

Sur l'homologie \mathfrak{sl}_3 des enchevêtrements ; algèbres de Khovanov – Kuperberg

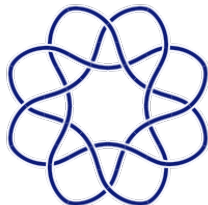
Louis-Hadrien Robert

Thèse dirigée par Christian Blanchet

- 1 Introduction
- 2 The Khovanov – Kuperberg algebras
- 3 Superficial webs
- 4 A Characterisation of indecomposable web-modules
- 5 Perspectives

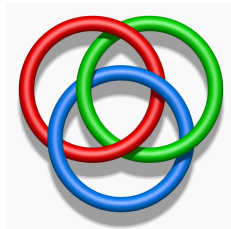
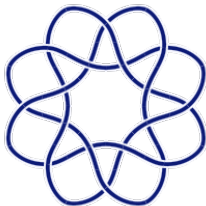
Definition

A *knot* is an embedding of \mathbb{S}^1 in \mathbb{S}^3 .



Definition

A *link* is an embedding of $\sqcup \mathbb{S}^1$ in \mathbb{S}^3 .



Theorem (Freyd – Hoste – Lickorish – Millett – Ocneanu – Yetter, Przytycki – Traczyk)

There exists a unique Laurent polynomial invariant $\langle \cdot \rangle$ satisfying the following relations:

$$q^3 \langle \text{web} \rangle - q^{-3} \langle \text{web} \rangle = (q^1 - q^{-1}) \langle \text{web} \rangle \langle \text{web} \rangle$$

$$\langle \text{web} \rangle = \langle \text{web} \rangle = q^2 + 1 + q^{-2}$$

Theorem (Freyd – Hoste – Lickorish – Millett – Ocneanu – Yetter, Przytycki – Traczyk)

There exists a unique Laurent polynomial invariant $\langle \cdot \rangle$ satisfying the following relations:

$$q^3 \left\langle \begin{array}{c} \diagup \quad \diagdown \\ \diagdown \quad \diagup \end{array} \right\rangle - q^{-3} \left\langle \begin{array}{c} \diagdown \quad \diagup \\ \diagup \quad \diagdown \end{array} \right\rangle = (q^1 - q^{-1}) \left\langle \begin{array}{c} \diagup \\ \diagdown \end{array} \right\rangle \left\langle \begin{array}{c} \diagdown \\ \diagup \end{array} \right\rangle$$

$$\left\langle \begin{array}{c} \circlearrowleft \end{array} \right\rangle = \left\langle \begin{array}{c} \circlearrowright \end{array} \right\rangle = q^2 + 1 + q^{-2}$$

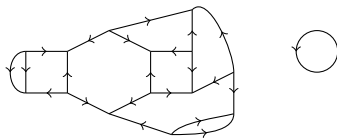
Alternative relations

$$\left\langle \begin{array}{c} \diagup \quad \diagdown \\ \diagdown \quad \diagup \end{array} \right\rangle = q^2 \left\langle \begin{array}{c} \diagup \\ \diagdown \end{array} \right\rangle \left\langle \begin{array}{c} \diagdown \\ \diagup \end{array} \right\rangle - q^3 \left\langle \begin{array}{c} \diagup \quad \diagdown \\ \diagdown \quad \diagup \\ \diagdown \quad \diagup \\ \diagup \quad \diagdown \end{array} \right\rangle,$$

$$\left\langle \begin{array}{c} \diagdown \quad \diagup \\ \diagup \quad \diagdown \end{array} \right\rangle = q^{-2} \left\langle \begin{array}{c} \diagup \\ \diagdown \end{array} \right\rangle \left\langle \begin{array}{c} \diagdown \\ \diagup \end{array} \right\rangle - q^{-3} \left\langle \begin{array}{c} \diagdown \quad \diagup \\ \diagup \quad \diagdown \\ \diagdown \quad \diagup \\ \diagup \quad \diagdown \end{array} \right\rangle.$$

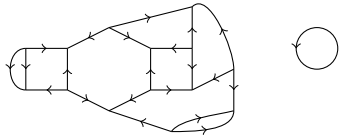
Definition (Kuperberg, 96)

A *web* is a plane oriented trivalent graph with possibly some vertexless circles.



