3)}) \cdot aa'_{(2)}a''_{(3)} \otimes b_{(3)}b'_{(2)}b'' \quad \text{[coass.]}
\end{aligned}$$

Similar considerations (one has to use that S is a anti-coalgebra homomorphism) show that $(a \otimes b)((a' \otimes b')(a'' \otimes b''))$ is equal to

$$\begin{aligned}
&\sigma(a''_{(1)} \otimes S(b'_{(1)})) \cdot \sigma(a''_{(5)} \otimes b'_{(3)}) \cdot \sigma(a'_{(1)} \otimes S(b_{(1)})) \cdot \sigma(a''_{(2)} \otimes S(b_{(2)})) \\
&\cdot \sigma(a'_{(3)} \otimes b_{(5)}) \cdot \sigma(a''_{(4)} \otimes b_{(4)}) \cdot aa'_{(2)}a''_{(3)} \otimes b_{(3)}b'_{(2)}b''
\end{aligned}$$

We show $(a \otimes b)(1 \otimes 1) = a \otimes b$

$$\sigma(1 \otimes S(b_{(1)})\sigma(1 \otimes b_{(3)})a1 \otimes b_{(2)}1 = \epsilon(S(b_{(1)}))\epsilon(b_{(3)})a \otimes b_{(2)} = a \otimes b.$$

Analogously one shows $(1 \otimes 1)(a \otimes b) = a \otimes b$. □

2. Show that $A \otimes B$ becomes a Hopf algebra with

$$\Delta(a \otimes b) := a_{(1)} \otimes b_{(1)} \otimes a_{(2)} \otimes b_{(2)} \qquad \epsilon(a \otimes b) := \epsilon(a) \cdot \epsilon(b)$$
$$S(a \otimes b) := \sigma(a_{(1)} \otimes b_{(1)}) \cdot \sigma(a_{(3)} \otimes S(b_{(3)}))S(a_{(2)}) \otimes S(b_{(2)}).$$

Solution. *Δ is clearly a coassociative comultiplication on $A \otimes B$ with counit ϵ so we still have to show the bialgebra axiom:*

$$\Delta((a \otimes b)(a' \otimes b)) = \Delta(a \otimes b)\Delta(a' \otimes b). \tag{1}$$

Note the equality

$$\sigma(a \otimes S(b_{(1)})b_{(2)}) = \sigma(a \otimes b_{(1)}S(b_{(2)})) = \sigma(a \otimes \epsilon(b)1) = \epsilon(a)\epsilon(b). \tag{2}$$

The right-hand side of (??) is equal to

$$\begin{aligned}
&\sigma(a'_{(1)} \otimes S(b_{(1)})) \cdot \sigma(a'_{(3)} \otimes b_{(3)}) \cdot \sigma(a'_{(4)} \otimes S(b_{(4)})) \cdot \sigma(a'_{(6)} \otimes b_{(6)}) \\
&a_{(1)}a'_{(2)} \otimes b_{(2)}b'_{(1)} \otimes a_{(2)}a'_{(5)} \otimes b_{(5)}b'_{(2)} \\
=&\sigma(a'_{(1)} \otimes S(b_{(1)})) \cdot \sigma(a'_{(3)} \otimes b_{(3)}S(b_{(4)})) \cdot \sigma(a'_{(5)} \otimes b_{(6)}) \\
&a_{(1)}a'_{(2)} \otimes b_{(2)}b'_{(1)} \otimes a_{(2)}a'_{(4)} \otimes b_{(5)}b'_{(2)} \quad \textit{[Hopf pairing]} \\
\overset{(??)}{=}&\sigma(a'_{(1)} \otimes S(b_{(1)})) \cdot \epsilon(a'_{(3)}) \cdot \epsilon(b_{(3)}) \cdot \sigma(a'_{(5)} \otimes b_{(5)}) \\
&a_{(1)}a'_{(2)} \otimes b_{(2)}b'_{(1)} \otimes a_{(2)}a'_{(4)} \otimes b_{(4)}b'_{(2)} \\
=&\sigma(a'_{(1)} \otimes S(b_{(1)})) \cdot \sigma(a'_{(4)} \otimes b_{(4)}) \\
&a_{(1)}a'_{(2)} \otimes b_{(2)}b'_{(1)} \otimes a_{(2)}a'_{(3)} \otimes b_{(3)}b'_{(2)} \quad \textit{[counits]}
\end{aligned}$$

