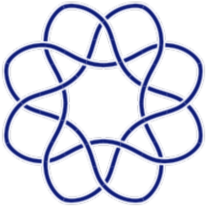
Sur l'homologie $\mathfrak{sl}_3$ des enchevêtrements ; algèbres de Khovanov – Kuperberg

Louis-Hadrien Robert

Thèse dirigée par Christian Blanchet

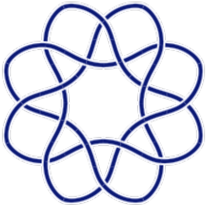
Definition

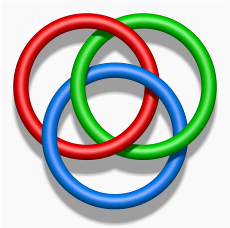
A *knot* is an embedding of $\mathbb{S}^1$ in $\mathbb{S}^3$.



Definition

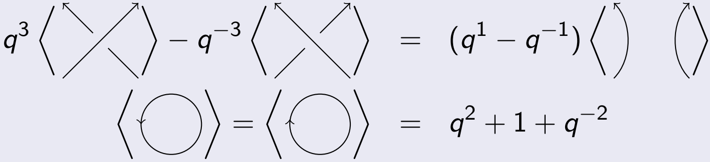
A *link* is an embedding of $\bigsqcup \mathbb{S}^1$ in $\mathbb{S}^3$.





Theorem (Freyd – Hoste – Lickorish – Millett – Ocneanu – Yetter, Przytycki – Traczyk)

There exists a unique Laurent polynomial invariant $\langle\cdot\rangle$ satisfying the following relations:



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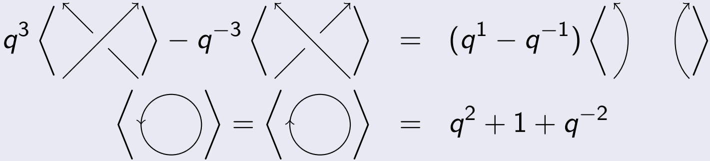
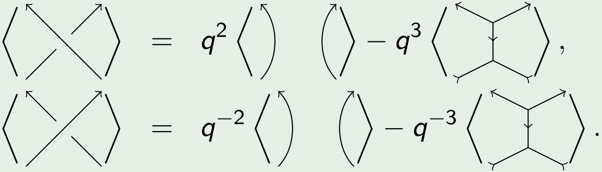
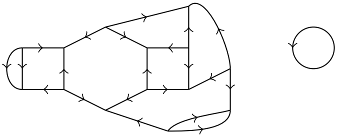


Alternative relations



Definition (Kuperberg, 96)

A *web* is a plane oriented trivalent graph with possibly some vertexless circles.



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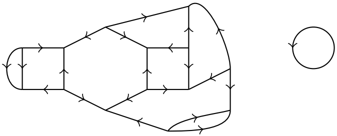
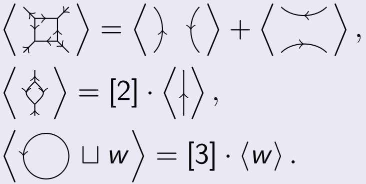


Kuperberg bracket



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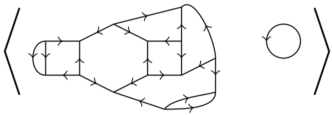

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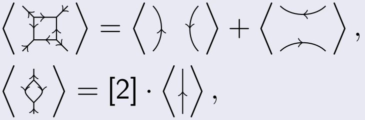

$$\left\langle \bigcirc \sqcup w \right\rangle = [3] \cdot \langle w \rangle .$$

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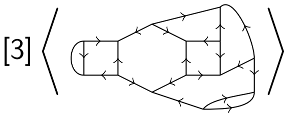

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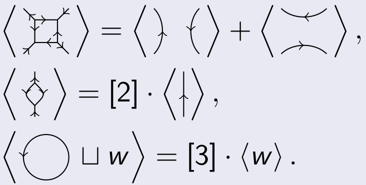

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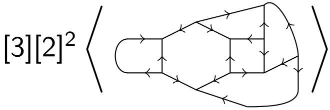

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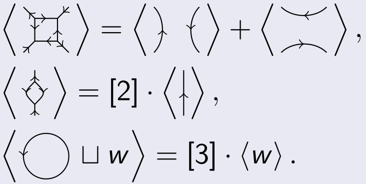

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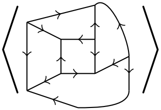
$[3][2]^3$ 