Definition (Kuperberg, 96)

A *web* is a plane oriented trivalent graph with possibly some vertexless circles.



Kuperberg bracket

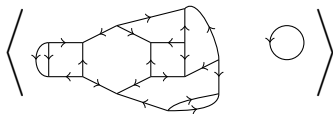
$$\langle \text{square with arrows} \rangle = \langle \text{right-pointing arrow} \rangle \langle \text{left-pointing arrow} \rangle + \langle \text{curved arrows} \rangle,$$

$$\langle \text{diamond with arrows} \rangle = [2] \cdot \langle \text{vertical arrow} \rangle,$$

$$\langle \text{circle with arrow} \sqcup w \rangle = [3] \cdot \langle w \rangle.$$

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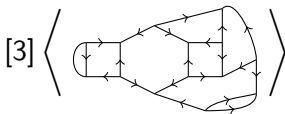
$$\langle \text{web with square} \rangle = \langle \text{web with arrow} \rangle \langle \text{web with arrow} \rangle + \langle \text{web with arrow} \rangle,$$

$$\langle \text{web with diamond} \rangle = [2] \cdot \langle \text{web with arrow} \rangle,$$

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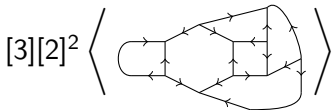
$$\langle \text{web with 4 vertices} \rangle = \langle \text{web with 3 vertices} \rangle \langle \text{web with 1 vertex} \rangle + \langle \text{web with 2 vertices} \rangle,$$

$$\langle \text{web with 1 vertex} \rangle = [2] \cdot \langle \text{web with 1 vertex} \rangle,$$

$$\langle \text{web with 1 vertex and a circle} \rangle = [3] \cdot \langle w \rangle.$$

Definition (Kuperberg, 96)

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Kuperberg bracket

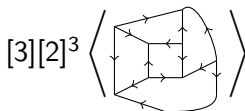
$$\langle \text{web with square and diamond} \rangle = \langle \text{web with square} \rangle \langle \text{web with diamond} \rangle + \langle \text{web with square and diamond} \rangle,$$

$$\langle \text{web with diamond} \rangle = [2] \cdot \langle \text{web with diamond} \rangle,$$

$$\langle \text{web with circle} \sqcup w \rangle = [3] \cdot \langle w \rangle.$$

Definition (Kuperberg, 96)

A *web* is a plane oriented trivalent graph with possibly some vertexless circles.



Kuperberg bracket

$$\langle \text{cube} \rangle = \langle \text{left} \rangle \langle \text{right} \rangle + \langle \text{other} \rangle,$$

$$\langle \text{diamond} \rangle = [2] \cdot \langle \text{two lines} \rangle,$$

$$\langle \text{circle} \sqcup w \rangle = [3] \cdot \langle w \rangle.$$

Definition (Kuperberg, 96)

A *web* is a plane oriented trivalent graph with possibly some vertexless circles.

$$\begin{aligned}
 & [3][2]^3 \langle \text{web}_1 \rangle \\
 & + [3][2]^3 \langle \text{web}_2 \rangle
 \end{aligned}$$

Kuperberg bracket

$$\langle \text{web}_1 \rangle = \langle \text{web}_2 \rangle + \langle \text{web}_3 \rangle,$$

$$\langle \text{web}_4 \rangle = [2] \cdot \langle \text{web}_5 \rangle,$$

$$\langle \text{web}_6 \sqcup w \rangle = [3] \cdot \langle w \rangle.$$

Definition (Kuperberg, 96)

A *web* is a plane oriented trivalent graph with possibly some vertexless circles.