The last line is equal to $\Delta((a \otimes b)(a' \otimes b))$.
The last thing we have to show is the antipode property, we will only show $S(a_{(1)} \otimes b_{(1)})(a_{(2)} \otimes b_{(2)}) = \epsilon(a)\epsilon(b)1 \otimes 1$. The left-hand side is by definition

$$\begin{aligned}
&\sigma(a_{(1)} \otimes b_{(1)}) \cdot \sigma(a_{(3)} \otimes S(b_{(3)}))(S(a_{(2)}) \otimes S(b_{(2)}))(a_{(4)} \otimes b_{(4)}) \\
=&\sigma(a_{(1)} \otimes b_{(1)}) \cdot \sigma(a_{(3)} \otimes S(b_{(5)})) \cdot \sigma(a_{(4)} \otimes S(S(b_{(4)}))) \cdot \sigma(a_{(6)} \otimes S(b_{(2)})) \\
&S(a_{(2)})a_{(5)} \otimes S(b_{(3)})b_{(6)} \quad \textit{[def. of mult. and } S \textit{ anti-alg. hom.]} \\
=&\sigma(a_{(1)} \otimes b_{(1)}) \cdot \sigma(a_{(3)} \otimes S(b_{(5)})S(S(b_{(4)}))) \cdot \sigma(a_{(5)} \otimes S(b_{(2)})) \\
&S(a_{(2)})a_{(4)} \otimes S(b_{(3)})b_{(6)} \quad \textit{[Hopf pairing]} \\
=&\sigma(a_{(1)} \otimes b_{(1)}) \cdot \sigma(a_{(3)} \otimes \epsilon(b_{(4)})1) \cdot \sigma(a_{(5)} \otimes S(b_{(2)})) \\
&S(a_{(2)})a_{(4)} \otimes S(b_{(3)})b_{(5)} \quad \textit{[antipode]} \\
=&\sigma(a_{(1)} \otimes b_{(1)}) \cdot \epsilon(a_{(3)})\epsilon(b_{(4)}) \cdot \sigma(a_{(5)} \otimes S(b_{(2)})) \\
&S(a_{(2)})a_{(4)} \otimes S(b_{(3)})b_{(5)} \quad \textit{[Hopf pairing]} \\
=&\sigma(a_{(1)} \otimes b_{(1)}) \cdot \sigma(a_{(4)} \otimes S(b_{(2)}))S(a_{(2)})a_{(3)} \otimes S(b_{(3)})b_{(4)} \quad \textit{[counits]} \\
=&\sigma(a_{(1)} \otimes b_{(1)}) \cdot \sigma(a_{(3)} \otimes S(b_{(2)}))\epsilon(a_{(2)})1 \otimes \epsilon(b_{(3)})1 \quad \textit{[antipodes]} \\
=&\sigma(a_{(1)} \otimes b_{(1)}) \cdot \sigma(a_{(2)} \otimes S(b_{(2)}))1 \otimes 1 \quad \textit{[counits]} \\
\overset{(??)}{=}&\epsilon(a)\epsilon(b)1 \otimes 1
\end{aligned}$$

□

3. Let H be a finite dimensional Hopf algebra over $\mathbb{K}$. Show that the evaluation $V^* \otimes V \to \mathbb{K}$ defines a Hopf pairing $\sigma : (H^{\mathrm{cop}})^* \otimes H \to \mathbb{K}$.

Solution. *Remember the definition of $\Delta : H^* \to H^* \otimes H^*$ and $\mu : H^* \otimes H^* \to H^*$, for $f, g \in H^*$ and $a, b \in H$ we have:*

$$\langle \Delta(f), a \otimes b \rangle := \langle f, a \rangle \langle f, b \rangle, \text{ i.e. } \Delta_{H^*} = \mu_H^*.$$
$$\langle f \cdot g, a \rangle := f(a_{(1)})g(h_{(2)})$$