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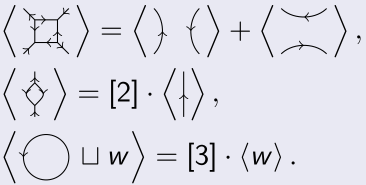

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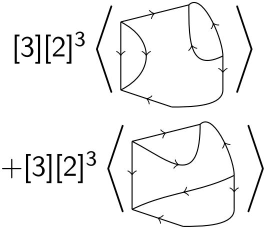

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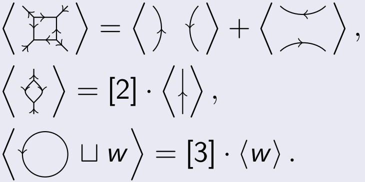

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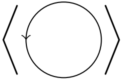
$2[3][2]^5$ 

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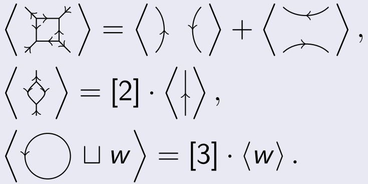

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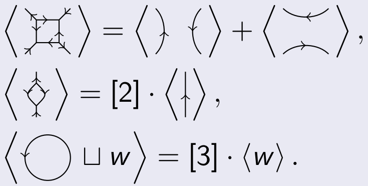
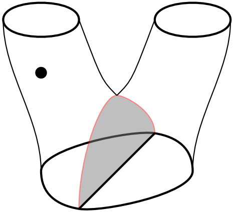


Theorem (Khovanov, 2003)

There exists a TQFT $\mathcal{F}$ from the category of webs and foams to the category of graded $\mathbb{Z}$-modules which categorifies the Kuperberg bracket. This TQFT allows to define the $\mathfrak{sl}_3$-homology which categories the $\mathfrak{sl}_3$ polynomial.



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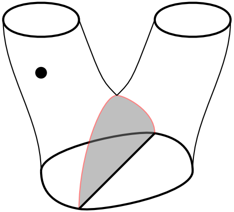
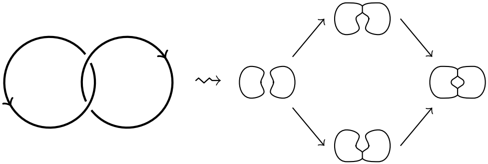
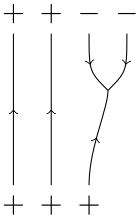




Sequences of signs ϵ

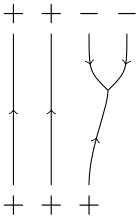
Web-tangles w



Foams (with corners) f

Sequences of signs $\quad \epsilon \quad \mapsto \quad$ Algebras

Web-tangles $\quad w \quad \mapsto \quad$ Bi-modules



Foams (with corners) $\quad f \quad \mapsto \quad$ Bi-modules maps

First attempt:

$$\widetilde{K^\epsilon} \stackrel{\text{def}}{=} \bigoplus_{w_1, w_2} \mathcal{F}(\overline{w_1} w_2)$$

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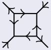
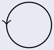
An ϵ-web is *non-elliptic* if it contains: NO , NO , and NO .

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For a given ϵ, there exist finitely many non-elliptic ϵ-webs.

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Definition (Mackaay – Pan – Tubbenhauer, R.)

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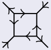
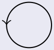
An ϵ-web is *non-elliptic* if it contains: NO , NO , and NO .

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w_0
w_1
w_1
w_2
w_0
w_2
$\epsilon \times I$
f
$\otimes$
g
$\mapsto$
f
g
$\epsilon \times I$

Definition (Mackaay – Pan – Tubbenhauer, R.)

$$\mathcal{F}(w) \stackrel{\text{def}}{=} \bigoplus_{\substack{w_1 \in NE(\epsilon_1) \\ w_2 \in NE(\epsilon_2)}} \mathcal{F}(\overline{w_1} w w_2) \qquad \mathcal{F}(u) \stackrel{\text{def}}{=} \sum_{\substack{w_1 \in NE(\epsilon_1) \\ w_2 \in NE(\epsilon_2)}} \mathcal{F}\left((\overline{w_1} \times I) u (w_2 \times I)\right)$$

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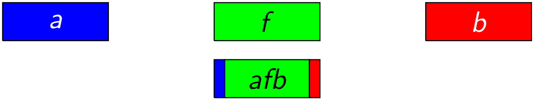

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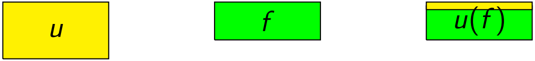

Theorem (Mackaay – Pan – Tubbenhauer, R.)