$$2[3][2]^5 \langle \text{circle} \rangle$$

Kuperberg bracket

$$\langle \text{square} \rangle = \langle \text{right} \rangle \langle \text{left} \rangle + \langle \text{top} \rangle \langle \text{bottom} \rangle,$$

$$\langle \text{diamond} \rangle = [2] \cdot \langle \text{vertical} \rangle,$$

$$\langle \text{circle} \sqcup w \rangle = [3] \cdot \langle w \rangle.$$

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Kuperberg bracket

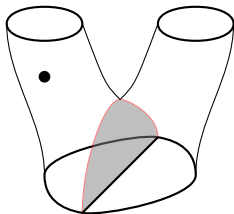
$$\langle \text{square with arrows} \rangle = \langle \text{trivalent vertex} \rangle + \langle \text{trivalent vertex with curved arrows} \rangle,$$

$$\langle \text{diamond with arrows} \rangle = [2] \cdot \langle \text{trivalent vertex} \rangle,$$

$$\langle \text{circle with arrow} \sqcup w \rangle = [3] \cdot \langle w \rangle.$$

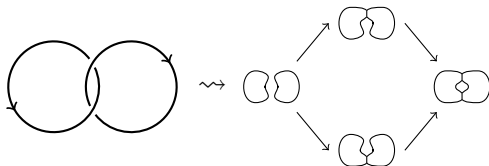
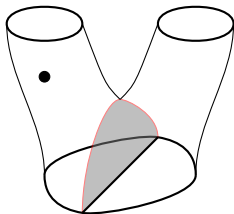
Theorem (Khovanov, 2003)

There exists a TQFT \mathcal{F} from the category of webs and foams to the category of graded \mathbb{Z} -modules which categorifies the Kuperberg bracket. This TQFT allows to define the \mathfrak{sl}_3 -homology which categorifies the \mathfrak{sl}_3 polynomial.



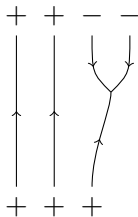
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Sequences of signs ϵ

Web-tangles w



Foams (with corners) f

Sequences of signs

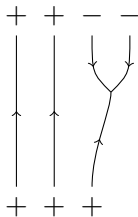
 $\epsilon \mapsto$

Algebras

Web-tangles

 $w \mapsto$

Bi-modules



Foams (with corners)

 $f \mapsto$

Bi-modules maps

First attempt:




$$\widetilde{K}^\epsilon \stackrel{\text{def}}{=} \bigoplus_{w_1, w_2} \mathcal{F}(\overline{w_1 w_2})$$

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$$\widetilde{K}^\epsilon \stackrel{\text{def}}{=} \bigoplus_{w_1, w_2} \mathcal{F}(\overline{w_1} w_2)$$

Definition (Kuperberg, 96)

An ϵ -web is *non-elliptic* if it

contains: NO , NO , and
NO .

Theorem (Kuperberg, 96)

For a given ϵ , there exist finitely many non-elliptic ϵ -webs.

First attempt:




$$\widetilde{K}^\epsilon \stackrel{\text{def}}{=} \bigoplus_{w_1, w_2} \mathcal{F}(\overline{w_1} w_2)$$

Definition (Mackaay – Pan – Tubbenhauer, R.)

$$K^\epsilon \stackrel{\text{def}}{=} \bigoplus_{\substack{w_1, w_2 \\ \text{non-elliptic}}} \mathcal{F}(\overline{w_1} w_2)$$

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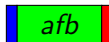
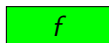
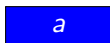
Definition (Mackaay – Pan – Tubbenhauer, R.)

$$\mathcal{F}(w) \stackrel{\text{def}}{=} \bigoplus_{\substack{w_1 \in NE(\epsilon_1) \\ w_2 \in NE(\epsilon_2)}} \mathcal{F}(\overline{w_1} w w_2) \quad \mathcal{F}(u) \stackrel{\text{def}}{=} \sum_{\substack{w_1 \in NE(\epsilon_1) \\ w_2 \in NE(\epsilon_2)}} \mathcal{F}((\overline{w_1} \times I) u (w_2 \times I))$$



Definition (Mackaay – Pan – Tubbenhauer, R.)

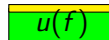
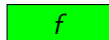
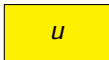
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Definition (Mackaay – Pan – Tubbenhauer, R.)

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Theorem (Mackaay – Pan – Tubbenhauer, R.)

This construction leads to a well-defined 2-functor from the 2-category of web-tangles to the 2-category of algebras.