Set $\sigma(f \otimes a) := \langle f, a \rangle$ and note that we consider $(H^{\mathrm{cop}})^*$, i.e. H^* with multiplication $\langle f \bullet g, a \rangle = f(a_{(2)})g(a_{(1)})$. Note also the convention $\langle f \otimes g, a \otimes b \rangle := \langle f, a \rangle \langle g, b \rangle$. We have

$$\begin{aligned}
\sigma(f \bullet g \otimes a) &= f(a_{(2)})g(a_{(1)}) = \sigma(f \otimes a_{(2)})\sigma(g \otimes a_{(1)}) \\
\sigma(f \otimes ab) &= \langle \mu_H^*(f), a \otimes b \rangle = \langle f_{(1)}, a \rangle \langle f_{(2)}, b \rangle = \sigma(f_{(1)} \otimes a)\sigma(f_{(2)} \otimes b) \\
\sigma(\epsilon \otimes a) &= \epsilon(a) \\
\sigma(f \otimes 1) &= f(1) = \epsilon(f)
\end{aligned}$$

so σ is indeed a Hopf pairing. □

Problem 3. Let G be a finite group and $D(G)$ the Drinfel'd double of the group (Hopf) algebra $\mathbb{K}[G]$ over a field $\mathbb{K}$. Assume also $|G| \nmid \operatorname{char} \mathbb{K}$. Due to this assumption, the category $\mathcal{C}$ of finite dimensional left $D(G)$-modules over $\mathbb{K}$ can be shown to be semisimple.[1]

1. Determine the isomorphism classes of simple objects in $\mathcal{C}$ for an abelian group G.

Solution. *Let us consider V a simple $D(G)$-module. We know from the lecture that a $D(G)$-module is the same as a G-graded vector space $V = \oplus_{g \in G} V_g$, together with a G-action, such that $g.v \in V_{ghg^{-1}}$ for $v \in V_h$. We suppose G abelian hence the G action preserves the V_g's. As V is simple, this shows that all the V_g's are trivial but one let us say for g_0. Once more, as V is simple, we have V_{g_0} is simple.*

We conclude that simple $D(G)$-modules are pairs (V, g) with $g \in G$ and V a simple G-module. As G is finite and abelian, we know that the simple G-modules are all 1-dimensional and hence described by their character in a very explicit way. □

2. Determine the isomorphism classes of simple objects in $\mathcal{C}$ for $G = S_3$, the symmetric group on three letters.

Solution. *Let V be a simple $D(G)$-module. We have $V = \oplus_{g \in G} V_g = \oplus_{c \in \Omega} \oplus_{g \in c} V_g = \oplus_{c \in \Omega} V_c$ where Ω is the set of conjugation classes of G. The same analysis as before shows that the V_c's are preserved by the G-action. So that all of them but one is trivial. $G = S_3$ has 3 conjugation classes: $c_1 := \{\mathrm{id}\}, c_2 := \{(12), (23), (13)\}$ and $c_3 := \{(123), (132)\}$.*

- *Suppose $V = V_{c_1}$, then as V is simple, V_{c_1} is a simple G-module (as we shall see there are three different isomorphism type of simple S_3-module: 1 of dimension 2 and 2 of dimension 1).*

[1]Equivalently one can consider the category ${}_H\mathcal{YD}^H$ of Yetter Drinfel'd module (left module and right comodule), where the compatibility condition reads: $h_{(1)}v_0 \otimes h_{(2)}v_1 = (h_{(2)}v)_{(0)} \otimes (h_{(2)}v)_{(1)}h_{(1)}$.

- *Suppose* $V = V_{c_2} = V_{(12)} \oplus V_{(23)} \oplus V_{(13)}$. *The action of elements of* G *provides isomorphisms between* $V_{(12)}$, $V_{(23)}$, $V_{(13)}$. *On the other hand, the subgroup* $G' := \{(12), \mathrm{id}\}$ *stabilize* $V_{(12)}$. *Hence* $V_{(12)}$ *can be seen as a* G'*-module. One easly see that* $V_{(12)}$ *has to be simple as a* G'*-module and that* V *is entirely determined by* $V_{(12)}$. *As we shall see there are exactly two simple* G'*-modules both of them are* 1*-dimensional.*
- *Suppose that* $V = V_{c_2} = V_{(123)} \oplus V_{(132)}$. *The same reasoning shows that* $V_{(123)}$ *is a simple* G''*-module with* $G'' = \{(123), (132), \mathrm{id}\}$. *As we shall see there are exactly three simple* G''*-module. All of them are* 1*-dimensional.*

To sum up we have 3+ 2 +3 = 8 simple Yetter-Drinfel'd S_3*-modules, 4 of dimension 1, 2 of dimension 2 and 2 of dimension 3.* ☐

3. Determine the isomorphism classes of simple objects in $\mathcal{C}$ for general G.

Solution. We can apply the same reasoning, the simple objects in $\mathcal{C}$ *are parametrized by a conjugation class* c *and a simple* $\mathrm{Stab}(g)$*-module where* g *is an element of* c. ☐