This construction leads to a well-defined 2-functor from the 2-category of web-tangles to the 2-category of algebras.

Theorem (Mackaay – Pan – Tubbenhauer, R.)

From this we can derived an extension of the $\mathfrak{sl}_3$-homology to tangles. It satisfies the following gluing relation at the level of complexes:

$$C(T_1 T_2) \simeq C(T_1) \otimes_{K_1^\epsilon} C(T_2).$$

Question

How are the projective indecomposable modules over K^ϵ ?

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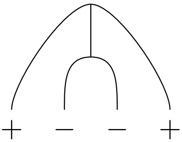
Proposition

If w_1 and w_2 are two ϵ-webs, the graded dimension of $\hom_{K^\epsilon}(\mathcal{F}(w_1), \mathcal{F}(w_2))$ is given by:

$$q^{l(\epsilon)} \cdot \langle \overline{w_1} w_2 \rangle \, .$$



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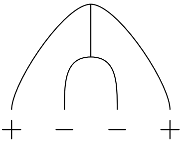
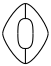
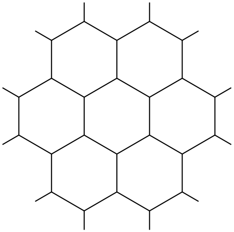


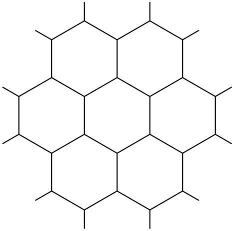
$q^4 \cdot \Big\langle$  $\Big\rangle = {\color{red}1} + 3q^2 + 4q^4 + 3q^6 + q^8$

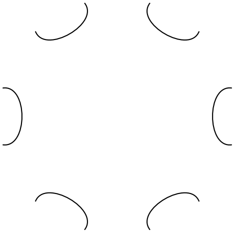
Consequence

This web-module is indecomposable.

$w =$ 

$$q^{12}\left\langle \overline{w}w \right\rangle = 2 + 80q^2 + 902q^4 + 4604q^6 + 13158q^8 + 23684q^{10} + 28612q^{12} + 23684q^{14} + 13158q^{16} + 4604q^{18} + 902q^{20} + 80q^{22} + 2q^{24}$$

$w =$ 

$w_0 =$ 

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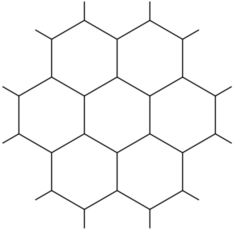
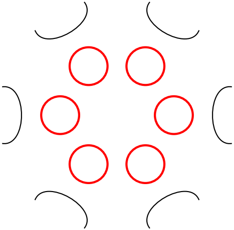
$w =$ 



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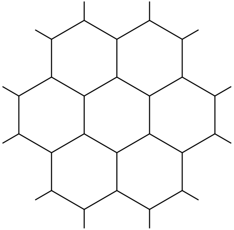
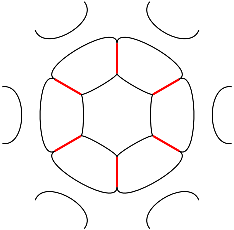
$w =$ 



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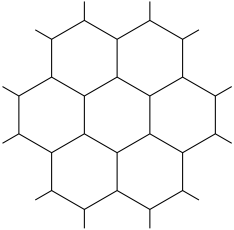
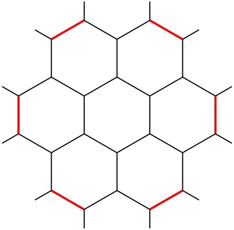
$w =$ 



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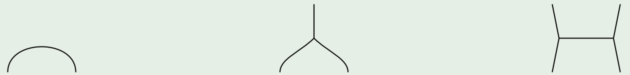
Definition (R.)

An ϵ-web is *superficial* if it has no nested face.

Theorem (R.)

If w be a superficial and non-elliptic ϵ-web, then the K^ϵ-module $\mathcal{F}(w)$ is indecomposable. If w' is another superficial and non-elliptic ϵ-web different from w, then $\mathcal{F}(w)$ and $\mathcal{F}(w')$ are not isomorphic as K^ϵ-modules.