Theorem (Mackaay – Pan – Tubbenhauer, R.)

From this we can derived an extension of the \mathfrak{sl}_3 -homology to tangles. It satisfies the following gluing relation at the level of complexes:

$$C(T_1 T_2) \simeq C(T_1) \otimes_{K_1^\epsilon} C(T_2).$$

Question

How are the projective indecomposable modules over K^ϵ ?

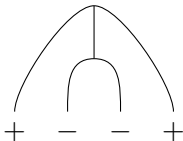
Question

How are the projective indecomposable modules over K^ϵ ?

Proposition

If w_1 and w_2 are two ϵ -webs, the graded dimension of $\text{hom}_{K^\epsilon}(\mathcal{F}(w_1), \mathcal{F}(w_2))$ is given by:

$$q^{l(\epsilon)} \cdot \langle \overline{w_1} w_2 \rangle .$$



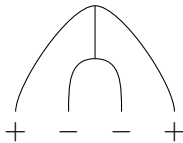
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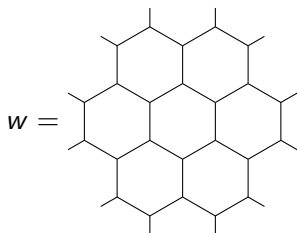
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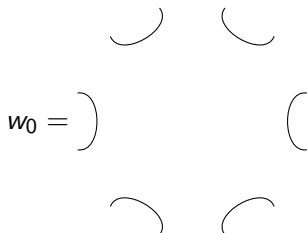
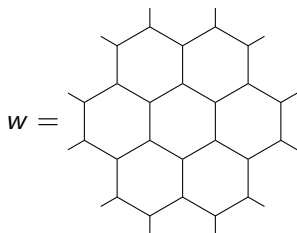
$$q^4 \cdot \left\langle \left(\begin{array}{c} \text{arch} \\ \text{loop} \\ \text{arch} \end{array} \right) \right\rangle = 1 + 3q^2 + 4q^4 + 3q^6 + q^8$$

Consequence

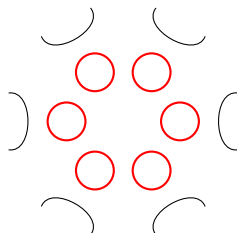
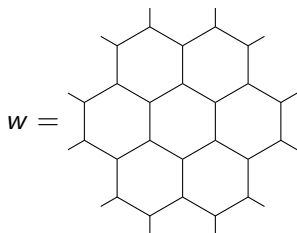
This web-module is indecomposable.



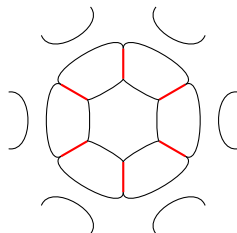
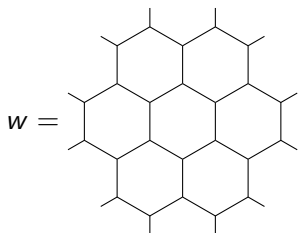
$$q^{12} \langle \bar{w}w \rangle = 2 + 80q^2 + 902q^4 + 4604q^6 + 13158q^8 + 23684q^{10} + 28612q^{12} + 23684q^{14} + 13158q^{16} + 4604q^{18} + 902q^{20} + 80q^{22} + 2q^{24}$$



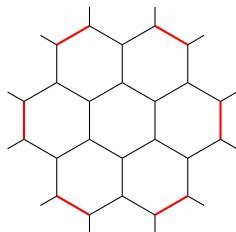
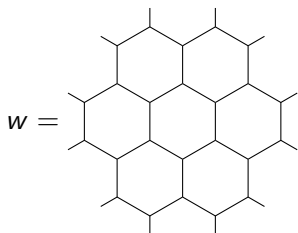
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Definition (R.)

An ϵ -web is *superficial* if it has no nested face.

Theorem (R.)

If w be a superficial and non-elliptic ϵ -web, then the K^ϵ -module $\mathcal{F}(w)$ is indecomposable. If w' is another superficial and non-elliptic ϵ -web different from w , then $\mathcal{F}(w)$ and $\mathcal{F}(w')$ are not isomorphic as K^ϵ -modules.

Around the boundary

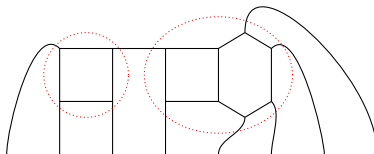


Around the boundary



Lemma

A semi-non-elliptic superficial ϵ -web is equal (in the skein module) to a sum of non-elliptic webs.



Proposition (last section)

Let w be an ϵ -web.

$\langle \overline{w}w \rangle$ is monic of degree $l(\epsilon) \Rightarrow \mathcal{F}(w)$ is indecomposable.

Theorem (R.)

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Theorem (R.)

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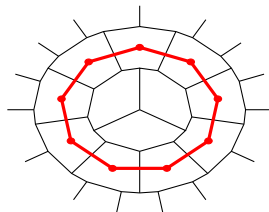
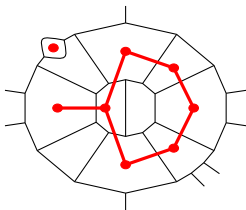
$\langle \bar{w}w \rangle$ is monic of degree $l(\epsilon) \Leftrightarrow \mathcal{F}(w)$ is indecomposable.

- Combinatorial data (red graphs) to idempotent foams,
- “Non-monicity” to red graphs.

Definition (R.)

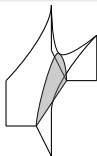
Let w be an ϵ -web, a red graph for G is an induced subgraph of the dual of w such that:

$$\begin{array}{c} \textcircled{f_2} \quad \textcircled{f_1} \\ \textcircled{f_3} \end{array} \Rightarrow \{f_1, f_2, f_3\} \not\subseteq V(G).$$



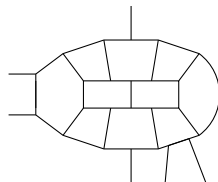
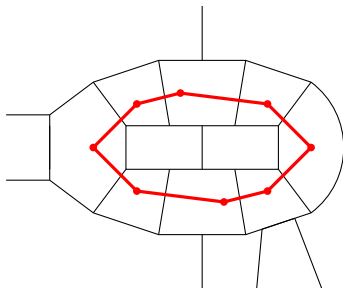
Proposition (R.)

From any exact red graph, one can construct a non-trivial idempotent.