Around the boundary



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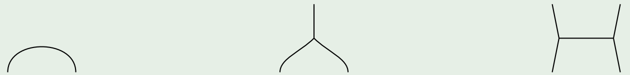
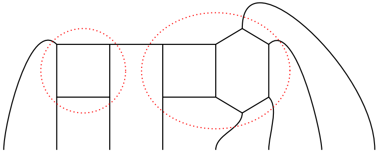


Lemma

A semi-non-elliptic superficial ϵ-web is equal (in the skein module) to a sum of non-elliptic webs.



Proposition (last section)

Let w be an ϵ-web.
$\langle \overline{w} w \rangle$ is monic of degree $l(\epsilon) \Rightarrow \mathcal{F}(w)$ is indecomposable.

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Let w be an ϵ-web.
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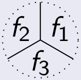
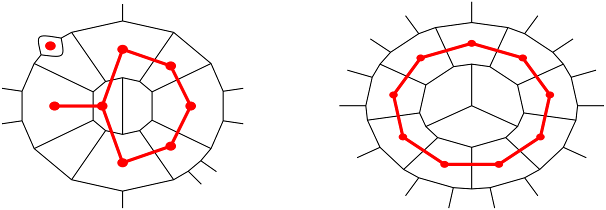
- Combinatorial data (red graphs) to idempotent foams,
- "Non-monicity" to red graphs.

Definition (R.)

Let w be an ϵ-web, a red graph for G is an induced subgraph of the dual of w such that:

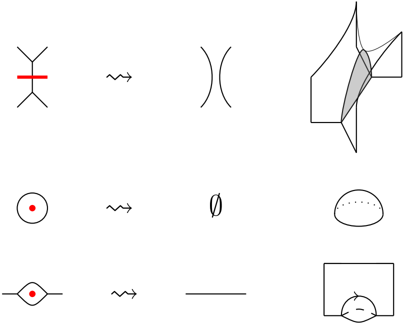


$$\Rightarrow \{f_1, f_2, f_3\} \not\subseteq V(G).$$



Proposition (R.)

From any exact red graph, one can construct a non-trivial idempotent.



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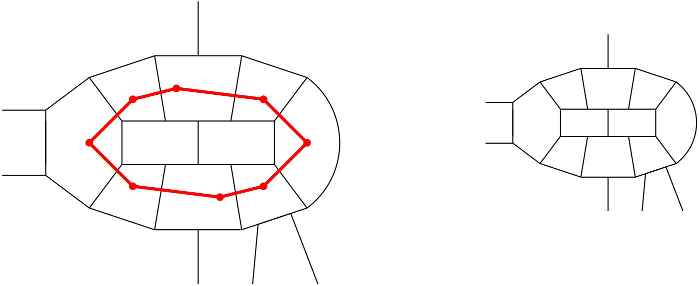

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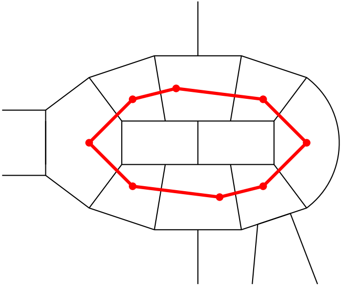

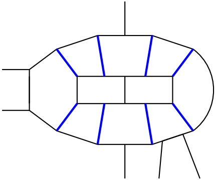

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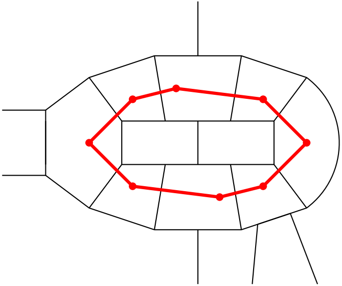

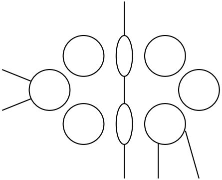

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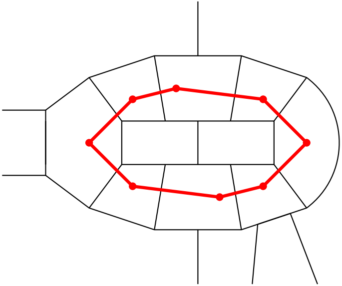

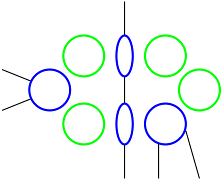

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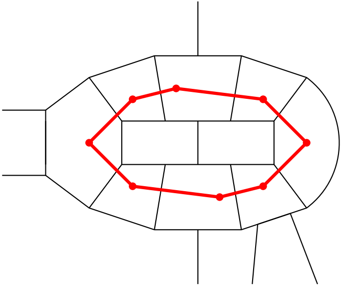