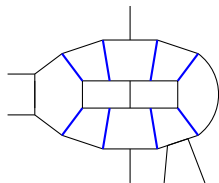
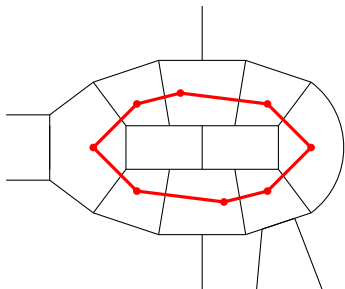
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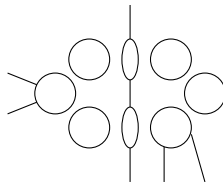
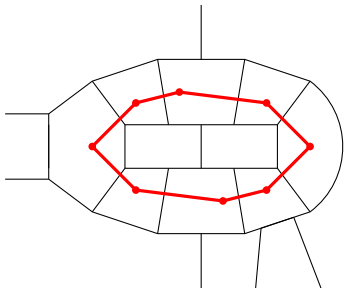
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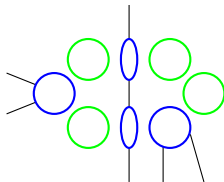
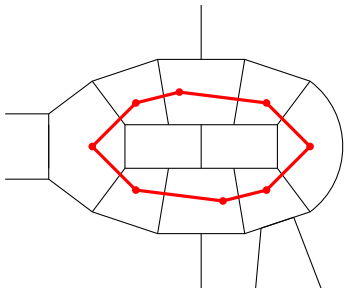
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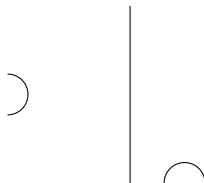
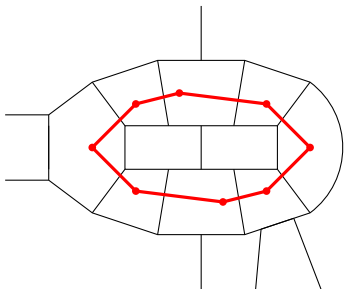
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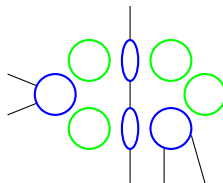
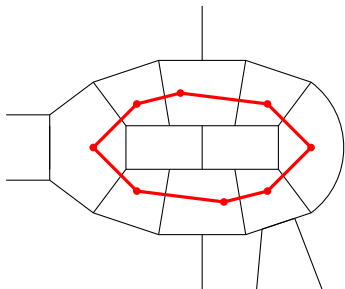
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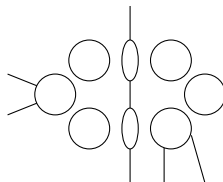
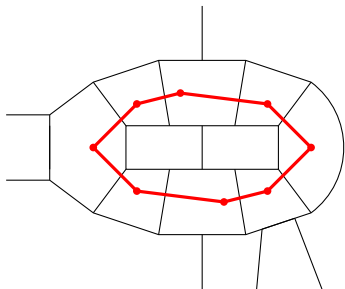
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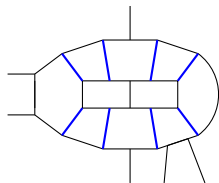
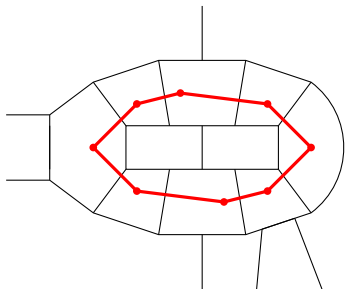
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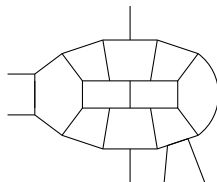
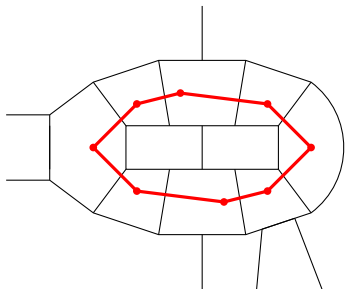
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- (Global) Exactness \Leftrightarrow foam of degree 0.
- Foam relations $\rightsquigarrow i \circ p = \lambda \text{id}_{W'}$.
- (Local) Exactness $\rightsquigarrow \lambda \neq 0$.
- Enhanced red graphs to encode foams.

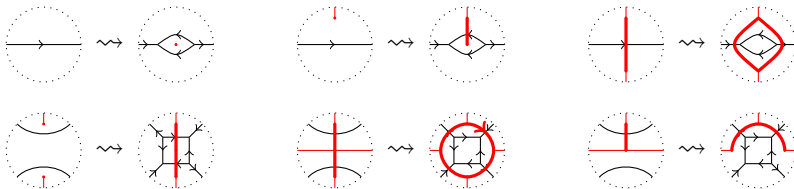


Lemma

If $\langle \overline{w}w \rangle$ is not monic of degree $l(\epsilon)$, then there exists an exact red graph for w .

Idea of the proof

Back-tracking the degree 0 contribution in the computation of the Kuperberg bracket.

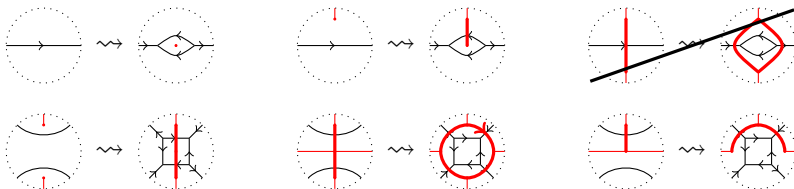


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Idea of the proof

Back-tracking the degree 0 contribution in the computation of the Kuperberg bracket.



- Web colourings to give a new definition of TQFT ?
- Web colourings with a geometrical meaning.
- Commutation of idempotents.
- There exists other kinds of webs: $\mathfrak{so}(5)$, $\mathfrak{so}(2n)$.