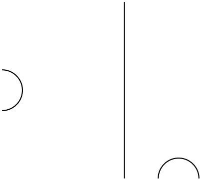

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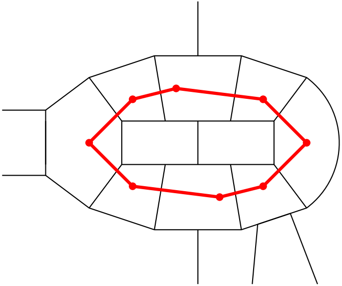

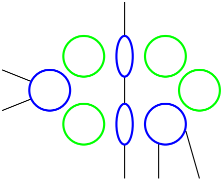

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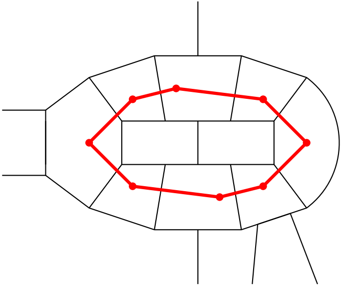

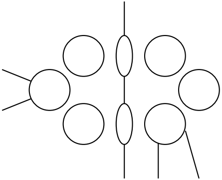

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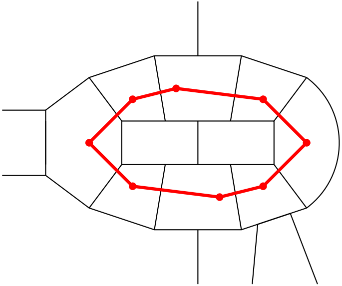

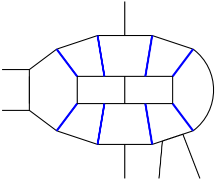

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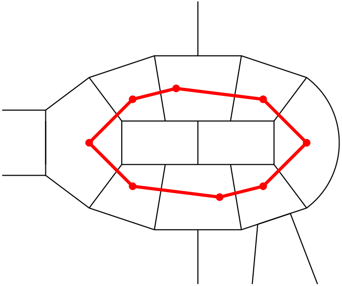
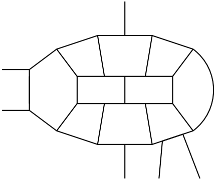
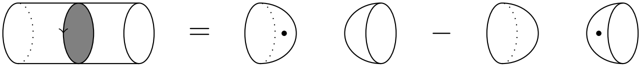




- (Global) Exactness $\leftrightsquigarrow$ foam of degree 0.
- Foam relations $\rightsquigarrow$ $i \circ p = \lambda \mathrm{id}_{w'}$.
- (Local) Exactness $\rightsquigarrow$ $\lambda \neq 0$.
- Enhanced red graphs to encode foams.

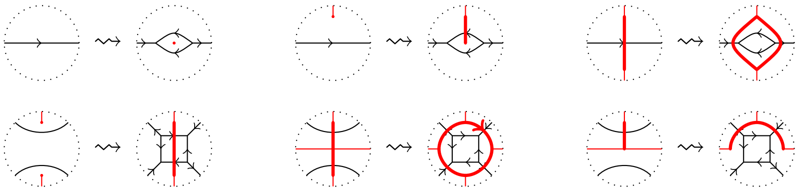


Lemma

If $\langle \overline{w} w \rangle$ is not monic of degree $l(\epsilon)$, then there exists an exact red graph for w.

Idea of the proof

Back-tracking the degree 0 contribution in the computation of the Kuperberg bracket.



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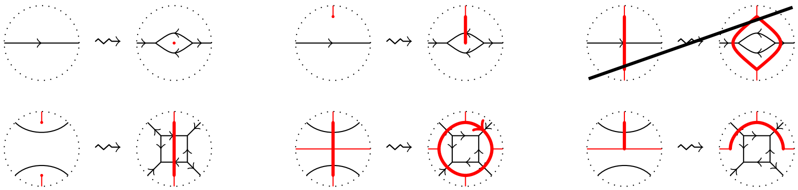

- Web colourings to give a new definition of TQFT ?
- Web colourings with a geometrical meaning.
- Commutation of idempotents.
- There exists other kinds of webs: $\mathfrak{so}(5)$, $\mathfrak{so}(2n)$.